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A remark on polyconvex envelopes of radially symmetric functions in dimension 2×2

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A REMARK ON POLYCONVEX ENVELOPES OF RADIALLY SYMMETRIC FUNCTIONS IN DIMENSION 2×2

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Abstract. We study polyconvex envelopes of a class of functions related to the function of Kohn and Strang introduced in [4]. We present an example of a function of this class for which the polyconvex envelope may be computed explicitly and we also point out some general features of the problem.

Keywords: polyconvex envelope, rank one convex envelope, lower semicontinuous functional

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I. Introduction

Consider the extremal problem

$$\text{(P)} \qquad I(u) = \int_\Omega f(\nabla u(x))\,\mathrm{d}x \to \min,\ u \in W_0^{1,\infty}(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u\colon \Omega \to \mathbb{R}^N$ (i.e., $\nabla u \in \mathbb{R}^{N\times n}$) and $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$.

In connection with the lower semicontinuity of the functional I over $W^{1,\infty}(\Omega)$, the following concepts play an important role:

Definition 1. i) The function f is said to be *quasiconvex* if

$$\int_\Omega f(A + \nabla\varphi(x))\,\mathrm{d}x \geqslant f(A)\operatorname{meas}\Omega$$

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for every $A \in \mathbb{R}^{N\times n}$ and every $\varphi \in W_0^{1,\infty}(\Omega)$.

ii) f is said to be *rank one convex* if the function

$$\mathbb{R} \ni t \longmapsto f(A + tB)$$

is convex for every $A \in \mathbb{R}^{N\times n}$ and $B \in \mathbb{R}^{N\times n}$ with rank $B \leqslant 1$.

iii) f is said to be *polyconvex* if there exists $g\colon \mathbb{R}^{\tau(n,N)} \to \mathbb{R}$ convex, such that

$$f(A) = g(T(A)),$$

where $T\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau}$, $\tau = \tau(n,N) := \sum\limits_{j=1}^{\min(n,N)} \binom{n}{j}\binom{N}{j}$,

$$T(A) = (A,\ \mathrm{adj}_2 A, \ldots,\ \mathrm{adj}_m A), \quad m := \min(n,N),$$

$\mathrm{adj}_k A$, $k = 2,\ldots,m$, being the matrix of all $k \times k$ minors of A.

The concept of quasiconvexity (note that in the convex analysis the concept of a quasiconvex function has a different meaning, it denotes a function whose level sets are convex) was introduced by Morrey [5] and it was proved that the functional $I(u)$ is lower semicontinuous over $W^{1,\infty}(\Omega)$ if and only if the function f is quasiconvex. However, in particular examples it is difficult to verify whether or not a given function is quasiconvex. For this reason, the concepts of polyconvexity and rank one convexity were introduced in [1]. Polyconvexity yields a sufficient and rank one convexity a necessary condition for a function to be quasiconvex. More precisely, we have the diagram

(1.1) $\qquad$ convex $\Longrightarrow$ polyconvex $\Longrightarrow$ quasiconvex $\Longrightarrow$ rank one convex.

When a function f in (P) fails to be quasiconvex, a useful information about this problem may be inferred from the so-called relaxation problem

$$\bar{I}(u) = \int\limits_{\Omega} Qf(\nabla u(x))\,\mathrm{d}x \to \min,\ u \in W_0^{1,\infty}(\Omega),$$

where

$$Qf = \sup\{g \leqslant f,\ g - \text{quasiconvex}\}$$

is the so-called quasiconvex envelope of f.

Similar to the problem of quasiconvexity, to find the quasiconvex envelope of a given function represents a difficult problem and for this reason the concepts of polyconvex and rank one convex envelopes have been introduced:

$$Pf = \sup\{g \leqslant f,\ g - \text{polyconvex}\}$$
$$Rf = \sup\{g \leqslant f,\ g - \text{rank one convex}\}.$$

Recall also the usual definition of the convex envelope

$$Cf = \sup\{g \leqslant f,\ g - \text{convex}\}.$$

According to (1.1) the following inequalities hold:

$$Cf \leqslant Pf \leqslant Qf \leqslant Rf, \tag{1.2}$$

particularly, if $Pf = Rf$, then $Pf = Qf = Rf$.

