

Applications of Mathematics

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Applications of Mathematics, Vol. 40 (1995), No. 1, 55--59

Persistent URL: <http://dml.cz/dmlcz/134278>

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ASYMPTOTICALLY NORMAL CONFIDENCE INTERVALS
FOR A DETERMINANT IN A GENERALIZED
MULTIVARIATE GAUSS-MARKOFF MODEL

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(Received June 29, 1993)

Summary. By using three theorems (Oktaba and Kieloch [3]) and Theorem 2.2 (Srivastava and Khatri [4]) three results are given in formulas (2.1), (2.8) and (2.11). They present asymptotically normal confidence intervals for the determinant $|\sigma^2 \Sigma|$ in the MGM model $(U, XB, \sigma^2 \Sigma \otimes V)$, $\Sigma > 0$, scalar $\sigma^2 > 0$, with a matrix $V \geq 0$. A known $n \times p$ random matrix U has the expected value $E(U) = XB$, where the $n \times d$ matrix X is a known matrix of an experimental design, B is an unknown $d \times p$ matrix of parameters and $\sigma^2 \Sigma \otimes V$ is the covariance matrix of U , \otimes being the symbol of the Kronecker product of matrices. A particular case of Srivastava and Khatri's [4] theorem 2.2 was published by Anderson [1], p. 173, Th. 7.5.4, when $V = I$, $\sigma^2 = 1$, $X = \mathbf{1}$ and $B = \mu' = [\mu_1, \dots, \mu_p]$ is a row vector.

Keywords: generalized multivariate Gauss-Markoff model, singular covariance matrix, determinant, asymptotically normal confidence interval, product of independent chi-squares, multivariate central limit theorem, Wishart distribution, matrix of product sums for error, hypothesis and "total"

AMS classification: 62H10

1. SOME THEOREMS

Theorem 1.1 Multivariate central limit theorem (Anderson [1] pp. 76-77).

Let $\mathbf{Z}(n)$ be an m -component random vector and \mathbf{b} a fixed vector. Assume $p \lim_{n \rightarrow \infty} \mathbf{Z}(n) = \mathbf{b}$ i.e. $\mathbf{Z}(n)$ converges stochastically to \mathbf{b} . Let $\mathbf{a} = \sqrt{n}[f(\mathbf{Z}(n)) - \mathbf{b}] \rightarrow N(\mathbf{0}, \mathbf{T}^*)$, i.e. \mathbf{a} is asymptotically distributed according to $N(\mathbf{0}, \mathbf{T}^*)$. Let $w = f(\mathbf{z})$ be a function of a vector \mathbf{z} with the first and second derivatives existing in a neighborhood of $\mathbf{z} = \mathbf{b}$. Let $\left. \frac{\partial f(\mathbf{z})}{\partial z_i} \right|_{\mathbf{z}=\mathbf{b}}$ be the i -th component of $\Phi_{\mathbf{b}}$. Then the limiting distribution of $n^{\frac{1}{2}}[f(\mathbf{Z}(n)) - f(\mathbf{b})]$ is

$$(1.1) \quad N(\mathbf{0}, \Phi_{\mathbf{b}}' \mathbf{T}^* \Phi_{\mathbf{b}}).$$

Srivastava nad Khatri [4] present the following Theorem 1.2 without proof. We give the proof using the idea of Anderson [1] p. 173, who proved it a special case (cf. Summary).

Theorem 1.2. *If $\nu K \sim W_p(\nu, \Sigma)$, then*

$$(1.2) \quad \nu^{\frac{1}{2}} \left[\frac{|K|}{|\Sigma|} - 1 \right] \rightarrow N(0, 2p)$$

as $\nu \rightarrow \infty$, where $|K|$ denotes the determinat of the $p \times p$ matrix K , $2p$ being the variance.

