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SHAPE OPTIMIZATION OF MATERIALLY NON-LINEAR BODIES IN CONTACT

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Abstract. Optimal shape design problem for a deformable body in contact with a rigid foundation is studied. The body is made from material obeying a nonlinear Hooke's law. We study the existence of an optimal shape as well as its approximation with the finite element method. Practical realization with nonlinear programming is discussed. A numerical example is included.

Keywords: shape optimization, sensitivity analysis

MSC 2000: 49K20, 49J20, 73K40

1. Introduction

Shape optimization is a branch of optimal control theory, in which the control variable is related to the shape of a structure. The aim is to find a shape in such a way that the structure behaves in an appropriate way. Shape optimization of systems, the behaviour of which is described by variational inequalities, deserves particular attention, as the resulting problem is *non-smooth*, in general. The so-called contact problems of deformable bodies are one of the most important applications of variational inequalities in mechanics of solids. The present paper analyzes shape optimization of bodies, materials of which obey a non-linear monotone Hooke's law, describing the so-called *deformation theory of plasticity.* The same approach can be used also in other problems where the constitutive laws are defined by monotone relations.

The paper is organized as follows: in Section 2, the non-linear state problem is defined and basic properties of the corresponding total potential energy are mentioned. An optimal shape design problem (P) is formulated. In Section 3, the existence of at least one solution of (P) is proved, provided the objective functional satisfies appropriate assumptions. Section 4 deals with the discretization of (P) which is based on the finite element approach. We prove that under reasonable assumptions the continuous and the discrete models are close on subsequences. Finally, the last Section is devoted to computational aspects and numerical results of model examples.

2. Setting of the problem

Let a deformable body be represented by a bounded plane domain $\Omega \subset \mathbb{R}^2$, the Lipschitz boundary of which will be decomposed as follows:

$$\partial\Omega = \bar{\Gamma}_U \cup \bar{\Gamma}_P \cup \bar{\Gamma}_C.$$

On each of these parts, different boundary conditions will be prescribed. Throughout the paper we assume that $\Gamma_U \neq \emptyset$ is open in $\partial\Omega$. The body will be made from a material obeying the theory of small elasto-plastic deformations, see [13] or [9]—Chapter 8. Plane strain situation is assumed troughout the paper. In this case the non-linear relation between the stress tensor $\sigma = (\sigma_{ij})_{i,j=1}^2$ and the linearized strain tensor $\varepsilon = (\varepsilon_{ij})_{i,j=1}^2$ is given by[1]

$$\sigma_{ij} = \kappa\varepsilon_{ll}\delta_{ij} + 2\mu(\gamma)\Big(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\Big), \tag{1}$$

where κ, μ respectively stand for the bulk and shear modulus and δ_{ij} is the Kronecker symbol. The shear modulus μ is assumed to be a function of the invariant $\gamma \equiv \gamma(\varepsilon_{ij})$, defined by

$$\gamma = \frac{1}{3}\left[(\varepsilon_{11} - \varepsilon_{22})^2 + \varepsilon_{11}^2 + \varepsilon_{22}^2 + 6\varepsilon_{12}^2\right]. \tag{2}$$

We shall assume that the functions $\kappa \equiv \kappa(x)$, $\mu \equiv \mu(t,x)$, $x \in \Omega$, $t \geqslant 0$, depend continuously on their arguments and μ is continuously differentiable with respect to t:

$$\kappa \in C(\bar{\Omega}),\ \mu \in C(\bar{\mathbb{R}}_+^1 \times \bar{\Omega}),\ \frac{\partial\mu}{\partial t} \in C(\bar{\mathbb{R}}_+^1 \times \bar{\Omega}).$$

Moreover, the following assumptions on κ, μ are made:

$$0 < \kappa_0 \leqslant \kappa(x) \leqslant \kappa_1 \quad \forall x \in \Omega; \tag{3}$$

$$0 < \mu_0 \leqslant \mu(t,x) \leqslant \frac{3}{2}\kappa(x) \quad \forall x \in \Omega,\ \forall t > 0; \tag{4}$$

$$0 < \theta_0 \leqslant \mu(t,x) + 2\frac{\partial\mu(t,x)}{\partial t}t \leqslant \theta_1 \quad \forall x \in \Omega,\ \forall t > 0, \tag{5}$$

[1] The summation convention is used in the paper.

where κ_0, κ_1, μ_0, θ_0 and θ_1 are given positive constants.

Now, we formulate the boundary conditions. Let Ω be *unilaterally supported* by a rigid half plane $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leqslant 0\}$, which supports Ω along Γ_C. Suppose that Γ_C is given by the graph of a non-negative function $\alpha\colon [a,b] \to \mathbb{R}^1$ (see Figure 1).

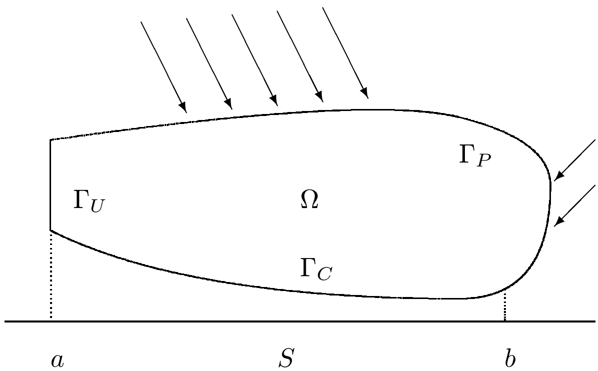

Fig. 1. Physical situation

On Γ_C, the classical contact conditions will be prescribed:

$$
\begin{cases} u_2(x_1, \alpha(x_1)) \geqslant -\alpha(x_1) \quad \forall x_1 \in (a,b), \\ T_2(u) \equiv \sigma_{2j}(u) n_j \geqslant 0, \quad T_2(u_2 + \alpha) = 0, \\ T_1(u) \equiv \sigma_{1j}(u) n_j = 0. \end{cases} \tag{6}
$$

For the mathematical justification of (6) see [11]—Section 2.1.2. Here $u = (u_1, u_2)$ denotes the displacement field, $n = (n_1, n_2)$ is the unit outward normal vector along $\partial\Omega$. On the remaining parts Γ_U and Γ_P, the body Ω is supposed to be fixed and subjected to surface tractions $P = (P_1, P_2)$, respectively:

$$
u_i = 0, \; i = 1,2 \quad \text{on } \Gamma_U, \tag{7}
$$

$$
T_i \equiv \sigma_{ij} n_j = P_i, \; i = 1,2 \quad \text{on } \Gamma_P. \tag{8}
$$

The body Ω is also subjected to a body force $F = (F_1, F_2)$.

By a *classical solution* of the Signorini problem for Ω, we mean any displacement field u satisfying boundary conditions (6)–(8) and the system of the equilibrium equations

$$
\frac{\partial \sigma_{ij}(u)}{\partial x_j} + F_i = 0 \quad i = 1,2 \quad \text{in } \Omega, \tag{9}
$$

where $(\sigma_{ij}(u))_{i,j=1}^2$ is the stress tensor related to the strain tensor $(\varepsilon_{ij}(u))_{i,j=1}^2$, $\varepsilon_{ij}(u) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ through the non-linear relation (1).

In order to give the weak formulation of this problem, we introduce the following notations

$$V = \{v \in (H^1(\Omega))^2 \mid v_i = 0 \text{ on } \Gamma_U,\ i = 1, 2\}, \tag{10}$$

$$K = \{v \in V \mid v_2(x_1, \alpha(x_1)) \geqslant -\alpha(x_1),\ x_1 \in]a, b[\}, \tag{11}$$

$$a_\Omega(u, v) \equiv (\sigma_{ij}(u), \varepsilon_{ij}(v))_{0,\Omega} \equiv \int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v)\,\mathrm{d}x, \tag{12}$$

$$L_\Omega(v) \equiv (F_i, v_i)_{0,\Omega} + (P_i, v_i)_{0,\Gamma_P} \equiv \int_\Omega F_i v_i\,\mathrm{d}x + \int_{\Gamma_P} P_i v_i\,\mathrm{d}s. \tag{13}$$

A *weak solution* of the problem is defined as an element $u \in K$ satisfying the variational inequality

$$a_\Omega(u, v - u) \geqslant L_\Omega(v - u) \quad \forall v \in K. \tag{$\mathscr{P}$}$$

Using Green's theorem, together with a suitable choice of test functions in ($\mathscr{P}$), we recover the conditions (6)–(8), as well as the system (9).

