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Mean square approximation by optimal periodic interpolation


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MEAN SQUARE APPROXIMATION
BY OPTIMAL PERIODIC INTERPOLATION

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Summary. Following the research of Babuška and Práger, the author studies the approximation power of periodic interpolation in the mean square norm thus extending his own former results.

Keywords: mean square approximation, periodic Hilbert space, exponential interpolants, optimal periodic interpolation

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0. Introduction

Babuška [1] introduced the concept of the periodic Hilbert space for studying optimal quadrature formulas, Práger [8] continued these investigations and related these problems to the minimum norm interpolation (optimal periodic interpolation) in periodic Hilbert spaces. These ideas have been further developed in a number of papers [2, 3, 4, 5, 6, 7]. In this paper we will study the approximation power of optimal periodic interpolation in the mean square norm an thereby extended results of [4].
1. Periodic Hilbert Spaces

We denote by $C$ the algebra of continuous complex-valued functions on $\mathbb{R}$ with period $2\pi$. $C$ is a Banach algebra with respect to the uniform norm $\|u\|_\infty = \sup_{x \in \mathbb{R}} |u(x)|$.

Next we denote by $L^2$ the Hilbert space of square integrable periodic functions with an inner product

$$ (u, v) = \frac{1}{2\pi} \int_0^{2\pi} u(x)v(x)^* \, dx. $$

The exponentials are given by $e_k(x) = e^{ikx}$, $k \in \mathbb{Z}$. The finite Fourier transform of $u \in L^2$ is given by

$$ (u, e_k) = \frac{1}{2\pi} \int_0^{2\pi} u(x)e_k(x)^* \, dx, \quad k \in \mathbb{Z}. $$

Its inversion is the Fourier series of $u$, $\sum_{k=-\infty}^{\infty} (u, e_k)e_k$, which converges in the $L^2$-norm to $u$.

Next we introduce the Wiener algebra $A$ of functions $u \in L^2$ having absolutely convergent Fourier series. $A$ is a Banach algebra with respect to the norm

$$ \|u\|_a = \sum_{k=-\infty}^{\infty} |(u, e_k)|. $$

Clearly, $A$ is a subalgebra of $C$. We have the estimates

$$ \|u\| \leq \|u\|_\infty \leq \|u\|_a \quad (x \in \mathbb{R}, u \in A) $$

and the inclusions

$$ A \subset C \subset L^2. $$

To introduce the periodic Hilbert space $H_d$ we need a biinfinite symmetric positive $l_1$-sequence $d = (d_k)$, i.e., we have

$$ d_{-k} = d_k > 0 \quad (k \in \mathbb{Z}), \quad \sum_{k=-\infty}^{\infty} d_k < \infty. $$

It is also convenient to assume the monotonicity condition

$$ d_k > d_{k+1} \quad (k \geq 0). $$
We define by $\mathcal{H}_d$ the linear space of all functions $u \in \mathcal{L}^2$ satisfying
\[
\sum_{k=-\infty}^{\infty} \frac{1}{d_k} |(u, e_k)|^2 < \infty.
\]
The inner product of $\mathcal{H}_d$ is defined by
\[
(u, v)_d = \sum_{k=-\infty}^{\infty} \frac{1}{d_k} (u, e_k)(v, e_k)^*.
\]
We list some properties of $\mathcal{H}_d$. Each $\mathcal{H}_d$ is a subspace of the Wiener algebra and therefore also of $\mathcal{C}$, the imbeddings being continuous:
\[
\mathcal{H}_d \subset \mathcal{A} \subset \mathcal{C}.
\]
We denote by $\tau$ the algebra of trigonometric polynomials. For $m \geq 0$ and $u \in \mathcal{L}^2$ we denote by
\[
S_m(u) = \sum_{k=-m}^{m} (u, e_k)e_k
\]
the Fourier partial sum polynomial of $u$ of order $m$. It is immediate that
\[
\tau \subset \mathcal{H}_d
\]
and
\[
\lim_{m \to \infty} \|u - S_m(u)\|_d = 0, \quad u \in \mathcal{H}_d.
\]
Moreover, $\mathcal{H}_d$ as well as its norm $\|u(\cdot - a)\|_d = \|u\|_d$ are translation invariant ($u \in \mathcal{H}_d$, $a \in \mathbb{R}$). The characterizing sequence $d = (d_k)$ defines the kernel function
\[
g(x) = \sum_{k=-\infty}^{\infty} d_k e_k(x)
\]
which is an element of $\mathcal{H}_d \cdot \mathcal{H}_d$ is a reproducing kernel Hilbert space with the kernel function $g(y, x) = K(y, x)$. We have
\[
u(x) = (u, g(\cdot - x))_d(u \in \mathcal{H}_d, \quad x \in \mathbb{R}),
\]
which implies the estimate
\[
|u(x)| \leq \|u\|_d \cdot \|g\|_d, \quad \|g\|_d = \sqrt{g(0)}.
\]
We conclude this section by presenting two examples.
Example 1. The sequence \( d \) is given by

