Chaitan P. Gupta; Sotiris K. Ntouyas; Panagiotis Ch. Tsamatos
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ON THE SOLVABILITY OF SOME MULTI-POINT
BOUNDARY VALUE PROBLEMS

CHAITAN P. GUPTA, Reno, S.K. NTOUYAS, P.CH. TSAMATOS, Ioannina

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Summary. Let \( f \colon [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) be a function satisfying Caratheodory's conditions and let \( e(t) \in L^1[0,1] \). Let \( \xi_i, \tau_j \in (0,1) \), \( c_i, a_j \in \mathbb{R} \), all of the \( c_i \)'s, (respectively, \( a_j \)'s) having the same sign, \( i = 1, 2, \ldots, m - 2 \), \( j = 1, 2, \ldots, n - 2 \), \( 0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1 \), \( 0 < \tau_1 < \tau_2 < \ldots < \tau_{n-2} < 1 \) be given. This paper is concerned with the problem of existence of a solution for the multi-point boundary value problems

\[
(E) \quad x''(t) = f(t, x(t), x'(t)) + e(t), \quad t \in (0,1)
\]
\[
(BC)_{mn} \quad x(0) = \sum_{i=1}^{m-2} c_i x'_{\xi_i}, \quad x(1) = \sum_{j=1}^{n-2} a_j x_{\tau_j}
\]

and

\[
(E) \quad x''(t) = f(t, x(t), x'(t)) + e(t), \quad t \in (0,1)
\]
\[
(BC)'_{mn} \quad x(0) = \sum_{i=1}^{m-2} c_i x'_{\xi_i}, \quad x'(1) = \sum_{j=1}^{n-2} a_j x'_{\tau_j},
\]

Conditions for the existence of a solution for the above boundary value problems are given using Leray-Schauder Continuation theorem.

Keywords: multi-point boundary value problems, four point boundary value problems, Leray-Schauder Continuation theorem, a priori bounds

AMS classification: 34B10, 34B15
1. Introduction

Let \( f: [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) be a function satisfying Caratheodory’s conditions and \( e: [0, 1] \to \mathbb{R} \) be a function in \( L^1[0, 1], c_i, a_j \in \mathbb{R}, \) with all of the \( c_i \)'s, (respectively, \( a_j \)'s), having the same sign, \( \xi_i, \tau_j \in (0, 1), i = 1, 2, \ldots, m - 2, j = 1, 2, \ldots, n - 2, \) \( 0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1, \) \( 0 < \tau_1 < \tau_2 < \ldots < \tau_{n-2} < 1. \) The main purpose of this paper is to get results on the solvability of the following boundary value problems (BVPs for short)

\[
(E) \quad x''(t) = f(t, x(t), x'(t)) + e(t), \quad t \in (0, 1)
\]

\[
(BC)_{mn} \quad x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \quad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)
\]

and

\[
(E) \quad x''(t) = f(t, x(t), x'(t)) + e(t), \quad t \in (0, 1)
\]

\[
(BC)'_{mn} \quad x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \quad x'(1) = \sum_{j=1}^{n-2} a_j x'(\tau_j)
\]

The results are motivated by the so called “nonlocal” BVPs studied by Il’in and Moiseev [5], [6]. Using the Mawhin’s version of “Leray-Schauder Continuation theorem” ([8]), we prove the existence of a solution of the BVPs \((E)-(BC)_{mn}\) and \((E)-(BC)'_{mn}\). This method reduces the problem of existence of solutions of a BVP to the problem of establishing a priori bounds for the set of solutions of a family of these problems. Hence our main purpose is to give conditions on \( f \) which imply the needed a priori bounds.

It is well known (see [5], [6]) that if a function \( x \in C^1 \) satisfies the boundary condition \((BC)_{mn}\) or \((BC)'_{mn}\) and \( c_i, a_j, i = 1, 2, \ldots, m - 2, j = 1, 2, \ldots, n - 2 \) are as above, then there exist \( \zeta \in [\xi_1, \xi_{m-2}], \eta \in [\tau_1, \tau_{n-2}] \) such that

\[
x(0) = \gamma x'(\zeta), \quad x(1) = \alpha x(\eta)
\]

or

\[
x(0) = \gamma x'(\zeta), \quad x'(1) = \alpha x'(\eta)
\]

respectively with \( \gamma = \sum_{i=1}^{m-2} c_i, \alpha = \sum_{j=1}^{n-2} a_j. \) Hence for every solution \( x \) of the BVPs \((E)-(BC)_{mn}\) or \((E)-(BC)'_{mn}\) there exist \( \zeta \in [\xi_1, \xi_{m-2}], \eta \in [\tau_1, \tau_{n-2}] \) such that \( x \) is a solution of the following four point BVPs

\[
(E) \quad x''(t) = f(t, x(t), x'(t)), \quad t \in [0, 1]
\]

\[
(BC)_4 \quad x(0) = \gamma x'(\zeta), \quad x(1) = \alpha x(\eta).
\]
or

\[(E) \quad x''(t) = f(t, x(t), x'(t)), \quad t \in [0, 1] \]

\[(BC)_4 \quad x(0) = \gamma x'(\zeta), \quad x'(1) = \alpha x'(\eta) \]

respectively. We shall prove that all solutions of the BVPs \((E_\lambda)-(BC)_4\) and \((E_\lambda)-(BC)'_4\) are a priori bounded, with bounds independent of \(\zeta\) and \(\eta\), where \((E_\lambda)\) stands for the equation \(x'' = \lambda f + \lambda x\). Then, it is obvious, that these a priori bounds are also a priori bounds for the solutions of the BVP \((E)-(BC)_{mn}\) and \((E)-(BC)'_{mn}\). Recently Gupta, Ntouyas and Tsamatos studied in [3] and [4] the above BVP when \(\gamma = 0\). Here we extend the results for general \(\gamma\). For some recent results on the three point BVPs see [1], [2], [7].

