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Beloslav Riečan

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ON THE ENTROPY AND GENERATORS  
OF DYNAMICAL SYSTEMS

BELOSLAV RIEČAN, Bratislava

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*Summary.* Recently D. Dumitrescu ([4], [5]) introduced a new kind of entropy of dynamical systems using fuzzy partitions ([1], [6]) instead of usual partitions (see also [7], [11], [12]). In this article a representation theorem is proved expressing the entropy of the dynamical system by the entropy of a generating partition.

*Keywords:* entropy of dynamical systems, fuzzy partitions, entropy of generating partition

*AMS classification:* 28D20

## 0. INTRODUCTION

Given a probability space  $(\Omega, \mathcal{S}, P)$  and a measure preserving transformation  $T: \Omega \rightarrow \Omega$ , Kolmogorov and Sinaj constructed an invariant  $h(T)$  called the entropy of the dynamical system  $(\Omega, \mathcal{S}, P, T)$ . With help of the invariant they showed that there exist non-isomorphic Bernoulli schemes. Namely, if  $(\Omega_1, \mathcal{S}_1, P_1, T_1)$  and  $(\Omega_2, \mathcal{S}_2, P_2, T_2)$  are two isomorphic dynamical systems, then they have the same entropy  $h(T_1) = h(T_2)$ . Hence if in some case  $h(T_1) \neq h(T_2)$ , then  $(\Omega_1, \mathcal{S}_1, P_1, T_1)$ ,  $(\Omega_2, \mathcal{S}_2, P_2, T_2)$  cannot be isomorphic. (For references see [12].)

Of course, if  $h(T_1) = h(T_2)$ , then generally we cannot say anything about the isomorphism of the corresponding dynamical systems. Therefore we tried in [7] to construct a larger family of invariants  $h_{\mathcal{G}}(T)$  by substituting set partitions by fuzzy set partitions, i.e., collections  $\{f_1, \dots, f_n\}$  of functions  $f_i: \Omega \rightarrow \langle 0, 1 \rangle$  such that  $f_1 + \dots + f_n = 1$ .

The Dumitrescu approach ([4], [5] and also [12]) is a little more general. He does not assume the existence of a probability space and starts with a fuzzy probability

space which is a set  $\mathcal{F} \subset \langle 0, 1 \rangle^\Omega$  with a function  $m: \mathcal{F} \rightarrow \langle 0, 1 \rangle$  satisfying some conditions. Of course, using a representation theorem from [3], we obtain

$$m(f) = \int f \, dP$$

so that the Dumitrescu theory can be reduced to the case studied in [7]. Another approach using fuzzy sets theory has been developed by Markechová ([8], [9], see also Mesiar [10]).

Probably one of the most important results of the theory for practical purposes is the Kolmogorov-Sinaj theorem stating that

$$h(T) = h(T, \mathcal{A}),$$

whenever  $\mathcal{A}$  is a partition generating the given  $\sigma$ -algebra. An analogue of this theorem is contained in [7]:

$$h_{\mathcal{G}}(T) \leq h(T, \mathcal{A}) + K_{\mathcal{G}},$$

where  $K_{\mathcal{G}}$  is a constant depending on the family  $\mathcal{G} \subset \langle 0, 1 \rangle^\Omega$  determining  $h_{\mathcal{G}}(T)$ . (Of course, if  $\mathcal{G}$  consists of indicators  $\chi_E$  only, then  $K_{\mathcal{G}} = 0$ .) Here  $\mathcal{A}$  is a crisp partition generating the given algebra of sets. In this paper we present a variant of the Kolmogorov-Sinaj theorem in the form  $h_{\mathcal{G}}(T) = h(T, \mathcal{A})$ , of course, in the case that  $\mathcal{A}$  is a fuzzy partition generating the given fuzzy  $\sigma$ -algebra  $\mathcal{F}$ .

## 1. DEFINITIONS

We assume that a set  $\mathcal{F} \subset \langle 0, 1 \rangle^X$  is given satisfying the following conditions:

- (i)  $1_X \in \mathcal{F}$ ,
- (ii)  $f, g \in \mathcal{F} \Rightarrow g - \min(f, g) \in \mathcal{F}$ ,
- (iii)  $f, g \in \mathcal{F}, f + g \leq 1 \Rightarrow f + g \in \mathcal{F}$ ,
- (iv)  $f, g \in \mathcal{F} \Rightarrow f \cdot g \in \mathcal{F}$ .