\[
d_0 = 1, \quad d_k = k^{-2r} \quad (k \neq 0).
\]

Here \( r \) is a positive integer. The periodic Hilbert space is the periodic Sobolev space \( \mathcal{W}^r \). The kernel function is given by

\[
g(x) = 1 + (-1)^r B_{2r}(x)
\]

where \( B_q(x) \) denotes the Bernoulli function (polynomial) of degree \( q \).

Example 2. In this case the sequence \( d \) is given by

\[
d_k = e^{-|k|b} \quad (k \in \mathbb{Z})
\]

where \( b \) is a positive real number. The periodic Hilbert space consists of restrictions of functions to the real axis which are holomorphic in the strip \( |\text{Im}(z)| < b \). The kernel function is in this case the well known Poisson kernel

\[
g(x) = \frac{\sin h(b)}{\cos h(b) - \cos(x)}.
\]

2. Optimal Periodic Interpolation

We first treat the problem of interpolation with shifts of the kernel function \( g \) which are related to the knots \( x_j, 0 \leq j < n \), satisfying

\[
0 \leq x_0 < x_1 < \ldots < x_{n-1} < 2\pi.
\]

It follows from the theory of trigonometric interpolation that for any data \( y_j, 0 \leq j < n \), there is a trigonometric polynomial \( w \in \tau \) satisfying the interpolation conditions

\[
w(x_j) = y_j, \quad 0 \leq j < n.
\]

If \( n = 2m + 1 \), then \( w \) can be made unique by assuming that \( w \) has order \( m \), i.e., \( w \in \tau_m \). If \( n = 2m \) then \( w \in \tau_m \) exists and can be made unique by deleting \( \cos(mx) \) or \( \sin(mx) \) depending on the position of knots.

The space of interpolating functions related to \( g \) is given by

\[
\langle g(\cdot - x_0), g(\cdot - x_1), \ldots, g(\cdot - x_{n-1}) \rangle.
\]
**Proposition 2.1.** The space $\langle g(-x_0), g(-x_1), \ldots, g(-x_{n-1}) \rangle$ has dimension $n$.

**Proof.** Recall that

$$u(x) = (u, g(-x))_d \quad (u \in \mathcal{H}_d, \ x \in \mathbb{R}).$$

Let $w_k \in \tau_m$ be a trigonometric polynomial satisfying

$$w_k(x_j) = \delta_{j,k}, \quad 0 \leq j, k < n.$$

Thus

$$(w_k, g(-x_j))_d = \delta_{j,k}, \quad 0 \leq j, k < n,$$

which yields the linear independence of the shifted functions $g(-x_0), g(-x_1), \ldots, g(-x_{n-1})$. \hfill \square

Let $Q_n$ denote the unique orthogonal projector having the $n$-dimensional space $\langle g(-x_0), g(-x_1), \ldots, g(-x_{n-1}) \rangle$ as its range:

$$\mathcal{R}(Q_n) = \langle g(-x_0), g(-x_1), \ldots, g(-x_{n-1}) \rangle.$$

For $u \in \mathcal{H}_d$ the function $Q_n(u)$ is the unique best approximation of $u$ in $\langle g(-x_0), g(-x_1), \ldots, g(-x_{n-1}) \rangle$.