We use the classical spaces \(C[0, 1], C^k[0, 1], L^k[0, 1], \) and \(L^\infty[0, 1]\) of continuous, \(k\)-times continuously differentiable, measurable real valued functions whose \(k\)-th power of the absolute value is Lebesgue integrable on \([0, 1]\), or measurable functions that are essentially bounded on \([0, 1]\). We also use the Sobolev space \(W^{2,k}(0, 1)\), \(k = 1, 2\) defined by

\[W^{2,k}(0, 1) = \{x: [0, 1] \to \mathbb{R} \mid x, x' \text{ abs. cont. on } [0, 1] \text{ with } x'' \in L^k[0, 1]\}\]

with the usual norm. We denote the norm in \(L^k[0, 1]\) by \(\|\cdot\|_k\), and the norm in \(L^\infty[0, 1]\) by \(\|\cdot\|_\infty\).

2. MAIN RESULTS

2A. THE BOUNDARY VALUE PROBLEM \((E)-(BC)_{mn}\)

We study first the BVP \((E)-(BC)_{mn}\). We begin with the following definition:

**Definition 2.1.** A function \(f: [0, 1] \times \mathbb{R}^2 \to \mathbb{R}\) satisfies Caratheodory’s conditions if (i) for each \((x, y) \in \mathbb{R}^2\), the function \(t \in [0, 1] \to f(t, x, y) \in \mathbb{R}\) is measurable on \([0, 1]\), (ii) for a.e. \(t \in [0, 1]\), the function \((x, y) \in \mathbb{R}^2 \to f(t, x, y) \in \mathbb{R}\) is continuous on \(\mathbb{R}^2\), and for each \(r > 0\), there exists \(g_r \in L^1[0, 1]\) such that \(|f(t, x, y)| \leq g_r(t)\) for a.e. \(t \in [0, 1]\) and \((x, y) \in \mathbb{R}^2\) with \(\sqrt{x^2 + y^2} \leq r\).

**Lemma 2.2.** Let \(\zeta, \eta \in (0, 1)\) be given and \(x(t) \in W^{2,1}(0, 1)\) be such that \(x(0) = \gamma x'(\zeta), x(1) = \alpha x'(\eta)\). Then

\[\|x\|_\infty \leq A\|x'\|_\infty, \quad \|x'\|_\infty \leq B\|x''\|_1\]
where

\[ A = \begin{cases} 
1, & \text{if } \alpha \leq 0 \\
L, & \text{if } \alpha > 0, \alpha \neq 1, \\
1 + |\gamma|, & \text{if } \alpha = 1 
\end{cases} \]

and

\[ B = \begin{cases} 
1, & \text{if } \alpha \leq 0, \gamma = 0 \\
\frac{1}{1-Q}, & \text{if } \alpha \leq 0, \gamma \neq 0 \\
\frac{1}{1-S}, & \text{if } \alpha > 0, \alpha \neq 1 \\
1, & \text{if } \alpha = 1 
\end{cases} \]

where for \( \alpha > 0, \alpha \neq 1, M = \min\{\frac{1}{1-M}, 1 + \frac{1-\eta}{|1-L|}, 1 + \frac{|\alpha(1-\eta)|}{|1-L|}, 1 + |\gamma|\}, S = \min\{\frac{1-\alpha}{1-\eta} L, \frac{1-\alpha}{\alpha(1-\eta)} L, \frac{1}{|\gamma|} L\}, Q = \min\{\frac{1-\alpha}{1-\eta}, \frac{1-\alpha}{|\alpha(1-\eta)|}, \frac{1}{|\gamma|}\} \) if \( \alpha < 0 \) and for \( \alpha = 0, Q = \frac{1}{|\gamma|} \) provided \( Q < 1, \) and \( S < 1. \)

**Proof.** We consider the following cases:

**Case 1.** \( \alpha \leq 0. \) In this case \( x(1) \cdot x(\eta) \leq 0 \) and accordingly there exists a \( \theta \in [\eta, 1] \) such that \( x(\theta) = 0. \) Hence it follows that \( \|x\|_\infty \leq \|x\|_\infty. \) Also if \( \gamma = 0, \) we have from \( x(0) = 0 \) and \( x(\theta) = 0 \) that there exists a \( z \in (0, \theta) \) such that \( x'(z) = 0. \) Accordingly, we get that \( \|x\|_\infty \leq \|x\|_1. \) Suppose, now, \( \alpha < 0 \) and \( \gamma \neq 0. \) Next we see from Mean Value Theorem there exists an \( \omega \in (\eta, 1) \) such that

\[ (\alpha - 1)x(\eta) = x(1) - x(\eta) = (1 - \eta)x'(\omega) \]

and hence

\[ x(\eta) = \frac{1-\eta}{\alpha-1} x'(\omega). \]