Even if to compute polyconvex and rank one envelopes of a given function is generally easier than to find the quasiconvex envelope, the computation of Pf and Rf is not easy even in such a simple case as $f(A) = g(\|A\|)$, where $\|A\| = \Big(\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{n} a_{ij}^2\Big)^{1/2}$ is the usual Euclidean matrix norm and $g\colon \mathbb{R} \to \mathbb{R}$. The most frequently cited example of a radially symmetric function for which $Rf = Qf = Pf \neq Cf$ is the example of Kohn and Strang [4], where $n = 2$ and $f(A) = g(\|A\|)$ with

$$g(r) = \begin{cases} 1 + r^2 & \text{if } r > 0, \\ 0 & \text{if } r = 0. \end{cases} \tag{1.3}$$

In this case

$$Cf(A) = \begin{cases} f(A) & \text{if } \|A\| \geqslant 1, \\ 2\|A\| & \text{if } \|A\| \leqslant 1, \end{cases} \tag{1.4$_1$}$$

(1.4$_2$)
$$Rf(A) = Pf(A) = \begin{cases} f(A) & \text{if } \|A\|^2 + 2|\mathrm{adj}_2 A| \geqslant 1, \\ 2\sqrt{\|A\|^2 + 2|\mathrm{adj}_2 A|} - 2|\ \mathrm{adj}_2 A| & \text{if } \|A\|^2 + 2|\mathrm{adj}_2 A| \leqslant 1. \end{cases}$$

In this paper we deal with another function whose polyconvex envelope may be computed explicitly, namely with the function

$$f_\infty(A) = \begin{cases} 0 & \text{if } A = 0, \\ 1 & \text{if } 0 < \|A\| \leqslant 1, \\ \infty & \text{if } \|A\| > 1. \end{cases} \tag{1.5}$$

This function and (1.3) are special cases of a general class of functions

$$f(A) = \begin{cases} 1 + g(\|A\|) & \text{if } A \neq 0, \\ 0 & \text{if } A = 0, \end{cases} \tag{1.6}$$

where $g(0) = 0$ and $\lim\limits_{r\to\infty} \frac{g(r)}{r^2} > 0$. The case (1.5) we treat here separately since it provides a good insight into the problem and is sufficiently representative for the understanding of problems which one meets when investigating polyconvex envelopes of radially symmetric functions. All our computations are performed for 2×2 matrices.

Unfortunately, in contrast to the quadratic example of Kohn and Strang, we have not been able till now to compute explicitly the rank one convex envelope Rf_∞ of (1.5) (for more detail see the last section). So we have done only a half of the step leading to the computation of Qf_∞ and the problem whether $Pf_\infty = Rf_\infty$ remains open.

The paper is organized as follows. In the next section we present some preparatory statements concerning the general theory of convex and polyconvex envelopes of radially symmetric functions. In Section III the properties of a certain important function are established and Section IV deals with the explicit computation of the polyconvex envelope of the function given by (1.5). The last section is devoted to some remarks and comments concerning possible extensions and modifications of the results of the paper.

II. Auxiliary results

First of all observe that a function $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ of the form

$$f(A) = g(\|A\|),$$

where $g\colon \mathbb{R} \to \mathbb{R}$, is radially symmetric, i.e.

$$f(A) = f(UAV) \tag{2.1}$$

for any orthogonal matrices U, V. Hence, in view of [3], it is sufficient in computing Pf and Cf to restrict ourselves to diagonal matrices. More precisely, it suffices to perform all computations for matrices of the form $A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ and in the results obtained, in order to get the formula for general 2×2 matrices A, to replace $x^2 + y^2$ by $\|A\|^2 = \sum\limits_{i,j=1}^{2} a_{ij}^2$ and xy by $|\det A|$. Here x and y satisfying $0 \leqslant y \leqslant x$ should be regarded as the singular values of the matrix A, i.e.,

$$\begin{aligned} x &= \frac{1}{2}\left[\sqrt{\|A\|^2 + 2|\det A|} + \sqrt{\|A\|^2 - 2|\det A|}\right], \\ y &= \frac{1}{2}\left[\sqrt{\|A\|^2 + 2|\det A|} - \sqrt{\|A\|^2 - 2|\det A|}\right]. \end{aligned} \tag{2.2}$$