We say that a family satisfying the conditions stated above is a fuzzy algebra. Evidently  $0_X \in \mathcal{F}$ . It is not difficult to show that  $\mathcal{F}$  is closed under the maximum and the minimum. Indeed, if  $f, g \in \mathcal{F}$ , then

$$fS_{\infty}g = \min(f + g, 1) = 1 - (1 - f - \min(1 - f, g)) \in \mathcal{F},$$

$$fT_{\infty}g = \max(f + g - 1, 0) = g - \min(1 - f, g) \in \mathcal{F},$$

and finally

$$\begin{aligned}
 \min(f, g) &= \max(f + \min(g - f, 0), 0) \\
 &= \max(f + \min((1 - f) + g, 1) - 1, 0) \\
 &= fT_\infty((1 - f)S_\infty g) \in \mathcal{F}, \\
 \max(f, g) &= 1 - \min(1 - f, 1 - g) \in \mathcal{F}.
 \end{aligned}$$

Further we assume that a function  $m: \mathcal{F} \rightarrow \langle 0, \infty \rangle$  is given satisfying the following conditions:

- (i)  $f, g \in \mathcal{F}, f \leq g \Rightarrow m(f) \leq m(g)$ ,
  - (ii)  $f, g, h \in \mathcal{F}, f = g + h \Rightarrow m(f) = m(g) + m(h)$ .
- As an easy consequence of (i) and (ii) we obtain
- (iii)  $f \in \mathcal{F}, f_i \in \mathcal{F} (i = 1, \dots, n), f \leq \sum_{i=1}^n f_i \Rightarrow m(f) \leq \sum_{i=1}^n m(f_i)$ .

Indeed, if  $f, g, h \in \mathcal{F}, f \leq g + h$ , then also  $f \leq \min(g + h, 1) \in \mathcal{F}$ , hence

$$\begin{aligned}
 m(f) &\leq m(\min(g + h, 1)) = m(g) + m(\min(g + h, 1) - g) \\
 &= m(g) + m(\min(h, 1 - g)) \leq m(g) + m(h).
 \end{aligned}$$

The third part of our assumptions is concerned with a mapping  $U: \mathcal{F} \rightarrow \mathcal{F}$  satisfying the following conditions:

- (i)  $f, g, h \in \mathcal{F}, f = g + h \Rightarrow U(f) = U(g) + U(h)$ ,
- (ii)  $U(1_X) = 1_X$ ,
- (iii)  $f \in \mathcal{F} \Rightarrow m(U(f)) = m(f)$ .

As a special case of the previous definition one can consider a mapping  $U: \mathcal{F} \rightarrow \mathcal{F}$  induced by a transformation  $T: X \rightarrow X$  by the formula  $Uf(x) = f(T(x))$ . In fact, this is the classical case.

As we have mentioned, a fuzzy partition is a finite collection  $\mathcal{A} = \{f_1, \dots, f_n\}$  of members of  $\mathcal{F}$  such that  $\sum_{i=1}^n f_i(x) = 1$  for all  $x \in X$ . If  $\mathcal{A} = \{f_1, \dots, f_n\}$  is a fuzzy partition, then we define  $U\mathcal{A} = \{Uf_1, \dots, Uf_n\}$ . Evidently  $U\mathcal{A}$  is a fuzzy partition, too. If  $\mathcal{A} = \{f_1, \dots, f_n\}, \mathcal{B} = \{g_1, \dots, g_m\}$  are two partitions, then we put

$$\mathcal{A} \vee \mathcal{B} = \{f_i \cdot g_j; i = 1, \dots, n, j = 1, \dots, m\}.$$

Also  $\mathcal{A} \vee \mathcal{B}$  is a partition. The entropy of the partition  $\mathcal{A}$  is defined by the formula

$$H(\mathcal{A}) = \sum_{i=1}^n \varphi(m(f_i)),$$

where

$$\varphi(x) = -x \log x, \quad \text{if } x > 0, \varphi(0) = 0.$$

The entropy of the dynamical system  $(X, \mathcal{F}, m, U)$  is defined as follows. If  $\mathcal{A}$  is a partition, then

$$h(U, \mathcal{A}) = \lim_{n \rightarrow \infty} \frac{1}{n} H(\mathcal{A} \vee U\mathcal{A} \vee \dots \vee U^{(n-1)}\mathcal{A}).$$

(The existence of the limit follows by the subadditivity property ( $a_{n+m} \leq a_n + a_m$ ) of the sequence  $(a_n)_n$ , where  $a_n = H(\mathcal{A} \vee U\mathcal{A} \vee \dots \vee U^{n-1}\mathcal{A})$ .) Finally, if  $\emptyset \neq \mathcal{G} \subset \mathcal{F}$ , then

$$H_{\mathcal{G}}(U) = \sup \{h(U, \mathcal{A}); \mathcal{A} \subset \mathcal{G}, \mathcal{A} \text{ is a partition}\}.$$

## 2. THEOREM

In order to be able to formulate the main result of the article we need the following notation. If  $f, g: X \rightarrow \langle 0, 1 \rangle$ , then

$$f \Delta g = f(1 - g) + g(1 - f).$$

(Of course, if  $f, g$  are indicators,  $f = \chi_A, g = \chi_B$ , then  $f \Delta g = \chi_{A \Delta B}$ .)

**Theorem.** *If  $\mathcal{C} \subset \mathcal{G}$  is a generator (i.e., such a partition that for every  $\lambda > 0$  and every  $f \in \mathcal{G}$  there is  $g \in \bigcup_{i=0}^{\infty} U^i \mathcal{C}$  such that  $m(f \Delta g) < \lambda$ ), then*

$$h_{\mathcal{G}}(U) = h(U, \mathcal{C}).$$

## 3. PROOF

The main idea of the proof can be found in [2]. We shall divide it into a sequence of lemmas. Denote by  $s(\mathcal{A})$  the fuzzy algebra generated by  $\mathcal{A}$ .

**Lemma 1.** *For a given partition  $\mathcal{A} = \{f_1, \dots, f_n\}$  and every  $\delta > 0$  there exists  $\lambda > 0$  such that for every family  $\mathcal{B} = \{g_1, \dots, g_n\}$  with  $m(f_i \Delta g_i) < \lambda$  ( $i = 1, 2, \dots, n$ ) there is a partition  $\{h_1, \dots, h_n\} \subset s(\mathcal{B})$  such that  $m(f_i \Delta h_i) < \delta$  ( $i = 1, 2, \dots, n$ ).*

Proof. Put

$$\begin{aligned} h_1 &= g_1, \\ h_i &= \min \left( g_i, 1 - \sum_{j=1}^{i-1} h_j \right), \quad i = 2, \dots, n-1, \\ h_n &= 1 - \sum_{j=1}^{n-1} h_j. \end{aligned}$$

Evidently  $\{h_1, \dots, h_n\}$  is a partition included in  $s(\mathcal{B})$ . Let  $1 < i < n$ . Then

$$(*) \quad m(f_i \Delta h_i) \leq \sum_{j=1}^{i-1} m(h_j \Delta f_j) + m(f_i \Delta g_i).$$

Indeed, if  $h_i(x) = g_i(x)$ , then

$$h_i \Delta f_i(x) = f_i \Delta g_i(x).$$

If  $h_i(x) = 1 - \sum_{j=1}^{i-1} h_j(x) \leq g_i(x)$ , then

$$\begin{aligned} h_i(x)(1 - f_i(x)) &\leq g_i(x)(1 - f_i(x)) \leq f_i \Delta g_i(x), \\ f_i(x)(1 - h_i(x)) &= f_i(x) \sum_{j=1}^{i-1} h_j(x) \leq \sum_{j=1}^{i-1} (1 - f_j(x))h_j(x) \leq \sum_{j=1}^{i-1} h_j \Delta f_j(x) \end{aligned}$$

so that

$$h_i \Delta f_i \leq \sum_{j=1}^{i-1} h_j \Delta f_j + f_i \Delta g_i,$$

which implies (\*). Now let  $i = n$ . Then

$$\begin{aligned} (1 - h_n(x))f_n(x) &= \sum_{j=1}^{n-1} h_j(x)f_n(x) \leq \sum_{j=1}^{n-1} h_j(x)(1 - f_j(x)) \leq \sum_{j=1}^{n-1} h_j \Delta f_j(x), \\ (1 - f_n(x))h_n(x) &= \sum_{j=1}^{n-1} f_j(x)h_n(x) \leq \sum_{j=1}^{n-1} f_j(x)(1 - h_j(x)) \leq \sum_{j=1}^{n-1} f_j \Delta h_j(x), \end{aligned}$$

hence

$$(**) \quad m(f_n \Delta h_n) \leq \sum_{j=1}^{n-1} m(h_j \Delta f_j).$$

Since  $m(f_i \Delta g_i) < \lambda$ , we obtain by (\*) and (\*\*) that

$$m(h_i \Delta f_i) < 2^{i-1} \lambda \quad (i = 1, 2, \dots, n).$$

Therefore we can put  $\lambda = \frac{\delta}{2^{n-1}}$ . □

For any partitions  $\mathcal{A} = \{f_1, \dots, f_n\}$ ,  $\mathcal{B} = \{g_1, \dots, g_m\}$  we define the conditional entropy

$$H(\mathcal{A}|\mathcal{B}) = \sum_{i=1}^n \sum_{\substack{j=1 \\ m(h_j) > 0}}^m m(h_j) \varphi \left( \frac{m(f_i h_j)}{m(h_j)} \right).$$

**Lemma 2.** For every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $H(\mathcal{A}|\mathcal{B}) < \varepsilon$  for any partitions  $\mathcal{A} = \{f_1, \dots, f_n\}$ ,  $\mathcal{B} = \{h_1, \dots, h_n\}$  satisfying the condition  $m(f_i \Delta h_i) < \delta$  ( $i = 1, \dots, n$ ).

*Proof.* First choose  $\delta_0 \in (0, 1)$  such that  $\varphi(t) < \frac{\varepsilon}{n}$  for every  $t \notin (\delta_0, 1 - \delta_0)$  and put

$$\delta = \min \left\{ \frac{\delta_0}{2} m(f_i); m(f_i) > 0 \right\}.$$

Then

$$\begin{aligned} m(f_i) &\leq m(f_i \Delta h_i) + m(h_i) < \delta + m(h_i) < \delta_0 \frac{m(f_i)}{2} + m(h_i), \\ \frac{m(f_i)}{2} &< m(f_i) - \delta_0 \frac{m(f_i)}{2} < m(h_i), \\ m(h_i) - m(f_i h_i) &\leq m(h_i \Delta f_i) \leq \delta \leq \delta_0 m(h_i). \end{aligned}$$

If we consider such an  $i$  that  $m(h_i) > 0$ , then

$$\frac{m(f_i h_i)}{m(h_i)} > 1 - \delta_0,$$

hence

$$\begin{aligned} \varphi \left( \frac{m(f_i h_i)}{m(h_i)} \right) &< \frac{\varepsilon}{n}, \\ H(\mathcal{A}|\mathcal{B}) &= \sum_i \sum_j m(h_i) \varphi \left( \frac{m(f_i h_i)}{m(h_i)} \right) < \sum_i \sum_j m(h_i) \frac{\varepsilon}{n} = \sum_j \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

□

**Lemma 3.**  $h(U, \mathcal{A}) \leq h(U, \mathcal{C}) + H(\mathcal{A}|\mathcal{C})$  for any partitions  $\mathcal{A}, \mathcal{C}$ .