Also, since \( x(1) = \alpha x(\eta) \) we get

\[ x(1) = \frac{\alpha(1-\eta)}{\alpha-1} x'(\omega). \]

It then follows from the relations

\[ x'(t) = x'(\omega) + \int_\omega^t x''(s) \, ds = \frac{\alpha-1}{1-\eta} x(\eta) + \int_\omega^t x''(s) \, ds, \]

\[ x'(t) = x'(\omega) + \int_\omega^t x''(s) \, ds = \frac{\alpha-1}{\alpha(1-\eta)} x(1) + \int_\omega^t x''(s) \, ds \]

and

\[ x'(t) = x'(\zeta) + \int_0^t x''(s) \, ds = \frac{1}{\gamma} x(0) + \int_0^t x''(s) \, ds \]

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that

\begin{equation}
\|x'\|_\infty \leq \frac{1}{1 - Q} \|x''\|_1,
\end{equation}

where \( Q = \min\left\{ \frac{1 - \alpha}{1 - \eta}, \frac{1 - \alpha}{|\alpha|(1 - \eta)}, \frac{1}{|\gamma|} \right\} \) if \( Q < 1 \). Finally, for \( \alpha = 0, \gamma \neq 0 \) it is easy to see from (2.4), (2.6) that \( Q = \frac{1}{|\gamma|} \) since we require that \( Q < 1 \) and \( \frac{1}{1 - \eta} > 1 \).

**Case 2.** \( \alpha > 0, \alpha \neq 1 \). We first consider the relations

\[
x(t) = x(1) + \int_1^t x'(s) \, ds = \alpha x(\eta) + \int_1^\eta x'(s) \, ds
\]

and

\[
x(t) = x(\eta) + \int_\eta^t x'(s) \, ds = \frac{1}{\alpha} x(1) + \int_\eta^t x'(s) \, ds.
\]

Since, now, \( M = \min\{\alpha, \frac{1}{\alpha}\} < 1 \), we get from the above relations that

\[
\|x\|_\infty \leq \frac{1}{1 - M} \|x'\|_\infty.
\]

Next, we use the equations (2.2) and (2.3) to get the relations

\[
x(t) = x(1) + \int_1^t x'(s) \, ds = \frac{\alpha(1 - \eta)}{\alpha - 1} x'(\omega) + \int_1^t x'(s) \, ds
\]

and

\[
x(t) = x(\eta) + \int_\eta^t x'(s) \, ds = \frac{1 - \eta}{\alpha - 1} x'(\omega) + \int_\eta^t x'(s) \, ds.
\]

Also

\[
x(t) = x(0) + \int_0^t x'(s) \, ds = \gamma x'(\zeta) + \int_0^t x'(s) \, ds.
\]

It is then immediate that

\[
\|x\|_\infty \leq L \|x'\|_\infty,
\]

where \( L = \min\{\frac{1}{1 - M}, 1 + \frac{1}{|\alpha - 1|}, 1 + \frac{|\alpha|(1 - \eta)}{|\alpha - 1|}, 1 + |\gamma|\} \).

Further, we see using the relations (2.4), (2.5) and (2.6) that

\begin{equation}
\|x''\|_\infty \leq \frac{1}{1 - S} \|x''\|_1,
\end{equation}

where \( S = \min\{\frac{1 - \alpha}{1 - \eta}, \frac{1 - \alpha}{|\alpha|(1 - \eta)}, \frac{1}{|\gamma|}, L\} \) if \( S < 1 \).

**Case 3.** \( \alpha = 1 \). Since \( x(1) = x(\eta) \) there exists an \( \omega \in (\eta, 1) \) with \( x'(\omega) = 0 \). It is then immediate that \( \|x''\|_\infty \leq \|x''\|_1 \). Also since \( x(t) = x(0) + \int_0^t x'(s) \, ds = \gamma x'(\zeta) + \int_0^t x'(s) \, ds \), it is immediate that \( \|x\|_\infty \leq (1 + |\gamma|) \|x'\|_\infty \).

This completes the proof of the lemma.
Theorem 2.3. Let \( f: [0,1] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function satisfying Caratheodory's conditions. Assume that there exist functions \( p(t), q(t), r(t) \) in \( L^1[0,1] \) such that

\[
|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t)
\]

for a.e. \( t \in [0,1] \) and all \((u,v) \in \mathbb{R}^2\). Also let \( \eta \in (0,1) \) be given and \( \alpha, \gamma \in \mathbb{R} \) with \( 1 + \gamma \neq \alpha(\gamma + \eta) \). Moreover we assume that \( Q < 1 \) and \( S < 1 \).