In the sequel, for the convenience of notation, we write, for a diagonal matrix $A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, $f(x, y)$ instead of $f\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right)$. The basic tool for computation of the convex and polyconvex envelopes is provided by the following theorem.

Theorem 2.1. [2] *Denote*

$$\begin{aligned} f^P(\alpha, \beta, \gamma) &= \sup_{x,y\in\mathbb{R}^2} \{\alpha x + \beta y + \gamma xy - f(x, y)\}, \\ f^*(\alpha, \beta) &= \sup_{x,y\in\mathbb{R}^2} \{\alpha x + \beta y - f(x, y)\} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} f^{PP}(x, y) &= \sup_{\alpha,\beta,\gamma\in\mathbb{R}^3} \{\alpha x + \beta y + \gamma xy - f^P(\alpha, \beta, \gamma)\}, \\ f^{**}(x, y) &= \sup_{\alpha,\beta\in\mathbb{R}^2} \{\alpha x + \beta y - f^*(\alpha, \beta)\}. \end{aligned} \tag{2.4}$$

Then

$$Pf(x, y) = f^{PP}(x, y), \quad Cf(x, y) = f^{**}(x, y). \tag{2.5}$$

Moreover, if f is convex then

$$Cf(x, y) = Pf(x, y) = f(x, y). \tag{2.6}$$

Now let us turn our attention to some general features of computation of Pf and Cf for functions of the form (1.6), where $g\colon \mathbb{R} \to \mathbb{R}$ is a convex function for which $0 = g(0) = \inf_{r>0} g(r)$. According to (2.3)

$$\begin{aligned} f^P(\alpha, \beta, \gamma) &= \max\left\{0, \sup_{x,y}\{\alpha x + \beta y + \gamma xy - g(r)\} - 1\right\} \\ &= \max\{0, \tilde{f}^P(\alpha, \beta, \gamma) - 1\}, \end{aligned}$$

where $\tilde{f}(x, y) = g(r) = g(\sqrt{x^2 + y^2})$ and

$$\begin{aligned} Pf(x, y) = \max\Big\{&\sup\{\alpha x + \beta y + \gamma xy \mid \tilde{f}^P(\alpha, \beta, \gamma) \leqslant 1\}, \\ &\sup\{\alpha x, +\beta y + \gamma xy - \tilde{f}^P(\alpha, \beta, \gamma) + 1 \mid, \tilde{f}^P(\alpha, \beta, \gamma) > 1\}\Big\}. \end{aligned} \tag{2.7}$$

Since the function $\alpha x + \beta y + \gamma xy$ is linear in α, β, γ, the first supremum in (2.7) is attained at the boundary, hence the first extremal problem in (2.7) is equivalent to

$$\alpha x + \beta y + \gamma xy \to \max, \ \ \tilde{f}^P(\alpha, \beta, \gamma) = 1. \tag{2.8}$$

Concerning the second supremum, if x, y are such that supremum is attained at a point $[\alpha, \beta, \gamma]$ for which $\tilde{f}^P(\alpha, \beta, \gamma) > 1$, i.e., the constrain in this extremal problem does not come into power, according to (2.6), this supremum is $f(x, y)$.

Consequently, the solution of (2.8) is the only possibility which may cause the inequality $Pf(x, y) > Cf(x, y)$ for some x, y. Observe also that concerning the computation of the convex envelope, a similar role is played by the problem

$$\alpha x + \beta y \to \max, \ \ \tilde{f}^*(\alpha, \beta) = \tilde{f}^P(\alpha, \beta, 0) = 1. \tag{2.9}$$

If the function g grows near ∞ less then quadratically, then $\tilde{f}^P(\alpha, \beta, \gamma) = +\infty$ for $\gamma \neq 0$, hence in this case (2.8), (2.9) are equivalent and we have the equality $Pf = Cf$.