**Proposition 2.2.** Given $u \in \mathcal{H}_d$, the best approximant $Q_n(u)$ of $u$ in $\langle g(-x_0), g(-x_1), \ldots, g(-x_{n-1}) \rangle$ is also the unique interpolant of $u$ at $x_j$, $0 \leq j < n$, with the minimum norm in $\mathcal{H}_d$:

(i) $Q_n(u)(x_j) = u(x_j)$, $0 \leq j < n$;

(ii) $\|Q_n(u)\|_d = \min\{\|v\|_d : v(x_j) = u(x_j), \ 0 \leq j < n\}$.

**Proof.** The characterization of the best approximation in Hilbert spaces yields the equation

$$(u - Q_n(u), g(-x_j))_d = 0 \quad (0 \leq j < n).$$

Taking into account

$$u(x) = (u, g(-x))_d$$

we obtain

$$Q_n(u)(x_j) = u(x_j), \quad 0 \leq j < n.$$

This proves (i).
As a consequence there exists a Lagrange basis

\[ h_0, \ldots, h_{n-1} \]

of

\[ \langle g(\cdot - x_0), g(\cdot - x_1), \ldots, g(\cdot - x_{n-1}) \rangle : \]

\[ \langle g(\cdot - x_0), \ldots, g(x_{n-1}) \rangle = \langle h_0, \ldots, h_{n-1} \rangle, \]

\[ h_k(x_j) = \delta_{j,k}, \quad 0 \leq j, k < n. \]

Using the Lagrange basis of \( \langle g(\cdot - x_0), \ldots, g(\cdot - x_{n-1}) \rangle \) it follows from \( u(x_j) = v(x_j), \)
\[ 0 \leq j < n, \] that \( Q_n(u) = Q_n(v). \) Taking into account

\[ (v - Q_n(v), Q_n(v))_d = 0 \]

we can conclude

\[ (v, v)_d = (Q_n(v), Q_n(v))_d + (v - Q_n(v), v - Q_n(v))_d \geq (Q_n(u), Q_n(u))_d \]

with equality if and only if \( v = Q_n(u). \) This proves (ii). \( \square \)

3. OPTIMAL PERIODIC INTERPOLATION ON UNIFORM MESHES

In this section we treat the much more explicit case of uniformly distributed knots:

\[ x_j = \frac{2\pi}{n} j, \quad 0 \leq j < n. \]

It is easily established that the space of interpolants

\[ \langle g(\cdot - x_0), \ldots, g(\cdot - x_{n-1}) \rangle \]

is translation invariant with respect to the mesh size \( \frac{2\pi}{n} : \)

\[ w \in \langle g(\cdot - x_0), \ldots, g(\cdot - x_{n-1}) \rangle \Rightarrow w\left( \cdot - \frac{2\pi}{n} \right) \in \langle g(\cdot - x_0), \ldots, g(\cdot - x_{n-1}) \rangle. \]

Thus the Lagrange basis \( h_0, \ldots, h_{n-1} \) of \( \langle g(\cdot - x_0), \ldots, g(\cdot - x_{n-1}) \rangle \) is obtained by translation:

\[ h_k = h_0(\cdot - x_k), \quad 0 \leq k < n. \]

Thus the interpolation projector \( Q_n \) possesses the Lagrange representation

\[ Q_n(u)(x) = \sum_{j=0}^{n-1} u(x_j)h_0(x - x_k). \]
We will now use the discrete Fourier transform to obtain an alternative representation of the optimal periodic interpolant $Q_n(u)$. Recall that the discrete Fourier transform of $u$ is defined by

$$c_{k,n}(u) = \frac{1}{n} \sum_{j=0}^{n-1} u(x_j) e_k(x_j)^*, \quad 0 \leq k < n.$$  

