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WEIGHT MINIMIZATION OF ELASTIC PLATES USING REISSNER-MINDLIN MODEL AND MIXED-INTERPOLATED ELEMENTS

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Summary. The problem to find an optimal thickness of the plate in a set of bounded Lipschitz continuous functions is considered. Mean values of the intensity of shear stresses must not exceed a given value. Using a penalty method and finite element spaces with interpolation to overcome the "locking" effect, an approximate optimization problem is proposed. We prove its solvability and present some convergence analysis.

Keywords: Reissner-Mindlin plate model, mixed-interpolated elements, weight minimization, penalty method

AMS classification: 49A22, 65N30, 73K40

INTRODUCTION

We consider a weight minimization problem for an elastic plate, the bending of which is described by means of the Reissner-Mindlin model. The constraints are given in terms of the intensity of shear stresses. The role of design variable is played by the function of thickness, belonging to a class of Lipschitz-continuous functions.

We consider two cases of boundary conditions, namely those for (i) hard clamped or (ii) hard simply supported edges of the plate. The finite element method proposed by Brezzi, Fortin, Bathe and Stenberg (see [3] and the literature therein) was extended to plates of variable thickness in a recent paper [1]. Here we employ penalty method and some results of [1] to introduce an approximate optimal design problem, to prove its solvability and to present some convergence analysis. The existence of an optimal thickness function for the weight minimization problem is proved, as well.

1. Setting of the optimal design problem

Throughout the paper we consider an elastic homogeneous anisotropic plate, which occupies a domain $\Omega \times (-t(x_1, x_2), t(x_1, x_2))$, where Ω is a bounded, simply connected domain in $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$ and t belongs to the set

$$\mathscr{U}_{ad} = \{t \in C^{(0),1}(\bar{\Omega}) \text{ (i.e., Lipschitz functions) } | \\ t_{\min} \leqslant t(x_1, x_2) \leqslant t_{\max},\ |\partial t/\partial x_i| \leqslant C_i,\ i = 1, 2\}.$$

Here $t_{\min}$, $t_{\max}$, C_1, C_2 are given positive constants, $t_{\min} < t_{\max}$.

Let the transverse displacement w (deflection) of the midplane belong to $H_0^1(\Omega)$ and let the rotation vector β of fibers normal to the midplane belong either (i) to $[H_0^1(\Omega)]^2$ for hard clamped plate or (ii) to $V = \{\beta \in [H^1(\Omega)]^2 \mid \beta \cdot \tau = 0 \text{ on } \partial\Omega\}$ for hard simply supported plate. Here τ is the unit vector tangential to the boundary.

The components of the small strain tensor are

$$e_{\alpha\beta} = -x_3 \frac{1}{2}\Big(\frac{\partial \beta_\alpha}{\partial x_\beta} + \frac{\partial \beta_\beta}{\partial x_\alpha}\Big), \quad \alpha, \beta = 1, 2,$$
$$e_{\alpha 3} = \frac{1}{2}\Big(\frac{\partial w}{\partial x_\alpha} - \beta_\alpha\Big), \quad \alpha = 1, 2,\ e_{33} = 0.$$

Henceforth, we use Greek subscripts within the range $\{1, 2\}$ and the summation convention for repeated subscripts.

The following stress-strain relations are considered

$$\sigma_{\alpha\beta} = c_{\alpha\beta\gamma\delta} e_{\gamma\delta}, \\ \sigma_{\alpha 3} = \mathscr{E}_{\alpha\beta} e_{\beta 3}, \tag{1.1}$$

where the coefficients $c_{\alpha\beta\gamma\delta}$, $\mathscr{E}_{\alpha\beta}$ are constant,

$$c_{\alpha\beta\gamma\delta} = c_{\gamma\delta\alpha\beta} = c_{\beta\alpha\gamma\delta} \\ c_{\alpha\beta\gamma\delta}\tau_{\alpha\beta}\tau_{\gamma\delta} \geqslant c_0 \tau_{\alpha\beta}\tau_{\alpha\beta} \tag{1.2}$$

holds for all symmetric matrices $(\tau_{\alpha\beta})$ with some positive c_0; $\mathscr{E}$ is a diagonal matrix with positive entries.

Assume that body forces are zero and an external surface load $\boldsymbol{f} = (0, 0, f)^{\mathrm{T}}$ acts on the upper surface $x_3 = t(x_1, x_2)$.

The total potential energy is

$$\Pi(\beta, w) = \tfrac{1}{2}\tilde{a}(t; \beta, \beta) + \tfrac{1}{2}{}_t[\nabla w - \beta, \nabla w - \beta] - (f, w), \tag{1.3}$$

where

$$\tilde{a}(t;\beta,\eta) = \frac{2}{3}\int_\Omega t^3 c_{\alpha\beta\gamma\delta}(\partial\beta_\alpha/\partial x_\beta)(\partial\eta_\gamma/\partial x_\delta)\,\mathrm{d}x, \tag{1.4}$$

$$_t[u,v] = \int_\Omega t(\mathscr{E}u)^{\mathrm{T}} v\,\mathrm{d}x,$$

$$(f,w) = \int_\Omega fw\,\mathrm{d}x.$$

The latter brackets will be used also for vector-functions from $[L^2(\Omega)]^2$, so that $_t[u,v] \equiv (t\mathscr{E}u, v)$.

