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HIGHER ORDER FINITE ELEMENT APPROXIMATION OF A QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEM OF A NON-MONOTONE TYPE

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Summary. A nonlinear elliptic partial differential equation with homogeneous Dirichlet boundary conditions is examined. The problem describes for instance a stationary heat conduction in nonlinear inhomogeneous and anisotropic media. For finite elements of degree $k \geqslant 1$ we prove the optimal rates of convergence $\mathcal{O}(h^k)$ in the H^1-norm and $\mathcal{O}(h^{k+1})$ in the L^2-norm provided the true solution is sufficiently smooth. Considerations are restricted to domains with polyhedral boundaries. Numerical integration is not taken into account.

Keywords: nonlinear boundary value problem, finite elements, rate of convergence, anisotropic heat conduction

AMS classification: 65N30

1. INTRODUCTION

In this paper we deal with a quasilinear elliptic problem whose classical formulation reads:

Find $u \in C(\overline{\Omega})$ such that $u|_\Omega \in C^2(\Omega)$ and

$$-\operatorname{div}(A(x,u) \operatorname{grad} u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, is a bounded domain with a Lipschitz boundary, $f \in L^2(\Omega)$, $A = (a_{ij})_{i,j=1}^d$ is a bounded uniformly positive definite matrix, i.e.,

$$\max_{x\in\Omega} \max_{\xi\in\mathbb{R}} |a_{ij}(x,\xi)| \leqslant C \qquad \forall i,j \in \{1,\dots,d\}, \tag{1.3}$$

$$C_0 \eta^T \eta \leqslant \eta^T A(x,\xi)\eta \quad \forall \eta \in \mathbb{R}^d \quad \forall x \in \Omega \quad \forall \xi \in \mathbb{R}, \tag{1.4}$$

where $C_0 > 0$ and, moreover, we assume that the derivatives $\partial a_{ij}/\partial\xi$ and $\partial^2 a_{ij}/\partial\xi^2$ are bounded and continuous on $\overline{\Omega} \times \mathbb{R}$. The matrix A need not be symmetric.

The problem (1.1)–(1.2) for $d > 1$ cannot be converted, in general, by the well-known Kirchhoff transformation to a linear problem even if A is independent of x, since A is a matrix function.

The existence of a weak solution u is obtained as a weak limit of Galerkin approximations. The uniqueness of the classical and weak solutions is proved in [13] and [14], respectively. Several uniqueness and comparison theorems for similar problems can be found in [1, 5, 11, 16]. The existence of the weak solution for various kinds of boundary conditions (including (1.2)) is studied in [9, 11, 14, 21].

In [4], Douglas and Dupont derived an optimal rate of convergence of the finite element method for the problem (1.1)–(1.2) in the case that

$$A(x,u) = \lambda(x,u)\, I, \tag{1.5}$$

where I is the identity matrix and λ is a smooth scalar function. The main aim of this paper (see Theorem 4.1) is to generalize the result of [4] to any smooth uniformly positive definite matrix $A(x,u)$ satisfying (1.3) and (1.4). This represents a practically interesting case, since the problem (1.1)–(1.2) describes a steady-state heat conduction in nonlinear inhomogeneous anisotropic media (e.g., in magnetic cores of large transformers, see [17]). The unknown function u represents the temperature, A is the matrix of heat conductivities and f is the density of volume heat sources. In this case A is symmetric.

The finite element method for the case (1.5) has been considered by many other authors. For instance, in [22], the method of Douglas and Dupont from [4] is generalized to obtain an asymptotic error estimate in the L^∞-norm. An optimal rate of convergence in the L^p-norm is proved in [19] for a mixed finite element method. Similar results were also obtained in the paper [2].