The inverse Fourier transform is given by

$$u(x_j) = \sum_{k=0}^{n-1} c_{k,n}(u) e_k(x_j), \quad 0 \leq j < n.$$ 

The fundamental property of the discrete Fourier transform is the convolution theorem. Let $w$ be the discrete convolution of $u$ and $v$:

$$w(x_k) = \sum_{j=0}^{n-1} u(x_j)v(x_k - x_j) = \sum_{j=0}^{n-1} u(x_j)v(x_{k-j}).$$

Then the convolution theorem yields

$$n \cdot c_{k,n}(u)c_{k,n}(v) = c_{k,n}(w), \quad 0 \leq k < n.$$  

Next we are looking for a trigonometric polynomial

$$a \in \{e_0, \ldots, e_{n-1}\}$$

such that the optimal periodic interpolant $Q_n(u)$ is given by

$$Q_n(u)(x) = \sum_{j=0}^{n-1} a(x_j)g(x - x_j).$$

The interpolation conditions yield the convolution equation

$$u(x_k) = \sum_{j=0}^{n-1} a(x_j)g(x_k - x_j).$$

Now the convolution theorem implies

$$c_{k,n}(u) = n \cdot c_{k,n}(a)c_{k,n}(g).$$
Since \( g \in A \) aliasing is applicable and we get

\[
c_{k,n}(g) = \sum_{r=-\infty}^{\infty} d_{k+r}\eta > 0.
\]

Thus we have

\[
c_{k,n}(a) = \frac{c_{k,n}(u)}{n \cdot c_{k,n}(g)}, \quad 0 \leq k < n.
\]

Using the inverse discrete Fourier transform we obtain

\[
a(x_j) = \sum_{k=0}^{n-1} \frac{c_{k,n}(u)}{n \cdot c_{k,n}(g)} e_k(x_j), \quad 0 \leq j < n.
\]

It is obvious that the trigonometric polynomial is given by

\[
a(x) = \sum_{k=0}^{n-1} \frac{c_{k,n}(u)}{n \cdot c_{k,n}(g)} e_k(x).
\]

**Proposition 3.1.** Let \( u \in \mathcal{H}_d \). Then the optimal periodic interpolant \( Q_n(u) \) with respect to the uniform knots \( x_j = \frac{2\pi}{n} j, 0 \leq j < n \), is given by

\[
Q_n(u)(x) = \sum_{j=0}^{n-1} \left( \sum_{k=0}^{n-1} \frac{c_{k,n}(u)}{n \cdot c_{k,n}(g)} e_k(x_j) \right) g(x - x_j).
\]

Note that for \( u = h_0 \) we get

\[
c_{k,n}(h_0) = \frac{1}{n}, \quad 0 \leq k < n.
\]

This implies the representation formula for the fundamental Lagrange function \( h_0 \) of optimal periodic interpolation

\[
h_0(x) = \sum_{j=0}^{n-1} \left( \sum_{k=0}^{n-1} \frac{1}{n^2 \cdot c_{k,n}(g)} e_k(x_j) \right) g(x - x_j).
\]

**Remark 3.1.** It follows from the proof of Proposition 3.1 that this result remains valid for any \( g \in C \) if and only if

\[
c_{k,n}(g) \neq 0, \quad 0 \leq k < n.
\]

See Locher [7], Cheney [2]. If these conditions are violated for some \( k \), a modified approach is possible. See Delvos [3].
4. EXPONENTIAL INTERPOLANTS

We apply the representation formula of Proposition 3.1 to the exponentials \( e_k \). In this connection it is appropriate to list some properties of the projector \( Q_n \) of optimal periodic interpolation. Since the kernel \( g \) is real valued we have

\[
Q_n(f^*) = Q_n(f)^*.
\]

As a special case we have

\[
Q_n(e^{-k}) = Q_n(e^k)^* , \quad k \in \mathbb{Z}.
\]