Let a specific weight $\omega = \text{const}$ be given. Then one half of the weight of the plate is

$$j(t) = \int_\Omega t\omega\,\mathrm{d}x.$$

The optimal design will be constrained as follows. We choose the second invariant of the stress tensor deviator (intensity of shear stresses)

$$I_2 = \sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)$$

at the extreme fibers ($x_3 = \pm t$) of the plate or at the midplane ($x_3 = 0$) to play the decisive role. Inserting the relations (1.1) and realizing that $\sigma_{\alpha 3} = 0$ for $x_3 = \pm t$ by symmetry of stress tensor and $\sigma_{\alpha\beta} = 0$ for $x_3 = 0$, we obtain

$$I_2 = t^2 I_{21}(\nabla\beta) \quad \text{for } x_3 = \pm t \tag{1.5}$$

$$I_2 = I_{22}(\nabla w - \beta) \quad \text{for } x_3 = 0,$$

where I_{21}, I_{22} are homogeneous quadratic forms, namely

$$I_{21}(\nabla\beta) = (c_{11\gamma\delta}\partial\beta_\gamma/\partial x_\delta)^2 + \ldots + 3(c_{12\gamma\delta}\partial\beta_\gamma/\partial x_\delta)^2, \tag{1.6}$$

$$I_{22}(\nabla w - \beta) = \frac{3}{4}\{(\mathscr{E}_{1\gamma}(\partial w/\partial x_\gamma - \beta_\gamma))^2 + (\mathscr{E}_{2\gamma}(\partial w/\partial x_\gamma - \beta_\gamma))^2\}.$$

Let us define functions

$$\psi_K(\beta,w) = (\operatorname{meas}\Delta_K)^{-1}\int_{\Delta_K} t^2 I_{21}(\nabla\beta)\,\mathrm{d}x - \sigma_d^2, \quad K = 1,\ldots,s,$$

$$\psi_K(\beta,w) = (\operatorname{meas}\Delta_K)^{-1}\int_{\Delta_K} I_{22}(\nabla w - \beta)\,\mathrm{d}x - \tau_d^2, \quad K = s+1,\ldots,\overline{K} < +\infty,$$

where $\Delta_K \subset \bar{\Omega}$ are given subdomains and σ_d, τ_d are given constants.

We define the constraints

$$\psi_K(\beta, w) \leqslant 0, \quad K = 1, \dots, \overline{K} \tag{1.7}$$

and the set of statically admissible design variables

$$S_{ad} = \left\{ t \in \mathscr{U}_{ad} \mid \sum_{K=1}^{\overline{K}} \left[\psi_K\big(\beta(t), w(t)\big)\right]^+ = 0 \right\},$$

where $\{\beta(t), w(t)\}$ is the minimizer of the potential energy $\Pi(t; \beta, w)$ over (i) $[H_0^1(\Omega)]^2 \times H_0^1(\Omega)$ or (ii) $V \times H_0^1(\Omega)$. We shall consider the following *Optimal Design Problem*:

$$t_0 = \arg\min_{t \in S_{ad}} j(t). \tag{1.8}$$

2. Existence of an optimal thickness function

The solvability of the Optimal Design Problem (1.8) will be proved by means of a penalty method. To this end, we introduce a penalized cost functional

$$\mathscr{J}_\varepsilon\big(t; \beta(t), w(t)\big) = j(t) + \varepsilon^{-1} \sum_{K=1}^{\overline{K}} \left[\psi_K\big(\beta(t), w(t)\big)\right]^+$$

where $\varepsilon > 0$ is an arbitrary parameter. Then we define the following *penalized optimal design problem*

$$t_\varepsilon = \arg\min_{t \in \mathscr{U}_{ad}} \mathscr{J}_\varepsilon\big(t; \beta(t), w(t)\big). \tag{2.1}$$

To prove the solvability of the problem (2.1) we shall need a continuous dependence of the solution $\{\beta(t), w(t)\}$ on the function t. The standard norms and seminorms in $H^k(\Omega)$ will be denoted by $\|\cdot\|_k$ and $|\cdot|_k$, respectively, $k = 1, 2$.

Proposition 2.1. *Let $t_n \to t$ in $C(\overline{\Omega})$, as $n \to \infty$, $t_n \in \mathscr{U}_{ad}$. Then*

$$\|\beta(t_n) - \beta(t)\|_1 + |w(t_n) - w(t)|_1 \to 0.$$

Proof. For brevity, let us denote $U \equiv \{\beta, w\}$, $Z \equiv \{\eta, \zeta\}$, $\beta_n = \beta(t_n)$, $w_n = w(t_n)$. Recall that [1—Lemma 1.3] positive constants c_1, c_2, c_3, c_4 exist such that

$$\frac{c_1 t_{\min}^3}{c_2 + c_3 t_{\min}^2}(\|\beta\|_1^2 + |w|_1^2) \leqslant {}_t[U, U]_A \leqslant c_4(\|\beta\|_1^2 + |w|_1^2) \tag{2.2}$$

holds for all $\{\beta, w\} \in V \times H_0^1(\Omega)$ and all $t \in \mathscr{U}_{ad}$, where

$$_t[U, Z]_A = \tilde{a}(t; \beta, \eta) + {}_t[\nabla w - \beta, \nabla\zeta - \eta].$$