Note that an analogue of the well-known Céa's lemma holds for those nonlinear elliptic problems whose associated operators are strongly monotone and Lipschitz continuous (see [3, 17]). Hence, in this case it is easy to derive the rate of convergence $\mathcal{O}(h^k)$ in the H^1-norm for the Lagrange elements of degree k. However, the papers [9, 14] contain one-dimensional examples which illustrate that the problem (1.1)–(1.2) is of a non-monotone and non-potential type.

Finite element approximations of nonlinear elliptic problems of strongly monotone and also pseudomonotone type are profoundly studied in [7, 8, 27]. The authors consider the numerical integration as well as the approximation of a curved boundary. They obtain a linear rate of convergence in the H^1-norm for linear finite elements provided the true solution is sufficiently smooth. In [27], the rate of convergence

$\mathcal{O}(h^\varepsilon)$ is proved for $u \in H^{1+\varepsilon}(\Omega)$. However, the papers [7, 8, 27] do not deal with higher order elements and the optimal error estimates in the $L^2(\Omega)$-norm.

2. Weak formulation and finite element approximation

Throughout the paper we shall employ the standard Sobolev space notation (see [3, 20]). The norm in the product Sobolev space $(W_p^k(\Omega))^n$, $k \in \{0, 1, \ldots\}$, $p \in [1, \infty]$, $n \in \{1, 2, \ldots\}$, is denoted by $\|\cdot\|_{k,p}$. In particular, if $p = 2$ then we set $H^k(\Omega) = W_2^k(\Omega)$ and $\|\cdot\|_k = \|\cdot\|_{k,2}$. By $H_0^1(\Omega)$ we mean the space of functions from $H^1(\Omega)$ whose traces vanish on $\partial\Omega$. The symbol $(\cdot,\cdot)_0$ stands for the usual scalar product in $L^2(\Omega)$.

According to the Cauchy-Schwarz and Hölder inequalities, we get

$$\|v\|_{0,3}^3 \leqslant \|v\|_0 \|v^2\|_0 \leqslant \|v\|_0 \|v\|_{0,3} \|v\|_{0,6} \quad \forall v \in L^6(\Omega).$$

From here and the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ for $d \leqslant 3$ (see [3, p. 114]) we find the inequality which will be used later:

$$\|v\|_{0,3} \leqslant C(\|v\|_0 \|v\|_1)^{1/2} \quad \forall v \in H^1(\Omega). \tag{2.1}$$

The weak formulation of problem (1.1)–(1.2) consists in finding $u \in H_0^1(\Omega)$ such that

$$a(u; u, v) = (f, v)_0 \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(z; w, v) = \int_\Omega (\operatorname{grad} w)^T A(x, z) \operatorname{grad} v \, dx, \quad v,\, w \in H^1(\Omega),\, z \in L^2(\Omega).$$

The argument x will be sometimes omitted in what follows. From (1.4), (1.3) and Friedrichs' inequality we see that there exist positive constants C_0 and C_1 such that

$$a(z; v, v) \geqslant C_0 \|v\|_1^2 \quad \forall z \in L^2(\Omega) \quad \forall v \in H_0^1(\Omega)$$

and

$$|a(z; w, v)| \leqslant C_1 \|v\|_1 \|w\|_1 \quad \forall z \in L^2(\Omega) \quad \forall w, v \in H^1(\Omega).$$

This means that $a(\cdot;\cdot,\cdot)$ is uniformly $H_0^1(\Omega)$-elliptic and continuous.

Theorem 2.1. *The weak solution of* (2.2) *exists and is unique.*

The proof is given in [14]. □

From now on assume that $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, has a polyhedral boundary and let $\mathcal{T}_h$ be a standard triangulation of $\overline{\Omega}$ into polyhedral elements (see [3]). Let us introduce the approximate problem: Find $u_h \in V_h$ such that

$$a(u_h; u_h, v_h) = (f, v_h)_0 \quad \forall v_h \in V_h, \tag{2.3}$$

where

$$V_h = \{v_h \in H_0^1(\Omega) \mid v_h\big|_K \in P_K \quad \forall K \in \mathcal{T}_h\}$$

is the finite element space, P_K is a finite dimensional space such that $P_K \supseteq P_k(K)$, $k \geqslant 1$ is an integer and $P_k(K)$ is the space of all polynomials of degree at most k defined on K. The space V_h can be generated by the Lagrange elements (or Hermite elements for $k \geqslant 3$).

