Peter Dzurenda

The use of linear approximation scheme for solving the Stefan problem

Applications of Mathematics, Vol. 42 (1997), No. 1, 35--56

Persistent URL: `http://dml.cz/dmlcz/134343`

THE USE OF LINEAR APPROXIMATION SCHEME FOR SOLVING THE STEFAN PROBLEM

Peter Dzurenda, Košice

(Received July 10, 1995)

Abstract. This paper deals with the linear approximation scheme to approximate a singular parabolic problem: the two-phase Stefan problem on a domain consisting of two components with imperfect contact. The results of some numerical experiments and comparisons are presented. The method was used to determine the temperature of steel in the process of continuous casting.

Keywords: heat equation, Stefan problem, phase change, Rothe method

MSC 2000: 35K05, 65N30, 80A20

1. Introduction

The two-phase Stefan problem in enthalpy formulation is a non-linear singular parabolic problem. This problem has been of great interest recently from both the theoretical and the numerical point of view.

Desioles, Droux, Rapaz J. and Rapaz M. in [1] solved this problem by the finite element method and for the corresponding non-linear discrete problem they used Newton's method. Nochetto and Verdi in [8] used piecewise linear finite elements in space and a semiimplicit scheme in time to obtain a discrete problem. Then the non-linear Gauss-Seidel method was used.

Nochetto and Verdi in [7] used a linear approximation scheme to approximate the singular parabolic problem by a linear discrete problem. Kačur, Handlovičová and Kačurová in [4] replaced the parameter μ in the linear approximation scheme by a function $\mu(x)$. This modification allows to increase the step of discretization of the time axis.

In this paper the linear approximation scheme, introduced in [4], is used to solve the two-phase Stefan problem on a domain consisting of two components with imperfect contact. A modification of the iterative method to determine the function $\mu(x)$ is also presented.

Let $\Omega \in \mathbb{R}^N$ $(N = 1, 2, 3)$ be a bounded domain, Ω_1, $\Omega_2 \in \mathbb{R}^N$, $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega = \Omega_1 \cup \Omega_2 \cup \Lambda$ $(\operatorname{mes} \Lambda = \emptyset)$ with the boundary $\partial\hat{\Omega} = \partial\hat{\Omega}_1 \cup \partial\hat{\Omega}_2 \cup \Gamma$. $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$, $\partial\hat{\Omega}_1 = \partial\Omega_1 \setminus \Gamma$, $\partial\hat{\Omega}_2 = \partial\Omega_2 \setminus \Gamma$ are Lipschitz continuous. Let $I = (0, T)$, $0 < T < \infty$.

This paper deals with the problem of determining the functions $u^1\colon \Omega_1 \times I \to \mathbb{R}$, $u^2\colon \Omega_2 \times I \to \mathbb{R}$ which satisfy, in a weak sense, the differential equations

$$\frac{\partial u^1}{\partial t} = \triangle\beta^1(u^1), \qquad \frac{\partial u^2}{\partial t} = \triangle\beta^2(u^2) \tag{1}$$

together with the contact condition on Γ

$$\begin{aligned} -\frac{\partial\beta^1(u^1)}{\partial\nu} &= h(x, t, \beta^1(u^1), \beta^2(u^2))[F^1(\beta^1(u^1)) - F^2(\beta^2(u^2))], \\ \frac{\partial\beta^1(u^1)}{\partial\nu} &= \frac{\partial\beta^2(u^2)}{\partial\nu} \qquad \text{on} \quad \Gamma \times I \end{aligned} \tag{2}$$

and the conditions on $\partial\hat{\Omega}$

$$\begin{aligned} -\frac{\partial\beta^1(u^1)}{\partial\nu} &= d^1(\beta^1(u^1))[F_0^1(\beta^1(u^1)) - \varphi^1] \qquad \text{on} \quad \partial\hat{\Omega}_1 \times I, \\ -\frac{\partial\beta^2(u^2)}{\partial\nu} &= d^2(\beta^2(u^2))[F_0^2(\beta^2(u^2)) - \varphi^2] \qquad \text{on} \quad \partial\hat{\Omega}_2 \times I \end{aligned} \tag{3}$$

as well as with the initial conditions

$$u^1(x, 0) = u_0^1(x) \quad \text{on} \quad \Omega_1, \qquad u^2(x, 0) = u_0^2(x) \quad \text{on} \quad \Omega_2. \tag{4}$$

In (3) ν is the unit outward normal vector to $\partial\hat{\Omega}$ and in (2) ν refers to the unit outward normal vector to Γ, pointing from Ω_1 to Ω_2. The condition (2) describes the imperfect contact between the components Ω_1 and Ω_2.

The functions β^1, β^2, d^1, d^2, φ^1, φ^2, F_0^1, F_0^2, F^1, F^2, h, u_0^1, u_0^2 are sufficiently regular functions of their variables satisfying the following assumptions:

(H_1) $\beta^i\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing, Lipschitz continuous function with a Lipschitz constant L_{β^i},

$$\beta^i(0) = 0, \qquad |\beta^i(s)| \geqslant C_1|s| - C_2 \quad \forall s \in \mathbb{R}, \qquad i = 1, 2,$$

(H$_2$) $F^i\colon \mathbb{R}\to\mathbb{R}$, $F_0^i\colon \mathbb{R}\to\mathbb{R}$, $d^i\colon \mathbb{R}\to\mathbb{R}$ are continuous functions,

$$|F^i(s)| \leqslant C(1+|s|), \qquad |F_0^i(s)| \leqslant C(1+|s|),$$
$$|d^i(s)| \leqslant C \qquad \forall s\in\mathbb{R}, \quad i=1,2,$$

(H$_3$) $h\colon \Omega\times I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $|h(x,t,s,r)|\leqslant C \quad \forall (x,t,s,r)\in\Omega\times I\times\mathbb{R}\times\mathbb{R}$,

(H$_4$) $u_0^i\in L_3(\Omega_i)$, $\varphi^i\in L_2(\partial\hat\Omega_i\times I)$, $i=1,2$,
where C, C_1, C_2 are constants.

The problem (1)–(4) includes the free boundary problems (Stefan problems) in enthalpy formulation. If β is nondecreasing, then (1)–(4) models the heat transfer in the course of solidification of steel in a process of continuous casting with imperfect contact between the mold and the slab.

Let $H^1(\Omega_1)$, $H^1(\Omega_2)$ be the usual first order Sobolev spaces. We introduce the product space $V=H^1(\Omega_1)\times H^1(\Omega_2)$, the dual space of V is denoted by V^* and the duality pairing between $v\in V$ and $w\in V^*$ is written as $\langle v,w\rangle$. Also we set

$$(v,w)=\int_{\Omega_1} v^1w^1\,\mathrm{d}x+\int_{\Omega_2} v^2w^2\,\mathrm{d}x,$$
$$(v,w)_{\partial\Omega}=\int_{\partial\hat\Omega_1} v^1w^1\,\mathrm{d}s+\int_{\partial\hat\Omega_2} v^2w^2\,\mathrm{d}s,$$
$$(v^1,w^1)_\Gamma=\int_\Gamma v^1w^1\,\mathrm{d}s, \qquad (v^2,w^2)_\Gamma=\int_\Gamma v^2w^2\,\mathrm{d}s.$$

Definition 1. A function $u\in L_2(I,L_2)$ is a variational solution of (1)–(4), if $\beta(u)\in L_2(I,V)$, $\partial_t u\in L_2(I,V^*)$ and

$$\begin{aligned}\langle\partial_t u,w\rangle &+ \big(\nabla\beta(u),\nabla w\big)+\big(d(\beta(u))(F_0(\beta(u))-\varphi),w\big)_{\partial\Omega}\\ &+\big(h(x,t,\beta^1(u^1),\beta^2(u^2))[F^1(\beta^1(u^1))-F^2(\beta^2(u^2))],w^1-w^2\big)_\Gamma=0\end{aligned} \tag{5}$$

$\forall w\in V$ almost everywhere in I and

$$u(0)=u_0 \qquad \text{in } V^*. \tag{6}$$

Here

$$w=[w^1,w^2], \qquad u_0=[u_0^1,u_0^2], \qquad \nabla w=[\nabla w^1,\nabla w^2], \qquad \beta(u)=[\beta^1(u^1),\beta^2(u^2)],$$
$$d(\beta(u))(F_0(\beta(u))-\varphi)=[d^1(\beta^1(u^1))(F_0^1(\beta^1(u^1))-\varphi^1),d^2(\beta^2(u^2))(F_0^2(\beta^2(u^2))-\varphi^2)]$$

and $w^1|_\Gamma$, $w^2|_\Gamma$ are traces of the functions $w^1\in H^1(\Omega_1)$, $w^2\in H^1(\Omega_2)$ on Γ.

2. Linear approximation schemes

To solve (5)–(6) numerically by the Rothe method we apply a linear scheme introduced in [4].