For any $U_n \equiv \{\beta_n, w_n\}$ we have (cf. [1—(19), (20)])

$$(2.3) \qquad {}_{t_n}[U_n, Z]_A = (f, \zeta) \quad \forall Z \in V \times H_0^1(\Omega) \ ([H_0^1(\Omega)]^2 \times H_0^1(\Omega), \text{ resp.}).$$

Inserting $Z = U_n$ and using (2.2) we obtain

$$\|\beta_n\|_1^2 + |w_n|_1^2 \leqslant C\|f\|_0\|w_n\|_0 \leqslant \frac{C}{2\varepsilon_1}\|f\|_0 + \tfrac{1}{2}CC_F\varepsilon_1|w_n|_1^2,$$

where the Friedrichs inequality

$$\|w_n\|_0^2 \leqslant C_F|w_n|_1^2$$

has been employed. Choosing $\varepsilon_1 = (CC_F)^{-1}$,

$$(2.4) \qquad \|U_n\|_1^2 \equiv \|\beta_n\|_1^2 + |w_n|_1^2 \leqslant \tilde{C}\|f\|_0^2$$

follows for all n. Consequently, a subsequence $\{U_k\} \subset \{U_n\}$ and a function $U \in V \times H_0^1(\Omega)$, $([H_0^1(\Omega)]^2 \times H_0^1(\Omega)$, resp.) exist such that $U_k \rightharpoonup U$ (weakly) in the corresponding space.

We show that $U = U(t)$, i.e., U solves the problem for the limit t.

First we prove that

$$(2.5) \qquad \lim_{k\to\infty} {}_{t_k}[U_k, Z]_A = {}_t[U, Z]_A$$

holds for any Z. In fact, we may write

$$(2.6) \qquad |\tilde{a}(t_k; \beta_k, \eta) - \tilde{a}(t; \beta, \eta)| \leqslant |\tilde{a}(t_k; \beta_k, \eta) - \tilde{a}(t; \beta_k, \eta)| + |\tilde{a}(t; \beta_k - \beta, \eta)| = \mathscr{K}_1 + \mathscr{K}_2 \to 0,$$

since

$$\mathscr{K}_1 \leqslant C\|t_k^3 - t^3\|_\infty\|\beta_k\|_1\|\eta\|_1 \to 0$$

due to (2.4) and

$$\lim_{k\to\infty} \mathscr{K}_2 = 0$$

by virtue of the weak convergence of β_k.

An analogous argument yields that

$$(2.7)\qquad |(t_k\mathscr{E}(\nabla w_k-\beta_k),\delta)-(t\mathscr{E}(\nabla w-\beta),\delta)|\to 0\quad \forall\delta\in[L^2(\Omega)]^2.$$

Combining (2.6) and (2.7), we arrive at (2.5).

By (2.3),

$$_{t_k}[U_k,Z]_A=(f,\zeta)\quad \forall Z=(\eta,\zeta).$$

Passing to the limit with $k\to\infty$ and using (2.5), we arrive at

$$_t[U,Z]_A=(f,\zeta).$$

Since the solution of our boundary value problem is unique for any $t\in\mathscr{U}_{ad}$, $U=U(t)$ and the whole sequence $\{U_n\}$ tends weakly to $U(t)$.

It remains to prove *strong convergence.* First of all, we have

$$(2.8)\qquad \lim_{n\to\infty}{}_{t_n}[U_n,U_n]_A=\lim_{n\to\infty}(f,w_n)=(f,w)={}_t[U,U]_A.$$

Second, we can show that

$$(2.9)\qquad |_{t_n}[U_n,U_n]_A-{}_t[U_n,U_n]_A|\to 0,\quad \text{as } n\to\infty.$$

Indeed, by virtue of (2.4), we may write

$$|\tilde{a}(t_n;\beta_n,\beta_n)-\tilde{a}(t;\beta_n,\beta_n)|\leqslant C\|t_n^3-t^3\|_\infty\|\beta_n\|_1^2\to 0$$

and

$$\big|((t_n-t)\mathscr{E}(\nabla w_n-\beta_n),\nabla w_n-\beta_n)\big|\leqslant\|t_n-t\|_\infty C\|\nabla w_n-\beta_n\|_0^2\to 0,$$

so that (2.9) follows.

Then we have

$$\lim_{n\to\infty}{}_t[U_n,U_n]_A={}_t[U,U]_A,$$

since

$$\begin{aligned}|_t[U_n,U_n]_A-{}_t[U,U]_A|&\leqslant|_t[U_n,U_n]_A-{}_{t_n}[U_n,U_n]_A|\\&\quad+|_{t_n}[U_n,U_n]_A-{}_t[U,U]_A|\to 0.\end{aligned}$$

follows from (2.9) and (2.8).