To prove the existence and the uniqueness of the weak solution of ($\mathscr{P}$), one can use the theory of monotone operators or the tools of convex analysis. Here we use the latter. To this end, we introduce the *total potential energy* Φ_Ω of the problem:

$$\Phi_\Omega(v) \equiv \frac{1}{2}\int_\Omega \left(\kappa\,\varepsilon_{ll}^2(v) + \int_0^{\Gamma^2(v)} \mu(t)\,\mathrm{d}t\right)\mathrm{d}x - L_\Omega(v), \tag{14}$$

where $\Gamma^2(v) \equiv \Psi(v, v)$. Here $\Psi(u, v)$ is the bilinear form defined by

$$\Psi(u, v) = -\frac{2}{3}\varepsilon_{ii}(u)\varepsilon_{jj}(v) + 2\varepsilon_{ij}(u)\varepsilon_{ij}(v).$$

One can easily verify that the total potential energy Φ_Ω is Gateaux differentiable in V and the Gateaux derivative is given by

$$\begin{aligned} D\Phi_\Omega(u, v) &= \int_\Omega \big(\kappa\varepsilon_{ii}(u)\varepsilon_{jj}(v) + \mu(\Gamma^2(u))\Psi(u, v)\big)\,\mathrm{d}x - L_\Omega(v) \\ &= \int_\Omega \left[\Big(\kappa - \frac{2}{3}\mu(\Gamma^2(u))\Big)\varepsilon_{ii}(u)\varepsilon_{jj}(v) + 2\mu(\Gamma^2(u))\varepsilon_{ij}(u)\varepsilon_{ij}(v)\right]\mathrm{d}x - L_\Omega(v). \end{aligned}$$

By a *variational solution* of the problem we mean any function $u \in K$ satisfying

$$\Phi_\Omega(u) \leqslant \Phi_\Omega(v) \quad \forall v \in K. \tag{$\mathscr{P}'$}$$

Below, the basic properties of Φ_Ω are listed. From them, the existence and the uniqueness of the solution of $(\mathscr{P}')$ follows, as well as the equivalence between $(\mathscr{P})$ and $(\mathscr{P}')$ (for details, see [9]).

Lemma 1. *For every $u, v \in (H^1(\Omega))^2$ we have*

$$D\Phi_\Omega(u+v,v) - D\Phi_\Omega(u,v) \geqslant c_1(\varepsilon_{ij}(v), \varepsilon_{ij}(v))_{0,\Omega} \tag{15}$$

$$D^2\Phi_\Omega(u,v,v) \geqslant c_1(\varepsilon_{ij}(v), \varepsilon_{ij}(v))_{0,\Omega} \tag{16}$$

where the constant c_1 depends on μ_0, θ_0, only.

From the assumptions on the functions κ, μ and Lemma 1, the *lower semicontinuity* and the *strict convexity* of Φ_Ω follows. The functional Φ_Ω is coercive on V, as follows from the following Lemma.

Lemma 2. *We have*

$$\Phi_\Omega(v) \geqslant \frac{1}{2}c_1(\varepsilon_{ij}(v), \varepsilon_{ij}(v))_{0,\Omega} - c_2\|v\|_{0,\Omega} - c_3\|v\|_{0,\Gamma_P} \tag{17}$$

where c_1 is the same as in Lemma 1 and c_2, c_3 can be estimated from above by $\|F\|_{0,\Omega}$ and $\|P\|_{0,\Gamma_P}$, respectively.

P r o o f. We may write

$$\Phi_\Omega(v) = \int_0^1 D\Phi_\Omega(tv, v)\,\mathrm{d}t = \int_0^1 D\Phi_\Omega(tv, tv)\frac{1}{t}\,\mathrm{d}t.$$

The integrand can be estimated from below by inserting $u = 0$ (zero function) and $v := tv$ into (15). Then

$$\begin{aligned} D\Phi_\Omega(tv, tv) &\geqslant c_1 t^2(\varepsilon_{ij}(v), \varepsilon_{ij}(v))_{0,\Omega} + tD\Phi_\Omega(0, v) \\ &= c_1 t^2(\varepsilon_{ij}(v), \varepsilon_{ij}(v))_{0,\Omega} - tL_\Omega(v). \end{aligned}$$

Using the Schwarz inequality for estimating $L_\Omega(v)$, we arrive at (17). $\square$

Lemma 3. *The functional Φ_Ω is continuous and bounded on V:*

$$v_n \to v \quad \text{in} \quad V \Rightarrow \Phi_\Omega(v_n) \to \Phi_\Omega(v), \quad n \to \infty$$
$$|\Phi_\Omega(v)| \leqslant c_4\|v\|_1^2 + c_5\|v\|_1 \quad \forall v \in V,$$

where c_4 depends on κ_1, $\|F\|_{0,\Omega}$ and $\|P\|_{0,\Gamma_P}$, only.

R e m a r k 1. The fact that the constants c_1, c_2, c_3 and c_4 depend on Ω only as indicated is very important for our subsequent considerations.

Up to now we assumed that the shape of Ω was given. In optimal shape design problems, the boundary $\partial\Omega$ (or at least some part of it) plays the role of the control variable, by means of which we can change properties of the structure.

Let $\mathscr{O}$ denote a family of *admissible domains*, in which all possible candidates are included. For the sake of simplicity of the mathematical analysis, we shall assume the family $\mathscr{O}$ which contains domains with special shape, namely

$$\mathscr{O} = \{\Omega(\alpha) \mid \alpha \in U_{\text{ad}}\}, \tag{18}$$

where

$$U_{\text{ad}} = \{\alpha \in C^{0,1}([a,b]) \mid 0 \leqslant \alpha(x_1) \leqslant \gamma_0,\ |\alpha'(x_1)| \leqslant \gamma_1 \quad \text{in }]a,b[\,,\ \operatorname{meas}\Omega(\alpha) = \gamma_2\}$$

and

$$\Omega(\alpha) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in\ ]a,b[\,,\ \alpha(x_1) < x_2 < \bar{\gamma}\}$$

(see Figure 2).

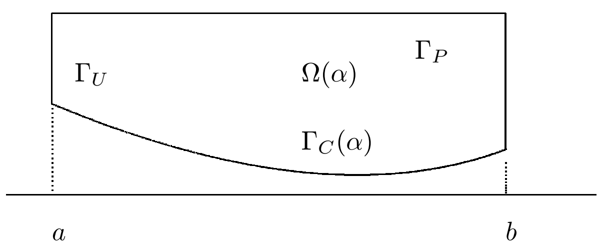

Fig. 2. Problem geometry

Here γ_0, γ_1, γ_2 and $\bar{\gamma}$ are given positive constants, chosen in such a way that $U_{\text{ad}} \neq \emptyset$. The contact part Γ_C (the goal of the optimization) is given by

$$\Gamma_C(\alpha) = \{(x_1,x_2) \mid x_2 = \alpha(x_1),\ x_1 \in\ ]a,b[\}.$$

In order to emphasize the dependence of the state problem on the design variable α, we shall write the symbol α as the argument wherever it will be necessary. So we shall use the following notation: $V(\alpha)$, $K(\alpha), a_{\Omega(\alpha)}$, $L_{\Omega(\alpha)}$. The definition is the same as before for a particular choice of Ω, we only indicate that $\Omega(\alpha) \in \mathscr{O}$ is variable now.

On each $\Omega(\alpha) \in \mathscr{O}$ we shall formulate the state problem $(\mathscr{P}(\alpha))$ (or $(\mathscr{P}(\alpha)'))$. In order to guarantee the existence and the uniqueness of $u(\alpha)$ solving $(\mathscr{P}(\alpha))$ for all

$\alpha \in U_{\mathrm{ad}}$, we shall suppose that the functions κ, μ satisfy the conditions (3)–(5) for all $x \in \hat{\Omega}$, $F \in (L^2(\hat{\Omega}))^2$ and $P \in (L^2(\partial\hat{\Omega}))^2$, where $\hat{\Omega} \equiv]a,b[\times]0,\bar{\gamma}[$. Observe that $\Omega(\alpha) \subset \hat{\Omega}$ $\forall\alpha \in U_{\mathrm{ad}}$. Moreover, let there exist $\delta > 0$ such that $\operatorname{meas}_1 \Gamma_U(\alpha) \geqslant \delta$ for any $\alpha \in U_{\mathrm{ad}}$. Here the symbol $\operatorname{meas}_1 \omega$ stands for the one-dimensional Lebesgue measure of ω.

Finally, let $I\colon V(\alpha)\times U_{\mathrm{ad}} \to \mathbb{R}^1$ be *a cost functional* and denote $J(\alpha) \equiv I(u(\alpha),\alpha)$, with $u(\alpha) \in K(\alpha)$ being the solution of $(\mathscr{P}(\alpha))$ on $\Omega(\alpha)$.

The optimal shape design problem now reads as follows:

$$\text{(P)} \qquad \begin{cases} \text{Find } \alpha^* \in U_{\mathrm{ad}} \quad \text{such that} \\ J(\alpha^*) \leqslant J(\alpha) \quad \forall\alpha \in U_{\mathrm{ad}}. \end{cases}$$

In the next part, the existence of at least one solution of (P) will be analyzed.