Let $m \in \mathbb{N}$, $\tau = \frac{T}{m}$, $t_j = j\tau$, $j = 0, 1, \ldots, m$. We denote

$$\begin{gathered}
u_j(x) = [u_j^1(x), u_j^2(x)] \approx [u^1(x,t_j), u^2(x,t_j)] = u(x,t_j),\\
\Theta_j(x) = [\Theta_j^1(x), \Theta_j^2(x)] \approx [\beta^1(u_j^1(x)), \beta^2(u_j^2(x))] = \beta(u_j(x)),\\
\varphi_j(x) = \frac{1}{\tau}\int_{t_{j-1}}^{t_j} \varphi(x,t)\,\mathrm{d}t,\\
\partial_t u(x,t_j) \approx \frac{u_j - u_{j-1}}{\tau}
\end{gathered}$$

and the linear approximation scheme will be in the form

$$\begin{aligned}
&\Big(\mu_j \frac{\Theta_j - \beta(u_{j-1})}{\tau}, w\Big) + (\nabla\Theta_j, \nabla w) + \big(d(\Theta_{j-1})(F_0(\Theta_{j-1}) - \varphi_j), w\big)_{\partial\Omega}\\
&\quad + \big(h(x,t_j,\Theta_{j-1}^1,\Theta_{j-1}^2)[F^1(\Theta_{j-1}^1) - F^2(\Theta_{j-1}^2)], w^1 - w^2\big)_\Gamma = 0 \quad \forall w \in V,
\end{aligned} \tag{7}$$

$$u_j = u_{j-1} + \mu_j(\Theta_j - \beta(u_{j-1})), \tag{8}$$

for $j = 1, 2, \ldots, m$.

Here $\mu_j = [\mu_j^1, \mu_j^2] \in L_\infty(\Omega_1) \times L_\infty(\Omega_2)$ and together with Θ_j it satisfies the convergence condition

$$|\beta(u_j) - \beta(u_{j-1})| \leqslant \alpha|\Theta_j - \beta(u_{j-1})| + o\Big(\frac{1}{m}\Big), \tag{9}$$

$$0 < \delta \leqslant \mu_j^1 \leqslant K, \qquad 0 < \delta \leqslant \mu_j^2 \leqslant K \tag{10}$$

for $j = 1, 2, \ldots, m$. δ, K are positive constants, $\alpha \in (0,1)$ and $mo(\frac{1}{m}) \to 0$ if $m \to \infty$.

According to (H_1) the functions β^1, β^2 are nondecreasing. If β^i is not strictly monotone, then we approximate it by β_m^i strictly monotone, that is Lipschitz continuous and $\|\beta_m^i - \beta^i\|_{L_\infty(R)} = o\big(\frac{1}{m}\big)$. If β^i is strictly monotone, we set $\beta_m^i = \beta^i$.

We determine the functions μ_j, Θ_j by iterations

$$\mu_{j,0} = \min\Big\{K, \frac{1}{{\beta_m}'(u_{j-1})}\Big\}, \tag{11}$$

$$\begin{aligned}
&\Big(\mu_{j,k-1} \frac{\Theta_{j,k} - \beta(u_{j-1})}{\tau}, w\Big) + (\nabla\Theta_{j,k}, \nabla w) + \big(d(\Theta_{j-1})(F_0(\Theta_{j-1}) - \varphi_j), w\big)_{\partial\Omega}\\
&\quad + \big(h(x,t_j,\Theta_{j-1}^1,\Theta_{j-1}^2)[F^1(\Theta_{j-1}^1) - F^2(\Theta_{j-1}^2)], w^1 - w^2\big)_\Gamma = 0 \qquad \forall w \in V,
\end{aligned} \tag{12}$$

$$
(13)\qquad \mu_{j,k} = \min\Big\{\mu_{j,k-1};\ \frac{\beta_m^{-1}(\beta_m(u_{j-1}) + \alpha(\Theta_{j,k} - \beta(u_{j-1}))) - u_{j-1}}{\Theta_{j,k} - \beta(u_{j-1})};\ \frac{\mu_{j,k-1}\alpha(\Theta_{j,k} - \beta(u_{j-1}))}{\beta_m(u_{j-1} + \mu_{j,k-1}(\Theta_{j,k} - \beta(u_{j-1}))) - \beta_m(u_{j-1})}\Big\}
$$

for $k = 1, 2, 3, \ldots$.

We use the functions Θ_j, μ_j, u_j to find the Rothe functions

$$
(14)\qquad u^m(t) = u_{j-1} + \frac{1}{\tau}(u_j - u_{j-1})(t - t_{j-1}), \qquad t_{j-1} \leqslant t \leqslant t_j,
$$

$$
(15)\qquad \Theta^m(t) = \Theta_{j-1} + \frac{t - t_{j-1}}{\tau}(\Theta_j - \Theta_{j-1}), \qquad t_{j-1} \leqslant t \leqslant t_j,
$$

$$
(16)\qquad \overline{\Theta}^m(t) = \Theta_j, \qquad \bar{u}^m(t) = u_j, \qquad t_{j-1} < t \leqslant t_j
$$

for $j = 1, 2, \ldots, m$.

We find that the iterative process (11)–(13) converges for $k \to \infty$ to Θ_j, μ_j satisfying (7)–(10), and the Rothe functions, defined in (14)–(16), converge to a weak solution of problem (1)–(4). First we prove some a priori estimates for the functions $\mu_{j,k}$, $\Theta_{j,k}$, defined in (11)–(13).

Lemma 1. *There exist $\delta > 0$, $C > 0$ such that for $k = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, m$ we have*

$$
\delta \leqslant \mu^1_{j,k} \leqslant C, \qquad \delta \leqslant \mu^2_{j,k} \leqslant C.
$$

P r o o f. From the construction of $\mu_{j,k}$ we find $\mu^i_{j,k} \leqslant C$, $i = 1, 2$, $k = 0, 1, 2, \ldots$ for $C = K$.

Let $\delta^1 = \frac{\alpha}{L_{\beta^1_m}}$. Then $\mu^1_{j,0} = \min\big\{K, \frac{1}{{\beta^1_m}'(u^1_{j-1})}\big\} > \delta^1$ because of ${\beta^1_m}'(s) \leqslant L_{\beta^1_m} \leqslant C_1$, $\alpha \in (0, 1)$.

Suppose that $\mu^1_{j,k-1} \geqslant \delta^1$. We find $\mu^1_{j,k} \geqslant \delta^1$.

If $\mu^1_{j,k} = \mu^1_{j,k-1}$, the assertion is true.

If

$$
\mu^1_{j,k} = \frac{(\beta^1_m)^{-1}(\beta^1_m(u^1_{j-1}) + \alpha(\Theta^1_{j,k} - \beta^1(u^1_{j-1}))) - u^1_{j-1}}{\Theta^1_{j,k} - \beta^1(u^1_{j-1})},
$$

then the monotocity and the Lipschitz continuity of β^1_m yields

$$
\begin{aligned}
\mu^1_{j,k} = |\mu^1_{j,k}| &= \frac{|(\beta^1_m)^{-1}(\beta^1_m(u^1_{j-1}) + \alpha(\Theta^1_{j,k} - \beta^1(u^1_{j-1}))) - u^1_{j-1}|}{|\Theta^1_{j,k} - \beta^1(u^1_{j-1})|} \\
&\geqslant \frac{1}{L_{\beta^1_m}} \frac{\alpha|\Theta^1_{j,k} - \beta^1(u^1_{j-1})|}{|\Theta^1_{j,k} - \beta^1(u^1_{j-1})|} = \delta^1.
\end{aligned}
$$

If

$$\mu^1_{j,k} = \frac{\mu^1_{j,k-1}\alpha(\Theta^1_{j,k} - \beta^1(u^1_{j-1}))}{\beta^1_m(u^1_{j-1} + \mu^1_{j,k-1}(\Theta^1_{j,k} - \beta^1(u^1_{j-1}))) - \beta^1_m(u^1_{j-1})},$$

then also

$$\begin{aligned}\mu^1_{j,k} = |\mu^1_{j,k}| &= \frac{|\mu^1_{j,k-1}\alpha(\Theta^1_{j,k} - \beta^1(u^1_{j-1}))|}{|\beta^1_m(u^1_{j-1} + \mu^1_{j,k-1}(\Theta^1_{j,k} - \beta^1(u^1_{j-1}))) - \beta^1_m(u^1_{j-1})|}\\ &\geqslant \frac{\alpha|\mu^1_{j,k-1}(\Theta^1_{j,k} - \beta^1(u^1_{j-1}))|}{L_{\beta^1_m}|\mu^1_{j,k-1}(\Theta^1_{j,k} - \beta^1(u^1_{j-1})|} = \delta^1,\end{aligned}$$

therefore $\mu^1_{j,k} \geqslant \delta^1$ for $\forall k = 0, 1, 2, \ldots$.