(I) If \( \alpha \leq 0, \gamma = 0 \) then the BVP (E)-(BC) has at least one solution in \( C^1[0,1] \) provided

\[
\|p\|_1 + \|q\|_1 < 1.
\]

(II) If \( \alpha \leq 0 \) and \( \gamma \neq 0 \) then the BVP (E)-(BC) has at least one solution in \( C^1[0,1] \) provided

\[
\|p\|_1 + \|q\|_1 < 1 - Q.
\]

(III) If \( \alpha > 0, \alpha \neq 1 \) then the BVP (E)-(BC) has at least one solution in \( C^1[0,1] \) provided

\[
L\|p\|_1 + \|q\|_1 < 1 - S
\]

(IV) If \( \alpha = 1 \) then the BVP (E)-(BC) has at least one solution in \( C^1[0,1] \) provided

\[
(1 + |\gamma|)\|p\|_1 + \|q\|_1 < 1.
\]

Proof. Let \( X \) be the Banach space \( C^1[0,1] \) and \( Y \) denote the Banach space \( L^1(0,1) \) with their usual norms. We denote a linear mapping \( L: D(L) \subset X \rightarrow Y \) by setting

\[
D(L) = \{ x \in W^{2,1}(0,1): x(0) = \gamma x'(\xi), x(1) = \alpha x(\eta) \},
\]

and for \( x \in D(L) \),

\[
Lx = x''.
\]

We also define a nonlinear mapping \( N: X \rightarrow Y \) by setting

\[
(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0,1].
\]

We note that \( N \) is a bounded mapping from \( X \) into \( Y \). Next, it is easy to see that the linear mapping \( L: D(L) \subset X \rightarrow Y \), is one-to-one mapping. Next, the linear
mapping $K: Y \to X$, defined for $y \in Y$ by

$$(K_y)(t) = \int_0^t (t - s)y(s) \, ds + \gamma \int_0^\zeta y(s) \, ds + \frac{\gamma + t}{1 + \gamma - \alpha(\gamma + \eta)} \left[ \alpha \int_0^\eta (\eta - s)y(s) \, ds - \int_0^1 (1 - s)y(s) \, ds + \gamma(\alpha - 1) \int_0^\zeta y(s) \, ds \right], \quad t \in [0, 1].$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LK_y = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela-Ascoli Theorem that $KN$ maps bounded subsets of $X$ into relatively compact subsets of $X$. Hence $KN: X \to X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the BVP (E)-(BC)$_4$ if and only if $x$ is a solution to the operator equation

$$Lx = Nx + e.$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke.$$

We apply the Leray-Schauder Continuation theorem (see, e.g. [8], Corollary IV.7) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the BVP (E)-(BC)$_4$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$(E)_\lambda \quad \quad x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \quad t \in (0,1)$$

$$x(0) = \gamma x'(\zeta), \quad x(1) = \alpha x(\eta)$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

(I) Assume that $\alpha \leq 0$, $\gamma = 0$. From Lemma 2.2 we have

$$\|x\|_\infty \leq \|x'\|_\infty \leq \|x''\|_1$$

Let, now, $x(t)$ be a solution of $(E)_\lambda$ for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = \gamma x'('), x(1) = \alpha x(\eta)$. We then get from $(E)_\lambda$ that

$$\|x''\|_1 = \lambda \|f(t, x(t), x'(t)) + e(t)\|_1$$

$$\leq \|p\|_1 \|x\|_\infty + \|q\|_1 \|x'\|_\infty + \|r\|_1 + \|e\|_1$$

$$\leq (\|p\|_1 + \|q\|_1) \|x''\|_1 + \|r\|_1 + \|e\|_1$$
It follows from the assumption (2.8) that there is a constant \( c \), independent of \( \lambda \in [0,1] \), such that
\[
\|x''\|_1 \leq c.
\]

It is now immediate that the set of solutions of the family of equations \((E_\lambda)\) is, a priori, bounded in \( C^1[0,1] \) by a constant independent of \( \lambda \in [0,1] \).

(II) Assume that \( \alpha \leq 0, \gamma \neq 0 \). Then we have, by Lemma 2.2 that
\[
\|x\|_\infty \leq \|x'\|_\infty, \|x''\|_1 \leq \frac{1}{1-Q} \|x''\|_1.
\]

We then get from \((E_\lambda)\) that
\[
\|x''\|_1 = \lambda \|f(t, x(t), x'(t)) + e(t)\|_1 \\
\leq \|p\|_1 \|x\|_\infty + \|q\|_1 \|x'\|_\infty + \|r\|_1 + \|e\|_1 \\
\leq \left(\|p\|_1 + \|q\|_1\right) \frac{1}{1-Q} \|x''\|_1 + \|r\|_1 + \|e\|_1.
\]

We proceed as in case (I).