Moreover, for the sequel it is convenient to assume

\[
n = 2m + 1.
\]

**Proposition 4.1.** The exponential interpolants are given by

\[
Q_n(e_k)(x) = \frac{1}{c_{k,n}(g)} \sum_{r=-\infty}^{\infty} d_{k+r,n} e_{k+r,n}(x), \quad 0 \leq k < n.
\]

**Proof.** Recall that for arbitrary \( u \) we have

\[
Q_n(u)(x) = \sum_{j=0}^{n-1} a(x_j) g(x - x_j)
\]

with

\[
a(x_j) = \sum_{l=0}^{n-1} \frac{c_{l,n}(u)}{n \cdot c_{l,n}(g)} e_l(x_j), \quad 0 \leq j < n.
\]

Replacing \( u \) by \( e_k \) the orthogonality properties of the discrete Fourier transform yield

\[
c_{l,n}(e_k) = 0, \quad l - k \notin n\mathbb{Z},
\]

\[
c_{l,n}(e_k) = 1, \quad l - k \in n\mathbb{Z}.
\]

Thus we get

\[
a(x_j) = \frac{1}{n \cdot c_{k,n}(g)} e_k(x_j), \quad 0 \leq j < n.
\]

Now we can conclude

\[
Q_n(e_k)(x) = \frac{1}{c_{k,n}(g)} \cdot \frac{1}{n} \sum_{j=0}^{n-1} g(x - x_j) e_k(x_j) = \frac{c_{-k,n}(g(x - \cdot))}{c_{k,n}(g)}.
\]
Recall the formula of aliasing for $w \in \mathcal{A}$:

$$c_{k,n}(w) = \sum_{r=-\infty}^{\infty} (w, e_{k+rn}).$$

Since

$$g(x-t) = \sum_{s=-\infty}^{\infty} d_s e_s(x) e_s(-t) = \sum_{s=-\infty}^{\infty} d_s e_s(x) e_s(t),$$

we get

$$c_{-k,n}(g(x-\cdot)) = \sum_{r=-\infty}^{\infty} d_{k-rn} e_{k-rn}(x) = \sum_{r=-\infty}^{\infty} d_{k+rn} e_{k+rn}(x),$$

which completes the proof in view of

$$Q_n(e_k)(x) = \frac{c_{-k,n}(g(x-\cdot))}{c_{k,n}(g)}.$$

\[\square\]

**Proposition 4.2.** Let $n = 2m + 1$. Then $Q_n(u)$ possesses the discrete Fourier representation

$$Q_n(u) = \sum_{k=-m}^{m} c_{k,n}(u) Q_n(e_k).$$

**Proof.** The trigonometric polynomial

$$T_m(u) = \sum_{k=-m}^{m} c_{k,n}(u) e_k$$

satisfies the interpolation conditions

$$v(x_j) = u(x_j), \quad 0 \leq j < n.$$ 

Since $Q_n(T_m(u)) = Q_n(u)$ the linearity of $Q_n$ completes the proof. \[\square\]

**Proposition 4.3.** The exponential interpolants are orthogonal:

(i) $(Q_n(e_k), Q_n(e_l))_d = 0$, $0 < |k - l| < n$.

(ii) $(Q_n(e_k), Q_n(e_l)) = 0$, $0 < |k - l| < n$.

Moreover, the relations
(iii) \((Q_n(e_k), Q_n(e_k))_d = \frac{1}{c_{k,n}(g)^2} \sum_{r=-\infty}^{\infty} d_{k+r} \cdot n, 0 \leq k < n,\)

(iv) \((Q_n(e_k), Q_n(e_k)) = \frac{1}{c_{k,n}(g)^2} \sum_{r=-\infty}^{\infty} (d_{k+r}^2), 0 \leq k < n\)

hold.