In the same way we find $\delta^2 \leqslant \mu^2_{j,k} \leqslant C$ for $\delta^2 = \frac{\alpha}{L_{\beta^2_m}}$, $C = K$. We set $\delta = \min\{\delta^1, \delta^2\}$ and the proof is complete. □

Lemma 2. *There exists C such that for $k = 1, 2, \ldots$ we have*

$$\|\Theta_{j,k}\|_V \leqslant C.$$

P r o o f. For $w = \Theta_{j,k}$ in (12) we obtain

$$\Big(\mu_{j,k-1}\frac{\Theta_{j,k} - \beta(u_{j-1})}{\tau}, \Theta_{j,k}\Big) + (\nabla\Theta_{j,k}, \nabla\Theta_{j,k}) + (d(\Theta_{j-1})(F_0(\Theta_{j-1}) - \varphi_j), \Theta_{j,k})_{\partial\Omega} \\ + (h[F^1(\Theta^1_{j-1}) - F^2(\Theta^2_{j-1})], \Theta^1_{j,k} - \Theta^2_{j,k})_\Gamma = 0.$$

Using the inequalities

$$(u, v) \leqslant \varepsilon\|u\|^2 + \frac{1}{\varepsilon}\|v\|^2, \qquad \varepsilon > 0, \tag{17}$$

$$\|u\|^2_{L_2(\partial\Omega)} \leqslant C\Big(\varepsilon\|\nabla u\|^2_{L_2(\Omega)} + \frac{1}{\varepsilon}\|u\|^2_{L_2(\Omega)}\Big), \qquad \varepsilon > 0 \tag{18}$$

and Lemma 1, we conclude

$$\begin{aligned}\delta\|\Theta_{j,k}\|^2_{L_2} + \tau\|\nabla\Theta_{j,k}\|^2_{L_2} &\leqslant (\mu_{j,k-1}\beta(u_{j-1}), \Theta_{j,k})\\ -\tau(d(\Theta_{j-1})(F_0(\Theta_{j-1}) - \varphi_j), \Theta_{j,k})_{\partial\Omega} &- \tau(h[F^1(\Theta^1_{j-1}) - F^2(\Theta^2_{j-1})], \Theta^1_{j,k} - \Theta^2_{j,k})_\Gamma\\ \leqslant \frac{K}{\varepsilon}\|2\beta(u_{j-1})\|^2_{L_2} &+ K\varepsilon\|\Theta_{j,k}\|^2_{L_2} + \frac{\tau}{\varepsilon}\|d(\Theta_{j-1})(F_0(\Theta_{j-1}) - \varphi_j)\|^2_{L_2(\partial\hat\Omega)}\\ + \frac{\tau}{\varepsilon}\|hF^1(\Theta^1_{j-1})\|^2_{L_2(\Gamma)} &+ \frac{\tau}{\varepsilon}\|hF^2(\Theta^2_{j-1})\|^2_{L_2(\Gamma)} + \tau\varepsilon C(\|\Theta_{j,k}\|^2_{L_2} + \|\nabla\Theta_{j,k}\|^2_{L_2}).\end{aligned}$$

For sufficiently small ε we deduce $\|\Theta_{j,k}\|_V \leqslant C$. □

The main result of this section reads as follows:

Theorem 1. *Iterations* (11)–(13) *converge for* $k \to \infty$, *that is* $\mu_{j,k} \to \mu_j$ *in* $L_p(\Omega_1) \times L_p(\Omega_2)$, $p \in \mathbb{N}$, $\Theta_{j,k} \to \Theta_j$ *in* $W_2^1(\Omega_1) \times W_2^1(\Omega_2)$, $\mu_j \in L_\infty(\Omega_1) \times L_\infty(\Omega_2)$, $\Theta_j \in W_2^1(\Omega_1) \times W_2^1(\Omega_2)$ *and* μ_j, Θ_j *satisfy conditions* (7), (9), (10).

P r o o f. The sequences $\{\mu^1_{j,k}\}_{k=0}^\infty$, $\{\mu^2_{j,k}\}_{k=0}^\infty$ are nonincreasing and bounded for $\forall x \in \Omega_1, \Omega_2$, so they converge pointwise to bounded functions μ_j^1, μ_j^2 and also $\mu_{j,k} \to \mu_j$ in $L_p(\Omega_1) \times L_p(\Omega_2)$, $1 \leqslant p < \infty$. In (12) we set $k = k$, $k = k-1$ and $w = \Theta_{j,k} - \Theta_{j,k-1}$. After subtraction we find

(19)
$$\Big(\mu_{j,k-1}\frac{\Theta_{j,k} - \beta(u_{j-1})}{\tau}, \Theta_{j,k} - \Theta_{j,k-1}\Big) - \Big(\mu_{j,k-2}\frac{\Theta_{j,k-1} - \beta(u_{j-1})}{\tau}, \Theta_{j,k} - \Theta_{j,k-1}\Big)$$
$$+(\nabla\Theta_{j,k} - \nabla\Theta_{j,k-1}, \nabla\Theta_{j,k} - \nabla\Theta_{j,k-1}) = 0$$

and

$$\Big(\frac{\mu_{j,k-2}}{\tau}(\Theta_{j,k} - \Theta_{j,k-1}), \Theta_{j,k} - \Theta_{j,k-1}\Big)$$
$$+\frac{1}{\tau}\big((\mu_{j,k-1} - \mu_{j,k-2})(\Theta_{j,k} - \beta(u_{j-1})), \Theta_{j,k} - \Theta_{j,k-1}\big) \leqslant 0,$$

where we have used $\|\nabla\Theta_{j,k} - \nabla\Theta_{j,k-1}\| \geqslant 0$.

According to Lemma 1 and Hölder's inequality we have

$$\|\Theta_{j,k} - \Theta_{j,k-1}\|^2_{L_2} \leqslant \frac{1}{\delta}\|\Theta_{j,k} - \Theta_{j,k-1}\|_{L_2} \cdot \|\mu_{j,k-1} - \mu_{j,k-2}\|_{L_6} \cdot \|\Theta_{j,k} - \beta(u_{j-1})\|_{L_3}.$$

The sequence $\{\Theta_{j,k}\}_{k=1}^\infty$ is bounded in V, $u_0 \in L_3(\Omega_1) \times L_3(\Omega_2)$ and β is Lipschitz continuous, therefore $\beta(u_{j-1}) \in L_3$ and $\|\Theta_{j,k} - \beta(u_{j-1})\|_{L_3}$ is bounded for $k = 1, 2, \ldots$. Since $\|\mu_{j,k-1} - \mu_{j,k-2}\|_{L_6} \to 0$, also $\|\Theta_{j,k} - \Theta_{j,k-1}\|_{L_2} \to 0$. By wirtue of (19) we also have $\|\nabla\Theta_{j,k} - \nabla\Theta_{j,k-1}\|_{L_2} \to 0$, therefore $\Theta_{j,k} \to \Theta_j$ in $W_2^1(\Omega_1) \times W_2^1(\Omega_2)$.

With respect to (13) $\forall x$, $\forall k$

$$\mu_{j,k} \leqslant \frac{\beta_m^{-1}(\beta_m(u_{j-1}) + \alpha(\Theta_{j,k} - \beta(u_{j-1}))) - u_{j-1}}{\Theta_{j,k} - \beta(u_{j-1})}.$$

We let $k \to \infty$ and invoke (8). Then

$$|\beta_m(u_j) - \beta_m(u_{j-1})| \leqslant \alpha|\Theta_j - \beta(u_{j-1})|.$$

Also

$$|\beta(u_j) - \beta(u_{j-1})| \leqslant |\beta(u_j) - \beta_m(u_j)| + |\beta_m(u_j) - \beta_m(u_{j-1})| + |\beta_m(u_{j-1}) - \beta(u_{j-1})|$$
$$\leqslant |\beta_m(u_j) - \beta_m(u_{j-1})| + o\Big(\frac{1}{m}\Big) \leqslant \alpha|\Theta_j - \beta(u_{j-1})| + o\Big(\frac{1}{m}\Big)$$

and the assertion (9) is proved. To obtain (7) we take the limit for $k \to \infty$ in (12). Thus the proof is complete. □

3. Convergence of linear approximation scheme

In this section we establish the convergence of Rothe functions defined in (14)–(16) to a weak solution of problem (1)–(4). First we prove some a priori estimates.

Lemma 3. *There exist C and τ_0 such that for all $\tau < \tau_0$,*

$$\max_{1\leqslant j\leqslant m} \|\beta(u_j)\|_{L_2} + \tau\sum_{j=1}^{m} \|\nabla\Theta_j\|^2_{L_2} + \sum_{j=1}^{m} \|u_j - u_{j-1}\|^2_{L_2} \leqslant C.$$

P r o o f. Rewriting (7) with respect to (8) in the form results in

$$\begin{aligned}(20)\qquad &\Big(\frac{u_j - u_{j-1}}{\tau}, w\Big) + (\nabla\Theta_j, \nabla w) + \big(d(\Theta_{j-1})(F_0(\Theta_{j-1}) - \varphi_j), w\big)_{\partial\Omega} + \\ &+ \big(h(t, x, \Theta^1_{j-1}, \Theta^2_{j-1})[F^1(\Theta^1_{j-1}) - F^2(\Theta^2_{j-1})], w^1 - w^2\big)_\Gamma = 0.\end{aligned}$$

Setting $w = \tau\Theta_j$ and summing (20) for $j = 1, 2, \ldots, k$ we obtain
(21)

$$\begin{aligned}&\sum_{j=1}^{k} (u_j - u_{j-1}, \Theta_j) + \sum_{j=1}^{k}(\nabla\Theta_j, \tau\nabla\Theta_j) + \sum_{j=1}^{k} \big(d(\Theta_{j-1})(F_0(\Theta_{j-1}) - \varphi_j), \tau\Theta_j\big)_{\partial\Omega} \\ &\quad + \sum_{j=1}^{k} \big(h(t, x, \Theta^1_{j-1}, \Theta^2_{j-1})[F^1(\Theta^1_{j-1}) - F^2(\Theta^2_{j-1})], \tau\Theta^1_j - \tau\Theta^2_j\big)_\Gamma = 0.\end{aligned}$$

Also from (8) we find

$$(22)\qquad \Theta_j = \frac{u_j - u_{j-1}}{\mu_j} + \beta(u_{j-1}),$$

therefore

$$\begin{aligned}\sum_{j=1}^{k}(u_j - u_{j-1}, \Theta_j) &= \sum_{j=1}^{k} \big(u_j - u_{j-1}, \frac{u_j - u_{j-1}}{\mu_j}\big) + \sum_{j=1}^{k} \big(u_j - u_{j-1}, \beta(u_j)\big) \\ &\quad - \sum_{j=1}^{k} \big(u_j - u_{j-1}, \beta(u_j) - \beta(u_{j-1})\big) \\ &\geqslant \sum_{j=1}^{k} \|u_j - u_{j-1}\|^2_{L_{2,\mu_j}} + \int_\Omega \Phi(u_k)\,\mathrm{d}x - \int \Phi(u_0)\,\mathrm{d}x \\ &\quad - \sum_{j=1}^{k} \|u_j - u_{j-1}\|_{L_{2,\mu_j}} . \|\mu_j(\beta(u_j) - \beta(u_{j-1}))\|_{L_{2,\mu_j}},\end{aligned}$$

where L_{2,μ_j} is the space L_2 with weight $\frac{1}{\mu_j}$ and $\Phi(s) = \int_0^s \beta(z)\,\mathrm{d}z$.