Remark 2.2. The existence of at least one solution u_h of (2.3) can be proved by the Brouwer fixed-point theorem (see [14, p. 174]). Some special sufficient conditions guaranteeing the uniqueness of u_h are given in [12, 14]. Nevertheless, the uniqueness of u_h, in general, has remained an open problem until now.

Remark 2.3. In [6], the existence of a discrete solution is proved in the case of linear elements, numerical integration and approximation of a piecewise curved boundary by a polygonal one. The proof is based on some results of [7, 8, 26]. A discrete maximum principle for the problem (1.1)–(1.2) in the case (1.5) is derived in [15]. The publications [17, 18, 24] are devoted to numerical calculation of real-life technical problems which are described by the equation (1.1).

Remark 2.4. The convergence of approximate solutions u_h to the weak solution u of (1.1)–(1.2) in the $H^1(\Omega)$-norm was proved in [14]. However, no attempt to derive any rate of convergence was made there.

Finally, we introduce an auxiliary lemma which will be used in Section 4.

Lemma 2.5. *Let α, β and γ be arbitrary real nonnegative numbers such that*

$$\alpha \leqslant C(\beta + \sqrt{\alpha\gamma}). \tag{2.4}$$

Then there exists a constant $C' > 0$ independent of α, β, γ such that

$$\alpha \leqslant C'(\beta + \gamma). \tag{2.5}$$

Proof. If $\alpha = 0$ then (2.5) holds. So let $\alpha \neq 0$. Then by (2.4)

$$C^2\gamma \geqslant \frac{(\alpha - C\beta)^2}{\alpha} \geqslant \alpha - 2C\beta.$$

□

3. Adjoint problem

In the next section we derive the optimal a priori asymptotic error estimate in the $H^1(\Omega)$-norm and also in the $L^2(\Omega)$-norm. In the latter case, we will employ the Aubin-Nitsche trick. To this end we shall utilize the weak solution φ of the linear problem

$$
\begin{aligned}
L^*\varphi \equiv -\operatorname{div}(A^T(x,u)\ \operatorname{grad}\varphi) + (\operatorname{grad} u)^T A_u^T(x,u)\ \operatorname{grad}\varphi &= \zeta \quad \text{in } \Omega, \\
\varphi &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{3.1}
$$

where u is the unique solution of (2.2), $\zeta \in L^2(\Omega)$, $A_u = ((a_{ij})_u)_{i,j=1}^d$ and the subscript u means the differentiation with respect to the last variable, i.e., $(a_{ij})_u = \partial a_{ij}(x,u)/\partial u$. In Theorem 3.1, we give a sufficient condition guaranteeing the existence and uniqueness of the weak (generalized) solution of the problem (3.1).

First we show how the above problem (3.1) can formally be obtained. Set

$$
\mathcal{L}(u) = -\operatorname{div}(A(u)\ \operatorname{grad} u)
$$

and choose $v \in H_0^1(\Omega) \cap H^2(\Omega)$ arbitrarily. Then for any real $t \neq 0$ we have

$$
\begin{aligned}
\frac{1}{t}(\mathcal{L}(u+tv) - \mathcal{L}(u)) &= -\frac{1}{t}\operatorname{div}(A(u+tv)\ \operatorname{grad}(u+tv) - A(u)\ \operatorname{grad} u \\
&\qquad - A(u)\ \operatorname{grad}(tv) + tA(u)\ \operatorname{grad} v) \\
&= -\operatorname{div}\Big(\frac{A(u+tv) - A(u)}{t}\ \operatorname{grad}(u+tv) + A(u)\ \operatorname{grad} v\Big).
\end{aligned}
$$

Letting $t \to 0$, we obtain the Gâteaux derivative of $\mathcal{L}$ at the point u and in the direction v

$$
Lv \equiv D\mathcal{L}(u;v) = -\operatorname{div}(A(x,u)\ \operatorname{grad} v + vA_u(x,u)\ \operatorname{grad} u).
$$

Notice that this operator is linear.