Proof. By virtue of $|\nu K| = \nu^p |K|$ and Oktaba [2], (2.1) we obtain

$$(1.3) \quad \begin{aligned} |W| &= \frac{|K|}{|\Sigma|} = \frac{\nu^p |K|}{|\Sigma| \nu^p} = \frac{|\nu K|}{|\Sigma| \nu^p} = \\ &= \frac{\chi_{\nu}^2 \cdot \chi_{\nu-1}^2 \cdots \chi_{\nu-p+1}^2}{\nu^p} = \frac{\chi_{\nu}^2}{\nu} \cdot \frac{\chi_{\nu-1}^2}{\nu} \cdots \frac{\chi_{\nu-p+1}^2}{\nu} \\ &= V_1(\nu) \cdot V_2(\nu) \cdots V_i(\nu) \cdots V_p(\nu), \end{aligned}$$

where

$$(1.4) \quad \nu \cdot V_i(\nu) = \chi_{\nu-p+i}^2, \quad i = 1, \dots, p.$$

are independent.

Note that the standardized variate

$$(1.5) \quad u_i = \frac{\chi_{\nu-p+i}^2 - E\chi_{\nu-p+i}^2}{\sqrt{\text{Var}(\chi_{\nu-p+i}^2)}} = \frac{\nu V_i(\nu) - (\nu - p + i)}{\sqrt{2(\nu - p + i)}} = \sqrt{\nu} \frac{V_i(\nu) - 1 + \frac{p-i}{\nu}}{\sqrt{2} \sqrt{1 - \frac{p-i}{\nu}}}$$

is asymptotically normal $N(0, 1)$. Thus

$$\sqrt{\nu}[V_i(\nu) - 1] \rightarrow N(0, 2).$$

If we replace z_i by v_i and $\mathbf{Z}(n)$ by $\mathbf{V}(\nu) = [V_1(\nu), \dots, V_p(\nu)]$ in the multivariate central limit theorem 1.1 then by virtue of $b' = [1, \dots, 1]$ we have

$$\begin{aligned} |W| &= \frac{|K|}{|\Sigma|} = f(\mathbf{V}(\nu)) = V_1(\nu) \cdots V_p(\nu), T^* = 2I_p, \\ \left. \frac{\partial f}{\partial V_i} \right|_{\mathbf{V}=\mathbf{b}} &= 1 \quad \text{and} \quad \Phi_b' T^* \Phi_b = 2p. \end{aligned}$$

Hence we get (1.2). □

2. ASYMPTOTICALLY NORMAL CONFIDENCE INTERVALS FOR $|\sigma^2\Sigma|$
IN THE MGM MODEL WITH A SINGULAR COVARIANCE MATRIX

We apply Theorem 1.2 to the MGM model (cf. Summary) with a singular covariance matrix and to three theorems (Oktaba and Kieloch [3]). In this way we get the following three theorems.

Theorem 2.1. *In the MGM model $(U, XB, \sigma^2\Sigma \otimes v)$ (cf. Summary), the $(1 - \alpha)$ percent asymptotically normal confidence interval for the determinant $|\sigma^2\Sigma|$ is of the form*

$$(2.1) \quad \frac{|S_e|}{\nu_e^p + u_\alpha \sqrt{2p\nu_e^{p-\frac{1}{2}}}} < |\sigma^2\Sigma| < \frac{|S_e|}{\nu_e^p - u_\alpha \sqrt{2p\nu_e^{p-\frac{1}{2}}}}$$

where

$$(2.2) \quad S_e = U' C_1 U,$$

$$(2.3) \quad \nu_e = r(V : X) - r(X),$$

$$(2.4) \quad C_1 = T^- - T^- X (X' T^- X)^- X' T,$$

$$(2.5) \quad T = V + X M X', \quad M = M' \quad \text{is such that} \quad R(X) \subset R(T),$$

u_α is obtained from the standard normal distribution $N(0, 1)$ and given in

$$(2.6) \quad p(-u_\alpha < u < u_\alpha) = 1 - \alpha$$

where $1 - \alpha$ is the confidence.

Proof. Using (1.2), $\nu K = S_e$ and replacing Σ by $\sigma^2\Sigma$ we obtain directly

$$(2.7) \quad P \left[-u_\alpha < \frac{\nu_e^{\frac{1}{2}}}{\sqrt{2p}} \left(\frac{|S_e|}{\nu_e^p |\sigma^2\Sigma|} - 1 \right) < u_\alpha \right] = 1 - \alpha,$$

where $S_e^- W_p(\nu_e, \sigma^2\Sigma)$ (Oktaba and Kieloch [3]). By solving the inequality in (2.7) we get (2.1). □

Theorem 2.2. *In the MGM model $(U, XB, \sigma^2\Sigma \otimes V)$, the $(1 - \alpha)$ percent asymptotically normal confidence interval for the determinant $|\sigma^2\Sigma|$, provided the hypothesis $H_0; L^*B = \psi$ is true, is of the form*

$$(2.8) \quad \frac{|S_H|}{\nu_H^p + u_\alpha \sqrt{2p\nu_H^{p-\frac{1}{2}}}} < |\sigma^2\Sigma| < \frac{|S_H|}{\nu_H^p - u_\alpha \sqrt{2p\nu_H^{p-\frac{1}{2}}}},$$

where (Oktaba nad Kieloch [3])

$$(2.9) \quad S_H = (L^* \hat{B} - \psi)' L^- (L^* \hat{B} - \psi), \quad \hat{B} = (X'T^-X)^- X'T^-U,$$

$$(2.10) \quad \nu_H = r(L), L = L^* C_4 L^*, C_4 = (X'T^-X)^- - M,$$

M and T being defined in (2.5).

Proof. We consider $\nu K = S_H$, replace Σ and ν by $\sigma^2 \Sigma$ and ν_H , respectively. We know (Oktaba nad Kieloch [3]) that

$$S_H^- W_p[r(L), \sigma^2 \Sigma].$$

Applying Theorem 1.2 we get (2.8). □

Theorem 2.3. In the MGM model $(1 - \alpha)$ percent asymptotically normal confidence interval for the determinant $|\sigma^2 \Sigma|$ can be presented as

$$(2.11) \quad \frac{|S_y|}{\nu_y^p + u_\alpha \sqrt{2p} \nu_y^{p-\frac{1}{2}}} < |\sigma^2 \Sigma| < \frac{|S_y|}{\nu_y^p - u_\alpha \sqrt{2p} \nu_y^{p-\frac{1}{2}}},$$

where

$$(2.12) \quad S_y = S_e + S_H, \quad \nu_y = \nu_e + \nu_H,$$

with S_e and S_H in (2.2) and (2.9), respectively and ν_e, ν_H in (2.3) and (2.10). u_α is defined in (2.6).

Proof. By virtue of $S_e^- W_p(\nu_e, \sigma^2 \Sigma), S_H^- W_p(\nu_H, \sigma^2 \Sigma)$ (Oktaba and Kieloch [3]) and additivity of the Wishart distribution we state that

$$S_e + S_H = S_y^- W_p(\nu_e + \nu_H, \sigma^2 \Sigma).$$

Applying Theorem 1.2 we get (2.11) analogously as in the proofs of Theorems 2.1 and 2.2. □

Particular case (Anderson [1]). In the standard multivariate model $(U, 1 \cdot \mu', \Sigma \otimes I_n)$ (cf. Oktaba [2]) the asymptotic normal confidence interval for the determinant $|\Sigma|$ can be obtained if we put: $X = \mathbf{1}$, $S_e = U' C_1 U$ in formula (2.1), where $C_1 = I_n - \frac{1}{n} \mathbf{1} \mathbf{1}'$, $\sigma^2 = 1$, $\nu_e = n - 1$. The symbol $\mathbf{1}$ is a column vector with ones, I_n denotes the $n \times n$ identity, μ' is a row vector.

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