For $\forall s \in \mathbb{R}$ we have

$$\frac{(\beta(s))^2}{2L_\beta} \leqslant \int_0^s \beta(z)\,\mathrm{d}z \leqslant \frac{L_\beta s^2}{2},$$

therefore

$$\int_\Omega \Phi(u_k)\,\mathrm{d}x \geqslant \int_\Omega \frac{(\beta(u_k))^2}{2L_\beta}\,\mathrm{d}x = C_1\|\beta(u_k)\|^2_{L_2},$$

$$\int_\Omega \Phi(u_0)\,\mathrm{d}x \leqslant \int_\Omega \frac{L_\beta u_0^2}{2}\,\mathrm{d}x = C_2.$$

By virtue of (9) we have

$$|\beta(u_j) - \beta(u_{j-1})| \leqslant \alpha|\Theta_j - \beta(u_{j-1})| + o\Big(\frac{1}{m}\Big) = \alpha\Big|\frac{1}{\mu_j}(u_j - u_{j-1})\Big| + o\Big(\frac{1}{m}\Big),$$

therefore

$$\|\mu_j(\beta(u_j) - \beta(u_{j-1}))\|_{L_{2,\mu_j}} \leqslant \alpha\|u_j - u_{j-1}\|_{L_{2,\mu_j}} + o\Big(\frac{1}{m}\Big).$$

We estimate the first term of (21) by

$$\sum_{j=1}^k (u_j - u_{j-1}, \Theta_j) \geqslant (1-\alpha)\sum_{j=1}^k \|u_j - u_{j-1}\|^2_{L_{2,\mu_j}} + C_1\|\beta(u_k)\|^2_{L_2} + C_2 - ko\Big(\frac{1}{m}\Big),$$

where $\alpha \in (0,1)$ and C_1, C_2 are independent of m.

The second term in (21) can be rewritten in the form

$$\sum_{j=1}^k (\nabla\Theta_j, \tau\nabla\Theta_j) = \tau\sum_{j=1}^k \|\nabla\Theta_j\|^2_{L_2}.$$

The third term in (21) satisfies

$$\begin{aligned}
&\Big|\sum_{j=1}^k \big(d(\Theta_{j-1})(F_0(\Theta_{j-1}) - \varphi_j), \tau\Theta_j\big)_{\partial\Omega}\Big| \\
&\qquad\leqslant \sum_{j=1}^k \int_{\partial\hat\Omega} |d(\Theta_{j-1})(F_0(\Theta_{j-1}) - \varphi_j)\tau\Theta_j|\,\mathrm{d}s \\
&\leqslant \sum_{j=1}^k \tau\int_{\partial\hat\Omega} C_3|(1 + |\Theta_{j-1}|)\Theta_j|\,\mathrm{d}s + \tau\sum_{j=1}^k \int_{\partial\hat\Omega} C_3|\varphi_j\Theta_j|\,\mathrm{d}s \\
&\qquad\leqslant C_4 + C_5\tau\sum_{j=1}^k \int_{\partial\hat\Omega} \Theta_j^2\,\mathrm{d}s.
\end{aligned}$$

The last term in (21) fulfils

$$\Big|\sum_{j=1}^{k}\big(h(t,x,\Theta^1_{j-1},\Theta^2_{j-1})[F^1(\Theta^1_{j-1})-F^2(\Theta^2_{j-1})],\tau(\Theta^1_j-\Theta^2_j)\big)\Big|$$
$$\leqslant \tau\sum_{j=1}^{k}C_6\int_\Gamma C_7(2+|\Theta^1_{j-1}|+|\Theta^2_{j-1}|)|\Theta^1_j-\Theta^2_j|\,\mathrm{d}s$$
$$\leqslant C_8+\tau C_9\sum_{j=1}^{k}\int_\Gamma(\Theta^1_j)^2\,\mathrm{d}s+\int_\Gamma(\Theta^2_j)^2\,\mathrm{d}s,$$

therefore the third and forth term in (21) is estimated by

$$C_{10}+C_{11}\tau\sum_{j=1}^{k}\Big[\int_{\partial\Omega_1}(\Theta^1_j)^2\,\mathrm{d}s+\int_{\partial\Omega_2}(\Theta^2_j)^2\,\mathrm{d}s\Big].$$

With the use of inequality (18) and an estimate that is a consequence of (22),

$$\|\Theta_j\|^2_{L_2}\leqslant\frac{1}{\delta}\|u_j-u_{j-1}\|^2_{L_2}+\|\beta(u_{j-1})\|^2_{L_2}, \tag{23}$$

we can rewrite

$$C_{10}+C_{11}\tau\sum_{j=1}^{k}\Big[\|\Theta^1_j\|^2_{L_2(\partial\Omega_1)}+\|\Theta^2_j\|^2_{L_2(\partial\Omega_2)}\Big]$$
$$\leqslant C_{10}+\varepsilon C_{12}\tau\sum_{j=1}^{k}\|\nabla\Theta_j\|^2_{L_2}+C_{12}\frac{\tau}{\varepsilon}\sum_{j=1}^{k}\|\beta(u_{j-1})\|^2_{L_2}+C_{12}\frac{\tau}{\delta\varepsilon}\sum_{j=1}^{k}\|u_j-u_{j-1}\|^2_{L_2}.$$

We substitute in (21):

$$C_{10}+\varepsilon C_{12}\tau\sum_{j=1}^{k}\|\nabla\Theta_j\|^2_{L_2}+C_{12}\frac{\tau}{\varepsilon}\sum_{j=1}^{k}\|\beta(u_{j-1})\|^2_{L_2}+C_{12}\frac{\tau}{\delta\varepsilon}\sum_{j=1}^{k}\|u_j-u_{j-1}\|^2_{L_2}$$
$$\geqslant(1-\alpha)\sum_{j=1}^{k}\|u_j-u_{j-1}\|^2_{L_{2,\mu_j}}+C_1\|\beta(u_k)\|^2_{L_2}+C_2+\tau\sum_{j=1}^{k}\|\nabla\Theta_j\|^2_{L_2}.$$

The norms L_2 and L_{2,μ_j} are equivalent, therefore

$$\Big[(1-\alpha)-C_{13}\frac{\tau}{\varepsilon}\Big]\sum_{j=1}^{k}\|u_j-u_{j-1}\|^2_{L_{2,\mu_j}}+(1-\varepsilon C_{12})\tau\sum_{j=1}^{k}\|\nabla\Theta_j\|^2_{L_2}+C_1\|\beta(u_k)\|^2_{L_2}$$
$$\leqslant C_{14}+C_{12}\frac{\tau}{\varepsilon}\sum_{j=1}^{k}\|\beta(u_{j-1})\|^2_{L_2}.$$

For sufficiently small ε and τ_0 we have

$$(1-\alpha) - C_{13}\frac{\tau_0}{\varepsilon} > 0, \qquad 1 - \varepsilon C_{12} > 0,$$

and by applying the Gronwall Lemma we prove the assertion of Lemma for $\tau < \tau_0$. □

Lemma 4. *There exists a constant C independent of m such that for $j = 1, 2, \ldots, m$*

$$\|\Theta_j\|_{L_2}^2 \leqslant C.$$

P r o o f. The assertion follows from (23) and Lemma 3. □

Lemma 5. *There exists C independent of m such that*

$$\|u^m - \bar{u}^m\|_{L_2(I,L_2)} \leqslant \frac{C}{\sqrt{m}}, \qquad \|\bar{u}^m(.+\tau) - \bar{u}^m\|_{L_2(I,L_2)} \leqslant \frac{C}{\sqrt{m}},$$
$$\|\overline{\Theta}^m - \beta(\bar{u}^m)\|_{L_2(I,L_2)} \leqslant \frac{C}{\sqrt{m}}, \qquad \|\overline{\Theta}^m - \Theta^m\|_{L_2(I,L_2)} \leqslant \frac{C}{\sqrt{m}}.$$

P r o o f. Lemma 3 yields

$$\begin{aligned}\|u^m - \bar{u}^m\|_{L_2(I,L_2)}^2 &= \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\int_\Omega \Big(u_{j-1} + \frac{t-t_{j-1}}{\tau}(u_j - u_{j-1}) - u_j\Big)^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{\tau}{3}\sum_{j=1}^{k}\|u_j - u_{j-1}\|_{L_2}^2 \leqslant \frac{CT}{3m},\end{aligned}$$
$$\|\bar{u}^m(.+\tau) - \bar{u}^m\|_{L_2(I,L_2)}^2 = \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\int_\Omega (u_{j+1} - u_j)^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{CT}{m},$$

because of $\tau = \tau_m = \frac{T}{m}$.

