AN IMMEDIATE ANALYSIS FOR GLOBAL SUPERCONVERGENCE FOR INTEGRODIFFERENTIAL EQUATIONS

Qun Lin, Shuhua Zhang, Beijing

(Received June 16, 1995)

Abstract. In this paper we study the finite element approximations to the parabolic and hyperbolic integrodifferential equations and present an immediate analysis for global superconvergence for these problems, without using the Ritz projection or its modified forms.

Keywords: integrodifferential equations, global superconvergence, immediate analysis

MSC 2000: 65B05, 65N30

1. Introduction

According to the conventional error analysis for FEMs of the time-dependent problems, e.g. parabolic problems, either the Ritz projection or its modified forms are necessary to be used as transitional tools. Especially, the interior pointwise superconvergence estimates were obtained skilfully in Thomée, Xu and Zhang [10] by using the Ritz projection for the parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f & \text{in} \quad \Omega \times (0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = v & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain. This result is essential. Also by means of the Ritz projection, optimal L^2 error estimates were derived by Thomée and Zhang in

[11] for the parabolic integrodifferential equation

$$(1.2)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u - \displaystyle\int_0^t \Delta u(x,s)\,\mathrm{d}s = f & \text{in} \quad \Omega \times (0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = v & \text{in} \quad \Omega. \end{cases}$$

The Ritz-Volterra projection, a modified Ritz projection, were first introduced by Lin, Thomée and Wahlbin in [8] to get error estimates for the problem (1.2), hyperbolic integrodifferential equations and the related differential equations. In addition, Y. P. Lin also considered the interior pointwise superconvergence for such problems (an unpublished manuscript) by the Ritz-Volterra projection. In a word, in the previous studies for time-dependent problems, the Ritz projection or its modified forms were indispensable in the error analysis of their FEMs. However, here we will use a new analysis from [3], i.e. an analysis for the "short side" in the FE-right triangle plus the sharp integral estimates of the "hypotenuse", rather than using the Ritz projection or the Ritz-Volterra projection, to get the global superconvergence for parabolic and hyperbolic integrodifferential equations, rather than the interior pointwise superconvergence. Our analysis sharpens the results and shortens the proofs appearing in the previous literature under the rectangular mesh assumption.

2. Parabolic integrodifferential equations with homogeneous boundary conditions

First of all, we discuss the model (1.2) which is simple to demonstrate our superconvergent analysis for FEMs. Here and below, we assume that Ω is a rectangular domain, T^h a rectangular partition over Ω with mesh size h and f, v sufficiently smooth given functions. The weak form of (1.2) consists in finding $u(.,t) \in H_0^1(\Omega)$ (the Sobolev space) for any fixed $t \in [0,T]$ such that

$$(2.1)\qquad \begin{cases} (u_t,\varphi) + (\nabla u, \nabla\varphi) + \displaystyle\int_0^t (\nabla u(s), \nabla\varphi)\,\mathrm{d}s = (f,\varphi) \quad \forall \varphi \in H_0^1(\Omega), \\ u(x,0) = v. \end{cases}$$

Let $S_0^h \subset H_0^1(\Omega)$ consist of piecewise bilinear functions. Thus, a continuous Galerkin approximation $u^h(x,t)\colon\ [0,T] \to S_0^h$ is defined such that

$$(2.2)\qquad \begin{cases} (u_t^h,\varphi) + (\nabla u^h, \nabla\varphi) + \displaystyle\int_0^t (\nabla u^h(s), \nabla\varphi)\,\mathrm{d}s = (f,\varphi) \quad \forall \varphi \in S_0^h(\Omega), \\ u^h(0) = i_h v, \end{cases}$$

where $i_h v \in S_0^h$ stands for the bilinear interpolation function of v. We need the following

Lemma 2.1.

$$|(\nabla(u - i_h u), \nabla\varphi)| \leqslant ch^2 \|u\|_4 \|\varphi\|_0 \quad \forall \varphi \in S_0^h.$$

Proof. For an arbitrary element $\tau \in T^h$, we assume that (x_τ, y_τ) is its center, s_1, s_3 are of length $2h_\tau$ and s_2, s_4 of length $2k_\tau$, its two sides being parallel to the x-axis and y-axis, respectively. Consequently,

$$\begin{aligned} s_1 &: \ y = y_\tau - k_\tau, \\ s_3 &: \ y = y_\tau + k_\tau, \\ s_2 &: \ x = x_\tau - h_\tau, \\ s_4 &: \ x = x_\tau + h_\tau. \end{aligned}$$

Define the error functions

$$E(x) = \frac{1}{2}[(x - x_\tau)^2 - h_\tau^2], \quad F(y) = \frac{1}{2}[(y - y_\tau)^2 - k_\tau^2].$$

In order to complete the proof of Lemma 2.1 we only need to prove for the first variable the inequality

$$|((u - i_h u)_x, \varphi_x)| \leqslant ch^2 \|u\|_4 \|\varphi\|_0.$$

From the definition of $F(y)$ we derive

$$F(y) = 0, \quad F'(y) = \text{constant on} \quad s_1, s_3 \tag{2.3}$$

$$F''(y) = 1, \quad (y - y_\tau) = \frac{1}{6}[F^2(y)]^{(3)}. \tag{2.4}$$

Moreover, we have by the definition of $i_h u$

$$\int_{s_1} (u - i_h u)_x \,\mathrm{d}x = 0, \qquad \int_{s_3} (u - i_h u)_x \,\mathrm{d}x = 0. \tag{2.5}$$

From Taylor's expansion of $\varphi_x(x, y)$ and from (2.4) we obtain

$$\begin{aligned} \int_\tau (u - i_h u)_x \varphi_x &= \int_\tau (u - i_h u)_x [\varphi_x(x, y_\tau) + (y - y_\tau)\varphi_{xy}(x, y)] \\ &= \int_\tau (u - i_h u)_x [F''(y)\varphi_x(x, y_\tau) + \frac{1}{6}(F^2(y))^{(3)}\varphi_{xy}(x, y)] \\ &\equiv I + II. \end{aligned}$$

It follows by integration by parts that

$$I = \left(\int_{s_3} - \int_{s_1}\right)(u - i_h u)_x F'(y)\varphi_x(x, y_\tau) - \int_\tau (u - i_h u)_{xy} F'(y)\varphi_x(x, y_\tau).$$
$$\equiv I_1 + I_2.$$

By (2.3) and (2.5)

$$I_1 = 0. \tag{2.6}$$

By integration by parts and by (2.3)

$$I_2 = \int_\tau F(u - i_h u)_{xyy}\varphi_x(x, y_\tau), \tag{2.7}$$

and thus

$$\begin{aligned} I &= \int_\tau F(u - i_h u)_{xyy}\varphi_x(x, y_\tau) \\ &= \int_\tau F u_{xyy}[\varphi_x(x, y) - F_y \varphi_{xy}(x, y)] \\ &= \int_\tau u_{xyy}[F\varphi_x(x, y) - \frac{1}{2}(F^2)'\varphi_{xy}(x, y)]. \end{aligned} \tag{2.8}$$