*Proof.* Since  $h(\mathcal{B} \vee \mathcal{D}) = H(\mathcal{B}) + H(\mathcal{D}|\mathcal{B})$  (see [5], [6]), we have

$$\begin{aligned} H \left( \bigvee_{i=0}^{n-1} U^i \mathcal{A} \right) &\leq H \left( \bigvee_{i=0}^{n-1} U^i \mathcal{A} \vee \bigvee_{j=0}^{n-1} U^j \mathcal{C} \right) \\ &= H \left( \bigvee_{j=0}^{n-1} U^j \mathcal{C} \right) + H \left( \bigvee_{i=0}^{n-1} U^i \mathcal{A} \middle| \bigvee_{j=0}^{n-1} U^j \mathcal{C} \right). \end{aligned}$$

Further  $H(\mathcal{D} \vee \mathcal{E} | \mathcal{B}) \leq H(\mathcal{D} | \mathcal{B}) + H(\mathcal{E} | \mathcal{B})$ ,  $H(\mathcal{D} | \mathcal{B} \vee \mathcal{E}) \leq H(\mathcal{D} | \mathcal{B})$  (see [5], [6]), hence

$$\begin{aligned} H\left(\bigvee_{i=0}^{n-1} U^i \mathcal{A} \middle| \bigvee_{j=0}^{n-1} U^j \mathcal{C}\right) &\leq \sum_{i=0}^{n-1} H\left(U^i \mathcal{A} \middle| \bigvee_{j=0}^{n-1} U^j \mathcal{C}\right) \\ &\leq \sum_{i=0}^{n-1} H(U^i \mathcal{A} | U^i \mathcal{C}) = nH(\mathcal{A} | \mathcal{C}). \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} U^i \mathcal{A}\right) \leq \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} U^j \mathcal{C}\right) + H(\mathcal{A} | \mathcal{C}).$$

□

**Lemma 4.**  $h(U, \mathcal{C}) = h\left(U, \bigvee_{j=0}^k U^j \mathcal{C}\right)$  for every  $k \in \mathbb{N}$  and any partition  $\mathcal{C}$ .

*Proof.* We immediately obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \left( \bigvee_{i=0}^{n-1} U^i \left( \bigvee_{j=0}^k U^j \mathcal{C} \right) \right) &= \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{t=0}^{n+k-1} U^t \mathcal{C}\right) \\ &= \lim_{p \rightarrow \infty} \frac{p}{p-k} \cdot \frac{1}{p} H\left(\bigvee_{t=0}^{p-1} U^t \mathcal{C}\right) = h(U, \mathcal{C}). \end{aligned}$$

□

*Proof of Theorem.* Let  $\varepsilon$  be an arbitrary positive number. Choose  $\delta > 0$  by Lemma 2 and  $\lambda > 0$  by Lemma 1. Now let  $\mathcal{A} = \{f_1, \dots, f_n\} \subset \mathcal{G}$  be any partition. For every  $i$  there are  $t_i \in \mathbb{N}$  and  $g_i \in U^{t_i} \mathcal{C}$  such that  $m(f_i \Delta g_i) < \lambda$  ( $i = 1, 2, \dots, n$ ). For sufficiently large  $k$  we have  $\{g_1, \dots, g_n\} \subset \bigcup_{j=0}^k U^j \mathcal{C}$ . By Lemma 1 there is a partition  $\mathcal{B} = \{h_1, \dots, h_n\}$  such that  $h_i \in \bigcup_{j=0}^k U^j \mathcal{C}$  and  $m(f_i \Delta h_i) < \delta$  ( $i = 1, \dots, n$ ). By Lemma 2 we obtain  $H(\mathcal{A} | \mathcal{B}) < \varepsilon$ , by Lemma 4 we have

$$h\left(U, \bigcup_{j=0}^k U^j \mathcal{C}\right) = H(U, \mathcal{C}).$$



Therefore by Lemma 3 we conclude

$$\begin{aligned} h(U, \mathcal{A}) &\leq h(U, \mathcal{B}) + H(\mathcal{A}|\mathcal{B}) < h(U, \mathcal{B}) + \varepsilon \\ &\leq h\left(U, \bigcup_{j=0}^k U^j \mathcal{C}\right) + \varepsilon = h(U, \mathcal{C}) + \varepsilon, \\ H_{\mathcal{G}}(U) = \sup\{H(U, \mathcal{A}); \mathcal{A} \subset \mathcal{G}\} &\leq h(U, \mathcal{C}). \end{aligned}$$

□

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*Author's address*: Beloslav Riečan, Matematický ústav SAV, Štefánikova 49, 814 73 Bratislava, Slovak Republic.