With the use of (8), (22) we find

$$
\begin{aligned}
\|\overline{\Theta}^m - \beta(\bar{u}^m)\|_{L_2(I,L_2)} &= \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\int_\Omega \Big(\Theta_j - \big(\Theta_{j+1} - \frac{1}{\mu_j}(u_{j+1}-u_j)\big)\Big)^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \tau\sum_{j=1}^{k}\Big\|\frac{1}{\mu_j}(u_{j+1}-u_j)\Big\|_{L_2}^2 + \tau\sum_{j=1}^{k}\|\Theta_{j+1}-\Theta_j\|_{L_2}^2 \\
&\leqslant \frac{CT}{\delta^2 m} + \tau\sum_{j=1}^{k}\Big\|\frac{u_{j+1}-u_j}{\mu_{j+1}} + \beta(u_j) - \frac{u_j - u_{j-1}}{\mu_j} - \beta(u_{j-1})\Big\|_{L_2}^2 \\
&\leqslant \frac{CT}{\delta^2 m} + \tau\sum_{j=1}^{k}\Big(\frac{1}{\delta^2}(\|u_{j+1}-u_j\|_{L_2}^2 + \|u_j - u_{j-1}\|_{L_2}^2) + L_\beta^2\|u_j-u_{j-1}\|_{L_2}^2\Big) \leqslant \frac{C_1}{m},
\end{aligned}
$$

$$
\begin{aligned}
\|\overline{\Theta}^m - \Theta^m\|_{L_2(I,L_2)} &= \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\int_\Omega \Big(\Theta_j - \Big(\Theta_{j-1} + \frac{t-t_{j-1}}{\tau}(\Theta_j - \Theta_{j-1})\Big)\Big)^2\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{\tau}{3}\sum_{j=1}^{k}\|\Theta_j - \Theta_{j-1}\|_{L_2}^2 \\
&\leqslant \frac{\tau}{3}\sum_{j=1}^{k}\Big(\frac{1}{\delta^2}(\|u_j - u_{j-1}\|_{L_2}^2 + \|u_{j-1}-u_{j-2}\|_{L_2}^2) + L_\beta^2\|u_{j-1}-u_{j-2}\|_{L_2}^2\Big) \leqslant \frac{C_2}{m},
\end{aligned}
$$

and the proof is complete. □

Lemma 7. *There exists C independent on m such that*

$$\|\partial_t u^m\|_{L_2(I,V^*)} \leqslant C.$$

P r o o f. According to (7) and with respect to (H_2), (H_3) and (18) we have

$$
\begin{aligned}
\sup_{\|w\|\leqslant 1}|(\partial_t u^m, w)| &\leqslant \sup_{\|w\|\leqslant 1}|(\nabla\overline{\Theta}^m,\nabla w)| + \sup_{\|w\|\leqslant 1}\big|\big(d(\overline{\Theta}_\tau^m)(F_0(\overline{\Theta}_\tau^m) - \bar{\varphi}^m, w\big)_{\partial\Omega}\big| \\
&\quad + \sup_{\|w\|\leqslant 1}\big|\big(h(x,t,\overline{\Theta}_\tau^{1m},\overline{\Theta}_\tau^{2m})[F^1(\overline{\Theta}_\tau^{1m}) - F^2(\overline{\Theta}_\tau^{2m})], w^1 - w^2\big)_\Gamma\big| \\
&\leqslant C_1 + C_2\|\overline{\Theta}^m\|_{L_2}^2 + C_3\|\nabla\overline{\Theta}^m\|_{L_2}^2,
\end{aligned}
$$

where $\overline{\Theta}^m(x,t-\tau) = \overline{\Theta}_\tau^m(x,t)$. Then the assertion of Lemma follows from Lemma 3 and Lemma 4. □

Lemma 8. *There exists C independent of m such that for $0 < z < z_0$ the inequality*

$$\int_0^{T-z}\|\overline{\Theta}^m(t+z) - \overline{\Theta}^m(t)\|_{L_2}^2\,\mathrm{d}t \leqslant \frac{C}{\sqrt{m}} + Cz$$

holds.

P r o o f.

$$\int_0^{T-z} \|\overline{\Theta}^m(t+z) - \overline{\Theta}^m(t)\|_{L_2}^2 \,\mathrm{d}t$$

$$= \int_0^{T-z} \Big\| \beta(\bar{u}^m(t+z-\tau)) + \frac{1}{\overline{\mu}^m(t+z)}(\bar{u}^m(t+z) - \bar{u}^m(t+z-\tau))$$

$$-\beta(\bar{u}^m(t-\tau)) - \frac{1}{\overline{\mu}^m(t)}(\bar{u}^m(t) - \bar{u}^m(t-\tau))\Big\|_{L_2}^2 \,\mathrm{d}t$$

$$\leqslant \int_0^{T-z} \|\beta(\bar{u}^m(t+z-\tau)) - \beta(\bar{u}^m(t-\tau))\|_{L_2}^2 \,\mathrm{d}t$$

$$+\frac{1}{\delta^2}\int_0^{T-z} \|\bar{u}^m(t+z) - \bar{u}^m(t+z-\tau)\|_{L_2}^2 + \|\bar{u}^m(t) - \bar{u}^m(t-\tau)\|_{L_2}^2 \,\mathrm{d}t.$$

The second term can be estimated by $\frac{C}{m}$ and the first with respect to the Lipschitz continuity of β can be estimated by

$$\int_0^{T-z-\tau} \|\beta(\bar{u}^m(t+z)) - \beta(\bar{u}^m(t))\|_{L_2}^2 \,\mathrm{d}t$$

$$\leqslant L_\beta \int_0^{T-z-\tau}\int_\Omega (\beta(\bar{u}^m(t+z)) - \beta(\bar{u}^m(t)))(\bar{u}^m(t+z) - \bar{u}^m(t)) \,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant L_\beta \int_0^{T-z-\tau}\int_\Omega (\overline{\Theta}^m(t+z-\tau) - \overline{\Theta}^m(t-\tau))(\bar{u}^m(t+z) - \bar{u}^m(t)) \,\mathrm{d}x\,\mathrm{d}t$$

$$+L_\beta \int_0^{T-z-\tau}\int_\Omega \bigg(\Big|\frac{1}{\overline{\mu}^m(t+z+\tau)}(\bar{u}^m(t+z+\tau) - \bar{u}^m(t+z))\Big|$$

$$+\Big|\frac{1}{\overline{\mu}^m(t+\tau)}(\bar{u}^m(t+\tau) - \bar{u}^m(t))\Big|\bigg)\big(|\bar{u}^m(t+z) - \bar{u}^m(t)|\big) \,\mathrm{d}x\,\mathrm{d}t,$$

where we have used (22) and the triangle inequality. From Lemma 4, 5 and 6 we have

$$L_\beta \int_0^{T-z-\tau}\int_\Omega (\overline{\Theta}^m(t+z-\tau) - \overline{\Theta}^m(t-\tau))(\bar{u}^m(t+z) - \bar{u}^m(t)) \,\mathrm{d}x\,\mathrm{d}t$$

$$= L_\beta \int_0^{T-z-\tau}\int_\Omega (\overline{\Theta}^m(t+z-\tau) - \overline{\Theta}^m(t-\tau))(\bar{u}^m(t+z) - u^m(t+z)$$

$$+\int_t^{t+z} \partial_t u^m \,\mathrm{d}s + u^m(t) - \bar{u}^m(t)) \,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant L_\beta \int_0^{T-z-\tau} (\|\overline{\Theta}^m(t+z-\tau)\|_V + \|\overline{\Theta}^m(t-\tau)\|_V)(\|\bar{u}^m(t+z) - u^m(t+z)\|_{L_2}$$

$$+\int_t^{t+z} \|\partial_t u^m(s)\|_{V^*} \,\mathrm{d}s + \|u^m(t) - \bar{u}^m(t)\|_{L_2}) \,\mathrm{d}t$$

$$\leqslant L_\beta \int_0^{T-z-\tau} 2C_1\Big(\frac{C_2}{\sqrt{m}} + \int_0^z \|\partial_t u^m(t+\xi)\|_{V^*} d\xi + \frac{C_2}{\sqrt{m}}\Big)\,\mathrm{d}t$$
$$\leqslant L_\beta\Big(2C_1 \int_0^z \|\partial_t u^m\|_{L_2(I,V^*)} d\xi + 4\frac{C_1C_2}{\sqrt{m}}T\Big) \leqslant C_3 z + \frac{C_4}{\sqrt{m}}.$$

Similarly

$$L_\beta \int_0^{T-z-\tau} \int_\Omega \Big(\Big|\frac{1}{\overline{\mu}^m(t+z+\tau)}(\bar{u}^m(t+z+\tau) - \bar{u}^m(t+z))\Big|$$
$$+\Big|\frac{1}{\overline{\mu}^m(t+\tau)}(\bar{u}^m(t+\tau) - \bar{u}^m(t))\Big|\Big)\big(|\bar{u}^m(t+z) - \bar{u}^m(t)|\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant L_\beta \cdot 2\frac{C_5}{\delta\sqrt{m}} 2C_6 \leqslant \frac{C_7}{\sqrt{m}},$$

because of $\|\bar{u}^m\|_{L_2} \leqslant C_6$. That means

$$\int_0^{T-z} \|\overline{\Theta}^m(t+z) - \overline{\Theta}^m(t)\|_{L_2}^2\,\mathrm{d}t \leqslant \frac{C}{m} + C_3 z + \frac{C_4}{\sqrt{m}} + \frac{C_7}{\sqrt{m}},$$

and the proof is complete. □

Theorem 2. *There exists $u \in L_2(I, L_2)$ with $\beta(u) \in L_2(I, V)$ such that $u^{m_k} \rightharpoonup u$ in $L_2(I, L_2)$, $\partial_t u^{m_k} \rightharpoonup \partial_t u$ in $L_2(I, V^*)$, $\Theta^{m_k} \rightharpoonup \beta(u)$ in $L_2(I, V)$ and $\beta(u^{m_k}) \to \beta(u)$, $\Theta^{m_k} \to \beta(u)$ in $L_2(I, L_2)$, $\{m_k\}$ is a suitable subsequence of $\{m\}$.*