Now choose $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ arbitrarily. Then, applying twice the Green theorem, we get

$$
\begin{aligned}
(Lv,\varphi)_0 &= -\int_\Omega \operatorname{div}(A(u)\ \operatorname{grad} v + vA_u(u)\ \operatorname{grad} u)\varphi\, dx \\
&= \int_\Omega (\operatorname{grad}\varphi)^T(A(u)\ \operatorname{grad} v + vA_u(u)\ \operatorname{grad} u)\, dx \\
&= \int_\Omega v(-\operatorname{div}(A^T(u)\ \operatorname{grad}\varphi) + (\operatorname{grad} u)^T A_u^T(u)\ \operatorname{grad}\varphi)\, dx \\
&= (v, L^*\varphi)_0,
\end{aligned}
$$

i.e., the linear operator L^* is adjoint to L. If $A(u)$ is independent of u then, of course, $A_u = 0$ and we get the standard adjoint problem like in [3, p. 138].

The weak formulation of (3.1) reads: Find $\varphi \in H_0^1(\Omega)$ such that

$$b(\varphi, v) = (\zeta, v)_0 \quad \forall v \in H_0^1(\Omega),$$

where

$$b(\varphi, v) = \int_\Omega [(\operatorname{grad} v)^T \mathcal{A}^T \operatorname{grad} \varphi + v c^T \operatorname{grad} \varphi] \,\mathrm{d}x,$$

$$\mathcal{A}(x) = A(x, u(x)),$$
$$c(x) = A_u(x, u(x)) \operatorname{grad} u(x)$$

for $x \in \Omega$ and $u \in H_0^1(\Omega)$ is the unique weak solution of (1.1)–(1.2) (compare Theorem 2.1).

Theorem 3.1. *Let $c \in (L^\infty(\Omega))^d$ and let (1.3) and (1.4) hold. Then there exists precisely one weak solution $\varphi \in H_0^1(\Omega)$ of the classical problem (3.1).*

Proof. By (1.3) and (1.4), the matrix $\mathcal{A}$ is bounded and uniformly positive definite. Since c is also bounded, the bilinear form $b(\cdot, \cdot)$ is continuous and the theorem directly follows from [11, p. 170]. (The proof of uniqueness of φ is based on the weak maximum principle and the existence of φ is a consequence of the Gårding inequality, the Fredholm alternative and the uniqueness.) □

Remark 3.2. If the weak solution u of the problem (1.1)–(1.2) belongs to the space of Lipschitz continuous functions $W_\infty^1(\Omega)$, then the assumption $c \in (L^\infty(\Omega))^d$ of the above Theorem 3.1 is obviously satisfied.

Remark 3.3. If $c \in (C^1(\overline{\Omega}))^d$ and $\operatorname{div} c \leqslant 0$ then for any $v \in H_0^1(\Omega)$ we get, by the Green theorem, that

$$(vc, \operatorname{grad} v)_0 = -(\operatorname{div}(cv), v)_0 = -(v \operatorname{div}\, c, v)_0 - (vc, \operatorname{grad} v)_0$$

and thus

$$(vc, \operatorname{grad} v)_0 = -\frac{1}{2}(\operatorname{div}\, c, v^2)_0 \geqslant 0.$$

Hence, the bilinear form is $H_0^1(\Omega)$-elliptic (see also [20, p. 44]),

$$b(v, v) \geqslant \int_\Omega (\operatorname{grad} v)^T \mathcal{A}^T \operatorname{grad} v \,\mathrm{d}x \geqslant C_0 \|v\|_1^2 \quad \forall v \in H_0^1(\Omega)$$

by the Friedrichs inequality and thus the well-known Lax-Milgram lemma [3, 17, 20] can be applied.