By virtue of (2.2) and the weak convergence of $\{U_n\}$ we obtain

$$C\|U_n-U\|_1^2\leqslant{}_t[U_n-U,U_n-U]_A={}_t[U_n,U_n]_A-2_t[U_n,U]_A+{}_t[U,U]_A\to 0.$$

□

Lemma 2.1. *Let $t_n \to t$ in $C(\overline{\Omega})$ as $n \to \infty$, $t_n \in \mathcal{U}_{ad}$. Then*

$$\lim_{n\to\infty} \left[\psi_K(\beta(t_n), w(t_n))\right]^+ = \left[\psi_K(\beta(t), w(t))\right]^+, \quad K = 1, \dots, \overline{K}.$$

Proof. We may write

$$\begin{aligned}
&\left|\left[\psi_K(\beta(t_n), w(t_n))\right]^+ - \left[\psi_K(\beta(t), w(t))\right]^+\right| \\
&\quad \leqslant \left|\psi_K(\beta(t_n), w(t_n)) - \psi_K(\beta(t), w(t))\right| \\
&\quad \leqslant (\operatorname{meas}\Delta_K)^{-1} \int_{\Delta_K} |t_n^2 I_{21}(\nabla\beta_n) - t^2 I_{21}(\nabla\beta) + \\
&\qquad + I_{22}(\nabla w_n - \beta_n) - I_{22}(\nabla w - \beta)| \, \mathrm{d}x \\
&\quad \leqslant (\operatorname{meas}\Delta_K)^{-1} \int_{\Delta_K} \left(t_{\max}^2 |I_{21}(\nabla\beta_n) - I_{21}(\nabla\beta)|\right. \\
&\qquad \left. + |t_n^2 - t^2||I_{21}(\nabla\beta)| + |I_{22}(\nabla w_n - \beta_n) - I_{22}(\nabla w - \beta)|\right) \mathrm{d}x.
\end{aligned}$$

Since I_{21} and I_{22} are homogeneous quadratic functions, the integral has a following upper bound

$$\begin{aligned}
&C\{\|\beta_n - \beta\|_1(\|\beta_n\|_1 + \|\beta\|_1) + \|t_n^2 - t^2\|_\infty \\
&\quad + (\|\nabla w - \beta\|_0 + \|\nabla w_n - \beta_n\|_0)(\|\nabla(w_n - w)\|_0 + \|\beta_n - \beta\|_0)\},
\end{aligned}$$

which tends to zero due to Proposition 2.1. □

Proposition 2.2. *The penalized problem* (2.1) *has a solution for any $\varepsilon > 0$.*

Proof. The functionals $j(t)$ and $\left[\psi_K(\beta(t), w(t))\right]^+$ are continuous in $\mathcal{U}_{ad}$ by virtue of Lemma 2.1. The set $\mathcal{U}_{ad}$ is compact in $C(\overline{\Omega})$. Hence a minimizer $t_\varepsilon \in \mathcal{U}_{ad}$ exists. □

Theorem 2.1. *Assume that $S_{ad} \neq \emptyset$. Let $\{\varepsilon\}$, $\varepsilon \to 0_+$, be a sequence and let $\{t_\varepsilon\}$ be a sequence of solutions to the penalized optimal design problem* (2.1), *$\{\beta(t_\varepsilon), w(t_\varepsilon)\}$ the sequence of corresponding rotation and deflection fields.*

Then there exist a subsequence $\{\tilde{\varepsilon}\} \subset \{\varepsilon\}$ and $t_0 \in S_{ad}$ such that

$$\begin{gathered}
t_{\tilde{\varepsilon}} \to t_0 \quad \text{in } C(\Omega), \\
\|\beta(t_{\tilde{\varepsilon}}) - \beta(t_0)\|_1 + |w(t_{\tilde{\varepsilon}}) - w(t_0)|_1 \to 0,
\end{gathered}$$

where t_0 is a solution of the Optimal Design Problem (1.8).

Proof follows from Proposition 2.1, Lemma 2.1 and the compactness of the set $\mathcal{U}_{ad}$. For the details, see an analogous proof of Theorem 2.1 in [2]. □

Corollary 2.2. *If $S_{ad} \neq \emptyset$, there exists at least one solution of the optimal design problem* (1.8).

Proof. An immediate consequence of Proposition 2.2 and Theorem 2.1. □

3. Approximate optimal design problem

We are going to define an approximate problem, combining the penalty method with a finite element discretization, by means of mixed-interpolated elements [3]. In contrast with [3], however, we need a more general method, which includes plates of variable thickness. Such a generalization was given by the author in [1] together with an error analysis, concerning a particular choice of piecewise polynomial finite element spaces. Here we employ the same finite elements and exploit some results of the above-mentioned paper.

Let us consider a *regular* family $\{\mathscr{T}_h\}$, $h \to 0_+$, of triangulations of the domain Ω.

We denote by $\mathscr{L}_s^k$ the space of piecewise polynomials on $\mathscr{T}_h$ of degree $\leqslant s$, which belong to $H^k(\Omega)$. Let B_3 be the space of "bubble functions" on $\mathscr{T}_h$ of the third degree, i.e.,

$$B_3 = \{v \mid v|_K \in P_3(K) \cap H_0^1(K) \quad \text{for all triangles } K \in \mathscr{T}_h\}.$$

Let H_h be the intersection of $(\mathscr{L}_2^1 \oplus B_3)^2$ with $[H_0^1(\Omega)]^2$ or V, respectively, (Crouzeix-Raviart elements), $W_h = \mathscr{L}_2^1 \cap H_0^1(\Omega)$.