Proof. The relations of Proposition 4.3 follow from Fourier expansions of the exponential interpolants and the definition of the inner product of the periodic Hilbert space.

\[ \square \]

5. CONVERGENCE OF THE EXPONENTIAL INTERPOLANTS

We start with investigating the approximation order of the exponential interpolant in the norm of \(H_d\). For this purpose we introduce the quantities

\[ D_{r,n} = c_{r,n}(g) - d_r = \sum_{l \neq 0} d_{r+ln}, \quad 0 \leq r \leq m, \quad n = 2m + 1. \]

Recall that

\[ d_0 = d_k > 0 \quad (k \in \mathbb{Z}), \quad d_k > d_{k+1} \quad (k \geq 0). \]

Then we have

\[ D_{r,n} \leq \sum_{l=1}^{\infty} d_{lm} =: D_n, \quad 0 \leq r \leq m, \quad n = 2m + 1. \]

It is obvious that

\[ D_n > D_{n+1}, \quad \lim_{n \to \infty} D_n = 0. \]

In many cases we have

\[ D_n \leq \alpha \cdot d_m \]

where \(\alpha\) is a constant independent of \(n\). In particular this is true for our examples.

Example 1.

\[ D_n = \frac{1}{m^{2r}} \sum_{s=1}^{\infty} s^{-2r} = d_m \sum_{s=1}^{\infty} s^{-2r}. \]

Example 2.

\[ D_n = \sum_{s=1}^{\infty} e^{-sm} \leq \frac{e^{-mb}}{1-e^{-b}} = d_m \frac{1}{1-e^{-b}}. \]
**Proposition 5.1.** The asymptotic relation

\[ \|e_k - Q_n(e_k)\|_d = \mathcal{O}(\sqrt{D_n}) \quad (n \to \infty) \]

holds.

**Proof.** We assume without loss of generality that \(0 \leq k \leq m, \ n = 2m + 1\). Recall that

\[ Q_n(e_k) = \frac{1}{c_{k,n}(g)} \sum_{r=-\infty}^{\infty} d_{k+r,n} e_{k+r,n}(x). \]

Then we obtain

\[
\begin{align*}
(\|e_k - Q_n(e_k)\|_d)^2 &= \frac{1}{d_k} \left| 1 - \frac{d_k}{c_{k,n}(g)} \right|^2 + \frac{1}{c_{k,n}(g)^2} \sum_{r \neq 0} d_{k+r,n} \\
&= \frac{1}{d_k} \left| \frac{D_{k,n}}{c_{k,n}(g)} \right|^2 + \frac{1}{c_{k,n}(g)^2} D_{k,n} \\
&\leq \frac{D_n^2}{d_k^3} + \frac{D_n}{d_k^2} = \mathcal{O}\left(\frac{D_n^2}{d_k^3}\right) = \mathcal{O}(D_n)
\end{align*}
\]

as \(n \to \infty\). \(\square\)

**Proposition 5.2.** The estimate

\[ \|e_k - Q_n(e_k)\| \leq \sqrt{2} \frac{D_n}{d_k} \]

holds for \(|k| \leq m, \ n = 2m + 1\). In particular we have

\[ \|e_k - Q_n(e_k)\| = \mathcal{O}(D_n)(n \to \infty). \]

**Proof.** As in the proof of Proposition 5.1 we have for \(0 \leq k \leq m\) and \(n = 2m+1\)

\[
\begin{align*}
\|e_k - Q_n(e_k)\|^2 &= \left| 1 - \frac{d_k}{c_{k,n}(g)} \right|^2 + \frac{1}{c_{k,n}(g)^2} \sum_{r \neq 0} (d_{k+r,n})^2 \\
&= \left| \frac{D_{k,n}}{c_{k,n}(g)} \right|^2 + \frac{1}{c_{k,n}(g)^2} (D_{k,n})^2 \\
&\leq \frac{(D_n)^2}{d_k^2} + \frac{1}{d_k^2} (D_n)^2 = 2 \cdot \frac{D_n^2}{d_k^2}.
\end{align*}
\]

\(\square\)
Using the discrete Fourier representation of the optimal periodic interpolant \( Q_n(u) \) we extend Proposition 5.2 to trigonometric polynomials.