P r o o f. Let $\overline{\Omega}^* \subset \Omega$ and $x+y \in \Omega\ \forall x \in \overline{\Omega}^*$. The sequence $\{\overline{\Theta}^m\}_{m=1}^\infty$ is bounded in $L_2(I, V)$, therefore

$$\int_{\Omega^*} (\overline{\Theta}^m(t, x+y) - \overline{\Theta}^m(t,x))^2\,\mathrm{d}x \leqslant |y| \int_\Omega (\nabla\overline{\Theta}^m(t,x))^2\,\mathrm{d}x \leqslant C|y|.$$

Let $|y| < \delta$, $0 < z < z_0$. Applying Lemma 7 we obtain

$$\int_0^{T-z_0}\int_{\Omega^*} (\overline{\Theta}^m(t+z, x+y) - \overline{\Theta}^m(t,x))^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \int_0^{T-z_0}\int_{\Omega^*} (\overline{\Theta}^m(t+z, x+y) - \overline{\Theta}^m(t, x+y))^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^{T-z_0}\int_{\Omega^*} (\overline{\Theta}^m(t, x+y) - \overline{\Theta}^m(t,x))^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{C}{\sqrt{m}} + Cz + C|y|.$$

$\{\overline{\Theta}^m\}_{m=1}^{\infty}$ in $L_2(I, L_2)$ is compact because of Lemma 7 and Kolmogorov's criterion. There exists a suitable subsequence $\{\overline{\Theta}^{m_k}\}_{k=0}^{\infty}$ which converges to ϑ in $L_2(I, L_2)$. Applying Lemma 5 we find that $\{\Theta^{m_k}\}$, $\{\beta(u^{m_k})\}$ tend to ϑ in $L_2(I, L_2)$. Because of $\|\bar{u}^{m_k}\|_{L_2(I,L_2)} \leqslant C$, a suitable subsequence $\{\bar{u}^{m_k}\}$ (also denoted by $\{\bar{u}^{m_k}\}$) converges in the weak sense to u in $L_2(I, L_2)$. $\partial_t u^{m_k} \rightharpoonup \chi$ in $L_2(I, V^*)$ because it is bounded in $L_2(I, V^*)$. We let $k \to \infty$ in the term

$$\int_0^{T'} (u^{m_k}(t), w)\,\mathrm{d}t = \int_0^{T'} \int_0^t (\partial_t u^{m_k}, w)\,\mathrm{d}s\,\mathrm{d}t = \int_0^{T'} \int_0^t \langle \partial_t u^{m_k}, w\rangle\ \mathrm{d}s\,\mathrm{d}t,$$

which implies $\chi = \partial_t u$.

$\beta(s)$ is monotone, therefore

$$\int_0^T (\beta(\bar{u}^{m_k}) - \beta(w), \bar{u}^{m_k} - w)\,\mathrm{d}t \geqslant 0, \qquad \forall w \in L_2(I, L_2).$$

Passing $k \to \infty$ we find

$$\int_0^T (\vartheta - \beta(w), u - w)\,\mathrm{d}t \geqslant 0, \qquad \forall w \in L_2(I, L_2).$$

Setting $w = u \pm \varepsilon z$, $z \in L_\infty(\Omega_1 \times I) \times L_\infty(\Omega_2 \times I)$, $\varepsilon > 0$ we arrive at

$$\int_0^T (\vartheta - \beta(u - \varepsilon z), \varepsilon z)\,\mathrm{d}t \leqslant 0, \qquad \int_0^T (\vartheta - \beta(u + \varepsilon z), \varepsilon z)\,\mathrm{d}t \geqslant 0.$$

Passing $\varepsilon \to 0$ we find $\vartheta = \beta(u)$.

Because of Lemma 3 and Lemma 4

$$\begin{aligned}
\|\Theta^m\|^2_{L_2(I,V)} &= \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \bigg[\int_\Omega \Big(\Theta_j + \frac{t - t_{j-1}}{\tau}(\Theta_j - \Theta_{j-1})\Big)^2 \mathrm{d}x \\
&\quad + \int_\Omega \Big(\nabla\Theta_j + \frac{t - t_{j-1}}{\tau}(\nabla\Theta_j - \nabla\Theta_{j-1})\Big)^2 \mathrm{d}x \bigg] \mathrm{d}t \\
&\leqslant \sum_{j=1}^m \frac{4\tau}{3}\|\Theta_j\|^2_{L_2} + \frac{\tau}{3}\|\Theta_{j-1}\|^2_{L_2} + \frac{4\tau}{3}\|\nabla\Theta_j\|^2_{L_2} + \frac{\tau}{3}\|\nabla\Theta_{j-1}\|^2_{L_2} \\
&\leqslant \frac{5}{3}C_1 m\tau + \frac{5}{3}C_2 = \frac{5}{3}(C_1 T + C_2),
\end{aligned}$$

therefore $\{\Theta^m\}$ is bounded in $L_2(I, V)$, and $\{\Theta^{m_k}\}$ converges in the weak sense to $\beta(u)$ in $L_2(I, V)$, because $L_2(I, V) \subset L_2(I, L_2)$. □

Lemma 9. *Let $u^{m_k} \rightharpoonup u$, then $\overline{\Theta}^{m_k} \to \beta(u)$ in $L_2(I,V)$.*

Proof. Rewrite (7) in the form

$$\int_0^T \langle \partial_t u^{m_k}, w\rangle \,\mathrm{d}t + \int_0^T (\nabla\overline{\Theta}^{m_k}, \nabla w)\,\mathrm{d}t + \int_0^T \big(d(\overline{\Theta}_\tau^{m_k})(F(\overline{\Theta}_\tau^{m_k}) - \overline{\varphi}^{m_k}), w\big)_{\partial\Omega}\,\mathrm{d}t$$

$$+ \int_0^T \big(h(x,t,\overline{\Theta}_\tau^{1m_k}, \overline{\Theta}_\tau^{2m_k})[F_0^1(\overline{\Theta}_\tau^{1m_k}) - F_0^2(\overline{\Theta}_\tau^{2m_k})], w^1 - w^2\big)_\Gamma\,\mathrm{d}t = 0, \tag{24}$$

where $\overline{\varphi}^{m_k} = \varphi(t_j)$, $t_{j-1} < t \leqslant t_j$, $j = 1,2,\ldots,m_k$. Let $w = \overline{\Theta}^{m_k} - \beta(u)$. Using

$$\lim_{k\to\infty} \int_0^t \big\langle \partial_t u^{m_k}, \overline{\Theta}^{m_k}\big\rangle \,\mathrm{d}t \geqslant \int_\Omega \Phi(u(t))\,\mathrm{d}x - \int_\Omega \Phi(u_0)\,\mathrm{d}x,$$

$$\int_0^t \langle \partial_t u, \beta(u)\rangle \,\mathrm{d}t = \int_\Omega \Phi(u(t))\,\mathrm{d}x - \int_\Omega \Phi(u_0)\,\mathrm{d}x,$$

(see [5]) we find

$$\lim_{k\to\infty} \int_0^T \big\langle \partial_t u^{m_k}, \overline{\Theta}^{m_k} - \beta(u)\big\rangle \,\mathrm{d}t \geqslant 0.$$

The second term in (24) can be rewritten in the form

$$\int_0^T (\nabla\overline{\Theta}^{m_k}, \nabla\overline{\Theta}^{m_k} - \nabla\beta(u))\,\mathrm{d}t = \int_0^T \|\nabla\overline{\Theta}^{m_k} - \nabla\beta(u)\|_{L_2}^2\,\mathrm{d}t + \int_0^T (\nabla\overline{\Theta}^{m_k} - \nabla\beta(u), \nabla\beta(u))\,\mathrm{d}t.$$

The sequence $\overline{\Theta}^{m_k}$ converges to $\beta(u)$ in $L_2(I,L_2)$, is bounded in $L_2(I,V)$, and a suitable subsequence converges in the weak sense to $\beta(u)$ in $L_2(I,V)$. Therefore

$$\int_0^T (\nabla\overline{\Theta}^{m_k} - \nabla\beta(u), \nabla\beta(u))\,\mathrm{d}t \to 0$$

for $k\to\infty$, and $\Theta^{1,m_k} \to \beta^1(u^1)$ in $L_2(I,L_2(\partial\Omega_1))$, $\Theta^{2,m_k} \to \beta^2(u^2)$ in $L_2(I,L_2(\partial\Omega_2))$ (see [5]), therefore

$$\int_0^T \big(d(\overline{\Theta}_\tau^{m_k})(F(\overline{\Theta}_\tau^{m_k}) - \overline{\varphi}^{m_k}), \overline{\Theta}^{m_k} - \beta(u)\big)_{\partial\Omega}\,\mathrm{d}t \to 0.$$

Similarly

$$\int_0^T \big(h(x,t,\overline{\Theta}_\tau^{1,m_k}, \overline{\Theta}_\tau^{2,m_k})[F_0^1(\overline{\Theta}_\tau^{1,m_k}) - F_0^2(\overline{\Theta}_\tau^{2,m_k})], \overline{\Theta}^{1,m_k} - \beta^1(u^1) - \overline{\Theta}^{2,m_k} + \beta^2(u^2)\big)_\Gamma\,\mathrm{d}t \to 0.$$

Using (24) we find

$$0 \geqslant \int_0^T \|\nabla\overline{\Theta}^{m_k} - \nabla\beta(u)\|_{L_2}^2 \, dt + o(1),$$

which implies $\overline{\Theta}^{m_k} \to \beta(u)$ in $L_2(I, V)$. □

The main result of this section reads as follows:

Theorem 3. *There exists a variational solution of problem* (1)–(4) *and subsequences* $\{\Theta^{m_k}\}_{k=0}^{\infty}$, $\{u^{m_k}\}_{k=0}^{\infty}$ *such that* $u^{m_k} \rightharpoonup u$ *in* $L_2(I, L_2)$, $\Theta^{m_k} \to \Theta \equiv \beta(u)$ *in* $L_2(I, V)$. *The functions* Θ^m, u^m *satisfy the conditions* (8)–(9). *If the variational solution of* (1)–(4) *is unique, the original sequences* $\{\Theta^m\}$, $\{u^m\}$ *are convergent.*

Proof. Theorem 2 and Lemma 9 imply

$$\partial_t u^{m_k} \rightharpoonup \partial_t u \quad \text{in } L_2(I, V^*), \qquad \overline{\Theta}^{m_k} \to \beta(u) \quad \text{in } L_2(I, V).$$

For $\tau \to 0$ we have

$$\int_0^t \|\overline{\Theta}_\tau^{m_k} - \beta(u_\tau)\|_{L_2}^2 \, dt \to 0, \qquad \int_0^t \|\nabla\overline{\Theta}_\tau^{m_k} - \nabla\beta(u_\tau)\|_{L_2}^2 \, dt \to 0,$$
$$\int_0^t \|\beta(u_\tau) - \beta(u)\|_{L_2}^2 \, dt \to 0, \qquad \int_0^t \|\nabla\beta(u_\tau) - \nabla\beta(u)\|_{L_2}^2 \, dt \to 0,$$

and $\overline{\Theta}_\tau^{m_k} \to \beta(u)$ almost everywhere in $I \times \partial\Omega_1$, $I \times \partial\Omega_2$. For more details see [4]. Therefore in (24) we can pass to $k \to \infty$, which implies the assertion of Theorem 3. Let u be the unique solution of problem (1)–(4). Suppose that $\{\Theta^m\}_{m=0}^{\infty}$ has two convergent subsequences, which tend to different functions $\Theta = \beta(u)$, $\tilde{\Theta} = \beta(\tilde{u})$. Then u, $\tilde{u}$ are variational solutions of (1)–(4), which contradicts with the uniqueness of u. Therefore the original sequence is convergent. □

4. Full discretization scheme

In this section we consider the full discretization scheme. Let $\{V_\lambda,\ \lambda \in \Lambda\}$ be a sequence of finite dimensional subspaces of V satisfying

(H$_5$) $\forall \{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_n \to 0$ for $n \to \infty$ and for $\forall v \in L_2(I, V) \exists \{v^{\lambda_n}\}_{n=1}^{\infty} \in L_2(I, V_{\lambda_n})$ such that $v^{\lambda_n} \to v$ in $L_2(I, V)$.

The corresponding full discretization scheme of (8)–(10) reads as follows:

$$u_j = u_{j-1} + \mu_j(\Theta_j^\lambda - \beta(u_{j-1})), \tag{25}$$

$$\begin{aligned}(26)\qquad &\Big(\mu_j\frac{\Theta_j^\lambda-\beta(u_{j-1})}{\tau},w\Big)+(\nabla\Theta_j^\lambda,\nabla w)+\big(d(\Theta_{j-1}^\lambda)(F_0(\Theta_{j-1}^\lambda)-\varphi_j),w\big)_{\partial\Omega}\\ &+\big(h(x,t_j,\Theta_{j-1}^{1,\lambda},\Theta_{j-1}^{2,\lambda})[F^1(\Theta_{j-1}^{1,\lambda})-F^2(\Theta_{j-1}^{2,\lambda})],w^1-w^2\big)_\Gamma=0\end{aligned}$$

for $\forall\, w\in V_\lambda$, $j=1,2,\dots,m$, where the functions $\mu_j=[\mu_j^1,\mu_j^2]\in L_\infty(\Omega_1)\times L_\infty(\Omega_2)$, $\Theta_j^\lambda\in V_\lambda$, satisfy the convergence conditions

$$(27)\qquad |\beta(u_j)-\beta(u_{j-1})|\leqslant\alpha|\Theta_j^\lambda-\beta(u_{j-1})|+o\Big(\frac{1}{m}\Big),$$

$$(28)\qquad 0<\delta\leqslant\mu_j^1\leqslant K,\qquad 0<\delta\leqslant\mu_j^2\leqslant K.$$

Analogously to the previous section μ_j, Θ_j^λ will be obtained by iterations

$$(29)\qquad \mu_{j,0}=\min\Big\{K,\frac{1}{{\beta_m}'(u_{j-1})}\Big\},$$

$$\begin{aligned}(30)\qquad &\Big(\mu_{j,k-1}\frac{\Theta_{j,k}^\lambda-\beta(u_{j-1})}{\tau},w\Big)+(\nabla\Theta_{j,k}^\lambda,\nabla w)+\big(d(\Theta_{j-1}^\lambda)(F_0(\Theta_{j-1}^\lambda)-\varphi_j),w\big)_{\partial\Omega}\\ &+\big(h(x,t_j,\Theta_{j-1}^{1,\lambda},\Theta_{j-1}^{2,\lambda})[F^1(\Theta_{j-1}^{1,\lambda})-F^2(\Theta_{j-1}^{2,\lambda})],w^1-w^2\big)_\Gamma=0\quad\forall w\in V_\lambda,\end{aligned}$$

$$\begin{aligned}(31)\qquad \mu_{j,k}=\min\Big\{&\mu_{j,k-1};\frac{\beta_m^{-1}(\beta_m(u_{j-1})+\alpha(\Theta_{j,k}^\lambda-\beta(u_{j-1})))-u_{j-1}}{\Theta_{j,k}^\lambda-\beta(u_{j-1})};\\ &\frac{\mu_{j,k-1}\alpha(\Theta_{j,k}^\lambda-\beta(u_{j-1}))}{\beta_m(u_{j-1}+\mu_{j,k-1}(\Theta_{j,k}^\lambda-\beta(u_{j-1})))-\beta_m(u_{j-1})}\Big\},\end{aligned}$$

for $k=1,2,3,\dots$.

Let $\gamma=(\tau,\lambda)$, $\tau=\frac{T}{m}$ We denote

$$\begin{aligned}(32)\qquad &\Theta^\gamma(t)=\Theta_{j-1}^\lambda+\frac{t-t_{j-1}}{\tau}(\Theta_j^\lambda-\Theta_{j-1}^\lambda),\qquad \bar\Theta^\gamma(t)=\Theta_{j-1}^\lambda,\\ &u^\gamma(t)=u_{j-1}+\frac{t-t_{j-1}}{\tau}(u_j-u_{j-1}),\qquad \bar u^\gamma(t)=u_{j-1}\end{aligned}$$

for $t_{j-1}<t\leqslant t_j$, $j=1,2,\dots,m$. The subsequence of γ, $\gamma\to(0,0)$, will be denoted by $\{\bar\gamma\}$.

Our main results of this section are

Theorem 5. *The iteration process* (29)–(31) *converges for $k\to\infty$, that means $\mu_{j,k}\to\mu_j$ in $L_p(\Omega_1)\times L_p(\Omega_2)$, $p\in\mathbb{N}$, $\Theta_{j,k}^\lambda\to\Theta_j^\lambda$ in $L_2(\Omega_1)\times L_2(\Omega_2)$, and μ_j, Θ_j^λ satisfies the conditions* (26), (28).

Theorem 6. *Let $\tau \to 0$, $\lambda \to 0$ and let the assumptions (H_1)–(H_5) be satisfied. Then there exists $\{\bar{\gamma}\}$ and a variational solution u of (1)–(4) such that $u^{\bar{\gamma}} \rightharpoonup u$ in $L_2(I, L_2)$, $\Theta^{\bar{\gamma}} \to \beta(u)$ in $L_2(I, V)$. The functions $\Theta^{\bar{\gamma}}$, $u^{\bar{\gamma}}$ are defined in (32) and Θ^{λ}_{j-1}, u_{j-1} satisfy (25)–(28). If the variational solution of (1)–(4) is unique, then the original sequences $\{\Theta^{\gamma}\}$, $\{u^{\gamma}\}$ are convergent.*

The proofs of Theorems 5 and 6 are similar to those of Theorems 2 and 4.