In the next Section 4 we shall, moreover, require the regularity

$$\|\varphi\|_2 \leqslant C\|\zeta\|_0, \tag{3.2}$$

where $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ is the weak solution of (3.1).

4. Rate of convergence

Throughout this section we assume that the family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of triangulations is regular, i.e., there exists a constant $\varkappa > 0$ such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any element $K \in \mathcal{T}_h$ there exists a ball B_K of radius ϱ_K such that $B_K \subset K$ and

$$\varkappa \operatorname{diam} K \leqslant \varrho_K. \tag{4.1}$$

Theorem 4.1. *Let $u \in H^{k+1}(\Omega)$ for $k \geqslant 1$ be the weak solution of* (1.1)–(1.2) *and let* (3.2) *hold. If u_h is a solution of* (2.3), *then there exist C, $h_0 > 0$ such that for any $h \in (0, h_0)$ we have*

$$\|u - u_h\|_0 + h\|u - u_h\|_1 \leqslant Ch^{k+1}, \tag{4.2}$$

where C depends on $\|u\|_{k+1}$.

Proof. Since $d \leqslant 3$, we have $H^2(\Omega) \hookrightarrow W_6^1(\Omega)$ (see [3, p. 114]) and thus $\|u\|_{1,6}$ is finite. According to [3, p. 123], for the solution $u \in H^{k+1}(\Omega)$ of (1.1)–(1.2) and sufficiently small h we obtain by the regularity of $\mathcal{F}$ (see (4.1)) that

$$\|u - \pi_h u\|_1 + h\|u - \pi_h u\|_{1,6} \leqslant Ch^k \|u\|_{k+1}, \tag{4.3}$$

where $\pi_h u \in V_h$ is the V_h-interpolant of u. In particular,

$$\|\pi_h u\|_{1,6} \leqslant \|u - \pi_h u\|_{1,6} + \|u\|_{1,6} \leqslant C\|u\|_{k+1}. \tag{4.4}$$

By the uniform $H_0^1(\Omega)$-ellipticity of $a(\cdot;\cdot,\cdot)$, (2.2), (2.3) and the Hölder inequality, we arrive at

$$\begin{aligned}
&C_0\|u_h - \pi_h u\|_1^2 \leqslant a(u_h; u_h - \pi_h u, u_h - \pi_h u)\\
&= a(u_h; u_h, u_h - \pi_h u) - a(u_h; \pi_h u, u_h - \pi_h u)\\
&= a(u; u, u_h - \pi_h u) - a(u_h; \pi_h u, u_h - \pi_h u)\\
&\leqslant |a(u; u - \pi_h u, u_h - \pi_h u)| + |a(u; \pi_h u, u_h - \pi_h u) - a(u_h; \pi_h u, u_h - \pi_h u)|\\
&\leqslant C_1\|u - \pi_h u\|_1\|u_h - \pi_h u\|_1 + C_2\|A(u) - A(u_h)\|_{0,3}\|\operatorname{grad} \pi_h u\|_{0,6}\|u_h - \pi_h u\|_1.
\end{aligned}$$

This, (4.4) and (4.3) imply that

$$
\begin{aligned}
C_0\|u_h - \pi_h u\|_1 &\leqslant C_1\|u - \pi_h u\|_1 + C_3\|A(u) - A(u_h)\|_{0,3} \\
&\leqslant C(h^k\|u\|_{k+1} + \|A(u) - A(u_h)\|_{0,3}),
\end{aligned}
\tag{4.5}
$$

where C_3 depends on u. Since the entries $a_{ij} = a_{ij}(x, \xi)$ are Lipschitz continuous with respect to the last variable ξ, we get by (2.1) that

$$
\|A(u) - A(u_h)\|_{0,3} \leqslant C_1\|u - u_h\|_{0,3} \leqslant C_2(\|u - u_h\|_0\|u - u_h\|_1)^{1/2}.
$$

From here, (4.3) and (4.5) it follows that

$$
\|u - u_h\|_1 \leqslant \|u - \pi_h u\|_1 + \|u_h - \pi_h u\|_1 \leqslant C(h^k\|u\|_{k+1} + \|u - u_h\|_0^{1/2}\|u - u_h\|_1^{1/2}).
$$

Setting

$$
\zeta \equiv u - u_h \in L^2(\Omega),
$$

we see by Lemma 2.5 that

$$
\|\zeta\|_1 \leqslant C(h^k\|u\|_{k+1} + \|\zeta\|_0). \tag{4.6}
$$

In order to bound $\|\zeta\|_0 = \|u - u_h\|_0$, we use a duality argument (see [4, 23]) based on the Aubin-Nitsche trick. Let $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ be the weak solution of the linear adjoint problem (3.1). Then, by the Green theorem,

$$
\begin{aligned}
\|\zeta\|_0^2 &= \int_\Omega \zeta^2 \,\mathrm{d}x = \int_\Omega \zeta(L^*\varphi)\,\mathrm{d}x \\
&= \int_\Omega [(\operatorname{grad}\zeta)^T A^T(u) \operatorname{grad}\varphi + \zeta\,(\operatorname{grad}u)^T A_u^T \operatorname{grad}\varphi]\,\mathrm{d}x \\
&= \int_\Omega [(\operatorname{grad}\varphi)^T A(u) \operatorname{grad}u - (\operatorname{grad}\varphi)^T A(u) \operatorname{grad}u_h \\
&\quad + \zeta\,(\operatorname{grad}u)^T A_u^T \operatorname{grad}\varphi]\,\mathrm{d}x \\
&= \int_\Omega [(\operatorname{grad}\varphi)^T A(u) \operatorname{grad}u - (\operatorname{grad}\varphi)^T A(u_h) \operatorname{grad}u_h \\
&\quad + (\operatorname{grad}\varphi)^T (A(u_h) - A(u)) \operatorname{grad}u_h + \zeta\,(\operatorname{grad}\varphi)^T A_u \operatorname{grad}u]\,\mathrm{d}x.
\end{aligned}
\tag{4.7}
$$

For any $x \in \Omega$ we have, by the mean value theorem,

$$
\begin{aligned}
A(x, u_h) - A(x, u) &= \int_0^1 A_u(x, u + t(u_h - u))(u_h - u)\,\mathrm{d}t \\
&= -\zeta \int_0^1 A_u(x, u - t\zeta)\,\mathrm{d}t = -\zeta\overline{A}_u(x),
\end{aligned}
$$

where $\overline{A}_u = ((\overline{a}_{ij})_u)_{i,j=1}^d$, and $(\overline{a}_{ij})_u = (a_{ij})_u(u - \theta_{ij}\zeta)$ for some $\theta_{ij} = \theta_{ij}^h(x) \in [0,1]$. Hence, for any $v_h \in V_h$ we obtain by (4.7), (2.2) and (2.3) that