**Proposition 5.3.** Let \( u \in \tau_m \) be a trigonometric polynomial of order \( m \). Then the estimate

\[
\|u - Q_n(u)\| \leq \sqrt{2} D_n \|u\|_{d^2}
\]

holds with \( n = 2m + 1 \) and \((d^2)_k = (d_k)^2\).

**Proof.** It follows from Proposition 4.1 that

\[
(e_k - Q_n(e_k), e_1 - Q_n(e_1)) = 0, \quad 0 < |k - 1| < n.
\]

Taking into account Proposition 4.2 and Proposition 5.2 we can conclude

\[
\|u - Q_n(u)\|^2 = \sum_{k=-m}^{m} (u, e_k)[e_k - Q_n(e_k)]^2
\]

\[
= \sum_{k=-m}^{m} |(u, e_k)|^2 \|e_k - Q_n(e_k)\|^2
\]

\[
\leq \left[ \sum_{k=-m}^{m} \frac{1}{d_k^2} |(u, e_k)|^2 \right] \cdot 2 \cdot D_n^2,
\]

i.e., we have shown

\[
\|u - Q_n(u)\|^2 \leq 2 \cdot D_n^2 \|u\|_{d^2}.
\]

\[\square\]

6. Convergence in Periodic Hilbert Spaces

We start with a qualitative result. Recall first that

\[
S_m(u) = \sum_{k=-m}^{m} (u, e_k)e_k
\]

satisfies

\[
\lim_{m \to \infty} \|u - S_m(u)\|_d = 0, \quad u \in \mathcal{H}_d.
\]

**Proposition 6.1.** Let \( u \in \mathcal{H}_d \). Then

\[
\lim_{n \to \infty} \|u - Q_n(u)\|_d = 0.
\]
Proof. Given \( \epsilon > 0 \) there exists \( r \in \mathbb{N} \) such that

\[
\|u - S_r(u)\|_d < \epsilon.
\]

It follows from Proposition 5.3, that there exists a \( q \in \mathbb{N} \) depending only on \( r \) such that

\[
\|S_r(u) - Q_n(S_r(u))\|_d < \epsilon, \quad n \geq q.
\]

Since \( Q_n \) is an orthogonal projector on \( \mathcal{H}_d \) we can conclude

\[
\|u - Q_n(u)\|_d \leq \|u - S_r(u)\|_d + \|S_r(u) - Q_n(S_r(u))\|_d + \|Q_n(u - S_r(u))\|_d \\
\leq 2\|u - S_r(u)\|_d + \|S_r(u) - Q_n(S_r(u))\|_d.
\]

i.e., we have

\[
\|u - Q_n(u)\|_d \leq 3\epsilon, \quad n \geq q.
\]

This completes the proof of Proposition 6.1. \( \square \)

Proposition 6.2. Let \( u \in \mathcal{H}_d \) and \( n = 2m + 1 \). Then

\[
\|Q_n(u - S_m(u))\| \leq D_n\|u - S_m(u)\|_d^2.
\]

Proof. Put

\[
v = u - S_m(u).
\]

Then we have

\[
(v, e_k) = 0, \quad |k| \leq m.
\]

Since

\[
Q_n(v) = \sum_{k=-m}^{m} c_{k,n}(v)Q_n(e_k)
\]

it follows from Proposition 4.3 that

\[
\|Q_n(v)\|^2 = \sum_{k=-m}^{m} |c_{k,n}(v)|^2 \|Q_n(e_k)\|^2.
\]

Recall that

\[
(Q_n(e_k), Q_n(e_k)) = \frac{1}{c_{k,n}(g)^2} \sum_{r=-\infty}^{\infty} (d_{k+rn})^2, \quad 0 \leq k < n,
\]