The process for the other cases is similar to the previous cases and we omit the details. This completes the proof of the theorem. \( \square \)

**Theorem 2.4.** Let \( f:[0,1] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function satisfying Caratheodory’s conditions. Assume that there exist functions \( p(t), q(t), r(t) \) in \( L^1[0,1] \) such that
\[
|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t)
\]
for a.e. \( t \in [0,1] \) and all \( (u,v) \in \mathbb{R}^2 \). Let \( c_i, a_j \in \mathbb{R} \), with all of the \( c_i \)'s, (respectively, \( a_j \)'s), having the same sign, \( \xi_i, \tau_j \in (0,1), i = 1,2,\ldots,m-2, j = 1,2,\ldots,n-2, 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, 0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1 \) be given. Suppose that
\[
1 + \left(\sum_{i=1}^{m-2} c_i\right) \left(1 - \sum_{j=1}^{n-2} a_j\right) - \sum_{j=1}^{n-2} a_j \tau_j \neq 0. \quad \text{Let } \gamma = \sum_{i=1}^{m-2} c_i \text{ and } \alpha = \sum_{j=1}^{n-2} a_j. \quad \text{Moreover we assume that } Q^{mn} < 1, \text{ and } S^{mn} < 1, \text{ where } M = \min\{\alpha, \frac{1}{\alpha}\} < 1,
\]

\[
L^{mn} = \min \left\{ \frac{1}{1-M}, 1 + \frac{1 - \tau_1}{|1 - \alpha|}, 1 + |\alpha|(1 - \tau_1), 1 + |\gamma| \right\},
\]
\[
S^{mn} = \min \left\{ \frac{|1 - \alpha|}{1 - \tau_{n-2}} L, \frac{|1 - \alpha|}{\alpha(1 - \tau_{n-2})} L, \frac{1}{|\gamma|} \right\},
\]
\[
Q^{mn} = \min \left\{ \frac{1}{1 - \tau_{n-2}}, |\alpha|(1 - \tau_{n-2}), |\gamma| \right\}.
\]
(I) If $\alpha \leq 0$, $\gamma = 0$ then the BVP $(E)-(BC)_{mn}$ has at least one solution in $C^1[0,1]$ provided

\begin{equation}
\|p\|_1 + \|q\|_1 < 1. \tag{2.14}
\end{equation}

(II) If $\alpha \leq 0$ and $\gamma \neq 0$ then the BVP $(E)-(BC)_4$ has at least one solution in $C^1[0,1]$ provided

\begin{equation}
\|p\|_1 + \|q\|_1 < 1 - Q^{mn}. \tag{2.15}
\end{equation}

(III) If $\alpha > 0$, $\alpha \neq 1$ then the BVP $(E)-(BC)_{mn}$ has at least one solution in $C^1[0,1]$ provided

\begin{equation}
L^{mn}\|p\|_1 + \|q\|_1 < 1 - S^{mn}. \tag{2.16}
\end{equation}

(IV) If $\alpha = 1$ then the BVP $(E)-(BC)_{mn}$ has at least one solution in $C^1[0,1]$ provided

\begin{equation}
(1 + |\gamma|)\|p\|_1 + \|q\|_1 < 1. \tag{2.17}
\end{equation}

Proof. As we have remarked in the introduction, we study the multi-point BVP using the a priori estimates that can be obtained for a four-point BVP. This is because for every solution $x(t)$ of the BVP $(E)-(BC)_{mn}$, there exist $\eta \in [\xi_1, \xi_{m-2}]$, $\zeta \in [\tau_1, \tau_{n-2}]$, depending on, $x(t)$, such that $x(t)$ is also a solution of the BVP $(E)-(BC)_4$ with $\gamma = \sum_{i=1}^{m-2} c_i$ and $\alpha = \sum_{j=1}^{n-2} a_j$. The proof is quite similar to the proof of Theorem 2.3 and uses the a priori estimates obtained in the proof of Theorem 2.3 for the set of solutions of the family of equations $(E_\lambda)-(BC)_4$. We note that it was shown that the set of solutions of the family of equations $(E_\lambda)-(BC)_4$ was, a priori, bounded by a constant independent of $\lambda \in [0,1]$ and both $\eta, \zeta \in (0,1)$, and this fact is the key point needed in the proof of Theorem 2.4.

Let $X$ be the Banach space $C^1[0,1]$ and $Y$ denote the Banach space $L^1(0,1)$ with their usual norms. We denote a linear mapping $L: D(L) \subset X \to Y$ by setting

\[ D(L) = \left\{ x \in W^{2,1}(0,1) : x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \quad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j) \right\}, \]

and for $x \in D(L)$,

\[ Lx = x''. \]
We also define a nonlinear mapping \( N : X \to Y \) by setting
\[
(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0, 1].
\]