Notice that

$$(F^2)''(y) = 2k_\tau^2, \quad (F^2)'(y) = 0 \quad \text{on} \quad s_1, s_3.$$

Therefore, we have again by integration by parts and (2.5)

$$\begin{aligned} II &= \frac{1}{6}\left(\int_{s_3} - \int_{s_1}\right)(F^2)''(u - i_h u)_x \varphi_{xy}(x, y) \\ &- \frac{1}{6}\int_\tau (F^2)''(u - i_h u)_{xy}\varphi_{xy}(x, y) \\ &= -\frac{1}{6}\int_\tau (F^2)''(u - i_h u)_{xy}\varphi_{xy}(x, y) \\ &= -\frac{1}{6}\left(\int_{s_3} - \int_{s_1}\right)(F^2)'(u - i_h u)_{xy}\varphi_{xy}(x, y) \\ &+ \frac{1}{6}\int_\tau (F^2)'(u - i_h u)_{xyy}\varphi_{xy} \\ &= \frac{1}{6}\int_\tau (F^2)' u_{xyy}\varphi_{xy}. \end{aligned} \tag{2.9}$$

Then, by (2.8) and (2.9),

$$\int_\tau (u - i_h u)_x \varphi_x = \int_\tau u_{xyy}[F\varphi_x - \frac{1}{3}(F^2)'\varphi_{xy}]. \tag{2.10}$$

Because φ vanishes on $\partial\Omega$ and $u_{xyy}F\varphi$, $u_{xyy}(F^2)'\varphi_y$ are continuous across s_2 and s_4, we gain by further integration by parts with respect to (2.10) and summation over $\tau \in T^h$

$$\begin{aligned}\int_\Omega (u - i_h u)_x \varphi_x &= \sum_{\tau \in T^h} \left(\int_{s_4} - \int_{s_2} \right) u_{xyy}[F\varphi - \frac{1}{3}(F^2)'\varphi_y] \\ &\quad - \int_\Omega u_{xxyy}[F\varphi - \frac{1}{3}(F^2)'\varphi_y] \\ &= - \int_\Omega u_{xxyy}[F\varphi - \frac{1}{3}(F^2)'\varphi_y].\end{aligned}$$

And thus, we get by the inverse estimates of FEM

$$|((u - i_h u)_x, \varphi_x)| \leqslant Ch^2 \|u\|_4 \|\varphi\|_0.$$

□

Theorem 2.1. *For sufficiently smooth u and u_t we have*

$$\|u^h - i_h u\|_1 \leqslant ch^2 \left[\int_0^t (\|u_t\|_2 + \|u\|_4 + \int_0^s \|u(\tau)\|_4 \, \mathrm{d}\tau)^2 \, \mathrm{d}s \right]^{1/2}.$$

P r o o f. Let $\theta(x,t) = u^h(x,t) - i_h u(x,t)$. Then

(2.11)
$$\begin{aligned}(\theta_t, \varphi) + (\nabla\theta, \nabla\varphi) + \int_0^t (\nabla\theta(s), \nabla\varphi) \, \mathrm{d}s &= (u_t - i_h u_t, \varphi) + (\nabla(u - i_h u), \nabla\varphi) \\ &\quad + \int_0^t (\nabla(u(s) - i_h u(s)), \nabla\varphi) \, \mathrm{d}s.\end{aligned}$$

Taking $\varphi = \theta_t$, we have according to Lemma 2.1

$$\begin{aligned}&\|\theta_t\|_0^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_1^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\nabla\theta(s), \nabla\theta(t)) \, \mathrm{d}s - |\theta|_1^2 \\ &\leqslant ch^2 \left(\|u_t\|_2 + \|u\|_4 + \int_0^t \|u(s)\|_4 \, \mathrm{d}s \right) \|\theta_t\|_0,\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_1^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\nabla\theta(s), \nabla\theta(t)) \, \mathrm{d}s - |\theta|_1^2 \\ &\leqslant ch^4 \left(\|u_t\|_2 + \|u\|_4 + \int_0^t \|u(s)\|_4 \, \mathrm{d}s \right)^2.\end{aligned}$$

Integrating with respect to t, we conclude from $\theta(x,0)=0$ that

$$|\theta|_1^2 \leqslant c\int_0^t |\theta|_1^2\,\mathrm{d}s + ch^4\int_0^t (\|u_t\|_2 + \|u\|_4 + \int_0^s \|u(\tau)\|_4\,\mathrm{d}\tau)^2\,\mathrm{d}s.$$

By virtue of Gronwall's Lemma, we finally obtain

$$|\theta|_1^2 \leqslant ch^4\int_0^t (\|u_t\|_2 + \|u\|_4 + \int_0^s \|u(\tau)\|_4 d\tau)^2\,\mathrm{d}s.$$

And thus, Theorem 2.1 follows. □

Theorem 2.2. *For sufficiently smooth u, u_t and u_{tt} we have*

$$\|u_t - u_t^h\|_0 \leqslant ch^2\Big\{[(\|u_t(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2\,\mathrm{d}s]^{1/2} + \|u_t\|_2\Big\}.$$

Proof. Differentiating (2.11) with respect to t, we have, for $\varphi \in S_0^h$,

$$(\theta_{tt},\varphi) + (\nabla\theta_t,\nabla\varphi) + (\nabla\theta(t),\nabla\varphi) = (u_{tt} - i_h u_{tt},\varphi) + (\nabla(u_t - i_h u_t),\nabla\varphi) + (\nabla(u - i_h u),\nabla\varphi),$$

and hence, with $\varphi = \theta_t$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_t\|_0^2 + c\|\theta_t\|_1^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_1^2 \leqslant ch^4(\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 + \frac{c}{2}\|\theta_t\|_0^2$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\theta_t\|_0^2 + |\theta|_1^2) \leqslant ch^4(\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2.$$

Integrating with respect to t, we obtain from $\theta(0)=0$ that

$$\|\theta_t\|_0^2 + |\theta|_1^2 \leqslant \|\theta_t(0)\|_0^2 + ch^4\int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2\,\mathrm{d}s.$$

Let $t=0$ and $\varphi = \theta_t(0)$ in (2.3). Then

$$\|\theta_t(0)\|_0 \leqslant ch^2(\|u_t(0)\|_2 + \|u(0)\|_4),$$

and thus

$$\|\theta_t\|_0 \leqslant ch^2 \Big[(\|u_t(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \, \mathrm{d}s\Big]^{1/2}. \tag{2.12}$$

Therefore, Theorem 2.2 follows from the triangle inequality

$$\|u_t^h - u_t\|_0 \leqslant \|u_t^h - i_h u_t\|_0 + \|i_h u_t - u_t\|_0.$$

□

In order to derive the L^∞ superconvergence, like in [7] (or [10]) we introduce the discrete Green's function $G_z^h \in S_0^h$ at any point $z \in \overline{\Omega}$ such that for $\varphi \in S_0^h$

$$(\nabla G_z^h, \nabla \varphi) = \varphi(z), \qquad (\nabla \partial_z G_z^h, \nabla \varphi) = \partial_z \varphi(z).$$

Then, the following lemma holds (see Lemma 2.1, 2.3 and 3.3 in [10]).