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which yields

$$\|Q_n(e_k)\|^2 \leq 1.$$  

This shows

$$\|Q_n(v)\|^2 \leq \sum_{k=-m}^{m} |c_{k,n}(v)|^2.$$  

Due to

$$(v,e_k) = 0, \quad |k| \leq m$$  

we obtain

$$\sum_{k=-m}^{m} |c_{k,n}(v)|^2 = \sum_{k=-m}^{m} \left| \sum_{r \neq 0} (v,e_{k+rn}) d_{k+rn} \right|^2$$  

$$\leq \sum_{k=-m}^{m} \left( \sum_{r \neq 0} \frac{|(v,e_{k+rn})|^2}{(d_{k+rn})^2} \cdot \sum_{s \neq 0} (d_{k+sn})^2 \right)$$  

$$= \sum_{k=-m}^{m} \sum_{r \neq 0} \frac{|(v,e_{k+rn})|^2}{(d_{k+rn})^2} (D_n)^2$$  

$$= (\|v\|_{d^2})^2 (D_n)^2,$$

i.e., we have shown that

$$\|Q_n(v)\|^2 \leq \sum_{k=-m}^{m} |c_{k,n}(v)|^2 \leq (\|v\|_{d^2})^2 (D_n)^2.$$  

\[\square\]

**Proposition 6.3.** Let $u \in \mathcal{H}_{d^2}$. Then

$$\|u - S_m(u)\| \leq D_n \|u - S_m(u)\|_{d^2}.$$  

**Proof.** If $u \in \mathcal{H}_{d^2}$ then

$$\|u - S_m(u)\|^2 = \sum_{|k| > m} \frac{|(u,e_k)|^2}{(d_k)^2} (d_k)^2$$  

$$\leq \sum_{|k| > m} \frac{|(u,e_k)|^2}{(d_k)^2} (d_m)^2$$  

$$\leq \sum_{|k| > m} \frac{|(u,e_k)|^2}{(d_k)^2} (D_n)^2,$$  

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i.e., we have shown
\[ \|u - S_m(u)\| \leq D_n \|u - S_m(u)\|_{d2}. \]

We conclude with the main quantitative result which extends Proposition 5.3 to the periodic Hilbert space \( \mathcal{H}_{d2} \).

**Proposition 6.4.** Let \( u \in \mathcal{H}_{d2} \). Then
\[ \|u - Q_n(u)\| \leq 4D_n \|u\|_{d2}. \]

**Proof.** Taking into account Proposition 5.3 and Proposition 6.2 we can conclude
\[
\begin{align*}
\|u - Q_n(u)\| &\leq \|u - S_m(u)\| + \|S_m(u) - Q_n(S_m(u))\| + \|Q_n(u - S_m(u))\| \\
&\leq D_n \|u - S_m(u)\|_{d2} + \sqrt{2}D_n \|S_m(u)\|_{d2} + D_n \|u - S_m(u)\|_{d2} \\
&\leq 4D_n \|u\|_{d2}.
\end{align*}
\]

We apply Proposition 6.4 to obtain quantitative bounds for the mean square error of optimal periodic interpolation in our specific examples.

**Example 1.**
\[
D_n = \frac{1}{m^{2r}} \sum_{s=1}^{\infty} s^{-2r} = d_m \sum_{s=1}^{\infty} s^{-2r}.
\]
If \( u \) is a function of the periodic Sobolev space \( \mathcal{W}^{2r} \) then Proposition 6.4 yields
\[
\|u - Q_n(u)\| = \mathcal{O}(m^{-2r}), \quad n = 2m + 1 \to \infty.
\]

**Example 2.**
\[
D_n = \sum_{s=1}^{\infty} e^{-smb} \leq \frac{e^{-mb}}{1 - e^{-b}} = d_m \frac{1}{1 - e^{-b}}.
\]
In this case \( u \) has to satisfy the condition
\[
\sum_{k=-\infty}^{\infty} |(u, e_k)|^2 \cdot e^{2b|k|} < \infty.
\]
Then Proposition 6.4 implies
\[
\|u - Q_n(u)\| = \mathcal{O}(e^{-mb}), \quad n = 2m + 1 \to \infty.
\]
References


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