$$\begin{aligned} \|\zeta\|_0^2 = &\int_\Omega [(\operatorname{grad}(\varphi - v_h))^T A(u) \operatorname{grad} u - (\operatorname{grad}(\varphi - v_h))^T A(u_h) \operatorname{grad} u_h \\ &\quad + \zeta(\operatorname{grad}\varphi)^T \overline{A}_u \operatorname{grad}(\zeta - u) + \zeta(\operatorname{grad}\varphi)^T A_u \operatorname{grad} u]\,\mathrm{d}x \\ = &\int_\Omega (\operatorname{grad}(\varphi - v_h))^T (A(u) - A(u_h)) \operatorname{grad} u\,\mathrm{d}x \\ &+ \int_\Omega (\operatorname{grad}(\varphi - v_h))^T A(u_h) \operatorname{grad}(u - u_h)\,\mathrm{d}x \\ &+ \int_\Omega \zeta\,(\operatorname{grad}\varphi)^T \overline{A}_u \operatorname{grad}\zeta\,\mathrm{d}x + \int_\Omega \zeta\,(\operatorname{grad}\varphi)^T (A_u - \overline{A}_u) \operatorname{grad} u\,\mathrm{d}x. \end{aligned} \tag{4.8}$$

Using similar arguments as before, the differentiability of A and the substitution $z = st$, we find for any $x \in \Omega$ that

$$\begin{aligned} A_u(x,u) - \overline{A}_u(x) &= \int_0^1 [A_u(x,u) - A_u(x, u + t(u_h - u))]\,\mathrm{d}t \\ &= \int_0^1 \Big(\int_0^1 A_{uu}(x, u + st(u_h - u)) t\zeta\, ds \Big)\,\mathrm{d}t \\ &= -\zeta \int_0^1 \Big(\int_0^t A_{uu}(x, u - \zeta z)\,\mathrm{d}z \Big)\,\mathrm{d}t \\ &= -\zeta \int_0^1 \Big(\int_z^1 A_{uu}(x, u - \zeta z)\,\mathrm{d}t \Big)\,\mathrm{d}z \\ &= -\zeta \int_0^1 (1 - z) A_{uu}(x, u - \zeta z)\,\mathrm{d}z = -\zeta \overline{A}_{uu}(x). \end{aligned}$$

Hence, since the derivatives of a_{ij} up to order two are bounded and since

$$\|\zeta\|_{0,3} \leqslant C\|\zeta\|_1$$

and $H^2(\Omega) \hookrightarrow W_6^1(\Omega)$ for $n \leqslant 3$, we have by (4.8) and the Hölder inequality that

$$\begin{aligned} \|\zeta\|_0^2 = &\int_\Omega \zeta(\operatorname{grad}(\varphi - v_h))^T \overline{A}_u \operatorname{grad} u\,\mathrm{d}x \\ &+ \int_\Omega (\operatorname{grad}(\varphi - v_h))^T A(u_h) \operatorname{grad}(u - u_h)\,\mathrm{d}x \\ &+ \int_\Omega \zeta\,(\operatorname{grad}\varphi)^T \overline{A}_u \operatorname{grad}\zeta\,\mathrm{d}x - \int_\Omega \zeta^2\,(\operatorname{grad}\varphi)^T \overline{A}_{uu} \operatorname{grad} u\,\mathrm{d}x \\ \leqslant &\; C\|\zeta\|_{0,3}\|\operatorname{grad}(\varphi - v_h)\|_0\|\operatorname{grad} u\|_{0,6} + C\|\varphi - v_h\|_1\|\zeta\|_1 \\ &+ C\|\zeta\|_{0,3}\|\operatorname{grad}\varphi\|_{0,6}\|\operatorname{grad}\zeta\|_0 + C\|\zeta\|_{0,3}^2\|\operatorname{grad}\varphi\|_{0,6}\|\operatorname{grad} u\|_{0,6} \\ \leqslant &\; C(u)(\|\varphi - v_h\|_1 + \|\zeta\|_1\|\varphi\|_2)\|\zeta\|_1 \end{aligned} \tag{4.9}$$

for any $v_h \in V_h$. Now choose $v_h \in V_h$ such that

$$\|\varphi - v_h\|_1 + h\|\varphi - v_h\|_{1,6} \leqslant Ch\|\varphi\|_2. \tag{4.10}$$