5. Numerical experiments

The scheme introduced above was tested by solving numerically the example

$$\frac{\partial u}{\partial t} = \triangle \beta(u), \tag{33}$$

$$-\frac{\partial \beta(u)}{\partial \nu}\bigg|_{x=a} = q_4(y,t), \qquad -\frac{\partial \beta(u)}{\partial \nu}\bigg|_{x=b} = q_2(y,t), \tag{34}$$
$$-\frac{\partial \beta(u)}{\partial \nu}\bigg|_{y=c} = q_1(y,t), \qquad -\frac{\partial \beta(u)}{\partial \nu}\bigg|_{y=d} = q_3(y,t),$$

$$u(x,y,0) = u_0(x,y), \tag{35}$$

where $\Omega = \langle 0, 0.25\rangle \times \langle 0, 0.5\rangle$, $I = (0, 0.4)$ and

$$\beta(s) = \begin{cases} s & s \leqslant 0 \\ 0 & 0 \leqslant s \leqslant 1 \\ 10s - 10 & 1 \leqslant s \end{cases}$$

with the exact solution

$$\tilde{u}(x,y,t) = \begin{cases} 2e^{0.1\Phi} - 1 & \Phi \geqslant 0 \\ \\ e^{\Phi} - 1 & \Phi < 0 \end{cases} \tag{36}$$

Here $\Phi = 0.1 - x - y + 2t = 0$ is the exact free boundary and q_1, q_2, q_3, q_4 are the derivatives of the exact solution.

The domain Ω was discretized by 1352 squares 0.01×0.01 and the time variable was gradually discretized with the steps $\tau = 0.1$, 0.02, 0.01, 0.004, 0.001, 0.0004, 0.0001.

We set $e(x,y,t) = |u(x,y,t) - \tilde{u}(x,y,t)|$. Figure 1 shows the results obtained with $\tau = 0.001$ at the time $t = 0.1$. The values of e are greater only in the neighbourhood of the free boundary.

The scheme (8)–(13) (Scheme 1) was also compared with the linear approximation schemes introduced in [4] (Scheme 2) and in [7] (Scheme 3). We set

$$E = \int_0^T \int_a^b \int_c^d (u(x,y,t) - \tilde{u}(x,y,t))^2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t. \tag{37}$$

Table 1 shows the results provided by various schemes for various values of τ. The value of E, obtained by Scheme 1, monotonely decreases with decreasing τ. The results of Scheme 2 are for greater τ very good, but with smaller τ the error grows. The results of Scheme 3 are not so good as those of Scheme 1 and Scheme 2.

The average numbers of iterations, which are made in every time step, are also of interest. Scheme 3 needs only 1 iteration for computation at one time layer, Scheme 2 needs 4 iterations. Scheme 1 needs more iterations, therefore this scheme demands more time for computation.

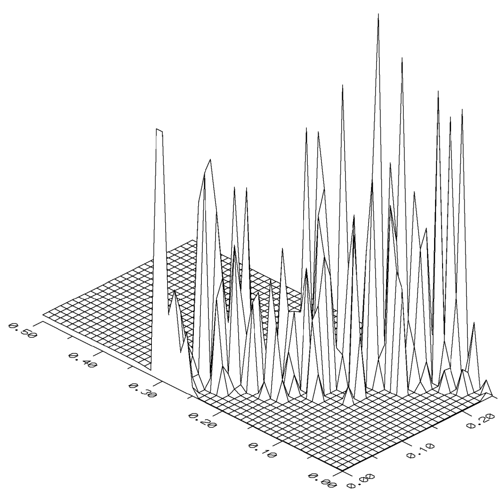

Figure 1

	Scheme 1		Scheme 2		Scheme 3	
τ	E	iter.	E	iter.	E	iter.
0.1	$5.64e-03$	11	$2.77e-03$	4	$5.28e-03$	1
0.02	$3.12e-03$	17	$1.13e-03$	4	$3.75e-03$	1
0.01	$1.81e-03$	22	$1.08e-03$	4	$3.02e-03$	1
0.004	$8.63e-04$	18	$2.73e-04$	4	$2.06e-03$	1
0.001	$2.79e-04$	18	$2.38e-04$	4	$1.07e-03$	1
0.0004	$1.00e-04$	19	$3.99e-04$	4	$6.62e-04$	1
0.0001	$8.83e-05$	19	$5.77e-04$	4	$3.10e-04$	1

Table 1

When solving the Stefan problem, the problem of determining the free boundary is very important. In Figure 2 we show the results of determining the free boundary with use of Scheme 1 (Figure 2.a), Scheme 2 (Figure 2.b) and Scheme 3 (Figure 2.c) at time $t = 0.1$ computed with $\tau = 0.01$. The boundary is determined as the set of $[x, y]$ that satisfy the inequality $0 \leqslant u(x, y, t) \leqslant 1$, therefore the boundary is not a single line but a zone. The best results are produced by Scheme 2, because the zone of the free boundary is tin, but for a greater time (for example $t = 0.3$) the determined zone does not include the exact free boundary. The zone produced by Scheme 3 is wide, so the approximation of the exact free boundary is not excellent. The zone produced by Scheme 1 is not so wide as in Scheme 3, the exact free boundary is well approximated also for a greater time.

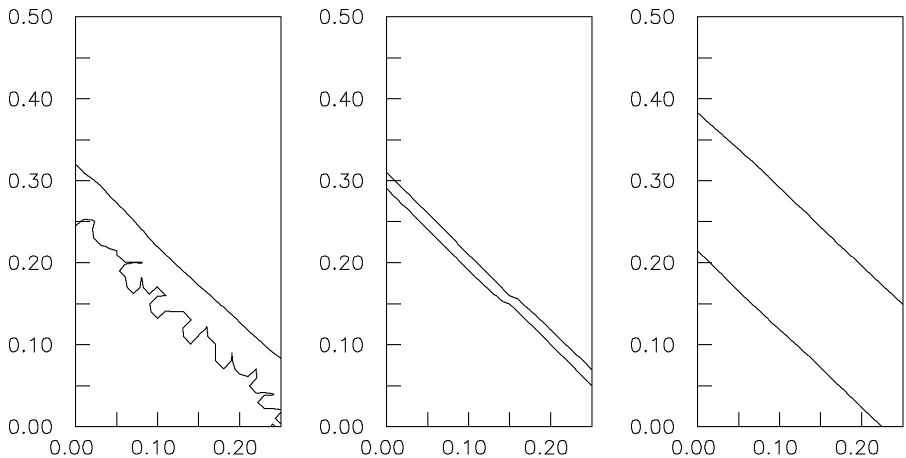

Figure 2a Figure 2b Figure 2c

6. The temperature of steel in the process of continuous casting

The scheme (8)–(13) was used to determine the temperature of steel in the process of continuous casting. We used the slab in the form of a rectangle 0.207m × 1.666m and the mold 0.271m × 1.730m. The contact between the mold and the slab was imperfect due to the presence of an air slot and was described by the condition (3). The speed of motion of the slab in the mold was constant, therefore we replaced the z-axis of the mold by the time-axis. The temperature was computed in 85 time layers with use of 14 625 finite elements of rectangular form. The results at the time when the slab leaves the mold, are shown in Figure 3.

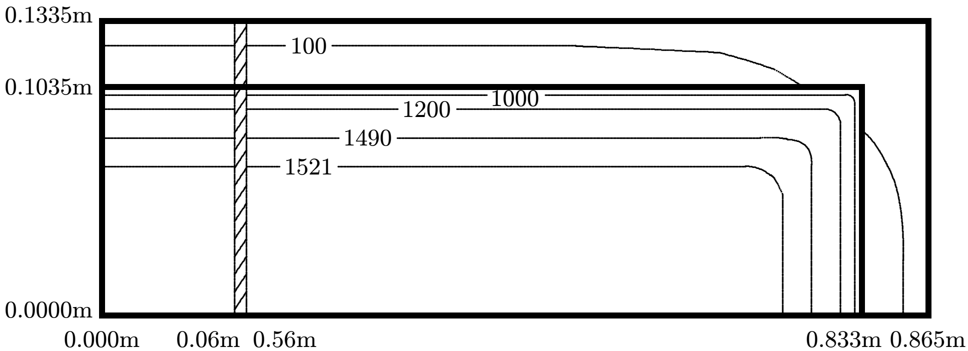

Figure 3

References

[1] *J. L. Desbiolles, J. J. Droux, J. Rapaz, M. Rapaz*: Simulation of solidification of alloys by the finite element method. An advanced course on the Finite Element Methods in Physics. European Association of Physics, Lausanne (Switzerland), 1987.

[2] *A. Handlovičová*: Errors estimates of a linear approximation schemes for nonlinear difussion problems. Acta Math. Univ. Comenianae *LXI, 1* (1992), 27–39.

[3] *W. Jäger, J. Kačur*: Solution of porous medium type systems by linear approximation schemes. Preprint 540, Universität Heidelberg SFB 1233, 1989.

[4] *J. Kačur, A. Handlovičová, M. Kačurová*: Solution of nonlinear diffusion problems by linear approximation schemes. SIAM Num. Anal. *30* (1993), 1703–1722.

[5] *J. Kačur, S. Luckhaus*: Approximation of degenerate parabolic systems by nondegenerate elliptic parabolic systems. Preprint M2-91, Comenius University, Bratislava.

[6] *R. H. Nochetto*: Error estimates for multidimensional Stefan problems with general boundary conditions. Research Notes in Math. 120, Pitman, Boston, 1985, pp. 50–60.

[7] *R. H. Nochetto, C. Verdi*: An efficent linear scheme to approximate parabolic free boundary problems: error estimates and implementation. American Math. Society, 1988, pp. 27–53.

[8] *R. H. Nochetto, C. Verdi*: Approximation of degenerate parabolic problems using numerical integration. SIAM J. numer. anal. *35* (1988), no. 4, 784–814.

Author's address: *Peter Dzurenda*, Katedra matematiky FEI, Technická univerzita, Hlavná 6, 042 00 Košice, Slovakia, e-mail: `dzurenda@ccsun.tuke.sk`.