Moreover, we use the space RT_1 of Raviart-Thomas elements of the first degree. Recall that (see [3])

$$RT_1(K) = \left(P_1(K)\right)^2 + \underline{x}P_1(K) \quad \forall K \in \mathscr{T}_h$$

and $RT_1 \subset H(\text{div};\Omega)$, i.e., the degrees of freedom are chosen in order to ensure continuity of the flux at interelement boundaries. Let $(RT_1)^\perp$ denote the rotation of vector-functions from RT_1 by $\pi/2$, defined by $\underline{a}^\perp = (-a_2, a_1)^{\mathrm{T}}$. We define the interpolation $\Pi_h\colon H_h \to (RT_1)^\perp$ by means of

$$\int_e (\eta_h - \Pi_h\eta_h)\cdot\tau\mu \,\mathrm{d}s = 0 \quad \forall \mu \in P_1(e)$$

for all sides $e \in \partial K \in \mathscr{T}_h$, and

$$\int_K (\eta_h - \Pi_h\eta_h)\,\mathrm{d}x = 0 \quad \forall K \in \mathscr{T}_h.$$

Here $P_1(e)$ denotes the space of linear polynomials on the side e.

Instead of $\mathscr{U}_{ad}$ we introduce an internal approximation

$$\mathscr{U}_{ad}^h = \mathscr{U}_{ad} \cap \mathscr{L}_1^1.$$

We define the *Approximate Optimal Design Problem*

$$t_h^\varepsilon = \operatorname*{arg\,min}_{t_h \in \mathscr{U}_{ad}^h} \mathscr{J}_\varepsilon\big(t_h; \beta_h(t_h), w_h(t_h)\big), \tag{3.1}$$

where

$$\{\beta_h(t_h), w_h(t_h)\} = \operatorname*{arg\,min}_{\{\eta_h,\zeta_h\}\in H_h\times W_h} \{\tfrac{1}{2}\tilde{a}(t_h;\eta_h,\eta_h) + \tfrac{1}{2}(t_h\mathscr{E}(\nabla\zeta_h - \Pi_h\eta_h), \nabla\zeta_h - \Pi_h\eta_h) - (f,\zeta_h)\}. \tag{3.2}$$

Next, let us prove the solvability of the problem (3.1). To this end we first establish

Lemma 3.1. *Let $\mathscr{T}_h$ be fixed and let $t_h^n \in \mathscr{U}_{ad}^h$, $n = 1, 2\ldots,$*

$$t_h^n \to t_h, \quad \text{as } n \to \infty.$$

Then

$$\beta_h(t_h^n) \to \beta_h(t_h),$$
$$w_h(t_h^n) \to w_h(t_h).$$

Proof. We shall drop out the subscript "h", in what follows. Let us denote $\beta_n = \beta(t^n)$, $w_n = w(t^n)$, $\beta = \beta(t)$, $w = w(t)$, $U^n = \{\beta_n, w_n\}$, $U = \{\beta, w\}$, $Z = \{\eta, \zeta\}$,

$$_t[U,Z]_A^h = \tilde{a}(t;\beta,\eta) + (t\mathscr{E}(\nabla w - \Pi\beta), \nabla\zeta - \Pi\eta). \tag{3.3}$$

The definition (3.2) implies that

$$_{t^n}[U^n,Z]_A^h = (f,\zeta) \quad \forall Z \in H_h \times W_h \tag{3.4}$$

Let us denote

$$\|U\| = (\|\beta\|_1^2 + |w|_1^2)^{1/2},$$
$$\Delta_n = U^n - U = \{\beta_n - \beta, w_n - w\}.$$

Using some results of [1—(16), (27)], we can prove that the form ${}_t[U,Z]_A^h$ is uniformly elliptic, i.e.,

$$(3.5) \qquad {}_t[U,U]_A^h \geqslant c\|U\|^2$$

holds for all $U \in H_h \times W_h$, $h \in (0,1]$ and $t \in \mathscr{U}_{ad}$, where $c > 0$ is independent of U, h, t. (The proof is parallel to that of [1—Lemma 1.3]).

Since $\mathscr{U}_{ad}^h$ is closed, $t \in \mathscr{U}_{ad}^h$. By definition, we have

$$(3.6) \qquad {}_t[U,Z]_A^h = (f,\zeta) \quad \forall Z \in H_h \times W_h.$$

From (3.5) and (3.4) we get

$$c\|U^n\|^2 \leqslant {}_{t^n}[U^n,U^n]_A^h = (f,w_n) \leqslant C\|f\|_0\|U^n\|$$

so that the sequence $\{\|U^n\|\}$ is bounded.