Then, by (4.9), we obtain

$$\|\zeta\|_0^2 \leqslant C(h + \|\zeta\|_1)\|\zeta\|_1\|\varphi\|_2,$$

where C depends on $\|u\|_2$. Therefore, the inequality (3.2) implies that

$$\|\zeta\|_0 \leqslant C(h\|\zeta\|_1 + \|\zeta\|_1^2).$$

Utilizing (4.6), we get

$$\|\zeta\|_0 \leqslant C(h^{k+1} + h\|\zeta\|_0 + h^{2k} + \|\zeta\|_0^2),$$

where C depends on $\|u\|_{k+1}$. Using (4.6) once again, we find that

$$\|\zeta\|_0 + h\|\zeta\|_1 \leqslant C(h^{k+1} + h\|\zeta\|_0 + h^{2k} + \|\zeta\|_0^2).$$

Consequently, for $k \geqslant 1$ and sufficiently small h we have

$$\|\zeta\|_0 + h\|\zeta\|_1 \leqslant C'(h^{k+1} + \|\zeta\|_0^2). \tag{4.11}$$

This inequality proves the theorem provided we can show that $\|\zeta\|_0 \to 0$ as $h \to 0$ (see also [14]).

From (4.3), (4.5) and the boundedness of A, we see that

$$\|u - u_h\|_1 \leqslant \|u - \pi_h u\|_1 + \|\pi_h u - u_h\|_1 \leqslant Ch^k\|u\|_{k+1} + C\alpha_1 \leqslant C.$$

Hence,

$$\|u_h\|_1 \leqslant C.$$

As a consequence of the Eberlein-Schmulyan theorem (see [25, Chap. V]) there exist an element $\omega \in H^1(\Omega)$ and a subsequence of $\{u_h\}$, denoted again by $\{u_h\}$, such that $u_h \rightharpoonup \omega$ in $H^1(\Omega)$. By the Rellich theorem (see [20, p. 17]), $u_h \to \omega$ in $L^2(\Omega)$. We wish to demonstrate that $\omega \equiv u$. To do that let $v \in C_0^\infty(\Omega)$. Then $\pi_h v \in V_h$ and we have $\|v - \pi_h v\|_1 \to 0$ as $h \to 0$. Therefore, by the relations (2.2), (2.3) and the Lipschitz continuity of a_{ij}, we get

$$\begin{aligned}|a(\omega;\omega,v) - (f,v)_0| &\leqslant |a(\omega;\omega - u_h, v)| + |a(\omega;u_h,v) - a(u_h;u_h,v)| \\ &\quad + |a(u_h;u_h,v-\pi_h v)| + |(f,\pi_h v - v)_0| \\ &\leqslant |a(\omega;\omega - u_h,v)| + C(v)\|\omega - u_h\|_0\|u_h\|_1 \\ &\quad + C_1\|u_h\|_1\|v - \pi_h v\|_1 + C_2\|v - \pi_h v\|_1 \to 0\end{aligned}$$

as $h \to 0$ due to the convergence of u_h to ω and $\pi_h v$ to v. By the density $\overline{C_0^\infty(\Omega)} = H_0^1(\Omega)$ we get that

$$a(\omega; \omega, v) = (f, v)_0 \quad \forall v \in H_0^1(\Omega).$$

Hence ω is the weak solution of (1.1)–(1.2). From the uniqueness of the weak solution of (1.1)–(1.2) it follows that $\omega \equiv u$ (see Theorem 2.1). It is easy to see that the whole original sequence $\{u_h\}$ converges to u. Hence, $\|\zeta\|_0 \to 0$ as $h \to 0$ and for h sufficiently small we obtain

$$C'\|\zeta\|_0^2 \leqslant \frac{1}{2}\|\zeta\|_0.$$

From (4.11) the inequality (4.2) follows. □

Remark 4.2. Asymptotic $L^\infty(\Omega)$-error estimates for quasilinear elliptic boundary value problems are established in [10, 22].

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