We note that \( N \) is a bounded mapping from \( X \) into \( Y \). Next, it is easy to see that
the linear mapping \( L : D(L) \subset X \to Y \), is one-to-one mapping. Next, the linear
mapping \( K : Y \to X \), defined for \( y \in Y \) by
\[
(Ky)(t) = \int_0^t (t - s)y(s) \, ds + ct + k, \quad t \in [0, 1]
\]
where \( c \) and \( k \) are given by,
\[
\left[ 1 + \left( \sum_{i=1}^{m-2} c_i \right) \left( 1 - \sum_{j=1}^{n-2} a_j \right) - \sum_{j=1}^{n-2} a_j \tau_j \right] c = \left( \sum_{j=1}^{n-2} a_j - 1 \right) \left( \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} y(s) \, ds \right)
\]
\[
+ \sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s) \, ds - \int_0^1 (1 - s)y(s) \, ds
\]
and
\[
\left[ 1 + \left( \sum_{i=1}^{m-2} c_i \right) \left( 1 - \sum_{j=1}^{n-2} a_j \right) - \sum_{j=1}^{n-2} a_j \tau_j \right] k = \sum_{i=1}^{m-2} c_i \sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s) \, ds
\]
\[
- \sum_{i=1}^{m-2} c_i \int_0^1 (1 - s)y(s) \, ds + \left( 1 - \sum_{j=1}^{n-2} a_j \tau_j \right) \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} y(s) \, ds
\]
is such that for \( y \in Y \), \( Ky \in D(L) \)and \( LK y = y \); and for \( u \in D(L) \), \( KL u = u \). Furthermore, it follows easily using the Arzela-Ascoli Theorem that \( KN \) maps
bounded subsets of \( X \) into relatively compact subsets of \( X \). Hence \( KN : X \to X \) is
a compact mapping.

We, next, note that \( x \in C^1[0, 1] \) is a solution of the BVP \((E)-(BC)_mn\) if and only
if \( x \) is a solution to the operator equation
\[
Lx = Nx + e.
\]
Now, the operator equation \( Lx = Nx + e \) is equivalent to the equation
\[
x = KNx + Ke.
\]
We apply the Leray-Schauder Continuation theorem (see, e.g. [8], Corollary IV.7) to
obtain the existence of a solution for \( x = KNx + Ke \) or equivalently to the BVP
\((E)-(BC)_mn\).
To do this, it suffices to verify that the set of all possible solutions of the family of equations

\[(E)\]
\[
  x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \quad t \in (0, 1)
\]

\[(BC)_{mn}\]
\[
  x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \quad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)
\]

is, a priori, bounded in $C^1[0, 1]$ by a constant independent of $\lambda \in [0, 1]$.

Let, now, $x(t)$ be a solution of $(E_{\lambda})-(BC)_{mn}$ for some $\lambda \in [0, 1]$, so that $x \in W^{2,1}(0, 1)$ with $x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i)$, $x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$. Accordingly, there exist $\zeta \in [\xi_1, \xi_{m-2}]$ and $\eta \in [\tau_1, \tau_{n-2}]$ depending on $x(t)$, such that $x(t)$ is a solution of the four point BVP

\[
  x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \quad t \in (0, 1)
\]

\[
  x(0) = \gamma x'(\zeta), \quad x(1) = \alpha x(\eta)
\]

It then follows, as in the proof of Theorem 2.4 that there is a constant $c$, independent of $\lambda \in [0, 1]$, and $\eta \in [\xi_1, \xi_{m-2}]$, $\zeta \in [\tau_1, \tau_{n-2}]$ such that

\[
  \|x\|_\infty \leq c_1 \|x'\|_\infty \leq c_2 \|x''\|_1 \leq c
\]

where $c_1$, $c_2$ are constants independent of $\lambda$, $\eta$, $\zeta$ according to the cases (I), (II) or (III). Thus the set of solutions of the family of equations $(E_{\lambda})-(BC)_{mn}$ is, a priori, bounded in $C^1[0, 1]$ by a constant, independent of $\lambda \in [0, 1]$.

It is important to remark that the assumptions of Theorem 2.4, ensure that the needed a priori bounds are independent of $\zeta \in [\xi_1, \xi_{m-2}]$ and $\eta \in [\tau_1, \tau_{n-2}]$. This completes the proof of the theorem. \qed

2B. THE BOUNDARY VALUE PROBLEM $(E)-(BC)'_{mn}$.

In this section we study, by a similar way, the BVP $(E)-(BC)'_{mn}$.

**Lemma 2.5.** Let $\eta \in (0, 1)$ and $\gamma, \alpha \in \mathbb{R}$ be given. Let $x(t) \in W^{2,1}(0, 1)$ be such that $x(0) = \gamma x'(\zeta), x'(1) = \alpha x'(\eta)$. Then

\[
  \|x\|_\infty \leq (1 + |\gamma|)\|x'\|_\infty, \quad \|x'\|_\infty \leq A\|x''\|_1,
\]
where
\[ A = \begin{cases} 
1, & \text{if } \alpha \leq 0 \\
\frac{1}{1-M}, & \text{if } \alpha > 0, \alpha \neq 1
\end{cases} \]
and \( M = \min\{\alpha, \frac{1}{\alpha}\} < 1. \)

**Proof.** First we have from the relation
\[ x(t) = x(0) + \int_0^1 x'(s) \, ds = \gamma x'(\zeta) + \int_0^1 x'(s) \, ds \]
that
\[ \|x\|_{\infty} \leq (1 + |\gamma|)\|x'\|_{\infty} \]
Next, when \( \alpha \leq 0 \) there exists a \( \theta \in [\eta, 1] \) such that \( x'(\theta) = 0 \) from which we get that \( \|x'\|_{\infty} \leq \|x''\|_1 \). Now, if \( \alpha > 0 \) and \( \alpha \neq 1 \) we see from the relations
\[ x'(t) = x'(1) + \int_1^t x''(s) \, ds = \alpha x'(\eta) + \int_1^t x''(s) \, ds \]
\[ x'(t) = x'(\eta) + \int_\eta^t x''(s) \, ds = \frac{1}{\alpha} x'(1) + \int_\eta^t x''(s) \, ds \]
that
\[ \|x'\|_{\infty} \leq M\|x'\|_{\infty} + \|x''\|_1 \]
and hence
\[ \|x'\|_{\infty} \leq \frac{1}{1-M}\|x''\|_1. \]
This completes the proof of the lemma. \( \square \)