3. Existence result for (P)

First of all we present some auxiliary results, which will be needed in what follows.

Lemma 4. *The family $\mathscr{O}$, defined by* (18), *possesses the so called uniform extension property, i.e. there exists a linear extension mapping*

$$p_{\Omega(\alpha)} \in \mathscr{L}(V(\alpha), (H^1(\hat{\Omega}))^2),$$

the norm of which can be estimated independently of $\Omega(\alpha) \in \mathscr{O}$.

For the proof, see [2].

Lemma 5. *Let $\alpha_n \rightrightarrows \alpha$ (uniformly) in $[a,b]$, where $\alpha_n, \alpha \in U_{\mathrm{ad}}$ and let $\varphi \in K(\alpha)$ be given. Then there exist a sequence $\{\varphi_j\}$, $\varphi_j \in (H^1(\hat{\Omega}))^2$ and a subsequence $\{\alpha_{n_j}\} \subset \{\alpha_n\}$ such that $\varphi_j|_{\Omega(\alpha_{n_j})} \in K(\alpha_{n_j})$ and*

$$\varphi_j \to \tilde{\varphi} \equiv p_{\Omega(\alpha)}\varphi \quad \text{in } (H^1(\hat{\Omega}))^2.$$

P r o o f. See Lemma 7.1, p. 125 in [5]. □

Finally, we shall need the following Lemma:

Lemma 6. *Let a sequence $\{y_n\}$, where $y_n \in H^1(\hat{\Omega})$ be such that*

$$y_n \rightharpoonup y \quad \text{(weakly) in } H^1(\hat{\Omega}).$$

Let $\alpha_n \rightrightarrows \alpha$ *in* $[a,b]$, *where* $\alpha_n, \alpha \in U_{\rm ad}$. *Then*

$$(\Pi(y_n), \xi)_{\alpha_n} \to (\Pi(y), \xi)_\alpha \quad \forall \xi \in C^\infty(\bar{\hat{\Omega}}),$$

where

$$(\Pi(y), \xi)_\alpha \equiv \int_a^b [y(x_1, \alpha(x_1)) + \alpha(x_1)]^- \xi(x_1, \alpha(x_1))\,\mathrm{d}x_1$$

and the symbol $[\ ]^-$ *stands for the negative part of a real number.*

P r o o f. See Lemma 1.2, p. 23, in [5]. □

Now we are ready to prove the basic result, showing that the solution of $(\mathscr{P}(\alpha))$ depends continuously on changes of $\Omega(\alpha) \in \mathscr{O}$.

Lemma 7. *Let* $\alpha_n \rightrightarrows \alpha$ *in* $[a,b]$, *where* $\alpha_n, \alpha \in U_{\rm ad}$. *Let* $u_n \equiv u(\alpha_n)$ *be the solution of* $(\mathscr{P}(\alpha_n))$. *Then there exists a subsequence of* $\{u_n\}$ *(denoted by the same sequence) and an element* $\tilde{u} \in (H^1(\hat{\Omega}))^2$ *such that*

$$p_{\Omega_n} u_n \rightharpoonup \tilde{u} \quad \text{in } (H^1(\hat{\Omega}))^2$$

and $u \equiv \tilde{u}\big|_{\Omega(\alpha)}$ *solves* $(\mathscr{P}(\alpha))$.

P r o o f. The proof will be done in several steps:

(i) The sequence $\{u_n\}$ is bounded in the sense that

$$\|u_n\|_{1,\Omega_n} \leqslant c, \tag{19}$$

where $\Omega_n \equiv \Omega(\alpha_n)$ and $c > 0$ does not depend on $\alpha \in U_{\rm ad}$ as we see. Indeed, (17) and Remark 1 imply that

$$\Phi_{\Omega_n}(u_n) \geqslant \frac{1}{2} c_1 (\varepsilon_{ij}(u_n), \varepsilon_{ij}(u_n))_{0,\Omega_n} - c_2 \|u_n\|_{0,\Omega_n} - c_3 \|u_n\|_{0,\Gamma_P^n}, \tag{20}$$

where c_1, c_2 and c_3 do not depend on $\alpha \in U_{\rm ad}$ (the constants c_2, c_3 can be estimated from above by $\|F\|_{0,\hat{\Omega}}$ and $\|P\|_{0,\partial\hat{\Omega}}$) and the symbol Γ_P^n denotes a part of $\partial\Omega_n$ where surface tractions are prescribed. In order to estimate the first term on the right hand side of (20), we use Korn's inequality

$$(\varepsilon_{ij}(v), \varepsilon_{ij}(v))_{0,\Omega(\alpha)} \geqslant c\|v\|^2_{1,\Omega(\alpha)} \quad \forall v \in V(\alpha) \tag{21}$$

and in particular an important fact that the constant c on the right hand side of (21) can be chosen independently on $\Omega(\alpha) \in \mathscr{O}$ (for the proof see [7], [6]). Also the last term on the right hand side of (20) can be estimated independently of $\alpha \in U_{\rm ad}$ using the trace theorem:

$$\|v\|_{0,\Gamma_P^n} \leqslant c\|v\|_{1,\Omega_n} \quad \forall v \in V(\alpha_n). \tag{22}$$

This follows easily from the definition of the family $\mathscr{O}$ and the proof of the trace theorem (see [9], p. 73).

From (20), (21) and (22) we see that there exists a constant $c > 0$, which does not depend on $\alpha \in U_{\rm ad}$, such that[2]

$$\Phi_{\Omega_n}(u_n) \geqslant c\|u_n\|^2_{1,\Omega_n} - c\|u_n\|_{1,\Omega_n}.$$

On the other hand, $\Phi_{\Omega_n}(u_n)$ is bounded from above, since

$$\Phi_{\Omega_n}(u_n) \leqslant \Phi_{\Omega_n}(0) \quad \text{and } 0 \in K(\alpha_n)\ \forall n.$$

This proves (19).

(ii) The construction of a function $\tilde{u}$:

Let $\tilde{u}_n \equiv p_{\Omega_n} u_n$, i.e. $\tilde{u}_n$ is the extension of u_n from Ω_n to $\hat{\Omega}$, introduced in Lemma 4. Then on the basis of the same lemma

$$\|\tilde{u}_n\|_{1,\hat{\Omega}} \leqslant c, \tag{23}$$

where c does not depend on $\alpha \in U_{\rm ad}$, again. Therefore, there exists a subsequence of $\{\tilde{u}_n\}$ (denoted by the same sequence) and an element $\tilde{u} \in (H^1(\hat{\Omega}))^2$ such that

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in } (H^1(\hat{\Omega}))^2. \tag{24}$$

(iii) Define $u \equiv \tilde{u}\big|_{\Omega(\alpha)}$. We prove that u solves $(\mathscr{P}'(\alpha))$. The fact that $u \in K(\alpha)$, especially that the unilateral conditions are satisfied for the second component u_2, follows immediately from Lemma 6. It remains to show that u is a minimizer of $\Phi_{\Omega(\alpha)}$ over $K(\alpha)$:

$$\Phi_{\Omega(\alpha)}(u) \leqslant \Phi_{\Omega(\alpha)}(v) \quad \forall v \in K(\alpha).$$

We can split Ω_n as follows:

$$\Omega_n = G_m \cup (\Omega_n \setminus \Omega(\alpha)) \cup ((\Omega(\alpha) \setminus G_m) \cap \Omega_n)), \tag{25}$$

[2] In the sequel, the symbol c denotes a generic constant, taking different values at different places.

where

$$G_m \equiv G_m(\alpha) = \left\{ (x_1, x_2) \in \Omega(\alpha) \mid x_1 \in]a, b[\,,\ \alpha(x_1) + \frac{1}{m} < x_2 < \bar{\gamma} \right\},$$

and m is a positive integer which is sufficiently large. Then the total potential energy Φ_{Ω_n} can be written as a sum of the contributions, corresponding to the decomposition (25):

$$\Phi_{\Omega_n}(u_n) = \Phi_{G_m}(u_n) + \Phi_{\Omega_n \setminus \Omega(\alpha)}(u_n) + \Phi_{(\Omega(\alpha) \setminus G_m) \cap \Omega_n}(u_n). \tag{26}$$