In the example of Kohn and Strang [4] the set of $[\alpha, \beta, \gamma] \in \mathbb{R}^3$ for which

$$\tilde{f}^P(\alpha, \beta, \gamma) = 1 \tag{2.10}$$

consists of a "smooth" part

$$M = \left\{ [\alpha, \beta, \gamma] \mid \frac{\alpha^2 + \beta^2 + \alpha\beta\gamma}{4 - \gamma^2} = 1, \ |\gamma| < 2 \right\}$$

and four "vertex" points $[2, 2, -2]$, $[2, -2, -2]$, $[-2, -2, -2]$, $[-2, 2, 2]$ and for the value of Pf for small x, y just these four points are important, see [4].

For our function f given by (1.5) we have

$$\tilde{f}^P(\alpha, \beta, \gamma) = \sup_{x^2 + y^2 = 1} \{\alpha x + \beta y + \gamma xy\}. \tag{2.11}$$

In contrast to the quadratic example (1.3), the set of $[\alpha, \beta, \gamma] \in \mathbb{R}^3$ satisfying (2.10) cannot be described explicitly in the case (2.11). Nevertheless, a closer examination of the extremal problem (2.8) reveals the fact that the explicit description of this set of $[\alpha, \beta, \gamma]$ is not necessary since only the values $[\alpha, \beta, \gamma]$ with $\alpha = \pm\beta$ are important and just for these values we have the explicit formula for a solution of (2.11).

III. Properties of the function $H(\alpha,\beta,\gamma,r)$

The function H in the title of the section is defined by

$$H(\alpha,\beta,\gamma,r)=\max_{x^2+y^2\leqslant r^2}\{\alpha x+\beta y+\gamma xy\}. \tag{H}$$

Even if we need the function H only for $r=1$ in this paper, we investigate also its dependence on r since it makes no difficulties and the results are useful when dealing with the polyconvex envelope of the function given by (1.6).

The explicit evaluation of the value of H leads to an algebraic equation of the fourth order and the application of formulae for the roots of this equation does not give satisfactory results[1]. For this reason, in this section we present some qualitative properties of this function.

We start with results which are trivial.

Lemma 3.1. *Concerning the function $H(\alpha,\beta,\gamma,r)$, the following hold:*
i) *The maximum in (H) is always attained on the boundary $x^2+y^2=r^2$, i.e.,*

$$H(\alpha,\beta,\gamma,r)=\max_{x^2+y^2=r^2}\{\alpha x+\beta y+\gamma xy\}.$$

ii) *For every $[\alpha,\beta,\gamma]\in\mathbb{R}^3$*

$$H(\alpha,\beta,\gamma,r)=H(\beta,\alpha,\gamma,r), \tag{3.1}$$

$$H(\alpha,\beta,-\gamma,r)=H(-\alpha,\beta,\gamma,r)=H(\alpha,-\beta,\gamma,r).$$

iii) *In the polar coordinates $x=r\cos\varphi$, $y=r\sin\varphi$, $\alpha=\varrho\cos\vartheta$, $\beta=\varrho\sin\vartheta$*

$$H(\varrho,\vartheta,\gamma,r)=r\varrho\cos(\tilde\varphi-\vartheta)+\frac{1}{2}\gamma r\sin 2\tilde\varphi, \tag{3.2}$$

where $\tilde\varphi=\tilde\varphi(\varrho,\vartheta,\gamma,r)$ is a solution of

$$-\varrho\sin(\varphi-\vartheta)+\gamma r\cos 2\varphi=0 \tag{3.3}$$

[1] Even in the simple case $\beta=0$, which may be treated explicitly, we have obtained a rather complex expression (for $\alpha>0$)

$$H(\alpha,0,\gamma,r)=\frac{1}{8\sqrt{2}|\gamma|}\left(\alpha+\sqrt{\alpha^2+3\gamma^2r^2}\right)\sqrt{4\gamma^2r^2-\alpha^2+\alpha\sqrt{\alpha^2+3\gamma^2r^2}}$$

which realizes the maximum of H.