**Theorem 2.6.** Let \( f: [0,1] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function satisfying Caratheodory’s conditions. Assume that there exist functions \( p(t), q(t), r(t) \) in \( L^1[0,1] \) such that
\[ |f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t) \]
for a.e. \( t \in [0,1] \) and all \( (u,v) \in \mathbb{R}^2 \). Also let \( \eta, \zeta \in (0,1) \) be given and \( \alpha, \gamma \in \mathbb{R} \) with \( \alpha \neq 1. \)

(I) If \( \alpha \leq 0 \) then the BVP (E)-(BC) has at least one solution in \( C^1[0,1] \) provided
\[ (1 + |\gamma|)\|p\|_1 + \|q\|_1 < 1. \]  

(II) If \( \alpha > 0, \alpha \neq 1 \) then the BVP (E)-(BC) has at least one solution in \( C^1[0,1] \) provided
\[ (1 + |\gamma|)\|p\|_1 + \|q\|_1 < 1 - M. \]
Proof. Let $X$ be the Banach space $C^1[0,1]$ and $Y$ denote the Banach space $L^1(0,1)$ with their usual norms. We denote a linear mapping $L: D(L) \subset X \rightarrow Y$ by setting

$$D(L) = \{ x \in W^{2,1}(0,1) : x(0) = \gamma x'(\zeta), x'(1) = \alpha x'(\eta) \},$$

and for $x \in D(L)$,

$$Lx = x''.$$

We also define a nonlinear mapping $N: X \rightarrow Y$ by setting

$$(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0,1].$$

We note that $N$ is a bounded mapping from $X$ into $Y$. Next, it is easy to see that the linear mapping $L: D(L) \subset X \rightarrow Y$, is one-to-one mapping. Next, the linear mapping $K: Y \rightarrow X$, defined for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t-s)y(s) \, ds + \frac{\gamma + t}{1-\alpha} \left[ \alpha \int_0^\zeta y(s) \, ds - \int_0^1 y(s) \, ds \right] + \gamma \int_0^\zeta y(s) \, ds, \quad t \in [0,1].$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LKy = y$, and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela-Ascoli Theorem that $KN$ maps bounded subsets of $X$ into relatively compact subsets of $X$. Hence $KN: X \rightarrow X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the BVP (E)-(BC) if and only if $x$ is a solution to the operator equation

$$Lx = Nx + e.$$ 

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$\tau = KNx + Ke.$$

We apply the Leray-Schauder Continuation theorem (see, e.g. [8], Corollary IV.7) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the BVP (E)-(BC). 

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$(E)\begin{align*}
     x''(t) &= \lambda f(t, x(t), x'(t)) + \lambda e(t), \quad t \in (0,1) \\
     x(0) &= \gamma x'(\zeta), \quad x'(1) = \alpha x'(\eta)
\end{align*}$$
is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$. Assume that $\alpha \leq 0$. From Lemma 2.5 we have

$$\|x\|_\infty \leq (1 + |\gamma|)\|x'\|_\infty, \|x'\|_\infty \leq \|x''\|_1$$

Let, now, $x(t)$ be a solution of $(E_{\lambda})$ for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = \gamma x'(\zeta), x'(1) = \alpha x'(\eta)$. We then get from $(E_{\lambda})$ that

$$\|x''\|_1 = \lambda\|f(t, x(t), x'(t)) + e(t)\|_1 \\
\leq \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|r\|_1 + \|e\|_1 \\
\leq ((1 + |\gamma|)\|p\|_1 + \|q\|_1)\|x''\|_1 + \|r\|_1 + \|e\|_1$$

It follows from the assumption (2.13) that there is a constant $c$, independent of $\lambda \in [0,1]$, such that

$$\|x''\|_1 \leq c.$$

It is now immediate that the set of solutions of the family of equations $(E_{\lambda})$ is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

The case $\alpha > 0, \alpha \neq 1$ is similar and simple.

This completes the proof of the theorem. \[\square\]

**Theorem 2.7.** Let $f: [0,1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function satisfying Caratheodory’s conditions. Assume that there exist functions $p(t), q(t), r(t)$ in $L^1[0,1]$ such that

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t)$$

for a.e. $t \in [0,1]$ and all $(u,v) \in \mathbb{R}^2$. Let $c_i, a_j \in \mathbb{R}$, with all of the $c_i$’s (respectively, $a_j$’s), having the same sign, $\xi_i, \tau_j \in (0,1), i = 1,2,\ldots,m-2, j = 1,2,\ldots,n-2, 0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1, 0 < \tau_1 < \tau_2 < \ldots < \tau_{n-2} < 1$ be given. Suppose that

$$1 - \sum_{j=1}^{n-2} a_j \neq 0.$$

Then for any given $e(t)$ in $L^1(0,1)$, the $mn$-point BVP $(E)-(BC)'_{mn}$ has at least one solution in $C^1[0,1]$.