We shall analyze each term on the right hand side of (26) separately. Let m be fixed and n sufficiently large. Then

$$\liminf_{n \to \infty} \Phi_{G_m}(u_n) \geqslant \Phi_{G_m}(u) \tag{27}$$

because of (24) and the fact that Φ_{G_m} is weakly lower semicontinuous in $V(G_m)$. Furthermore

$$\begin{aligned}\Phi_{\Omega_n \setminus \Omega(\alpha)}(u_n) = {} & \frac{1}{2} \int_{\Omega_n \setminus \Omega(\alpha)} \left(\kappa \varepsilon_{ll}^2(u_n) + \int_0^{\Gamma^2(u_n)} \mu(t)\,\mathrm{d}t \right) \mathrm{d}x - L_{\Omega_n \setminus \Omega(\alpha)}(u_n) \\ \geqslant {} & - L_{\Omega_n \setminus \Omega(\alpha)}(u_n)\end{aligned}$$

because κ and μ are non-negative. A direct computation shows that

$$\lim_{n \to \infty} L_{\Omega_n \setminus \Omega(\alpha)}(u_n) = 0$$

and consequently

$$\liminf_{n \to \infty} \Phi_{\Omega_n \setminus \Omega(\alpha)}(u_n) \geqslant 0. \tag{28}$$

Using the same argument, the third term on the right hand side of (26) can be estimated from below:

$$\Phi_{(\Omega(\alpha) \setminus G_m) \cap \Omega_n}(u_n) \geqslant -L_{(\Omega(\alpha) \setminus G_m) \cap \Omega_n}(u_n)$$

and also

$$|L_{(\Omega(\alpha) \setminus G_m) \cap \Omega_n}(u_n)| \leqslant c(m), \tag{29}$$

where $c(m)$ is such that $\lim_{m \to \infty} c(m) = 0$. Taking into account (27), (28) and (29) we see that

$$\liminf_{n \to \infty} \Phi_{\Omega_n}(u_n) \geqslant \Phi_{G_m}(u) - c(m)$$

holds for any m. Letting $m \to \infty$ we arrive at

$$\liminf_{n\to\infty} \Phi_{\Omega_n}(u_n) \geqslant \Phi_{\Omega(\alpha)}(u). \tag{30}$$

Let $v \in K(\alpha)$ be given. Then Lemma 5 implies the existence of $v_j \in (H^1(\hat{\Omega}))^2$ such that

$$v_j\big|_{\Omega_{n_j}} \in K(\alpha_{n_j}) \tag{31}$$

$$v_j \to p_{\Omega(\alpha)}v,\ j \to \infty \quad \text{in } (H^1(\hat{\Omega}))^2. \tag{32}$$

Taking into account the definition of $(\mathscr{P}(\alpha_{n_j}))$ and the fact that $v_j\big|_{\Omega_{n_j}} \in K(\alpha_{n_j})$, we have

$$\Phi_{\Omega_{n_j}}(u_{n_j}) \leqslant \Phi_{\Omega_{n_j}}(v_j). \tag{33}$$

Next, we shall show that

$$\Phi_{\Omega_{n_j}}(v_j) \to \Phi_{\Omega(\alpha)}(v),\ j \to \infty. \tag{34}$$

Indeed: as before

$$\Phi_{\Omega_{n_j}}(v_j) = \Phi_{G_m}(v_j) + \Phi_{\Omega_{n_j}\setminus\Omega(\alpha)}(v_j) + \Phi_{(\Omega(\alpha)\setminus G_m)\cap\Omega_{n_j}}(v_j). \tag{35}$$

Since Φ_{G_m} is continuous on $V(G_m)$, we have

$$\lim_{j\to\infty} \Phi_{G_m}(v_j) = \Phi_{G_m}(v). \tag{36}$$

For the second term on the right hand side of (35) we have

$$\begin{aligned} |\Phi_{\Omega_{n_j}\setminus\Omega(\alpha)}(v_j)| &\leqslant c\sum_{l=1}^{2} \|v_j\|^l_{1,\Omega_{n_j}\setminus\Omega(\alpha)} \\ &\leqslant c\Big(\sum_{l=1}^{2} \|v_j - p_{\Omega(\alpha)}v\|^l_{1,\hat{\Omega}} + \sum_{l=1}^{2} \|p_{\Omega(\alpha)}v\|^l_{1,\Omega_{n_j}\setminus\Omega(\alpha)}\Big) \to 0 \end{aligned} \tag{37}$$

when j, $n_j \to \infty$, making use of Lemma 3 and (32).

Finally,

$$\begin{aligned} |\Phi_{(\Omega(\alpha)\setminus G_m)\cap\Omega_{n_j}}(v_j)| &\leqslant c\sum_{l=1}^{2} \|v_j\|^l_{1,\Omega(\alpha)\setminus G_m} \\ &\leqslant c\Big(\sum_{l=1}^{2} \|v_j - v\|^l_{1,\Omega(\alpha)} + \sum_{l=1}^{2} \|v\|^l_{1,\Omega(\alpha)\setminus G_m}\Big) \end{aligned}$$

and therefore

$$\limsup_{n_j\to\infty} |\Phi_{(\Omega(\alpha)\setminus G_m)\cap\Omega_{n_j}}(v_j)| \leqslant c(m) \tag{38}$$

where $c(m)\to 0$ when $m\to\infty$. From (35)–(38) we finally obtain (34). The assertion of lemma now easily follows from (30), (33) and (34). □

R e m a r k 2. Now we show that a subsequence $\{u_{n_j}\}$ of $\{u_n\}$, where $\{u_n\}$ satisfies (24), tends strongly to u on any compact subset $Q\subset\Omega(\alpha)$:

$$\|u_{n_j}-u\|_{1,Q}\to 0,\quad n_j\to\infty.$$

Indeed, let $Q\subset\subset\Omega(\alpha)$ be given and choose $G_m(\alpha)$ in such a way that $G_m(\alpha)\supseteq Q$. Using the Taylor expansion of Φ_{G_m} at the point u, we can write for some $\theta\in\,]0,1[$:

$$\begin{aligned}\Phi_{G_m}(u_n) &= \Phi_{G_m}(u)+D\Phi_{G_m}(u,u_n-u)\\ &\quad+\frac{1}{2}D^2\Phi_{G_m}(u+\theta(u_n-u),u_n-u,u_n-u)\\ &\geqslant \Phi_{G_m}(u)+D\Phi_{G_m}(u,u_n-u)+c\|u_n-u\|^2_{1,G_m},\end{aligned}\tag{39}$$

making use of Lemma 1 and the fact that Korn's inequality is uniform with respect to m and $\alpha\in U_{\rm ad}$. Let m be fixed and n sufficiently large, such that $\Omega_n\supset\overline{G}_m$. Then

$$\Phi_{\Omega_n}(u_n)=\Phi_{G_m}(u_n)+\Phi_{\Omega_n\setminus G_m}(u_n),$$

from which

$$\Phi_{G_m}(u_n)=\Phi_{\Omega_n}(u_n)-\Phi_{\Omega_n\setminus G_m}(u_n)\leqslant\Phi_{\Omega_n}(u_n)+L_{\Omega_n\setminus G_m}(u_n) \tag{40}$$

follows. Replacing the left hand side of (39) by (40) we obtain

$$\begin{aligned}&\Phi_{G_m}(u)+D\Phi_{G_m}(u,u_n-u)+c\|u_n-u\|^2_{1,G_m}\\ &\leqslant\Phi_{\Omega_n}(u_n)+L_{\Omega_n\setminus G_m}(u_n)\leqslant\Phi_{\Omega_n}(v)+L_{\Omega_n\setminus G_m}(u_n)\quad\forall v\in K(\alpha_n).\end{aligned}\tag{41}$$

Let $\{\bar v_j\}$, $\bar v_j\in(H^1(\hat\Omega))^2$ be a sequence tending strongly to $p_{\Omega(\alpha)}u$ in $(H^1(\hat\Omega))^2$ and such that $\bar v_j\big|_{\Omega_{n_j}}\in K(\alpha_{n_j})$, where $\{\alpha_{n_j}\}\subset\{\alpha_n\}$ (see Lemma 5). Replacing v by $\bar v_j$ in the last inequality in (41) considered on Ω_{n_j} we obtain:

$$\begin{aligned}\|u_{n_j}-u\|^2_{1,Q}&\leqslant\|u_{n_j}-u\|^2_{1,G_m}\\ &\leqslant\Phi_{\Omega_{n_j}}(\bar v_j)+L_{\Omega_{n_j}\setminus G_m}(u_n)-\Phi_{G_m}(u)-D\Phi_{G_m}(u,u_{n_j}-u).\end{aligned}$$

Letting $n_j \to \infty$ we get

$$\limsup_{n_j\to\infty} \|u_{n_j} - u\|_{1,Q}^2 \leqslant \Phi_{\Omega(\alpha)}(u) - \Phi_{G_m}(u) + c(m),$$

where $c(m) \to 0$ as $m \to \infty$. Here (24) and (34) have been used. Finally, letting $m \to \infty$ we arrive at the assertion.