Lemma 2.2.

$$\|G_z^h\|_0 \leqslant c, \qquad \|\partial_z G_z^h\|_0 \leqslant c(\log \frac{1}{h})^{1/2}, \qquad \|\partial_z G_z^h\|_{1,1} \leqslant c \log \frac{1}{h}.$$

Theorem 2.3. *For sufficiently smooth u and u_t we have*

$$\|u^h - u\|_{0,\infty} \leqslant ch^2 \Big\{ \|u\|_{2,\infty} + \|u_t\|_2 + \|u\|_4 + \int_0^t \|u(s)\|_4 \, \mathrm{d}s + [(\|u_t(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \, \mathrm{d}s]^{1/2} \Big\}.$$

P r o o f. Setting $\varphi = G_z^h$ in (2.11), we have

$$\theta(z) = -\int_0^t \theta(z,s)\, \mathrm{d}s - (\theta_t, G_z^h) + (u_t - i_h u_t, G_z^h) + (\nabla(u - i_h u), \nabla G_z^h) + \int_0^t (\nabla(u(s) - i_h u(s)), \nabla G_z^h)\, \mathrm{d}s,$$

and it follows from Lemma 2.1, 2.2 and (2.12) that

$$|\theta(z)| \leqslant \int_0^t |\theta(z,s)|\, \mathrm{d}s + ch^2 \Big\{ \|u_t\|_2 + \|u\|_4 + \int_0^t \|u(s)\|_4 \, \mathrm{d}s + \Big[(\|u_t(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \, \mathrm{d}s\Big]^{1/2} \Big\}.$$

According to Gronwall's Lemma, we finally get

$$|\theta(z)| \leqslant ch^2 \left\{ \|u_t\|_2 + \|u\|_4 + \int_0^t \|u(s)\|_4 \, \mathrm{d}s \right.$$
$$\left. + \left[(\|u_t(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \, \mathrm{d}s \right]^{1/2} \right\},$$

and Theorem 2.3 follows from the well-known estimate for $\|u - i_h u\|_{0,\infty}$ and the triangle inequality

$$\|u - u^h\|_{0,\infty} \leqslant \|u - i_h u\|_{0,\infty} + \|i_h u - u^h\|_{0,\infty}.$$

□

Theorem 2.4. *For sufficiently smooth u, u_t and u_{tt}, we have*

$$\|u^h - i_h u\|_{1,\infty} \leqslant ch^2 \Big(\log \frac{1}{h}\Big)^{1/2} \left\{ \|u_t\|_2 + \|u\|_4 \right.$$
$$+ \int_0^t \|u(s)\|_4 \, \mathrm{d}s + \left[(\|u_t(0)\|_2 + \|u(0)\|_4)^2 \right.$$
$$\left. \left. + \int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \, \mathrm{d}s \right]^{1/2} \right\}.$$

Proof. Taking $\varphi = \partial_z G_z^h$ in (2.11), we have

$$\partial_z \theta(z) = -\int_0^t \partial_z \theta(z,s) \, \mathrm{d}s - (\theta_t, \partial_z G_z^h) + (u_t - i_h u_t, \partial_z G_z^h)$$
$$+ (\nabla(u - i_h u), \nabla \partial_z G_z^h) + \int_0^t (\nabla(u(s) - i_h u(s)), \nabla \partial_z G_z^h) \, \mathrm{d}s,$$

and it follows from Lemmas 2.1, 2.2 and (2.12) that

$$|\partial_z \theta(z)| \leqslant \int_0^t |\partial_z \theta(z,s)| \, \mathrm{d}s + ch^2 \Big(\log \frac{1}{h}\Big)^{1/2} \left\{ \|u_t\|_2 + \|u\|_4 + \int_0^t \|u(s)\|_4 \, \mathrm{d}s \right.$$
$$\left. + \left[(\|u_t(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \, \mathrm{d}s \right]^{1/2} \right\}.$$

Again by Gronwall's Lemma, we finally get Theorem 2.4. □

Theorems 2.1 and 2.4 are the basis for the global superconvergence. Numerical analysts are used to utilize the averaging technique to get the interior pointwise superconvergence. Instead, we use an interpolation postprocessing technique from [4] (or [5]) to obtain the global superconvergence. For this purpose, we assume that T^h was obtained from T^{2h} by subdividing each element of T^{2h} into four congruent elements. Thus, we can define a nodal biquadratic interpolation operator I_{2h}^2 associated with T^{2h} of mesh size $2h$. It is easy to check that

$$I_{2h}^2 i_h = I_{2h}^2, \quad \|I_{2h}^2 \varphi\|_{1,p} \leqslant c\|\varphi\|_{1,p} \qquad \forall \varphi \in S_0^h \ (p = 2, \infty),$$

$$\|I_{2h}^2 \varphi - \varphi\|_{1,p} \leqslant ch^2 \|\varphi\|_{3,p} \quad (p = 2, \infty).$$

And thus, we have the following main

Theorem 2.5. *For sufficiently smooth u and u_t, we have*

$$\|I_{2h}^2 u^h - u\|_1 \leqslant ch^2 \left\{ \|u\|_3 + \left[\int_0^t \left(\|u_t\|_2 + \|u\|_4 + \int_0^s \|u(\tau)\|_4 \,\mathrm{d}\tau \right)^2 \mathrm{d}s \right]^{1/2} \right\}.$$

P r o o f. Due to the nature of I_{2h}^2, we have

$$I_{2h}^2 u^h - u = I_{2h}^2 (u^h - i_h u) + (I_{2h}^2 u - u).$$

And thus, according to Theorem 2.1,

$$\|I_{2h}^2 u^h - u\|_1 \leqslant ch^2 \left\{ \|u\|_3 + \left[\int_0^t \left(\|u_t\|_2 + \|u\|_4 + \int_0^s \|u(\tau)\|_4 \,\mathrm{d}\tau \right)^2 \mathrm{d}s \right]^{1/2} \right\}.$$

□

Analogously we have by Theorem 2.4

Theorem 2.6. *For sufficiently smooth u, u_t and u_{tt}, we have*

$$\begin{aligned} \|I_{2h}^2 u^h - u\|_{1,\infty} \leqslant ch^2 \left(\log \frac{1}{h} \right)^{1/2} \Bigg\{ & \|u\|_{3,\infty} + \|u_t\|_2 + \|u\|_4 \\ & + \int_0^t \|u(s)\|_4 \,\mathrm{d}s + \bigg[(\|u_t(0)\|_2 + \|u(0)\|_4)^2 \\ & + \int_0^t (\|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 \,\mathrm{d}s \bigg]^{1/2} \Bigg\}. \end{aligned}$$

3. Parabolic integrodifferential equations with boundary integral conditions

Now we will consider the semidiscrete Galerkin approximation to the problem

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u + f(x,t) & \text{in} \quad \Omega \times (0,T], \\
\dfrac{\partial u}{\partial n} = \displaystyle\int_0^t u(x,s)\,\mathrm{d}s & \text{on} \quad \partial\Omega \times (0,T], \\
u(x,0) = v(x) & \text{in} \quad \Omega,
\end{cases} \tag{3.1}
$$

where $n(x) = (n_1(x), n_2(x))$ is the outer-normal direction on $\partial\Omega$. The weak form of (3.1) reads as follows: Find $u(.,t) \in H^1(\Omega)$ such that

$$
\begin{cases}
(u_t, \varphi) + (\nabla u, \nabla\varphi) - \displaystyle\int_0^t \langle u(s), \varphi\rangle \,\mathrm{d}s = (f, \varphi) \quad \forall \varphi \in H^1(\Omega), \\
u(x,0) = v.
\end{cases} \tag{3.2}
$$

where

$$
\langle \varphi, \psi \rangle = \int_{\partial\Omega} \varphi\psi \,\mathrm{d}x.
$$

Assume that $S^h \subset H^1(\Omega)$ consists of piecewise bilinear functions without the zero boundary condition. Then, a semidiscrete Galerkin approximation $u^h(x,t)\colon\ [0,T] \to S^h$ is defined such that

$$
\begin{cases}
(u_t^h, \varphi) + (\nabla u^h, \nabla\varphi) - \displaystyle\int_0^t \left\langle u^h(s), \varphi\right\rangle \,\mathrm{d}s = (f, \varphi) \quad \forall \varphi \in S^h(\Omega), \\
u^h(0) = i_h v,
\end{cases} \tag{3.3}
$$

where $i_h v \in S^h$ is the bilinear interpolation function of v. We need the following (see [6])

Lemma 3.1.

$$
\|u\|_{0,\partial\Omega}^2 \leqslant \varepsilon |u|_1^2 + \frac{c}{4\varepsilon}\|u\|_0^2 \quad \forall \varepsilon > 0.
$$

Lemma 3.2. *For* $\varphi \in S^h$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \leqslant \infty$,

$$
|(\nabla(u - i_h u), \nabla\varphi)| \leqslant
\begin{cases}
ch^2 \|u\|_{3,p} |\varphi|_{1,q}, \\
ch^{1.5} \|u\|_{4,p} \|\varphi\|_{0,q}.
\end{cases}
$$

P r o o f. The proof of the lemma is similiar to that of Lemma 2.1 when we notice that for any $\varphi \in S^h(\Omega)$,

$$
\|\varphi\|_{0,q,\partial\tau} \leqslant ch^{-0.5} \|\varphi\|_{0,q,\tau} \quad \tau \in T^h.
$$

□

Lemma 3.3. *For sufficently smooth u and u_t we have*

$$\|u^h - u\|_0 \leqslant ch^2 \left\{ \|u\|_2 + \left[\int_0^t (\|u_t\|_2^2 + \|u\|_3^2 + \int_0^s \|u(\tau)\|_2^2 \,\mathrm{d}\tau) \,\mathrm{d}s \right]^{1/2} \right\}.$$

P r o o f. Let $\theta(x,t) = u^h(x,t) - i_h u(x,t)$. We have

$$\begin{aligned} (\theta_t, \varphi) + (\nabla\theta, \nabla\varphi) - \int_0^t \langle \theta(s), \varphi \rangle \,\mathrm{d}s &= (u_t - i_h u_t, \varphi) + (\nabla(u - i_h u), \nabla\varphi) \\ &\quad - \int_0^t \langle u(s) - i_h u(s), \varphi \rangle \,\mathrm{d}s \end{aligned} \tag{3.4}$$

and hence, with $\varphi = \theta$ and Lemmas 3.1, 3.2,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_0^2 + |\theta|_1^2 - \int_0^t \langle \theta(s), \theta(t) \rangle \,\mathrm{d}s &\leqslant ch^4\|u_t\|_2^2 + \|\theta\|_0^2 + ch^4\|u\|_3^2 + \frac{1}{6}|\theta|_1^2 \\ &\quad + ch^4 \int_0^t \|u\|_2^2 \,\mathrm{d}s + \frac{1}{6}|\theta|_1^2 + c\|\theta\|_0^2 \end{aligned}$$

or

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_0^2 + \frac{2}{3}|\theta|_1^2 &\leqslant c\int_0^t \|\theta(s)\|_1^2 \,\mathrm{d}s + \frac{1}{6}|\theta|_1^2 + c\|\theta\|_0^2 \\ &\quad + ch^4(\|u_t\|_2^2 + \|u\|_3^2 + \int_0^t \|u\|_2^2 \,\mathrm{d}s), \end{aligned}$$

that is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_0^2 + \|\theta\|_1^2 &\leqslant c\int_0^t \|\theta(s)\|_1^2 \,\mathrm{d}s + c\|\theta\|_0^2 \\ &\quad + ch^4 \left(\|u_t\|_2^2 + \|u\|_3^2 + \int_0^t \|u\|_2^2 \,\mathrm{d}s \right). \end{aligned}$$

Integrating with respect to t, we obtain from $\theta(0) = 0$ and Gronwall's Lemma that

$$\|\theta\|_0^2 + \int_0^t \|\theta\|_1^2 \,\mathrm{d}s \leqslant ch^4 \int_0^t \left(\|u_t\|_2^2 + \|u\|_3^2 + \int_0^s \|u\|_2^2 \,\mathrm{d}\tau \right) \mathrm{d}s$$

or

$$\|\theta\|_0 \leqslant ch^2 \left[\int_0^t \left(\|u_t\|_2^2 + \|u\|_3^2 + \int_0^s \|u\|_2^2 \,\mathrm{d}\tau \right) \mathrm{d}s \right]^{1/2}.$$

Then, Lemma 3.3 follows from the triangle inequality

$$\|u^h - u\|_0 \leqslant \|u^h - i_h u\|_0 + \|i_h u - u\|_0.$$

□

Theorem 3.1. *For sufficiently smooth u and u_t we have*

$$\|u^h - i_h u\|_1 \leqslant ch^2 \left[\|u\|_3^2 + \int_0^t \left(\|u\|_3^2 + \|u_t\|_3^2 + \int_0^s \|u\|_3^2 \,\mathrm{d}\tau\right) \mathrm{d}s\right]^{1/2}.$$

P r o o f. Taking $\varphi = \theta_t$ in (3.4), we get

$$\begin{aligned} &\|\theta_t\|_0^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_1^2 - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \langle\theta(s), \theta(t)\rangle \,\mathrm{d}s + \langle\theta, \theta\rangle \\ &= (u_t - i_h u_t, \theta_t) + \frac{\mathrm{d}}{\mathrm{d}t}(\nabla(u - i_h u), \nabla\theta) - (\nabla(u_t - i_h u_t), \nabla\theta) \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \langle u(s) - i_h u(s), \theta(t)\rangle \,\mathrm{d}s + \langle u - i_h u, \theta\rangle\,. \end{aligned}$$

In virtue of Lemmas 3.1, 3.2 and by integration with respect to t we have

$$|\theta|_1^2 \leqslant c\int_0^t \|\theta\|_1^2 \,\mathrm{d}s + c\|\theta\|_0^2 + ch^4\int_0^t (\|u\|_2^2 + \|u_t\|_3^2)\,\mathrm{d}s + ch^4\|u\|_3^2$$

or

$$\|\theta\|_1^2 \leqslant c\int_0^t \|\theta\|_1^2 \,\mathrm{d}s + c\|\theta\|_0^2 + ch^4\int_0^t (\|u\|_3^2 + \|u_t\|_3^2)\,\mathrm{d}s + ch^4\|u\|_3^2.$$

And thus, it follows from Lemma 3.3 and Gronwall's Lemma that

$$\|\theta\|_1 \leqslant ch^2 \left[\|u\|_3^2 + \int_0^t \left(\|u\|_3^2 + \|u_t\|_3^2 + \int_0^s \|u\|_3^2 \,\mathrm{d}\tau\right) \mathrm{d}s\right]^{1/2}.$$

□

In order to obtain L^∞ estimates we need the following (see [13] or see p. 67 Lemma 4 in [9] for the special case)

Lemma 3.4. *For a bounded domain $\Omega \subset \mathbb{R}^2$ satisfying the cone condition, i.e. there being a fixed cone G such that for all $x \in \Omega$ there exists a cone $G_x \subset \Omega$ congruent with G whose vertex is x, we have*

$$\|\varphi\|_{0,\infty} \leqslant c\left[\|\varphi\|_0 + \left(\log\frac{1}{h}\right)^{1/2}|\varphi|_1\right] \quad \forall\varphi \in S^h.$$

P r o o f. It is possible that [13] is not available. So, we write the proof again. For arbitrary $p > 2$, $v \in C^\infty(\Omega)$ and $x \in \Omega$, there exists a cone $G_x \subset \Omega$. By means of Sobolev's integral identity and simple calculations, we get

$$|v(x)| \leqslant (\mathrm{mes}\, G_x)^{1/p'}\|v\|_{0,p,G_x} + K\|\nabla v\|_{0,p,G_x},$$

where

$$K = c\left(\int_{G_x} |x-y|^{-p'}\,dy\right)^{1/p'} \qquad \text{and} \quad \frac{1}{p}+\frac{1}{p'}=1.$$

Let H be the height of G_x. Then

$$K \leqslant c\left(\frac{H^{2-p'}}{2-p'}\right)^{1/p'}.$$

and thus

$$|v(x)| \leqslant c\left\{\|v\|_{0,p} + (2-p')^{-1/p'}|v|_{1,p}\right\}.$$

Because $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$, we obtain

$$\|\varphi\|_{0,\infty} \leqslant c\left\{\|\varphi\|_{0,p} + (2-p')^{-1/p'}|\varphi|_{1,p}\right\} \quad \forall\varphi\in W^{1,p}(\Omega).$$

From the inverse estimates of FEM, we derive

$$|\varphi|_{k,p} \leqslant ch^{2(1/p-1/2)}|\varphi|_k \quad \forall\varphi\in S^h(k=0,1 \text{ and } p>2),$$

which, together with the above inequality, leads to

$$\|\varphi\|_{0,\infty} \leqslant c\left\{h^{2/p-1}\|\varphi\|_0 + (2-p')^{-1/p'}h^{2/p-1}|\varphi|_1\right\}.$$

Choosing $p>2$ such that $\varepsilon\equiv 1-\frac{2}{p} = \left(\log\frac{1}{h}\right)^{-1}$, we obtain

$$h^{2/p-1} = e(\text{constant}).$$

Furthermore,

$$2-p' = \frac{2\varepsilon}{1+\varepsilon} \approx 2\varepsilon,$$
$$\frac{1}{p'} = \frac{1+\varepsilon}{2} \approx \frac{1}{2}$$

and we finally get

$$\|\varphi\|_{0,\infty} \leqslant c\left[\|\varphi\|_0 + \left(\log\frac{1}{h}\right)^{1/2}|\varphi|_1\right] \quad \forall\varphi\in S^h.$$

□

Theorem 3.2. *For sufficiently smooth u and u_t we have*

$$\|u^h - u\|_{0,\infty} \leqslant ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\left\{\|u\|_{2,\infty} + \left[\|u\|_3^2 + \int_0^t (\|u\|_3^2 + \|u_t\|_3^2 + \int_0^s \|u\|_3^2\,\mathrm{d}\tau)\,\mathrm{d}s\right]^{1/2}\right\}.$$

P r o o f. The theorem follows from Lemmas 3.3, 3.4, Theorem 3.1 and the triangle inequality

$$\|u^h - u\|_{0,\infty} \leqslant \|u^h - i_h u\|_{0,\infty} + \|i_h u - u\|_{0,\infty}.$$

□

Theorem 3.3. *For sufficiently smooth u, u_t and u_{tt}, we have*

$$\|u_t^h - u_t\|_0 \leqslant ch^{1.5}\left\{\|u_t\|_2 + \|u_t(0)\|_2^2 + \|u(0)\|_4^2 + \|u\|_3^2 + \int_0^t \left(\|u_{tt}\|_2^2 + \|u_t\|_3^2 + \|u\|_3^2 + \int_0^s \|u\|_3^2\,\mathrm{d}\tau\right)\mathrm{d}s\right\}.$$

P r o o f. Differentiating (3.4) with respect to t, we have, for $\varphi \in S^h$,

$$(\theta_{tt},\varphi) + (\nabla\theta_t,\nabla\varphi) - \langle\theta(t),\varphi\rangle = (u_{tt} - i_h u_{tt},\varphi) + (\nabla(u_t - i_h u_t),\nabla\varphi) - \langle u - i_h u,\varphi\rangle,$$

and hence, with $\varphi = \theta_t$ and Lemmas 3.1, 3.2,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_t\|_0^2 + |\theta_t|_1^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{0,\partial\Omega}^2 \leqslant ch^4\|u_{tt}\|_2^2 + c\|\theta_t\|_0^2 + ch^4\|u_t\|_3^2 \\ \frac{1}{4}|\theta_t|_1^2 + ch^4\|u\|_3^2 + \frac{1}{4}|\theta_t|_1^2.$$

Integrating, we have

$$\|\theta_t\|_0^2 \leqslant \|\theta_t(0)\|_0^2 + c\|\theta\|_1^2 + c\int_0^t \|\theta_t\|_0^2\,\mathrm{d}s + ch^4\int_0^t (\|u_{tt}\|_2^2 + \|u_t\|_3^2 + \|u\|_3^2)\,\mathrm{d}s.$$

Let $t = 0$ and $\varphi = \theta_t(0)$ in (3.4), then to Lemma 3.2 yields

$$\|\theta_t(0)\|_0 \leqslant ch^{1.5}(\|u_t(0)\|_2 + \|u(0)\|_4),$$

which, together with Theorem 3.1 and Gronwall's Lemma, leads to Theorem 3.3 by the triangle inequality

$$\|u_t^h - u_t\|_0 \leqslant \|u_t^h - i_h u_t\|_0 + \|i_h u_t - u_t\|_0.$$

□

Theorem 3.4. *For sufficiently smooth u, u_t and u_{tt} we have*

$$\|u^h - i_h u\|_{1,\infty} \leqslant ch^{1.5}\left(\log\frac{1}{h}\right)^{1/2}\Big\{[\|u_t(0)\|_2^2 + \|u(0)\|_4^2 + \int_0^t \left(\|u_{tt}\|_2^2 + \|u_t\|_3^2 + \|u\|_3^2\int_0^s \|u\|_3^2\,\mathrm{d}\tau\right)\mathrm{d}s]^{1/2} + \|u_t\|_2 + \|u\|_{3,\infty} + \int_0^t \|u\|_{2,\infty}\,\mathrm{d}s + \int_0^t \mathrm{d}s\left[\|u\|_3^2 + \int_0^s \left(\|u_t\|_3^2 + \|u\|_3^2 + \int_0^\tau \|u\|_3^2\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right]^{1/2}\Big\}.$$

P r o o f. Setting $\varphi = \partial_z G_z^h$ in (3.4), we have

$$(\theta_t, \partial_z G_z^h) + \partial_z\theta(z,t) - \int_0^t \langle\theta(s), \partial_z G_z^h\rangle\,\mathrm{d}s = (u_t - i_h u_t, \partial_z G_z) + (\nabla(u - i_h u), \nabla\partial_z G_z^h) - \int_0^t \langle u(s) - i_h u(s), \partial_z G_z^h\rangle\,\mathrm{d}s.$$

By virtue of Lemmas 2.2 and 3.2

$$\begin{aligned}|\partial_z\theta(z,t)| &\leqslant c\Big(\log\frac{1}{h}\Big)^{1/2}\|\theta_t\|_0 + \int_0^t |\langle\theta(s), \partial_z G_z^h\rangle|\,\mathrm{d}s \\ &\quad + ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\|u_t\|_2 + ch^2\log\frac{1}{h}\left(\|u\|_{3,\infty} + \int_0^t \|u\|_{2,\infty}\,\mathrm{d}s\right) \\ &\leqslant c\Big(\log\frac{1}{h}\Big)^{1/2}\|\theta_t\|_0 + c\log\frac{1}{h}\int_0^t \|\theta\|_{0,\infty}\,\mathrm{d}s \\ &\quad + ch^2\log\frac{1}{h}\left(\|u_t\|_2 + \|u\|_{3,\infty} + \int_0^t \|u\|_{2,\infty}\,\mathrm{d}s\right),\end{aligned} \tag{3.5}$$

In Theorems 3.3 and 3.2 we have got respectively

$$\|\theta_t\|_0^2 \leqslant ch^3\Big[\|u_t(0)\|_2^2 + \|u(0)\|_4^2 + \|u\|_3^2 + \int_0^t (\|u\|_3^2 + \|u_t\|_3^2 + \|u_{tt}\|_2^2 + \int_0^s \|u\|_3^2\,\mathrm{d}\tau)\,\mathrm{d}s\Big], \tag{3.6}$$

$$\|\theta\|_{0,\infty} \leqslant ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\left[\|u\|_3^2 + \int_0^t \left(\|u\|_3^2 + \|u_t\|_3^2 + \int_0^s \|u\|_3^2\,\mathrm{d}\tau\right)\mathrm{d}s\right]^{1/2}. \tag{3.7}$$

From (3.5)–(3.7) we finally obtain Theorem 3.4. □

Similarly to Section 2, we can get the global superconvergence by means of an interpolation postprocessing technique, rather than using the averaging technique by which one can only get the interior pointwise superconvergence.

Theorem 3.5. *For sufficiently smooth u and u_t we have*

$$\|I_{2h}^2 u^h - u\|_1 \leqslant ch^2 \biggl\{ \|u\|_3 + \biggl[\|u\|_3^2 + \int_0^t (\|u\|_3^2 + \|u_t\|_3^2 + \int_0^s \|u\|_3^2 \,\mathrm{d}\tau) \,\mathrm{d}s \biggr]^{1/2} \biggr\}.$$

Theorem 3.6. *For sufficiently smooth u, u_t and u_{tt}, we have*

$$\begin{aligned} \|I_{2h}^2 u^h - u\|_{1,\infty} \leqslant ch^{1.5} \log \frac{1}{h} \biggl\{ & \|u\|_{3,\infty} + \biggl[\|u_t(0)\|_2^2 + \|u(0)\|_4^2 \\ & + \int_0^t \biggl(\|u_{tt}\|_2^2 + \|u_t\|_3^2 + \|u\|_3^2 \\ & \int_0^s \|u\|_3^2 \,\mathrm{d}\tau \biggr) \,\mathrm{d}s \biggr]^{1/2} \\ & + \|u_t\|_2 + \|u\|_{3,\infty} + \int_0^t \|u\|_{2,\infty} \,\mathrm{d}s \\ & + \int_0^t \mathrm{d}s \biggl[\int_0^s \biggl(\|u_t\|_3^2 + \|u\|_3^2 + \int_0^\tau \|u\|_3^2 \,\mathrm{d}\sigma \biggr) \,\mathrm{d}\tau \biggr]^{1/2} \biggr\}. \end{aligned}$$

4. Hyperbolic integrodifferential equations

In the conclusion we study the hyperbolic integrodifferential equation

$$\text{(4.1)} \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u - \displaystyle\int_0^t \Delta u(x,s) \,\mathrm{d}s = f & \text{in} \quad \Omega \times (0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = v, \ \dfrac{\partial u}{\partial t}(x,0) = w(x) & \text{in} \quad \Omega. \end{cases}$$

The weak form of (1.4) consists in finding $u(.,t) \in H_0^1(\Omega)$ such that

$$\text{(4.2)} \qquad \begin{cases} (u_{tt}, \varphi) + (\nabla u, \nabla \varphi) + \displaystyle\int_0^t (\nabla u(s), \nabla \varphi) \,\mathrm{d}s = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega), \\ u(x,0) = v, \qquad u_t(x,0) = w. \end{cases}$$

Thus, a continuous Galerkin approximation $u^h(x,t)\colon\ [0,T]\to S_0^h\subset H_0^1$ is defined such that

$$
(4.3)\qquad \begin{cases} (u_{tt}^h,\varphi)+(\nabla u^h,\nabla\varphi)+\displaystyle\int_0^t(\nabla u^h(s),\nabla\varphi)\,\mathrm{d}s=(f,\varphi) \quad \forall\varphi\in S_0^h(\Omega),\\ u^h(0)=i_hv,\qquad u_t^h(0)=i_hw. \end{cases}
$$

We have

Theorem 4.1. *For sufficiently smooth u, u_t and u_{tt} we have*

$$
\|u^h-i_hu\|_1+\|u_t^h-u_t\|_0\leqslant ch^2\left\{\|u_t\|_2+\left[\int_0^t\left(\|u_{tt}\|_2+\|u\|_4+\int_0^s\|u(\tau)\|_4\,\mathrm{d}\tau\right)^2\mathrm{d}s\right]^{1/2}\right\}.
$$

P r o o f. Setting $\theta(x,t)=u^h(x,t)-i_hu(x,t)$, we have

(4.4)

$$
(\theta_{tt},\varphi)+(\nabla\theta,\nabla\varphi)+\int_0^t(\nabla\theta(s),\nabla\varphi)\,\mathrm{d}s=(u_{tt}-i_hu_{tt},\varphi)+(\nabla(u-i_hu),\nabla\varphi)+\int_0^t(\nabla(u(s)-i_hu(s)),\nabla\varphi)\,\mathrm{d}s.
$$

Hence, with $\varphi=\theta_t$ and Lemma 2.1,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_t\|_0^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta|_1^2+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(\nabla\theta(s),\nabla\theta(t))\,\mathrm{d}s-|\theta|_1^2 \leqslant ch^4\left(\|u_{tt}\|_2+\|u\|_4+\int_0^t\|u(s)\|_4\,\mathrm{d}s\right)^2+\|\theta_t\|_0^2.
$$

Integrating with respect to t, we obtain from $\theta(0)=\theta_t(0)=0$ that

(4.5)

$$
\|\theta_t\|_0^2+|\theta|_1^2\leqslant c\int_0^t(\|\theta_t\|_0^2+|\theta|_1^2)\,\mathrm{d}s+ch^4\int_0^t\left(\|u_{tt}\|_2+\|u\|_4+\int_0^s\|u(\tau)\|_4d\tau\right)^2\mathrm{d}s,
$$

which yields Theorem 4.1 by virtue of Gronwall's Lemma and the triangle inequality

$$
\|u_t^h-u_t\|_0\leqslant\|u_t^h-i_hu_t\|_0+\|i_hu_t-u_t\|_0.
$$

□

Theorem 4.2. *For sufficiently smooth u, u_{tt} and u_{ttt} we have*

$$\|u_t^h - i_h u_t\|_1 + \|u_{tt}^h - u_{tt}\|_0 \leqslant ch^2 \Big\{ \|u_{tt}\|_2 + [(\|u_{tt}(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4 + \int_0^s \|u(\tau)\|_4 \, \mathrm{d}\tau)^2 \, \mathrm{d}s]^{1/2} \Big\}.$$

P r o o f. Differentiating (4.4) with respect to t, we have, for $\varphi \in S_0^h$,

$$(\theta_{ttt}, \varphi) + (\nabla\theta_t, \nabla\varphi) + (\nabla\theta(t), \nabla\varphi) = (u_{ttt} - i_h u_{ttt}, \varphi) + (\nabla(u_t - i_h u_t), \nabla\varphi) + (\nabla(u - i_h u), \nabla\varphi)$$

and hence, with $\varphi = \theta_{tt}$ and Lemma 2.1,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\theta_{tt}\|_0^2 + |\theta_t|_1^2) + \frac{\mathrm{d}}{\mathrm{d}t}(\nabla\theta(t), \nabla\theta_t(t)) - |\theta_t|_1^2 \leqslant ch^4(\|u_{ttt}\|_2 + \|u_t\|_4 + \|u\|_4)^2 + \|\theta_{tt}\|_0^2.$$

Thus, integrating with respect to t, we find from (4.5) that

(4.6)
$$\|\theta_{tt}\|_0^2 + |\theta_t|_1^2 \leqslant c\int_0^t (\|\theta_{tt}\|_0^2 + |\theta_t|_1^2) \, \mathrm{d}s + c\|\theta_{tt}(0)\|_0^2 + ch^4 \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4 + \int_0^s \|u\|_4 \, \mathrm{d}\tau)^2 \, \mathrm{d}s.$$

Let $t = 0$ and $\varphi = \theta_{tt}(0)$ in (4.4), then according to Lemma 2.1

$$\|\theta_{tt}(0)\|_0 \leqslant ch^2(\|u_{tt}(0)\|_2 + \|u(0)\|_4)$$

which, together with (4.6) leads to Theorem 4.2 by the triangle inequality

$$\|u_{tt}^h - u_{tt}\|_0 \leqslant \|u_{tt}^h - i_h u_{tt}\|_0 + \|i_h u_{tt} - u_{tt}\|_0.$$

□

Theorem 4.3. *For sufficiently smooth u, u_t and u_{tt} we have*

$$\|u - u^h\|_{0,\infty} \leqslant ch^2 \Big\{ \|u\|_{2,\infty} + \|u_{tt}\|_2 + \|u\|_4 + \Big[(\|u_{tt}(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4 + \int_0^s \|u\|_4 d\tau)^2 \, \mathrm{d}s \Big]^{1/2} \Big\}.$$

P r o o f. Taking $\varphi = G_z^h$ in (4.4), we get by Lemma 2.2

$$(\theta_{tt}, G_z^h) + \theta(z,t) + \int_0^t \theta(z,s)\,\mathrm{d}s \leqslant ch^2(\|u_{tt}\|_2 + \|u\|_4 + \int_0^t \|u(s)\|_4\,\mathrm{d}s).$$

Now it follows from Theorem 4.2 that

$$\begin{aligned}|\theta(z)| \leqslant \int_0^t |\theta(z,s)|\,\mathrm{d}s + ch^2\Big\{\|u_{tt}\|_2 + \|u\|_4 + \Big[(\|u_{tt}(0)\|_2 + \|u(0)\|_4)^2 \\ + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4 + \int_0^s \|u(\tau)\|_4\,\mathrm{d}\tau)^2\,\mathrm{d}s\Big]^{1/2}\Big\}.\end{aligned} \tag{4.7}$$

Then, the theorem holds by Gronwall's Lemma and the triangle inequality

$$\|u^h - u\|_{0,\infty} \leqslant \|u^h - i_h u\|_{0,\infty} + \|i_h u - u\|_{0,\infty}.$$

□

Theorem 4.4. *For sufficiently smooth u, u_t and u_{tt} we have*

$$\begin{aligned}\|u^h - i_h u\|_{1,\infty} \leqslant ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\Big\{\|u_{tt}\|_2 + \|u\|_4 + [(\|u_{tt}(0)\|_2 + \|u(0)\|_4)^2 \\ + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4 + \int_0^s \|u\|_4\,\mathrm{d}\tau)^2\,\mathrm{d}s]^{1/2}\Big\}.\end{aligned}$$

P r o o f. Setting $\varphi = \partial_z G_z^h$ in (4.4), we have in terms of Lemma 2.2

$$\begin{aligned}&(\theta_{tt}, \partial_z G_z^h) + \partial_z\theta(z,t) + \int_0^t \partial_z\theta(z,s)\,\mathrm{d}s \\ &\leqslant ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\Big(\|u_{tt}\|_2 + \|u\|_4 + \int_0^t \|u\|_4\,\mathrm{d}s\Big),\end{aligned}$$

or

$$\begin{aligned}|\partial_z\theta(z)| \leqslant \int_0^t |\partial_z\theta(z,s)|\,\mathrm{d}s + c\Big(\log\frac{1}{h}\Big)^{1/2}\|\theta_{tt}\|_0 \\ + ch^2\Big(\log\frac{1}{h}\Big)^{1/2}\Big(\|u_{tt}\|_2 + \|u\|_4 + \int_0^t \|u(s)\|_4\,\mathrm{d}s\Big).\end{aligned}$$

Theorem 4.4 follows from (4.6),(4.7) and Gronwall's Lemma. □

In the same way as in Sections 2 and 3, we have the main

Theorem 4.5. *For sufficiently smooth u and u_{tt} we have*

$$\|I_{2h}^2 u^h - u\|_1 \leqslant ch^2\Big\{\|u\|_3 + \Big[\int_0^t (\|u_{tt}\|_2 + \|u\|_4 + \int_0^s \|u(\tau)\|_4\,\mathrm{d}\tau)^2\,\mathrm{d}s\Big]^{1/2}\Big\}.$$

Theorem 4.6. *For sufficiently smooth u and u_{tt} we have*

$$\|I_{2h}^2 u^h - u\|_{1,\infty} \leqslant ch^2 = \Big(\log\frac{1}{h}\Big)^{1/2}\Big\{\|u\|_{3,\infty} + \|u_{tt}\|_2 + \|u\|_4 + \Big[(\|u_{tt}(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t (\|u_{ttt}\|_2 + \|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4 + \int_0^s \|u\|_4 \,\mathrm{d}\tau)^2 \,\mathrm{d}s\Big]^{1/2}\Big\}.$$

Theorem 4.7. *For sufficiently smooth u, u_t and u_{tt}, we have*

$$\|I_{2h}^2 u_t^h - u_t\|_1 \leqslant ch^2 \Big\{\|u_t\|_3 + \|u_{tt}\|_2 + \Big[(\|u_{tt}(0)\|_2 + \|u(0)\|_4)^2 + \int_0^t \Big(\|u_{ttt}\|_2 + \|u_{tt}\|_2 + \|u_t\|_4 + \|u\|_4 + \int_0^s \|u(\tau)\|_4 \,\mathrm{d}\tau\Big)^2 \,\mathrm{d}s\Big]^{1/2}\Big\}.$$

R e m a r k 1. In another paper, we shall discuss the case of elements which are of degree $k \geqslant 2$ for the above problems.

R e m a r k 2. When Ω is a convex quadrilateral domain, then the corresponding superconvergent results hold for such problems if the quadrilateral meshes are almost uniform and are constructed by connecting the equi-proportional points of two opposite boundaries (see [5]).

Acknowledgement. Professor Y. P. Lin generously gave us his papers about integrodifferential equations when the first author visited Canada in 1994. His outstanding work in this field aroused our interest in such problems. The authors would like to thank Professor M. Křížek whose comments improved the final version of the paper. The authors also thank an anonymous referee and the editor in chief for their many helpful suggestions.

References

[1] *J. Cannon, Y. Lin*: A Galerkin procedure for diffusion equations with boundary integral conditions. Int. J. Eng. Sci. *28* (1990), 579–587.

[2] *M. Křížek, P. Neittaanmäki*: On Finite Element Approximation of Variational Problems and Applications. Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific & Technical, Essex, 1989.

[3] *Q. Lin*: A new observation in FEM. Proc. Syst. Sci. & Syst. Eng., Great Wall (H.K.). Culture Publish Co., 1991, pp. 389–391.

[4] *Q. Lin, N. Yan, A. Zhou*: A rectangle test for interpolated finite elements, ibid.
[5] *Q. Lin, Q. Zhu*: The Preprocessing and Postprocessing for the Finite Element Method. Shanghai Scientific & Technical Publishers, 1994.
[6] *Y. Lin*: Galerkin methods for nonlinear parabolic integrodifferential equations with nonlinear boundary conditions. SIAM J. Numer. Anal. *27* (1990), 608–621.
[7] *Y. Lin, T. Zhang*: The stability of Ritz-Volterra projection and error estimates for finite element methods for a class of integro-differential equations of parabolic type. Applications of Mathematics *36* (1991), no. 2, 123–133.
[8] *Y. Lin, V. Thomée, L. Wahlbin*: Ritz-Volterra projection on finite element spaces and applications to integrodifferential and related equations. SIAM J. Numer. Anal. *28* (1991), 1047–1070.
[9] *V. Thomée*: Galerkin Finite Element Methods for Parabolic Problems. Lect. Notes in Math., 1054, 1984.
[10] *V. Thomée, J. Xu, N. Zhang*: Superconvergence of the gradient in piecewise linear finite element approximation to a parabolic problem. SIAM J. Numer. Anal. *26* (1989), 553–573.
[11] *V. Thomée, N. Zhang*: Error estimates for semidiscrete finite element methods for parabolic integrodifferential equations. Math. Comp. *53* (1989), 121–139.
[12] *M. Wheeler*: A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations. SIAM J. Numer. Anal. *10* (1973), 723–759.
[13] *Q. Zhu, Q. Lin*: Superconvergence Theory of the Finite Element Methods. Hunan Science Press, 1990.

Authors' address: *Qun Lin*, *Shuhua Zhang*, Institute of Systems Science, Academia Sinica, Beijing 100080, P. R. China.