**Proof.** As we have remarked in the introduction, we study the $mn$-point BVP using the a priori estimates that can be obtained for a four point BVP. This is because for every solution $x(t)$ of the BVP $(E)-(BC)_{mn}$ there exist $\zeta \in [\xi_1, \xi_{m-2}], \eta \in [\tau_1, \tau_{n-2}]$, depending on $x(t)$, such that $x(t)$ is also a solution of the BVP $(E)-(BC)'_4$ with $\gamma = \sum_{i=1}^{m-2} c_i$ and $\alpha = \sum_{j=1}^{n-2} a_j \neq 1$. The proof is quite similar to the proof of Theorem 2.6 and uses the a priori estimates obtained in the proof of
Theorem 2.3 for the set of solutions of the family of equations $(E_{\lambda})-(BC)^{4}$. We note that it was shown that the set of solutions of the family of equations $(E_{\lambda})-(BC)^{4}$ was, a priori, bounded by a constant independent of $\lambda \in [0,1]$ and $\eta \in (0,1)$, and this fact is the key point needed in the proof of Theorem 2.7.

Let $X$ be the Banach space $C^{1}[0,1]$ and $Y$ denote the Banach space $L^{1}(0,1)$ with their usual norms. We denote a linear mapping $L: D(L) \subset X \rightarrow Y$ by setting

$$D(L) = \left\{ x \in W^{2,1}(0,1): x(0) = \sum_{i=1}^{m-2} c_{i} x'(\xi_{i}), x'(1) = \sum_{j=1}^{n-2} a_{j} x'(\tau_{j}) \right\},$$

and for $x \in D(L)$,

$$Lx = x''.$$ 

We also define a nonlinear mapping $N: X \rightarrow Y$ by setting

$$(Nx)(t) = f(t,x(t),x'(t)), \quad t \in [0,1].$$

We note that $N$ is a bounded mapping from $X$ into $Y$. Next, it is easy to see that the linear mapping $L: D(L) \subset X \rightarrow Y$, is one-to-one mapping. Next, the linear mapping $K: Y \rightarrow X$, defined for $y \in Y$ by

$$K(y) = \frac{t + \sum_{i=1}^{m-2} c_{i} \int_{0}^{\xi_{i}} y(s) ds + \int_{0}^{t} y(s) ds}{1 - \sum_{j=1}^{n-2} a_{j} \int_{0}^{\tau_{j}} y(s) ds + \int_{0}^{t} (t-s)y(s) ds}, \quad t \in [0,1]$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela-Ascoli Theorem that $KN$ maps bounded subsets of $X$ into relatively compact subsets of $X$. Hence $KN: X \rightarrow X$ is a compact mapping.

We, next, note that $x \in C^{1}[0,1]$ is a solution of the BVP $(E)-(BC)^{4}$ if and only if $x$ is a solution to the operator equation

$$Lx = Nx + e.$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke.$$
We apply the Leray-Schauder Continuation theorem (see, e.g. [8], Corollary IV.7) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the BVP $(E) - (BC)'_{mn}$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$(E)_{\lambda} \quad \quad \quad \quad \quad \quad x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \quad t \in (0, 1)$$

$$(BC)'_{mn} \quad \quad \quad \quad \quad \quad x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \quad x'(1) = \sum_{j=1}^{n-2} a_j x'(\tau_j)$$

is, a priori, bounded in $C^1[0, 1]$ by a constant independent of $\lambda \in [0, 1]$.

Let, now, $x(t)$ be a solution of $(E_{\lambda}) - (BC)'_{mn}$ for some $\lambda \in [0, 1]$, so that $x \in W^{2,1}(0, 1)$ with $x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \quad x'(1) = \sum_{j=1}^{n-2} a_j x'(\tau_j)$. Accordingly, there exist $\zeta \in [\xi_1, \xi_{m-2}], \quad \eta \in [\tau_1, \tau_{n-2}]$ depending on $x(t)$, such that $x(t)$ is a solution of the four point BVP

$$x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \quad t \in (0, 1)$$

$$x(0) = \gamma x'(\zeta), \quad x'(1) = \alpha x'(\eta)$$

It then follows, as in the proof of Theorem 2.6 that there is a constant $c$, independent of $\lambda \in [0, 1]$, and $\zeta \in [\xi_1, \xi_{m-2}], \quad \eta \in [\tau_1, \tau_{n-2}]$ such that

$$\|x\|_{\infty} \leq c_1 \|x'\|_{\infty} \leq c_2 \|x''\|_1 \leq c.$$

Thus the set of solutions of the family of equations $(E_{\lambda}) - (BC)'_{mn}$ is, a priori, bounded in $C^1[0, 1]$ by a constant, independent of $\lambda \in [0, 1]$.

This completes the proof of the theorem. \hfill \Box

References


Authors' addresses: Chaitan P. Gupta, Department of Mathematics, University of Nevada, Reno, Reno, NV 89557; S.K. Ntouyas and P.Ch. Tsamatos, Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece.