Giovanni Cimatti; Eduard Feireisl

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A SIMPLE MODEL OF THERMOELECTRIC OSCILLATIONS

GIOVANNI CIMATTI, Pisa, EDUARD FEIREISL, Praha

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Summary. A system of ordinary differential equations modelling an electric circuit with a thermistor is considered. Qualitative properties of solution are studied, in particular, the existence and nonexistence of time-periodic solutions (the Hopf bifurcation).

Keywords: thermistor, time-periodic solution, Hopf bifurcation

AMS classification: 78A97, 34C25

1. INTRODUCTION

In the electric circuit of Figure 1 R_T is a special resistor called "thermistor" whose conductance σ is a function of the temperature u, R is an ordinary resistor and C a capacitor. We make the simplifying assumption that the temperature and the electric potential φ across the thermistor depend on time only. Let u_a be the room's temperature. By Newton's law of cooling we have, taking into account the Joule heating,

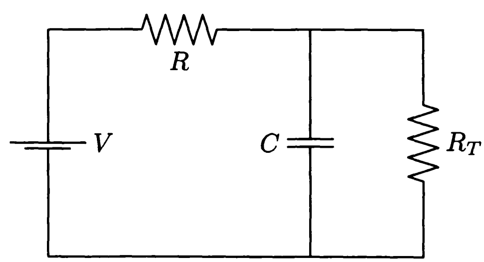

Fig. 1

$$H\frac{\mathrm{d}u}{\mathrm{d}t} = -k(u-u_a) + \sigma(u)\varphi^2 \tag{1.1}$$

where H and k are the incremental heat capacity and the incremental dissipation constant. Using Kirchhoff's principles we obtain

$$V = \varphi + \varrho\left(\varphi\sigma(u) + c\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) \tag{1.2}$$

where ϱ is the resistance of R, c the capacitance of C and V a fixed applied difference of potential. This circuit is interesting for two reasons: firstly, multiple states of equilibrium (typically three) can exist, secondly, low-frequency thermoelectric oscillations are observed experimentally. This effect gives rise to various practical applications, see e.g. [2]. In Section 2 we find conditions under which time periodic solutions cannot exist, in particular this will be the case when $\frac{d\sigma}{du} < 0$. Section 3 is devoted to the study of the local stability of stationary points and to the proof of the existence of periodic solutions using Hopf's bifurcation. Sections 4 and 5 deal with a more specific conductivity and with the corresponding bifurcation diagram. The physically crucial problem of the stability of periodic solutions is discussed in Section 5. Finally a more realistic, but more difficult, model is presented in the last part.

Let $s(y) \in C^2(\mathbb{R}^1)$ and suppose

$$s(y) \geqslant s_m > 0 \qquad \text{for all } y \geqslant 0. \tag{1.3}$$

We assume $\sigma(u)$ to be an empirically given conductivity of the form

$$\sigma(u) = \sigma_0 s\left(\frac{u - u_a}{u_a}\right). \tag{1.4}$$

Use will be made of the following dimensionless quantities

$$y = (u - u_a)/u_a, \quad x = \varphi/V, \quad c_1 = k\varrho c/H, \quad \tau = t/c\varrho,$$
$$\eta = (\sigma_0 \varrho c V^2)/H u_a, \quad \alpha = \varrho\sigma_0, \quad \beta = c_1/\eta.$$

System (1.1), (1.2) can be rewritten in dimensionless form as follows:

$$\dot{x} = F(x, y, \alpha) := 1 - x - \alpha x s(y) \tag{1.5}$$

$$\left(\dot{x} = \frac{dx}{d\tau}\right)$$

$$\dot{y} = G(x, y, \beta, \eta) := \eta(x^2 s(y) - \beta y). \tag{1.6}$$

By their physical meaning α, β and η must all be positive. Concerning the global behaviour of (1.5), (1.6) we have

Lemma 1.1. *There exists a constant $C = C(\beta, \eta)$ such that*

$$M = \left\{(x, y)\,;\, 0 < x < 1,\, 0 < y,\, \frac{x^2}{2} + \frac{\alpha y}{\eta} < C\right\}$$

forms an invariant absorbing set for the system (1.5), (1.6), *i.e. any solution* $(x(t), y(t))$ *of the problem reaches and remains inside* M *after a finite time.*

Proof. By direct inspection we find from (1.5), (1.6) that for any $(x(0), y(0)) \in \mathbb{R}^2$ there is $t_0 = t_0(|x(0)|, y(0)|)$ such that

$$0 < x(t) < 1 \quad 0 < y(t) \quad \text{for all} \quad t \geqslant t_0. \tag{1.7}$$

Now we can multiply (1.5) by $x(t)$ and (1.6) by $\frac{\alpha}{\eta}$ and, setting $z(t) = \frac{x^2}{2} + \frac{\alpha}{\eta}$, we obtain

$$\dot{z}(t) + \beta\eta z(t) = \frac{2x + (\beta\eta - 2)x^2}{2}. \tag{1.8}$$

Seeing that the right-hand side of (1.8) is bounded for $t \geqslant t_0$ and $\beta\eta > 0$, the standard decay estimates for first order O.D.E. yield

$$z(t) \leqslant C(\beta, \eta) \quad \text{for all} \quad t \geqslant t_1 \tag{1.9}$$

with $t_1 = t_1(|x(0)|, |y(0)|)$ which, along with (1.7), completes the proof. □

2. Nonexistence of periodic solutions

Lemma 2.1. *If*

$$\frac{\mathrm{d}s}{\mathrm{d}y} < \frac{1}{\eta}\big(1 + \alpha s(y) + \eta\beta\big) \quad \textit{for all} \quad y > 0, \tag{2.1}$$

then the only periodic solutions of system (1.5), (1.6) *are the stationary ones.*

Proof. By Lemma 1.1 all possible periodic solutions are contained in M. A direct computation shows that

$$F_x + G_y = -1 - \alpha s(y) + \eta x^2 \frac{\mathrm{d}s}{\mathrm{d}y} - \eta\beta.$$

Since $0 < x < 1$ on M, condition (2.1) implies $F_x + G_y < 0$. Hence, by the Bendixon criterion [1] no periodic solution can exist in M and therefore in the whole phase plane. □

Note that, by virtue of (1.3), the inequality (2.1) is satisfied in the physically relevant case $\mathrm{d}s/\mathrm{d}y < \beta$ for all $y \geqslant 0$.

3. Local stability

The stationary points of system (1.5), (1.6) are given by

$$X(x,y,\alpha) := 1 - x - \alpha x s(y) = 0, \tag{3.1}$$

$$Y(x,y,\beta) := x^2 s(y) - \beta y = 0. \tag{3.2}$$

From (3.1) we have

$$x = \delta(y,\alpha) := \frac{1}{1+\alpha s(y)}. \tag{3.3}$$

Substituting into (3.2) we find

$$\beta = H(y,\alpha) := \frac{s(y)}{y\big(1+\alpha s(y)\big)^2}. \tag{3.4}$$

If α is fixed, the plot of (3.4) gives the bifurcation diagram for the solutions of (3.1), (3.2). Let

$$(x,y) = \left(\frac{1}{1+\alpha s(y)}, y\right) \tag{3.5}$$

be any stationary point of (1.5), (1.6) and $A(y,\alpha)$ the 2×2 matrix of the corresponding linearized system. Note that $X_x Y_\beta = \big(1+\alpha s(y)\big)y \neq 0$ if (x,y) is a stationary point. Calculating $\operatorname{Det} A$ and using (3.3), (3.4) we have

$$\eta H_y(y,\alpha) = -\left[\frac{\operatorname{Det} A}{X_x Y_\beta}\right]_{x=\delta(y,\alpha)\beta=H(y,\alpha)}. \tag{3.6}$$

In view of (3.6) we have that the curve of the plane y,α in which $\operatorname{Det} A(y,\alpha)$ vanishes coincides with the locus of points in which $H_y(y,\alpha)$ is zero. The equation of this curve is given by

$$\alpha = f(y) := \frac{1}{s(y)} \frac{y(\mathrm{d}s/\mathrm{d}y) - s(y)}{y(\mathrm{d}s/\mathrm{d}y) + s(y)}. \tag{3.7}$$

The following lemma gives a partial information concerning the stability of the stationary points.

Lemma 3.1. *Let α, β and y satisfy* (3.4) *and assume $\alpha < f(y)$, $ys'(y) + s(y) > 0$. Then the corresponding stationary point* (3.5) *is a saddle point.*

Proof. Simply note that if $\alpha < f(y)$ we have $\operatorname{Det} A(y,\alpha) < 0$ by (3.6). □

R e m a r k 3.1. The degree of the mapping (F, G) evaluated with respect to ∂M is one. Therefore the existence of a saddle point implies the existence of at least one other stationary point. If, in particular, $\operatorname{Det} A \neq 0$, there must be at least two different critical points either attracting or repelling.

To make further progress in the study of the local stability we need the trace, $\operatorname{Tr} A$, of A which is given by

$$\operatorname{Tr} A = \frac{\eta\big(y(\mathrm{d}s/\mathrm{d}y) - s(y)\big) - y\big(1+\alpha s(y)\big)^3}{y\big(1+\alpha s(y)\big)^2}. \tag{3.8}$$

Using the well-known Hopf's bifurcation theorem (see [4]), we can prove

Lemma 3.2. *Suppose $\bar\alpha, \bar\beta$ and $\bar y$ satisfy* (3.4) *and*

$$\bar\eta = \frac{\bar y\big(1+\bar\alpha s(\bar y)\big)^3}{\bar y(\mathrm{d}s/\mathrm{d}y) - s(y)}. \tag{3.9}$$

Assume

$$\bar\alpha > f(\bar y) \tag{3.10}$$

and

$$\bar y(\mathrm{d}s/\mathrm{d}y) - s(\bar y) > 0. \tag{3.11}$$

Then there exist two continuous functions $\omega, \eta\colon (0,\xi_0) \to \mathbb{R}^+$, $\omega(\xi) \to \bar\omega$, $\eta(\xi) \to \bar\eta$ as $\xi \to 0_+$, and a branch of non-constant periodic solutions of system (1.5), (1.6) *(with $\alpha = \bar\alpha$, $\beta = \bar\beta$, $\eta = \eta(\xi)$) with period $T_\xi = 2\pi/\omega(\xi)$ for any $\xi \in (0,\xi_0)$.*

P r o o f. First of all, (3.11) implies $\bar\eta > 0$ and, therefore, the result is physically meaningful. The surface of the y, α, η space on which $\operatorname{Tr} A$ vanishes is (3.9). Condition (3.10) guarantees that $\operatorname{Det} A > 0$ by (3.6). Hence, $A(\bar y, \bar\alpha)$ has two purely imaginary complex conjugate eigenvalues $\lambda = \pm i\bar\omega$. By (3.11) we have

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \operatorname{Re} \lambda(\eta) > 0 \tag{3.12}$$

when η is given by (3.9). All hypotheses of Hopf's bifurcation theorem are satisfied and the lemma holds. □

To prove the stability of these periodic solutions is a more difficult task. A result in this direction is presented in Section 5.

4. A THREE SOLUTIONS CASE

The bifurcation diagram (3.7) may have many shapes corresponding to the possible choices of $s(y)$. We examine in detail the following case which makes it possible to predict the three solutions situation which is typical of the circuit. Assume

$$\frac{d^2 s}{dy^2} \geqslant m > 0 \quad \text{for all} \quad y \geqslant 0, \tag{4.1}$$

$$\frac{ds}{dy}(0) > 0. \tag{4.2}$$

The function $f(y)$ defined in (3.9) vanishes when $y(ds/dy) - s(y) = 0$. By (1.3) and (4.1) there exists a unique $y_0 > 0$ such that $f(y_0) = 0$. Suppose df/dy to vanish only when $y = y_M$ with $y_M > y_0$ (see Figure 2). This implies $f(y) > 0$ if $y > y_0$ and $f(y) \to 0$ when $y \to \infty$. Define $\alpha_M = f(y_M)$. We distinguish the following two cases:

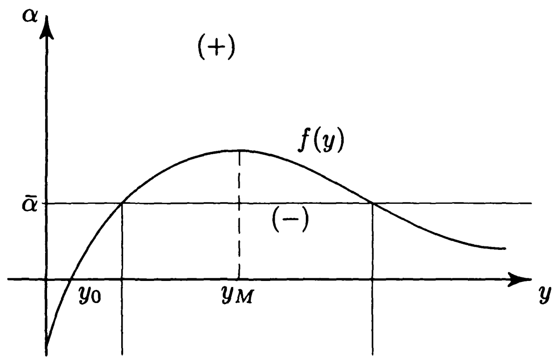

Fig. 2

Case (A). Let

$$0 < \alpha < \alpha_M. \tag{4.3}$$

The bifurcation diagram (3.4) has in this case the characteristic S shape of Figure 3. Moreover, $H(y, \alpha) \to \infty$ when $y \to 0+$ and $H(y, \alpha) \to 0$ when $y \to \infty$. Referring to Figure 2 we see that Det A is positive in region (+) and negative in region (−) by

(3.6). Let y_1, y_2 ($y_1 < y_2$) be the local minimum and the local maximum of $H(y, \alpha)$

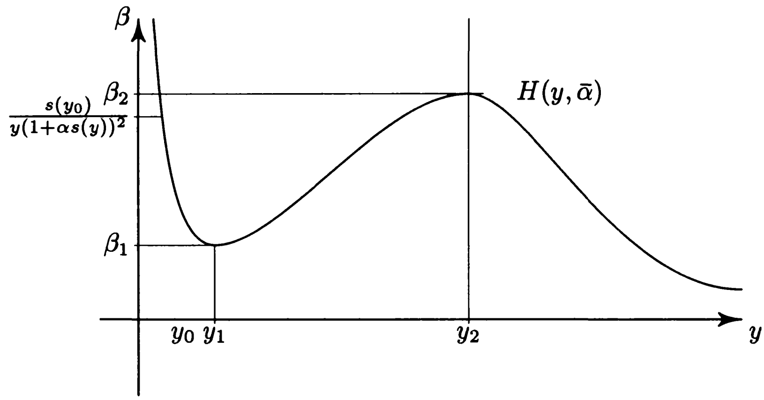

Fig. 3

and put $\beta_1 = H(y_1, \alpha)$, $\beta_2 = H(y_2, \alpha)$. Clearly $\beta_1 < \beta_2$ and $y_0 < y_1 < y_2$. To concentrate on a specific case, let us assume

$$\beta_2 > \frac{s(y_0)}{y_0\big(1 + \alpha s(y_0)\big)^2}. \tag{4.4}$$

We consider the following four sub-cases.
(A_I) If

$$\beta > \beta_2 \tag{4.5}$$

then there is only one fixed point $(\bar{x}_1, \bar{y}_1)$ with

$$\bar{y}_1 < y_0. \tag{4.6}$$

This implies $\operatorname{Det} A(\bar{y}_1, \alpha) > 0$ and $\operatorname{Tr} A(\bar{y}_1, \alpha) < 0$. Thus $(\bar{x}_1, \bar{y}_1)$ is asymptotically stable for every η.
(A_{II}) If

$$\beta_2 > \beta > \frac{s(y_0)}{y_0\big(1 + \alpha s(y_0)\big)^2} \tag{4.7}$$

then there are three fixed points $(\bar{x}_1, \bar{y}_1)$, $(\bar{x}_2, \bar{y}_2)$ and $(\bar{x}_3, \bar{y}_3)$ such that

$$\bar{y}_1 < y_0 < \bar{y}_2 < \bar{y}_3. \tag{4.8}$$

Since $\operatorname{Det} A(\bar{y}_1, \alpha) > 0$ and $\operatorname{Tr} A(\bar{y}_1, \alpha) < 0$, $(\bar{x}_1, \bar{y}_1)$ is asymptotically stable for all η. Moreover, $\operatorname{Det} A(\bar{y}_2, \alpha) < 0$ thus $(\bar{x}_2, \bar{y}_2)$ is a saddle point in accordance with Lemma 3.1. Finally, $\operatorname{Det} A(\bar{y}_3, \alpha) > 0$ and the sign of $\operatorname{Tr} A(\bar{y}_3, \alpha)$ (and therefore the stability) depends only on the numerical value of η.

(A_{III}) If

$$\frac{s(y_0)}{y_0\big(1+\alpha s(y_0)\big)^2} > \beta > \beta_1 \tag{4.9}$$

then there are again three stationary points, but now

$$y_0 < \bar{y}_1 < \bar{y}_2 < \bar{y}_3. \tag{4.10}$$

The stationary point $(\bar{x}_2, \bar{y}_2)$ is again a saddle point. Since $\operatorname{Det} A(\bar{x}_1, \bar{y}_1) > 0$ and $\operatorname{Det} A(\bar{x}_3, \bar{y}_3) > 0$, the stationary points $(\bar{x}_1, \bar{y}_1)$, $(\bar{x}_3, \bar{y}_3)$ can be asymptotically stable or not depending on the value of η.

(A_{IV}) When

$$\beta_1 < \beta, \tag{4.11}$$

then there is only one fixed point $(\bar{x}, \bar{y})$ with $\operatorname{Det} A(\bar{y}, \alpha) > 0$. The sign of $\operatorname{Tr} A$ and therefore the stability depends only on η.

Case(B). If

$$\alpha_M < \alpha \tag{4.12}$$

then there is one and only one stationary point for all $\beta > 0$ and $\operatorname{Det} A(\bar{y}, \alpha) > 0$. The asymptotic stability depends on the value of η.

Remark 4.1. It is interesting to note that the existence of three nondegenerate stationary points such that one of them is a saddle, while the other one is a repellor, necessarily brings about the existence of at least one closed trajectory, either a homoclinic orbit or a periodic one.

5. Stable periodic solutions

As a consequence of the Poincaré-Bendixon theorem [1] we have

Lemma 5.1. *If the invariant absorbing set M guaranteed by Lemma 2.1 contains exactly one unstable stationary point $(\bar{x}, \bar{y})$, this point is surrounded by (at least one) stable periodic orbit.*

To apply the above Lemma we note that a sufficient condition for the existence of a unique stationary point $(\bar{x}, \bar{y})$ is given by

$$\frac{\mathrm{d}s}{\mathrm{d}y} > 0 \quad \text{for all} \quad y > 0, \tag{5.1}$$

$$\alpha s(0) \geqslant 1. \tag{5.2}$$

Assume further that there exists an interval (y_1, y_2) such that

$$y\frac{\mathrm{d}s}{\mathrm{d}y} - s(y) > 0 \quad \text{for all} \quad y \in (y_1, y_2). \tag{5.3}$$

From (3.4) it follows that

$$\beta = \frac{s(y)}{y\big(1 + \alpha s(y)\big)^2}. \tag{5.4}$$

Hence, there exists an interval (β_1, β_2) such that for any $\beta \in (\beta_1, \beta_2)$ we have $\bar{y} \in (y_1, y_2)$. Thus, recalling the expression for $\mathrm{Tr}\, A$, i.e. (3.8), there exists $\eta_0 = \eta_0(\alpha, \beta, s)$ having the following properties: if $\eta < \eta_0$ the unique stationary point is locally stable, if $\eta > \eta_0$ the stationary point is unstable and therefore by Lemma 5.1, M contains a stable limit cycle.

Under the assumptions of Lemma 3.2 it is also possible to prove the existence of periodic solutions using the following version of Hopf's bifurcation.

Theorem 5.1. *Assume the system*

$$\dot{x} = X(x, y, \eta) \quad \dot{y} = Y(x, y, \eta) \tag{5.5}$$

has a fixed point $(\bar{x}, \bar{y})$ for all values of the parameter η. Furthermore, suppose the eigenvalues of the linearized system $\lambda_1(\eta)$, $\lambda_2(\eta)$ are purely imaginary complex conjugate when $\eta = \eta_0$. If the real part of the eigenvalues satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\mathrm{Re}[\lambda(\eta)] > 0 \quad \textit{when} \quad \eta = \eta_0 \tag{5.6}$$

and $(\bar{x}, \bar{y})$ is asymptotically stable when $\eta = \eta_0$, then

a) *there exists $\eta_1 < \eta_0$ such that when $\eta \in (\eta_1, \eta_0)$, $(\bar{x}, \bar{y})$ is a stable focus;*

b) *there exists $\eta_2 > \eta_0$ such that when $\eta \in (\eta_0, \eta_2)$, $(\bar{x}, \bar{y})$ is an unstable focus surrounded by a stable limit cycle.*