It is readily seen that

$$\begin{aligned}(3.7) \qquad & |{}_{t^n}[U,\Delta_n]_A^h - {}_t[U,\Delta_n]_A^h| \\ &\leqslant |\tilde{a}(t^n,\beta,\beta_n-\beta) - \tilde{a}(t;\beta,\beta_n-\beta)| \\ &\quad + \big|((t^n-t)\mathscr{E}(\nabla w - \Pi\beta), \nabla(w_n - w) - \Pi(\beta_n-\beta))\big| \\ &\leqslant C\big\{\|(t^n)^3 - t^3\|_\infty \|\beta\|_1 \|\beta_n - \beta\|_1 \\ &\quad + \|t^n - t\|_\infty (|w|_1 + \|\Pi\beta\|_0)\big(|w_n - w|_1 + \|\Pi(\beta_n-\beta)\|_0\big)\big\} \\ &\leqslant C\big(\|(t^n)^3 - t^3\|_\infty + \|t^n - t\|_\infty\big)\|U\|\|\Delta_n\| \to 0,\end{aligned}$$

since $\|\Delta_n\|$ are bounded and

$$\|\Pi\eta\|_0 \leqslant \|\eta\|_1 \quad \forall \eta \in H_h$$

(see [1—(27)]).

Finally, we may write

$$\begin{aligned} c\|\Delta_n\|^2 &\leqslant {}_{t^n}[U^n,\Delta_n]_A^h - {}_{t^n}[U,\Delta_n]_A^h \\ &= ({}_{t^n}[U^n,\Delta_n]_A^h - {}_t[U,\Delta_n]_A^h) + ({}_t[U,\Delta_n]_A^h - {}_{t^n}[U,\Delta_n]_A^h).\end{aligned}$$

Form (3.4) and (3.6) we see that the first term vanishes. The second term tends to zero by virtue of (3.7). □

Theorem 3.1. *The Approximate Optimal Design Problem has at least one solution for any (fixed) triangulation $\mathcal{T}_h$ and any positive ε.*

Proof. Making use of Lemma 3.1, we prove that the functions

$$t_h \to \left[\psi_K\big(t_h; \beta_h(t_h), w_h(t_h)\big)\right]^+$$

are continuous in $\mathcal{U}_{ad}^h$ (cf. the analogous proof of Lemma 2.1). Consequently, the function $t_h \to \mathcal{J}_\varepsilon\big(t_h; \beta_h(t_h), w_h(t_h)\big)$ in (3.1) is continuous, as well.

Obviously, we have

$$t_h \in \mathcal{U}_{ad}^h \iff \{t_h(Q_i)\}_{i=1}^{r_h} \in \mathcal{A}_h \subset \mathbb{R}^{r_h},$$

where Q_i are vertices of $\mathcal{T}_h$. The set $\mathcal{A}_h$ is compact, being bounded and closed. Hence the functional $\mathcal{J}_\varepsilon$ attains its minimum in $\mathcal{U}_{ad}^h$. □

4. Finite element analysis

A natural question arises, what happens if we keep the parameter ε fixed and refine the mesh size h. We can prove, that a subsequence of solutions $\{t_h^\varepsilon\}$, $h \to 0_+$, exists, which converges to a solution t_ε of the penalized optimal design problem.

Lemma 4.1. *Let us assume that the solution $\{\beta(t), w(t)\}$ is regular, so that*

$$\beta(t) \in [H^2(\Omega)]^2, \quad w(t) \in H^2(\Omega) \quad \forall t \in \mathcal{U}_{ad}, \tag{A1}$$

and there exists a constant $\bar{C} > 0$ such that

$$\|\beta(t)\|_2 + \|w(t)\|_2 \leqslant \bar{C} \quad \forall t \in \mathcal{U}_{ad}.$$

Then

$$\|\beta(t) - \beta_h(t)\|_1 + |w(t) - w_h(t)|_1 \leqslant Ch \quad \forall t \in \mathcal{U}_{ad}, \ \forall h \in (0, 1]$$

holds, where the constant C is independent of t and h.

Proof. Denote again $U = \{\beta, w\}$, $\mathcal{Z} = (\eta, \zeta)$,

$${}_t[U, \mathcal{Z}]_A = \tilde{a}(t; \beta, \eta) + {}_t[\nabla w - \beta, \nabla \eta - \zeta].$$

The solution $U \equiv U(t)$ satisfies the condition

$${}_t[U, \mathcal{Z}]_A = (f, \zeta) \quad \forall \mathcal{Z} \in [H_0^1(\Omega)]^2 \times H_0^1(\Omega) \quad (\text{or } V \times H_0^1(\Omega)). \tag{4.1}$$

The approximate solution $U_h \equiv U_h(t) = \{\beta_h(t), w_h(t)\}$ satisfies the analogous condition

$$(4.2) \qquad {}_t[U_h, \mathscr{Z}_h]_A^h = (f, \zeta_h) \quad \forall \mathscr{Z}_h \in H_h \times W_h$$

(cf. (3.2) and (3.3)).

Subtracting, we obtain

$$(4.3) \qquad {}_t[U, \mathscr{Z}_h]_A - {}_t[U_h, \mathscr{Z}_h]_A^h = 0 \quad \forall \mathscr{Z}_h \in H_h \times W_h.$$

For the time being, let $\mathscr{V}_h \in H_h \times W_h$ be arbitrary. From the uniform elipticity (3.5) of the second bilinear form and (4.3) we obtain

$$\begin{aligned}(4.4) c\|U_h - \mathscr{V}_h\|^2 &\leqslant {}_t[U_h - \mathscr{V}_h, U_h - \mathscr{V}_h]_A^h \\ &= {}_t[U_h - \mathscr{V}_h, U_h - \mathscr{V}_h]_A + {}_t[\mathscr{V}_h, U_h - \mathscr{V}_h]_A - {}_t[\mathscr{V}_h, U_h - \mathscr{V}_h]_A^h \\ &\leqslant C\|U - \mathscr{V}_h\|\|U_h - \mathscr{V}_h\| + {}_t[\mathscr{V}_h, U_h - \mathscr{V}_h]_A - {}_t[\mathscr{V}_h, U_h - \mathscr{V}_h]_A^h.\end{aligned}$$

We use the triangle inequality and (4.4) to derive that

$$(4.5) \quad \|U - U_h\| \leqslant C \inf_{\mathscr{V}_h \in H_h \times W_h} \left\{\|U - \mathscr{V}_h\| + \sup_{\mathscr{Z}_h \in H_h \times W_h} \frac{{}_t[\mathscr{V}_h, \mathscr{Z}_h]_A - {}_t[\mathscr{V}_h, \mathscr{Z}_h]_A^h}{\|\mathscr{Z}_h\|}\right\},$$

(which is a consequence of the First Strang Lemma—cf. [5—Th. 26.1, p. 192]).

Let us substitute $\mathscr{V}_h := \bar{U}_h$, i.e. the projection of U into $H_h \times W_h$ with respect to the inner product

$$(U_h, \mathscr{Z}_h)_{10} = (\beta_h, \eta_h)_1 + (\nabla w_h, \nabla \zeta_h)_0.$$

Thus we arrive at

$$(4.6) \qquad \|U - U_h\| \leqslant C\{\|U - \bar{U}_h\| + \sup_{\mathscr{Z}_h}({}_t[\bar{U}_h, \mathscr{Z}_h]_A - {}_t[\bar{U}_h, \mathscr{Z}_h]_A^h)/\|\mathscr{Z}_h\|\}.$$

The classical approximation theory yields

$$(4.7) \quad \|U - \bar{U}_h\| \leqslant \inf_{\{\eta_h, \zeta_h\} \in H_h \times W_h} \{\|\beta - \eta_h\|_1 + |w - \zeta_h|_1\} \leqslant Ch(|\beta(t)|_2 + |w(t)|_2)$$

Next, we may write

$$\begin{aligned}&{}_t[\bar{U}_h, \mathscr{Z}_h]_A - {}_t[\bar{U}_h, \mathscr{Z}_h]_A^h = [\nabla \bar{w}_h - \bar{\beta}_h, \nabla \zeta_h - \eta_h] - [\nabla \bar{w}_h - \Pi_h \bar{\beta}_h, \nabla \zeta_h - \Pi_h \eta_h] \\ &= [-\bar{\beta}_h + \Pi_h \bar{\beta}_h, \nabla \zeta_h] + [\nabla \bar{w}_h, -\eta_h + \Pi_h \eta_h] + [\bar{\beta}_h, \eta_h] - [\Pi_h \bar{\beta}_h, \Pi_h, \eta_h] \pm [\Pi_h \bar{\beta}_h, \eta_h] \\ &\leqslant C\{(|\zeta_h|_1 + \|\beta_h\|_0)\|\Pi_h \bar{\beta}_h - \bar{\beta}_h\|_0 + (|\bar{w}_h|_1 + \|\Pi_h \bar{\beta}_h\|_0)\|\Pi_h \eta_h - \eta_h\|_0\}.\end{aligned}$$

Making use of the estimates

$$\|\Pi_h\bar{\beta}_h - \bar{\beta}_h\|_0 \leqslant Ch\|\bar{\beta}_h\|_1, \quad \|\Pi_h\eta_h - \eta_h\|_0 \leqslant Ch\|\eta_h\|_1$$

and [1—(27)], we obtain

$$([U_h, \mathscr{Z}_h]_A - [\bar{U}_h, \mathscr{Z}_h]^h_A)/\|\mathscr{Z}_h\| \leqslant Ch(\|\bar{\beta}_h\|_1 + |\bar{w}_h|_1). \tag{4.8}$$

Since $\|\bar{U}_h\| \leqslant \|U\|$ follows by definition, we have

$$\|\bar{\beta}_h\|_1 + |\bar{w}_h|_1 \leqslant \sqrt{2}\,\|\bar{U}_h\| \leqslant \sqrt{2}(\|\beta(t)\|_1 + |w(t)|_1) \tag{4.9}$$

Inserting (4.7), (4.8) and (4.9) into (4.6), we arrive at

$$\|U - U_h\| \leqslant Ch(\|\beta(t)\|_2 + \|w(t)\|_2). \tag{4.10}$$

Consequently, Lemma 4.1 follows from (4.10) and (A1). □

Proposition 4.1. *Let the assumption* (A1) *of Lemma* 4.1 *be fulfilled and let* $\{t_h\}$, $h \to 0_+$ *be a sequence of* $t_h \in \mathscr{U}^h_{ad}$, *such that*

$$t_h \to t \quad \text{in } C(\bar{\Omega}).$$

Then

$$\|\beta_h(t_h) - \beta(t)\|_1 + |w_h(t_h) - w(t)|_1 \to 0 \quad \text{as } h \to 0_+.$$

P r o o f. By triangle inequality, Lemma 4.1 and Proposition 2.1 we have

$$\begin{aligned} \|U_h(t_h) - U(t)\| &\leqslant \|U_h(t_h) - U(t_h)\| + \|U(t_h) - U(t)\| \\ &\leqslant Ch + \|U(t_h) - U(t)\| \to 0. \end{aligned}$$

□

Proposition 4.2. *Let the assumptions of Proposition* 4.1 *be fulfilled. Then*

$$\mathscr{J}_\varepsilon\big(t_h; \beta_h(t_h), w_h(t)\big) \to \mathscr{J}_\varepsilon\big(t; \beta(t), w(t)\big), \quad \text{as } h \to 0_+.$$

P r o o f is analogous to that of Lemma 2.1. □

Lemma 4.2. *Let each triangle $K \in \mathcal{T}_h \in \{\mathcal{T}_h\}$, $h \to 0_+$, have two sides parallel with the coordinate axes. Then for any $t \in \mathcal{U}_{ad}$ there exists a sequence $\{t_h\}$, $h \to 0_+$, such that $t_h \in \mathcal{U}_{ad}^h$ and*

$$t_h \to t \quad in\ C(\bar{\Omega}).$$

For the *Proof*—see [2—Lemma 4.2].

Theorem 4.1. *Let the regular family of triangulations $\{\mathcal{T}_h\}$, $h \to 0_+$, satisfy the assumption of Lemma 4.2. Let the solutions $\{\beta(t), w(t)\}$ satisfy the assumption (A1) of Lemma 4.1.*

Assume that $\{t_h^\varepsilon\}$, $h \to 0_+$, is a sequence of solutions of the Approximate Optimal Design Problems (3.1).

Then there exists a subsequence $\{t_{\hat{h}}^\varepsilon\} \subset \{t_h^\varepsilon\}$ and an element $t_\varepsilon \in \mathcal{U}_{ad}$ such that

$$t_{\hat{h}}^\varepsilon \to t_\varepsilon \quad in\ C(\bar{\Omega}) \tag{4.11}$$

$$\|\beta_{\hat{h}}(t_{\hat{h}}^\varepsilon) - \beta(t_\varepsilon)\|_1 + |w_{\hat{h}}(t_{\hat{h}}^\varepsilon) - w(t_\varepsilon)|_1 \to 0, \tag{4.12}$$

where t_ε is a solution of the penalized optimal design problem (2.1).

Each uniformly convergent subsequence of $\{t_h^\varepsilon\}$ tends to a solution of the problem (2.1).

Proof. Since $\mathcal{U}_{ad}^h \subset \mathcal{U}_{ad}$ and $\mathcal{U}_{ad}$ is compact in $C(\bar{\Omega})$, a subsequence of $\{t_h^\varepsilon\}$ exists such that (4.11) holds with $t_\varepsilon \in \mathcal{U}_{ad}$. Then (4.12) follows from Proposition 4.1.

Consider any $t \in \mathcal{U}_{ad}$. By Lemma 4.2, there exists a sequence $\{\tau_{\hat{h}}\}$, $\tau_{\hat{h}} \in \mathcal{U}_{ad}^{\hat{h}}$, such that $\tau_{\hat{h}} \to t$ in $C(\bar{\Omega})$. By definition of the problem (3.1)

$$\mathcal{J}_\varepsilon\big(t_{\hat{h}}^\varepsilon; \beta_{\hat{h}}(t_{\hat{h}}^\varepsilon), w_{\hat{h}}(t_{\hat{h}}^\varepsilon)\big) \leqslant \mathcal{J}_\varepsilon\big(\tau_{\hat{h}}; \beta_{\hat{h}}(\tau_{\hat{h}}), w_{\hat{h}}(\tau_{\hat{h}})\big).$$

Passing to the limit with $\hat{h} \to 0_+$ and using Proposition 4.2 on both sides, we obtain

$$\mathcal{J}_\varepsilon\big(t_\varepsilon; \beta(t_\varepsilon), w(t_\varepsilon)\big) \leqslant \mathcal{J}\big(t; \beta(t), w(t)\big).$$

Consequently, t_ε is a solution of the problem (2.1). The rest of the theorem is obvious. □

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References

[1] *Hlaváček, I.*: Reissner-Mindlin model for plates of variable thickness. Solution by mixed-interpolated elements. Appl. Math. *41* (1996), 57–78.

[2] *Hlaváček, I.*: Weight minimization of an elastic plate with a unilateral inner obstacle by a mixed finite element method. Appl. Math. *39* (1994), 375–394.

[3] *Brezzi, F. – Fortin, M.*: Mixed and Hybrid Finite Element Methods. Springer-Verlag, New York, Berlin, 1991.

[4] *Brezzi, F. – Fortin, M. – Stenberg, R.*: Error analysis of mixed-interpolated elements for Reissner-Mindlin plates. Math. Models and Meth. in Appl. Sci. *1* (1991), 125–151.

[5] *Ciarlet, P.G.*: Basic error estimates for elliptic problems. Handbook of Numer. Analysis, ed. by P. G. Ciarlet and J. L. Lions. vol. II, North-Holland, Amsterdam, 1991, pp. 17–352.

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