In order to ensure the existence of at least one solution of (P), *the lower semicontinuity* of I has to be assumed. Let I satisfy at least one of the following two assumptions:

(A1) If $\alpha_n \rightrightarrows \alpha$ in $[a,b]$, where $\alpha_n, \alpha \in U_{\mathrm{ad}}$ and $y_n \rightharpoonup y$ in $(H^1(\hat\Omega))^2$, where $y_n, y \in (H^1(\hat\Omega))^2$ then

$$\liminf_{n\to\infty} I\big(y_n\big|_{\Omega_n}, \alpha_n\big) \geqslant I\big(y\big|_{\Omega(\alpha)}, \alpha\big),$$

or

(A2) If $\alpha_n \rightrightarrows \alpha$ in $[a,b]$, where $\alpha_n, \alpha \in U_{\mathrm{ad}}$ and $y_n \to y$ in $(H^1_{\mathrm{loc}}(\Omega(\alpha)))^2$, where $y_n \in V(\alpha_n)$, $y \in V(\alpha)$, then

$$\liminf_{n\to\infty} I(y_n, \alpha_n) \geqslant I(y, \alpha).$$

The main result of this part is given in the following theorem:

Theorem 1. *Let I satisfy* (A1) *or* (A2). *Then the problem* (P) *has at least one solution.*

P r o o f. Denote

$$q \equiv \inf_{\alpha\in U_{\mathrm{ad}}} I(u(\alpha), \alpha) = \lim_{n\to\infty} I(u_n, \alpha_n), \tag{42}$$

i.e. the sequence $\{\alpha_n\}$, $\alpha_n \in U_{\mathrm{ad}}$, is a minimizing sequence and u_n is the corresponding state. As U_{ad} is a compact subset of $C([a,b])$, we may assume that $\alpha_n \rightrightarrows \alpha^* \in U_{\mathrm{ad}}$ in $[a,b]$ and at the same time $p_{\Omega_n} u_n \rightharpoonup \tilde u$ in $(H^1(\hat\Omega))^2$, where $\tilde u$ is such that $u^* \equiv \tilde u\big|_{\Omega(\alpha^*)}$ solves $(\mathscr{P}(\alpha^*))$, as follows from Lemma 7. From this, (42) and (A1) we conclude that (u^*, α^*) is an optimal pair for (P). If (A2) is satisfied, then the result of Remark 2 will be used. □

As an example, which will be used in subsequent parts, let us consider

$$J(\alpha) \equiv I(u(\alpha), \alpha) = \Phi_{\Omega(\alpha)}(u(\alpha)), \tag{43}$$

i.e. J is equal to the total potential energy evaluated in the equilibrium state $u(\alpha)$. Such a choice of I satisfies (A1), as follows from (30), and consequently (P) has at least one solution.

R e m a r k 3. As mentioned in the introduction, optimal control problems the state of which is described by a variational inequality, are in general non-smooth. This is due to the fact that the mapping η: *control variable* $\rightarrow$ *state of the system* is only locally Lipschitz continuous. In order to overcome this difficulty, a regularization of the state problem can be used. In our case, we use a penalty approach. Define a functional $j_\alpha\colon V(\alpha) \rightarrow \mathbb{R}^1$ as follows:

$$j_\alpha(v) \equiv \frac{1}{3} \int_a^b \left([v_2(x_1, \alpha(x_1)) + \alpha(x_1)]^-\right)^3 \, \mathrm{d}x_1.$$

Instead of $(\mathscr{P}(\alpha))$ we define a new state problem as follows:

$$(\mathscr{P}(\alpha)_\varepsilon) \qquad \begin{cases} \text{Find } u_\varepsilon(\alpha) \in V(\alpha) \text{ such that} \\ \Phi_{\Omega(\alpha)}(u_\varepsilon(\alpha)) + \frac{1}{\varepsilon} j_\alpha(u_\varepsilon(\alpha)) \leqslant \Phi_{\Omega(\alpha)}(v) + \frac{1}{\varepsilon} j_\alpha(v) \quad \forall v \in V(\alpha), \end{cases}$$

where $\varepsilon > 0$ is the penalty parameter.

Now, we define a new shape optimization problem, in which the state problem $(\mathscr{P}(\alpha))$ is replaced by $(\mathscr{P}(\alpha)_\varepsilon)$:

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} \text{Find } \alpha_\varepsilon^* \in U_{\mathrm{ad}} \text{ such that} \\ I(u_\varepsilon(\alpha_\varepsilon^*), \alpha_\varepsilon^*) \leqslant I(u_\varepsilon(\alpha), \alpha) \quad \forall \alpha \in U_{\mathrm{ad}} \end{cases}$$

with $u_\varepsilon(\alpha)$ being the unique solution of $(\mathscr{P}(\alpha)_\varepsilon)$. Under the same assumptions formulated before, it is possible to prove the existence of at least one solution α_ε^*. Moreover, when $\varepsilon \rightarrow 0+$, then (P_ε) and (P) are close on subsequences. More precisely, one can prove the following theorem:

Theorem 2. *Let* $\alpha_\varepsilon^* \in U_{\mathrm{ad}}$ *be a solution of* (P_ε) *and let* $u_\varepsilon(\alpha_\varepsilon^*)$ *be the solution of* $(\mathscr{P}(\alpha_\varepsilon^*))$. *Then there exists a subsequence of* $\{\alpha_\varepsilon^*\}$ *and* $\{u_\varepsilon(\alpha_\varepsilon^*)\}$ *(still denoted by the same sequence) and elements* $\alpha^* \in U_{\mathrm{ad}}$, $\tilde{u} \in (H^1(\hat{\Omega}))^2$ *such that*

$$\begin{aligned} &\alpha_\varepsilon^* \rightrightarrows \alpha^* \quad \text{in } [a,b] \\ &p_{\Omega(\alpha_\varepsilon^*)} u_\varepsilon(\alpha_\varepsilon^*) \rightharpoonup \tilde{u} \quad \text{in } (H^1(\hat{\Omega}))^2, \ \varepsilon \rightarrow 0+. \end{aligned}$$

Moreover, α^* *solves* (P) *and* $u^* \equiv \tilde{u}\big|_{\Omega(\alpha^*)}$ *solves* $(\mathscr{P}(\alpha^*))$.

Proof for the case of elastic bodies is done in [6].

The main advantage of this approach is the fact that the variational inequality $(\mathscr{P}(\alpha))$ is now replaced by a system of variational equations $(\mathscr{P}(\alpha)_\varepsilon)$, for which the mapping η, introduced before, is continuously differentiable. In Section 5 we shall show that the cost functional J given by (43) is continuously differentiable despite the fact that the inner mapping η (see Remark 3) is not. In this case the regularization of $(\mathscr{P}(\alpha))$ is not necessary.

4. Approximation of (P)

In this section we describe the discretization of (P), which is based mainly on finite element approximation of the state problem.

Let $a \equiv a_0 < a_1 < \ldots < a_{D(h)} \equiv b$ be a partition of $]a,b[$. The admissible set $U_{\rm ad}$ will be approximated by piecewise linear functions as follows:

$$U_{\rm ad}^h = \Big\{\alpha_h \in C([a,b]) \mid \alpha_h\big|_{[a_{i-1},a_i]} \in P_1,\ i=1,\ldots,D\Big\} \cap U_{\rm ad},$$

where P_1 denotes the set of polynomials in one variable of the degree at most one. Let us observe that $U_{\rm ad}^h$ can be easily constructed, because of piecewise linearity of its elements. As $\Omega(\alpha_h)$, $\alpha_h \in U_{\rm ad}^h$, is a polygonal domain, one can construct its triangulation into elements, i.e. the finite element mesh $\mathscr{T}(h,\alpha_h)$ (now depending also on α_h). As well as the usual requirements on the mutual position of elements, belonging to $\mathscr{T}(h,\alpha_h)$, we shall suppose that for $h>0$ fixed, triangulations $\mathscr{T}(h,\alpha_h)$ are *topologically equivalent* for all $\alpha_h \in U_{\rm ad}^h$, i.e.:

(T1) $\mathscr{T}(h,\alpha_h)$ has the same number of nodes and the nodes have the same neighbours for all $\alpha_h \in U_{\rm ad}^h$.

(T2) The position of nodes in $\mathscr{T}(h,\alpha_h)$ depends continuously on α_h.

The family $\{\mathscr{T}(h,\alpha_h)\}$, $h \to 0+$, $\alpha_h \in U_{\rm ad}^h$ is uniformly regular, i.e.

(T3) There exists $\vartheta_0 > 0$ such that all interior angles are bounded from below by ϑ_0 for all $h>0$ and $\alpha_h \in U_{\rm ad}^h$.

Moreover, we shall construct $\mathscr{T}(h,\alpha_h)$ in such a way that any straight line segment of α_h is a side of one element only. The domain $\Omega(\alpha_h)$ with a given mesh $\mathscr{T}(h,\alpha_h)$ will be denoted as Ω_h, in what follows.