iv)

$$H(\alpha,\beta,0,r) = r\sqrt{\alpha^2+\beta^2}, \qquad (3.4)$$
$$H(0,0,\gamma,r) = \frac{1}{2}|\gamma| r^2.$$

v) *H is convex in (α,β) for every $\gamma \in \mathbb{R}$ since H is the conjugate function of $f_\gamma := -\gamma xy$.*

An explicit formula for H may be obtained in the case $\alpha = \pm\beta$. We formulate the result only for $\alpha = \beta$, the formula for $\alpha = -\beta$ we get from (3.1).

Lemma 3.2. *Let $\gamma \neq 0$. Then*

$$H(\alpha,\alpha,\gamma,r) = \begin{cases} -\frac{\alpha^2}{2\gamma} - \frac{\gamma r^2}{2} & \text{if } r \geqslant \frac{1}{\sqrt{2}}\left|\frac{\alpha}{\gamma}\right| \text{ and } \gamma < 0, \\ \sqrt{2}|\alpha| r + \frac{\gamma r^2}{2} & \text{if } r \leqslant \frac{1}{\sqrt{2}}\left|\frac{\alpha}{\gamma}\right| \text{ or } \gamma > 0, \end{cases} \qquad (3.5)$$

whereby in the first case the maximum is attained at

$$x = -\frac{\alpha}{2\gamma} \pm \frac{1}{2|\gamma|}\sqrt{2\gamma^2 r^2 - \alpha^2},\ y = -\frac{\alpha}{2\gamma} \mp \frac{1}{2|\gamma|}\sqrt{2\gamma^2 r^2 - \alpha^2}, \qquad (3.6)$$

and in the second case at

$$x = y = \pm\frac{r}{\sqrt{2}}. \qquad (3.7)$$

P r o o f. Consider the extremal problem

$$\alpha(x+y) + \gamma xy \to \max,\ x^2 + y^2 = r^2.$$

The extremal point $[x,y]$ is a solution of the system

$$\alpha + \gamma y = \lambda x, \qquad (3.8)$$
$$\alpha + \gamma x = \lambda y,$$
$$x^2 + y^2 - r^2 = 0,$$

where λ is a Lagrange multiplier. The first two equations give $(y-x)(\gamma+\lambda) = 0$. If $x = y$ then $x = y = \pm\frac{r}{\sqrt{2}}$. If $\lambda = -\gamma$ then $x + y = -\frac{\alpha}{\gamma}$ and substitution into the third equation in (3.8) gives

$$x = -\frac{\alpha}{2\gamma} \pm \frac{1}{2|\gamma|}\sqrt{2\gamma^2 r^2 - \alpha^2},\ y = -\frac{\alpha}{2\gamma} \mp \frac{1}{2|\gamma|}\sqrt{2\gamma^2 r^2 - \alpha^2},$$

i.e., this is possible only for $r \geqslant \frac{1}{\sqrt{2}}|\frac{\alpha}{\gamma}|$. Comparing the values of $\alpha(x+y)+\gamma xy$ for x, y given by (3.6) and (3.7) we get (3.5) whereby for $\gamma < 0$, $r \geqslant \frac{1}{\sqrt{2}}\left|\frac{\alpha}{\gamma}\right|$ the maximum is attained at (3.6) and for $\gamma > 0$ or $r \leqslant \frac{1}{\sqrt{2}}\left|\frac{\alpha}{\gamma}\right|$ at (3.7). $\square$

Lemma 3.3. *At all points $[\varrho,\vartheta,\gamma]\in\mathbb{R}^3$ where H is differentiable we have*

$$\begin{aligned}H'_\varrho(\varrho,\vartheta,\gamma,r)&=r\cos(\tilde\varphi-\vartheta),\\ H'_\gamma(\varrho,\vartheta,\gamma,r)&=\frac{1}{2}r^2\sin 2\tilde\varphi,\\ H'_\vartheta(\varrho,\vartheta,\gamma,r)&=\varrho r\sin(\tilde\varphi-\vartheta),\\ H'_r(\varrho,\vartheta,\gamma,r)&=\varrho\cos(\tilde\varphi-\vartheta)+\gamma r\sin 2\tilde\varphi,\end{aligned}\tag{3.9}$$

where $\tilde\varphi$ is a solution of (3.3) *which realizes the maximum in the definition of H.*

P r o o f. We prove only the first formula, the proof of the other ones is analogous.

$$\begin{aligned}H'_\varrho(\varrho,\vartheta,\gamma,r)&=\frac{\partial}{\partial\varrho}[r\varrho\cos(\tilde\varphi-\vartheta)+\frac{1}{2}\gamma r^2\sin 2\tilde\varphi]\\ &=r\cos(\tilde\varphi-\vartheta)+\tilde\varphi'_\varrho(-r\varrho\sin(\tilde\varphi-\vartheta)+\frac{1}{2}r^2\gamma\sin 2\tilde\varphi)=r\varrho\cos(\tilde\varphi-\vartheta).\end{aligned}$$

Here we have used the fact that $\tilde\varphi$ is a solution of (3.3). □

In the next section we shall need some information about the set

$$M=\{[\alpha,\beta,\gamma]\in\mathbb{R}^3 \mid H(\alpha,\beta,\gamma,1)=1\}.$$

For simplicity we write $H(\alpha,\beta,\gamma)$ instead of $H(\alpha,\beta,\gamma,1)$. According to the symmetry relations (3.1) it is sufficient to take into account $\gamma\leqslant 0$, $0\leqslant\beta\leqslant\alpha$. If $\alpha=\beta$, then (3.5) yields

$$\begin{aligned}H(\alpha,\alpha,\gamma)=1&\iff\alpha^2+\gamma^2+2\gamma=0\iff\frac{\varrho^2}{2}+\gamma^2+2\gamma=0,\\ &\qquad\qquad\text{if }\gamma\leqslant-\frac{|\alpha|}{\sqrt2}=-\frac{\varrho}{2},\\ &\iff\sqrt2|\alpha|+\frac{\gamma}{2}=1\iff\varrho+\frac{\gamma}{2}=1\\ &\qquad\qquad\text{if }\gamma\geqslant-\frac{|\alpha|}{\sqrt2}=-\frac{\varrho}{2}.\end{aligned}\tag{3.10}$$

Observe also that the straight line $\sqrt2|\alpha|+\frac{\gamma}{2}=1$ forms the tangent to the circle $\alpha^2+\gamma^2+2\gamma=0$ at the points $[\pm\frac{2\sqrt2}{3},-\frac{2}{3}]$, i.e., the curve $H(\alpha,\alpha,\gamma)=1$ is of the class C^1 (for $\gamma<2$).

Finally, look for the points where the tangent to the curve $H(\varrho,\vartheta,\gamma)=1$, $\vartheta=\vartheta_0=\text{const}$ is vertical. According to (3.9) this occurs for such triples $[\varrho,\vartheta,\gamma]$ that

$\sin 2\tilde{\varphi} = 0$, i.e. $\tilde{\varphi} = 0$ (since we consider $\vartheta \in [0, \frac{\pi}{4}]$). Then (3.3) and the equation $H(\alpha, \beta, \gamma) = 1$ read

$$-\varrho \sin \vartheta = \gamma,$$
$$\varrho \cos \vartheta = 1$$

or, equivalently,

$$\beta = -\gamma, \ \alpha = 1. \tag{3.11}$$

Consequently, the set of $[\alpha, \beta, \gamma] \in M$, $\gamma \leqslant 0$, $0 \leqslant \alpha \leqslant \beta$ for which the tangent is vertical is the intersection of the planes (3.11). By virtue of the symmetry of H relative to α, β, the set of these points in the octant $\alpha, \beta \geqslant 0$, $\gamma \leqslant 0$, when looking from the negative direction of the γ-axis, looks like the square $\alpha = 1$, $\beta \in [0, 1]$, $\beta = 1$, $\alpha \in [0, 1]$. Computing $H''_{\varrho\varrho}$ at these points, we may see that ϱ is here maximal, i.e., also the set M looks from the γ-direction like a square (particularly, H is not differentiable at the point $[1, 1, -1]$).