A proof of Theorem 5.1 is to Chaffee and can be found in [3], Theorems (3.1), (3.B4). Let the hypotheses of Lemma 3.2 hold. To apply Theorem 5.1 it remains to check if the fixed point $(\bar{x}, \bar{y})$ is, in certain cases, asymptotically stable when η is given by (3.9). We can use (see [3]) the following algorithm:

a) translate the fixed point to the origin,

b) find a non-singular matrix M such that

$$M^{-1}AM = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

where A is the matrix of the linearized system and $\lambda = \pm i\omega$,

c) transform the system by the change of variable $\mathbf{x} = M\mathbf{y}$, $\mathbf{x} = (x, y)$, $\mathbf{y} = (u, v)$ into

$$\dot{u} = -\omega v + f(u, v) \quad \dot{v} = \omega u + g(u, v)$$

with $f(0,0) = g(0,0) = 0$, $Df(0,0) = Dg(0,0) = 0$,

d) calculate the index

$$\begin{aligned} a = {} & f_{uuu} + f_{uuv} + g_{uuv} + g_{vvv} \\ & + (1/\omega)[f_{uu}(f_{uu} + f_{vv}) + g_{uu}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}] \end{aligned}$$

where all partial derivatives are computed at $(u, v) = (0, 0)$. If $a < 0$, $(\bar{x}, \bar{y})$ is asymptotically stable and the conclusion of Theorem 5.1 follows. The calculations involved in computing a in the present case are rather tedious but straightforward. A computer program using the language for symbolic manipulation MAXIMA has been written to obtain an explicit expression of a in terms of α, β. This formula is however too massive to be reported here. We simply use it in the following

Example. Let $s(y) = \exp(y)$, $\alpha = 1$ and $\beta = 15$. We find $(\bar{x}, \bar{y}) = (0.23305, 1.19142)$ and $\eta_0 = 149.471$. Using our MAXIMA code we find $a = -528\,634.05897$. The periodic solution given by Lemma 3.2 is therefore stable.

6. An open problem

To get a more accurate description we can treat the thermistor as a three-dimensional body, represented by a bounded domain Ω of $\mathbb{R}^3$. The metallic, and therefore equipotential electrodes of the thermistor are assumed to be two disjoint surfaces S and $\bar{S}$ such that $\partial\Omega = S \cup \bar{S}$. If $\mathbf{J}$ and $\mathbf{E}$ are the current density and the electric field in Ω we have, by Ohm's law,

$$\mathbf{J} = \sigma(u)\mathbf{E}. \tag{6.1}$$

If $\varphi(x,t), x \in \Omega$ is the electric potential inside Ω, we can write, assuming quasistatic conditions,

$$\mathbf{E} = -\nabla\varphi. \tag{6.2}$$

By the law of conservation of charge we have

$$\nabla \cdot \mathbf{J} = 0. \tag{6.3}$$

Let $i(t)$ be the current crossing R. We have

$$i(t) = c\dot{\Phi}(t) + \int_S \sigma(u)\frac{\mathrm{d}\varphi}{\mathrm{d}n}\,\mathrm{d}S \tag{6.4}$$

where $\mathbf{n}$ is the unit vector normal to S and

$$\varphi = \Phi(t) \quad \text{on } S, \quad \varphi = 0 \quad \text{on } \bar{S}. \tag{6.5}$$

Hence

$$V = \varrho i(t) + \Phi(t). \tag{6.6}$$

Assuming the usual heat equation to be valid and inserting (6.1) and (6.2) into (6.3), we arrive at the following problem:

To find a period T and three T-periodic functions $\varphi(x,t)$, $u(x,t)$ and $\Phi(t)$ such that

$$\nabla \cdot (\sigma(u)\nabla\psi) = 0 \tag{6.7}$$

$$\varphi = \Phi(t) \quad \text{on } S, \quad \varphi = 0 \quad \text{on } \bar{S}. \tag{6.8}$$

$$V = \varrho c\dot{\Phi}(t) + \varrho\int_S \sigma(u)\frac{\mathrm{d}\varphi}{\mathrm{d}n}\,\mathrm{d}S + \Phi(t) \tag{6.9}$$

$$u_t = a_1\Delta u + a_2\sigma(u)|\nabla\varphi|^2 \tag{6.10}$$

$$u = 0 \quad x \in \partial\Omega \tag{6.11}$$

where $a_2 = 1/dc_v$ (d mass density, c_v specific heat), whereas $a_1 = \kappa/a_2$ (κ thermal conductivity) is the diffusivity. However, the application of the theory of Hopf's bifurcation to problem (6.7)–(6.11) seems to present serious difficulties.

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Authors' address: *Giovanni Cimatti*, Department of Mathematics, Via Buonarroti, 3, 56100 Pisa, Italy; *Eduard Feireisl*, Institute of Mathematics AV ČR, Žitná 25, Praha 1, Czech Republic.

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