We start with the finite element approximation of the state problem. Let $\alpha_h \in U_{\rm ad}^h$ be given and let $V_h(\alpha_h)$ be the space of piecewise linear functions over $\mathscr{T}(h,\alpha_h)$:

$$\begin{aligned} V_h(\alpha_h) = \big\{ & v_h \in (C(\overline{\Omega}(\alpha_h)))^2 \mid v_h\big|_{T_i} \in (P_1)^2 \quad \forall T_i \in \mathscr{T}(h,\alpha_h), \\ & v_h = 0 \quad \text{on } \overline{\Gamma}_U(\alpha_h)\big\}. \end{aligned}$$

By $K_h(\alpha_h)$ we denote the closed convex subset of $V_h(\alpha_h)$, defined as follows:

$$\begin{aligned} K_h(\alpha_h) = \{ & v_h = (v_{h1}, v_{h2}) \in V_h(\alpha_h) \mid \\ & v_{h2}(a_i, \alpha_h(a_i)) \geqslant -\alpha_h(a_i),\ i \in \mathscr{C}\}, \end{aligned}$$

where $\mathscr{C}$ is the index set of all contact nodes $N_i \equiv (a_i, \alpha_h(a_i)) \in \overline{\Gamma}_C(\alpha_h) \setminus \overline{\Gamma}_U(\alpha_h)$. It is readily seen that $K_h(\alpha_h)$ is an *inner approximation* of $K(\alpha_h)$. The state problem now will be approximated by the classical Ritz method:

$$(\mathscr{P}(\alpha_h)'_h) \qquad \begin{cases} \text{Find } u_h \equiv u_h(\alpha_h) \quad \text{such that} \\ \Phi_{\Omega_h}(u_h) \leqslant \Phi_{\Omega_h}(v_h) \quad \forall v_h \in K_h(\alpha_h). \end{cases}$$

The approximation of the whole optimal shape design problem now reads as follows:

$$(\mathrm{P}_h) \qquad \begin{cases} \text{Find } \alpha_h^* \in U_{\mathrm{ad}}^h \quad \text{such that} \\ I_h(u_h(\alpha_h^*), \alpha_h^*) \leqslant I_h(u_h(\alpha_h), \alpha_h) \quad \forall \alpha_h \in U_{\mathrm{ad}}^h, \end{cases}$$

where $I_h\colon V_h(\alpha_h) \times U_{\mathrm{ad}}^h \to \mathbb{R}^1$ is an *approximation* of I and $u_h(\alpha_h)$ solves $(\mathscr{P}(\alpha_h)'_h)$.

In order to prove the existence of at least one solution of (P_h), we need the following hypothesis on lower semicontinuity of I_h (for $h > 0$ fixed):

(I1) If $\alpha_h^j \to \alpha_h$, $j \to \infty$ in $[a, b]$, where α_h^j, $\alpha_h \in U_{\mathrm{ad}}^h$ and $y_h^j \rightharpoonup y_h$, $j \to \infty$ in $(H^1(\hat{\Omega}))^2$, where $y_h^j\big|_{\Omega(\alpha_h^j)} \in V_h(\alpha_h^j)$, $y_h\big|_{\Omega(\alpha_h)} \in V_h(\alpha_h)$ then

$$\liminf_{j\to\infty} I_h\big(y_h^j\big|_{\Omega(\alpha_h^j)}, \alpha_h^j\big) \geqslant I_h\big(y_h\big|_{\Omega(\alpha_h)}, \alpha_h\big).$$

Theorem 3. *Let I_h satisfy* (I1). *Then* (P_h) *has at least one solution.*

P r o o f. First of all, for any $\alpha_h \in U_{\mathrm{ad}}^h$ fixed there exists a unique solution $u_h(\alpha_h) \in K_h(\alpha_h)$ of $(\mathscr{P}(\alpha_h)'_h)$. By virtue of (T1), $\dim V_h(\alpha_h)$ is the same for any $\alpha_h \in U_{\mathrm{ad}}^h$. Let $\alpha_h^j \to \alpha_h$ as $j \to \infty$. Arguing in the same way as in Lemma 7 it is possible to show that

$$\|u_h(\alpha_h^j)\|_{1,\Omega(\alpha_h^j)} \leqslant c,$$

where $c > 0$ does not depend on j, h. Denote by $p_{\Omega(\alpha_h^j)} u_h(\alpha_h^j)$ the extension of $u_h(\alpha_h^j)$ from $\Omega(\alpha_h^j)$ to $\hat{\Omega}$. Then also

$$\|p_{\Omega(\alpha_h^j)} u_h(\alpha_h^j)\|_{1,\Omega(\alpha_h^j)} \leqslant c.$$

Thus there exists a subsequence of $\{p_{\Omega(\alpha_h^j)} u_h(\alpha_h^j)\}$ (still denoted by the same sequence) and an element $\tilde{u} \in (H^1(\hat{\Omega}))^2$ such that

$$p_{\Omega(\alpha_h^j)} u_h(\alpha_h^j) \rightharpoonup \tilde{u} \quad \text{in } (H^1(\hat{\Omega}))^2.$$

At the same time one can assume that $\mathscr{T}(h, \alpha_h^j) \to \mathscr{T}(h, \alpha_h)$, $j \to \infty$, i.e. the nodes of $\mathscr{T}(h, \alpha_h^j)$ converge to the corresponding nodes of $\mathscr{T}(h, \alpha_h)$ (see (T2)). It is readily seen that the restriction $\tilde{u}\big|_{\Omega(\alpha_h)} \in V_h(\alpha_h)$, where $V_h(\alpha_h)$ is the space of linear elements constructed on $\mathscr{T}(h, \alpha_h)$. The fact that $u_h \equiv \tilde{u}\big|_{\Omega(\alpha_h)}$ solves $(\mathscr{P}(\alpha_h)'_h)$ can be proved in the same way as in Lemma 7. The rest of proof, namely that (P_h) has at least one solution, proceeds exactly in the same way as in Theorem 1. $\square$

Next, we shall study the mutual relation between (P_h) and (P), when $h \to 0+$. To this end we need an auxiliary result.

Lemma 8. *Let* $\alpha_h \rightrightarrows \alpha$ *in* $[a,b]$, *where* $\alpha_h \in U_{\mathrm{ad}}^h$, $\alpha \in U_{\mathrm{ad}}$. *Let* $\varphi \in K(\alpha)$ *be given. Then there exist a subsequence* $\{\alpha_{h_j}\} \subset \{\alpha_h\}$ *and a sequence* $\{\varphi_{h_j}\}$, $\varphi_{h_j} \in (H^1(\hat{\Omega}))^2$ *such that* $\varphi_{h_j}\big|_{\Omega(\alpha_{h_j})} \in K_{h_j}(\alpha_{h_j})$ *and* $\varphi_{h_j} \to p_{\Omega(\alpha)}\varphi$ *in* $(H^1(\hat{\Omega}))^2$.

Proof. See [5] (Lemma 7.3 and the proof of Lemma 7.4). □

On the basis of Lemma 8, the following important result will be proved.

Lemma 9. *Let* $\alpha_h \rightrightarrows \alpha$ *in* $[a,b]$, *where* $\alpha_h \in U_{\mathrm{ad}}^h$, $\alpha \in U_{\mathrm{ad}}$. *Let* $u_h \equiv u_h(\alpha_h)$ *be the solution of* $(\mathscr{P}(\alpha_h)_h)$. *Then there exists a subsequence of* $\{u_h\}$ *(still denoted by the same sequence) and a function* $\tilde{u} \in (H^1(\hat{\Omega}))^2$ *such that* $u \equiv \tilde{u}\big|_{\Omega(\alpha)} \in K(\alpha)$,

$$p_{\Omega_h} u_h \rightharpoonup \tilde{u} \quad \text{in } (H^1(\hat{\Omega}))^2$$

and $u \equiv \tilde{u}\big|_{\Omega(\alpha)}$ *solves* $(\mathscr{P}(\alpha))$.

Proof. The proof is parallel to that of Lemma 7, making use of Lemma 8. □

Remark 4. In a similar way as in Remark 2 one can show that there is a subsequence of $\{u_h\}$ (denoted by the same symbol) such that

$$u_h(\alpha_h) \to u \quad \text{in } H^1_{\mathrm{loc}}(\Omega(\alpha)).$$

Now, we are able to establish the main result of this section, analyzing the mutual relation between (P_h) and (P), when $h \to 0+$. To this end we shall suppose that at least one of the following two conditions is satisfied:

(I2) If $\alpha_h \rightrightarrows \alpha$ in $[a,b]$, where $\alpha_h \in U_{\mathrm{ad}}^h$, $\alpha \in U_{\mathrm{ad}}$ and $y_h \rightharpoonup y$ in $(H^1(\hat{\Omega}))^2$, where $y_h, y \in (H^1(\hat{\Omega}))^2$, $y_h\big|_{\Omega_h} \in V_h(\alpha_h)$, then

$$\lim_{h\to 0+} I_h\big(y_h\big|_{\Omega_h}, \alpha_h\big) = I\big(y\big|_{\Omega(\alpha)}, \alpha\big).$$

(I3) If $\alpha_h \rightrightarrows \alpha$ in $[a,b]$, where $\alpha_h \in U_{\mathrm{ad}}^h$, $\alpha \in U_{\mathrm{ad}}$ and $y_h \to y$ in $(H^1_{\mathrm{loc}}(\Omega(\alpha)))^2$, where $y_h \in V_h(\alpha_h)$, $y \in V(\alpha)$, then

$$\lim_{h\to 0+} I_h(y_h, \alpha_h) = I(y, \alpha).$$

Theorem 4. *Let* (I2) *or* (I3) *be satisfied. Let* $\alpha_h^* \in U_{\mathrm{ad}}^h$ *be a solution of* (P_h) *and* $u_h(\alpha_h^*)$ *the solution of* $(\mathscr{P}(\alpha_h^*)_h)'$. *Then there exist subsequences of* $\{\alpha_h^*\}$ *and* *of* $\{u_h(\alpha_h^*)\}$ *(denoted by the same symbols) and elements* $\alpha^* \in U_{\rm ad}$*,* $\tilde u \in (H^1(\hat\Omega))^2$ *such that*

$$\alpha_h^* \rightrightarrows \alpha^* \quad \textit{in } [a,b];$$
$$p_{\Omega_h} u_h(\alpha_h^*) \rightharpoonup \tilde u \quad \textit{in } (H^1(\hat\Omega))^2,\ h \to 0+ .$$

Moreoever, α^* *is a solution of* (P) *and* $u^* \equiv \tilde u\big|_{\Omega(\alpha^*)}$ *solves* $(\mathscr{P}(\alpha^*))$*.*

P r o o f. We may already suppose that

$$\alpha_h^* \rightrightarrows \alpha^* \quad \text{in } [a,b]$$

and

$$p_{\Omega_h} u_h(\alpha_h^*) \rightharpoonup \tilde u \quad \text{in} (H^1(\hat\Omega))^2.$$

The fact that $u^* \equiv \tilde u\big|_{\Omega(\alpha^*)}$ solves $(\mathscr{P}(\alpha^*))$ follows from Lemma 9. If (I2) is satisfied then

$$\lim_{h\to 0+} I_h(u_h(\alpha_h^*), \alpha_h^*) = I(u^*, \alpha^*). \tag{44}$$

If (I3) is satisfied, then (44) holds as well, by virtue of Remark 4.

Let $\alpha \in U_{\rm ad}$ be given. Then there exists a sequence $\{\alpha_h\}$, $\alpha_h \in U_{\rm ad}^h$ such that (see [1])

$$\alpha_h \rightrightarrows \alpha \quad \text{in } [a,b]. \tag{45}$$

Let $u_h(\alpha_h)$ be the solution of $(\mathscr{P}(\alpha_h))$ corresponding to the sequence with property (45). Then also

$$\lim_{h\to 0+} I_h(u_h(\alpha_h), \alpha_h) = I(u(\alpha), \alpha),$$

by the same argument as before. Also the fact that

$$I(u^*, \alpha^*) \leqslant I(u(\alpha), \alpha) \quad \forall \alpha \in U_{\rm ad}$$

is readily seen, i.e. α^* is a solution of (P). □

R e m a r k 5. In some cases, the conditions (I2) and (I3) are too strong. From the proof of the previous theorem we see that only solutions $u_h(\alpha_h)$ (and not general elements from $V_h(\alpha_h)$) enter our considerations. Therefore, instead of general $y_h \in V_h(\alpha_h)$ we can formulate new conditions (I2') and (I3') with $u_h(\alpha_h)$ replacing y_h, i.e. (I3') reads as follows:

(I3') Let $\alpha_h \rightrightarrows \alpha$ in $[a,b]$, where $\alpha_h \in U_{\rm ad}^h$, $\alpha \in U_{\rm ad}$ and let $u_h(\alpha_h) \to u(\alpha)$ in $H^1_{\rm loc}(\Omega(\alpha))$, where $u_h(\alpha_h)$, $u(\alpha)$ solves $(\mathscr{P}(\alpha_h)'_h)$, $(\mathscr{P}(\alpha)')$, respectively. Then

$$\lim_{h\to 0+} I_h(u_h(\alpha_h), \alpha_h) = I(u(\alpha), \alpha).$$

This will be useful in our next example.

Let us consider $J_h(\alpha_h) \equiv I_h(u_h(\alpha_h), \alpha_h) = \Phi_{\Omega_h}(u_h(\alpha_h))$. Then (I3') is satisfied. Indeed, the inequality

$$\liminf_{h\to 0+} \Phi_{\Omega_h}(u_h(\alpha_h)) \geqslant \Phi_{\Omega(\alpha)}(u(\alpha)) \tag{46}$$

has been already proved. From Lemma 8 it follows that there exists a subsequence $\{\alpha_{h_j}\} \subset \{\alpha_h\}$ and a sequence $\{\varphi_{h_j}\}$, $\varphi_{h_j} \in (H^1(\hat{\Omega}))^2$, such that $\varphi_{h_j}\big|_{\Omega(\alpha_{h_j})} \subset K_{h_j}(\alpha_{h_j})$ and

$$\varphi_{h_j} \to p_{\Omega(\alpha)} u(\alpha) \quad \text{in } (H^1(\hat{\Omega}))^2.$$

Using the definition of $(\mathscr{P}(\alpha_{h_j})'_{h_j})$, we have

$$\Phi_{\Omega_{h_j}}(u_{h_j}(\alpha_{h_j})) \leqslant \Phi_{\Omega_{h_j}}(\varphi_{h_j})$$

and consequently

$$\limsup_{h_j\to 0+} \Phi_{\Omega_{h_j}}(u_{h_j}(\alpha_{h_j})) \leqslant \limsup_{h_j\to 0+} \Phi_{\Omega_{h_j}}(\varphi_{h_j}) = \Phi_{\Omega(\alpha)}(u(\alpha)) \tag{47}$$

as follows from the proof of Lemma 7 (see (34)). From (46) and (47) we see that

$$\lim_{h_j\to 0+} \Phi_{h_j}(u_{h_j}(\alpha_{h_j})) = \Phi_{\Omega(\alpha)}(u(\alpha))$$

and consequently, Theorem 4 holds true for the choice $J_h(\alpha_h) \equiv \Phi_{\Omega_h}(u_h(\alpha_h))$.

5. Numerical realization

Taking into account the parametrization of $\Omega(\alpha_h)$, we find that the shape of $\Gamma_C(\alpha_h)$ (and hence also of $\Omega(\alpha_h)$) is uniquely determined by the x_2-coordinates of the nodes $N_i = (a_i, \alpha_h(a_i))$ defined on $\Gamma_C(\alpha_h)$. Consequently, the design (or control) variables are $d_i \equiv \alpha_h(a_i)$, $i = 0, \ldots, D$. We define the design vector

$$\mathbf{d} = (d_0, d_1, \ldots, d_D).$$

We can identify the set U^h_{ad} with a closed convex subset of $\mathbb{R}^{D+1}$

$$\mathscr{U} = \Big\{\mathbf{d} \in \mathbb{R}^{D+1} \mid 0 \leqslant d_i \leqslant \gamma_0, \tag{48}$$
$$-\gamma_1 \leqslant \frac{d_i - d_{i-1}}{a_i - a_{i-1}} \leqslant \gamma_1,\ \sum_{i=1}^{D} \frac{2\bar{\gamma} - d_i - d_{i-1}}{a_i - a_{i-1}} = 2\gamma_2\Big\}.$$

The discrete state problem $(\mathscr{P}(\alpha_h)'_h)$ is a general nonlinear programming problem. We prefer to solve it as a sequence of quadratic programming problems

$$u_h^{n+1} = \arg\min_{v_h \in K_h} \frac{1}{2} B_{\Omega_h}(u_h^n; v_h, v_h) - L_{\Omega_h}(v_h), \tag{49}$$

where

$$B_\Omega(u; v, w) \equiv \int_\Omega \Big[\Big(\kappa - \frac{2}{3}\mu(\Gamma^2(u))\Big)\varepsilon_{ii}(v)\varepsilon_{jj}(w) + 2\mu(\Gamma^2(u))\varepsilon_{ij}(v)\varepsilon_{ij}(w)\Big]\,\mathbf{d}x.$$

This approach is known as secant-modulus or Kachanov method ([9], [10]).

By the vector $\mathbf{q}(\mathbf{d}) = (q_1(\mathbf{d}), \ldots, q_n(\mathbf{d}))$ we denote the nodal values of the displacement field $u_h(\alpha_h)$. Let $\mathscr{S}_2$ be an index set containing the indices of x_2-components of the displacement field at the nodes of the contact boundary $\bar{\Gamma}_C \setminus \bar{\Gamma}_U$. Furthermore, the following convention is used: by q_{j_i}, $j_i \in \mathscr{S}_2$ we refer to the j_i-th component of the displacement vector which is the x_2-component of $u_h(\alpha_h)$ at the node N_i.

The problem (P_h) is equivalent to the nonlinear programming problem

$$\begin{cases} \text{Find } \mathbf{d}^* \in \mathscr{U} \text{ such that} \\ \mathscr{J}(\mathbf{d}^*) \leqslant \mathscr{J}(\mathbf{d}) \quad \forall \mathbf{d} \in \mathscr{U}, \end{cases} \tag{50}$$

where $\mathscr{J}(\mathbf{d}) \equiv \mathscr{I}(\mathbf{d}, \mathbf{q}(\mathbf{d}))$ and $\mathscr{I}(\mathbf{d}, \mathbf{q}(\mathbf{d}))$ denotes the matrix form of the cost function $I_h \equiv \Phi_{\Omega_h}$. To be able to use efficient nonlinear programming algorithms for the numerical solution of (50) we must perform the sensitivity analysis, i.e. to calculate the gradient of $\mathscr{J}(\mathbf{d})$.

Assumptions (T1) and (T2) imply that for fixed $\mathbf{q}$ the mapping $\mathbf{d} \mapsto \mathscr{I}(\mathbf{d}, \mathbf{q})$ is differentiable. Moreover, its gradient can be easily computed by applying the isoparametric technique described in [4]. On the other hand, the mapping $\mathbf{d} \mapsto \mathbf{q}(\mathbf{d})$ is *only directionally differentiable* in general, not continuously differentiable as follows from [12]. Consequently, it might seem that the mapping $\mathbf{d} \mapsto \mathscr{J}(\mathbf{d})$ is not continuously differentiable. However, our particular choice of the cost function leads to the differentiable case. Indeed, let $\mathscr{J}'(\mathbf{d}; \hat{\mathbf{d}})$ denote the directional derivative of $\mathscr{J}$ at $\mathbf{d}$ in the direction $\hat{\mathbf{d}}$. Then

$$\begin{aligned} \mathscr{J}'(\mathbf{d}; \hat{\mathbf{d}}) &= \lim_{t \to 0+} \frac{\mathscr{J}(\mathbf{d} + t\hat{\mathbf{d}}) - \mathscr{J}(\mathbf{d})}{t} \\ &= \Big(\nabla_{\mathbf{d}} \mathscr{I}(\mathbf{d}, \mathbf{q})\Big)^T \hat{\mathbf{d}} + \Big(\nabla_{\mathbf{q}} \mathscr{I}(\mathbf{d}, \mathbf{q})\Big)^T \mathbf{q}'(\mathbf{d}), \end{aligned} \tag{51}$$

where

$$\mathbf{q}'(\mathbf{d}) \equiv \lim_{t \to 0+} \frac{\mathbf{q}(\mathbf{d} + t\hat{\mathbf{d}}) - \mathbf{q}(\mathbf{d})}{t}.$$

We shall eliminate the directional derivative of $\mathbf{q}$ from (51). Components of the residual vector

$$\mathbf{r}(\mathbf{d}) \equiv \nabla_{\mathbf{q}} \mathscr{I}(\mathbf{d}, \mathbf{q}(\mathbf{d}))$$

are discrete analogues of the x_2-component of the stress vector along the contact part. Since $\mathbf{q}$ depends continuously on $\mathbf{d}$ and $\mathscr{I}$ is continuously differentiable with respect to $\mathbf{q}$, then $\mathbf{r}$ depends continuously on $\mathbf{d}$, as well. Therefore if $r_{i_j}(\mathbf{d}) \neq 0$ for some $i_j \in \mathscr{S}_2$, then $r_{i_j}(\mathbf{d} + t\hat{\mathbf{d}}) \neq 0$ for any $t > 0$ sufficiently small. This means that the corresponding node on the contact part remains in contact regardless of the small perturbations of $\Omega_h(\alpha_h)$:

$$q_{i_j}(\mathbf{d} + t\hat{\mathbf{d}}) = -d_j - t\hat{d}_j. \tag{52}$$

Consequently,

$$\frac{\partial q_{i_j}}{\partial d_k} = -\delta_{jk}, \quad i_j \in \mathscr{S}_2,\ k = 0, \ldots, D. \tag{53}$$

Now (52)–(53) yield

Theorem 5. *The mapping $\mathbf{d} \mapsto \mathscr{J}(\mathbf{d})$ is once continuously differentiable for all $\mathbf{d} \in \mathscr{U}$ and*

$$\frac{\partial \mathscr{J}(\mathbf{d})}{\partial d_j} = -r_{i_j} - \frac{\partial \mathscr{I}(\mathbf{d}, \mathbf{q})}{\partial d_j}, \quad j = 0, \ldots, D.$$

Example 6. Let $a = 0$, $b = 4$, $\bar{\gamma} = 1$ be the constants defining $\Omega(\alpha)$ and let $\gamma_0 = 0.2$, $\gamma_1 = 1$, $\gamma_2 = 3.8$ be the constants defining the set U_{ad}. We have assumed the nonlinear Hooke's law

$$\sigma_{ij} = \kappa \varepsilon_{ll} \delta_{ij} + 2\tilde{\mu}(e) \left(\varepsilon_{ij} - \frac{1}{3} \delta_{ij} \varepsilon_{ll} \right),$$

where $e = \sqrt{\gamma}$ (with γ defined in (2)) and

$$\tilde{\mu}(e) = \begin{cases} \mu_1, & e \leqslant e_0 \\ \mu_1 \frac{e_0}{e} \left(\log \dfrac{e}{e_0} + 1 \right), & e_0 < e \leqslant e_1 \\ \mu_1 \frac{e_0}{e_1} \left(\log \dfrac{e_1}{e_0} + 1 \right), & e > e_1 \end{cases}$$

where e_1 is sufficiently large. We have chosen $\mu_1 = E/(2 + 2\nu)$, $\kappa = E/(3 - 6\nu)$ and $e_0 = 0.001$ with Young's modulus $E = 1.0$ and Poisson's ratio $\nu = 0.3$. The body force F is assumed zero and the external load is of the form

$$P = \begin{cases} (0, 0.001), & x_2 = 1,\ x_1 \in]2, 4[\\ (0, 0), & \text{otherwise.} \end{cases}$$

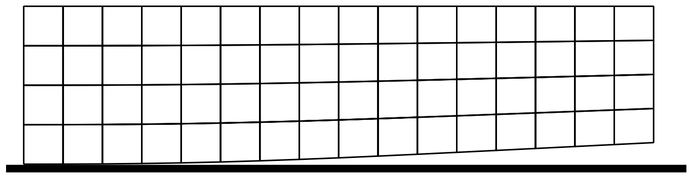

Fig. 3. Finite element mesh of the optimal structure

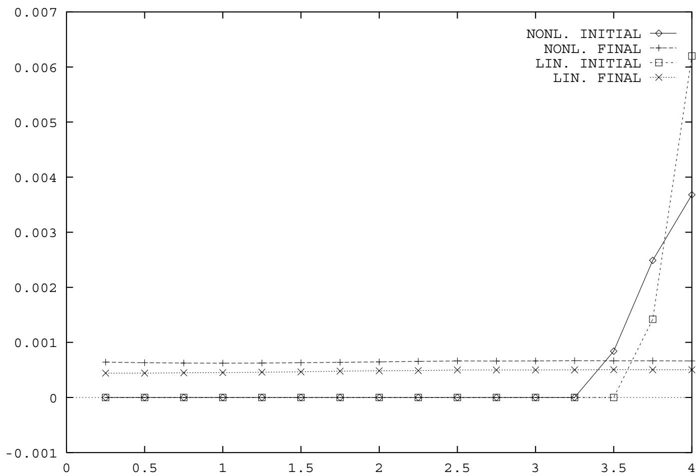

Fig. 4. Contact stress distribution

As our initial guess we have chosen $\alpha_h^0 \equiv 0.04$. Instead of linear triangular elements, we have used four-noded isoparametric elements. The initial cost is -7.68×10^{-5}. After 31 optimization iterations the cost was reduced to -1.49×10^{-4}. The finite element mesh corresponding to the final domain is shown in Figure 3. The initial and final contact stress distributions are plotted in Figure 4 together with the initial and final ones corresponding to the linear law $\tilde{\mu}(e) \equiv \mu_1$.

In optimization we have used the sequential quadratic programming subroutine E04UCF from the NAG library ([8]). The quadratic programming problem (49) has been solved using block SOR-method with projection. Computations have been done in double precision using a HP9000/710-workstation. The total CPU-time needed was approximately 540 seconds.

It was shown by [3] that in the linear case the minimizing total potential energy should yield "almost" constant contact stress distribution. Similar behaviour can be observed also in the nonlinear case.

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