In the sequel, the part of M in the octants $\alpha, \beta \geqslant 0$ above the broken line $\alpha = 1$, $\beta = -\gamma$, $\beta = 1$, $\alpha = -\gamma$ will be called the *northern part* of M and the part below this line the *southern part* of M.

Now, let us investigate in more detail the properties of the northern part of M. Note that in the first quadrant the southern part is immaterial for the extremal problem (2.8). Indeed, if $P_1 = [\alpha, \beta, \gamma_1]$ is on the southern part and $P_2 = [\alpha, \beta, \gamma_2]$ on the northern part (i.e., $\gamma_2 > \gamma_1$), then for $x, y \geqslant 0$

$$\alpha x + \beta y + \gamma_2 xy > \alpha x + \beta y + \gamma_1 xy.$$

Lemma 3.4. *For a given $\vartheta \in (0, \frac{\pi}{4})$, denote by $\varrho(\vartheta)$, $\gamma(\vartheta)$ that ϱ, γ for which $H(\varrho, \vartheta, \gamma) = 1$. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\varrho(\vartheta) = -\varrho(\vartheta)\tan(\tilde{\varphi} - \vartheta), \ \frac{\mathrm{d}}{\mathrm{d}\vartheta}\gamma(\vartheta) = \frac{2\varrho(\vartheta)\sin(\tilde{\varphi} - \vartheta)}{\sin 2\tilde{\varphi}}, \tag{3.12}$$

where $\tilde{\varphi}$ is a solution of (3.3) *realizing the maximum of H.*

Proof. Since $H(\varrho, \vartheta, \gamma) = \varrho\cos(\tilde{\varphi} - \vartheta) - \frac{1}{2}\gamma \sin 2\tilde{\varphi} = 1$, differentiating this equality and taking into account (3.3) we get the required statement. □

The lemmas given in this section give a rough picture of the set M in $[\alpha, \beta, \gamma]$ space. Consider the cylindric surface $H(\alpha, \beta, 0) = \sqrt{\alpha^2 + \beta^2} = 1$ and the prism $\max\{|\alpha|, |\beta|\} = 1$. Let us describe the shape of M for $\alpha, \beta \geqslant 0$ (the remaining part may be reconstructed using the symmetry relations (3.1)).

Let C_1 and C_2 be the line segments on the faces of the prism $\alpha = 1$, $\beta = 1$, respectively, which arise as intersections of these faces with the planes $\beta = -\gamma$ and $\alpha = \gamma$, respectively. Suppose that the cylinder $\alpha^2 + \beta^2 = 1$ is elastic and may be homeomorphically deformed in such a way that it is smoothly (this means that the vertical lines on the faces of the prism form tangents to the curves $M \cap \{[\varrho, \vartheta, \gamma] \mid \gamma = \gamma_0 = \text{ const}\}$) glued to the prism at C_1 and C_2. Below C_1, C_2 the cylindric surface is "closed" in such a way that the point $[0, 0, -2]$ (see (3.4)) form the "southern pole" of the surface M. For $\gamma > 0$ (and still $\alpha, \beta \geqslant 0$), M is inside the cylinder and is again attached to the γ-axis at the point $[0, 0, 2]$ which plays the role of the north pole.

The curve which is the intersection of M with a plane $\gamma = \gamma_0$ has maximal distance from the point $[0, 0, \gamma_0]$ in the direction $\alpha = \beta$ (i.e., $\vartheta = \frac{\pi}{4}$) and minimal (in the first (α, β)-quadrant) on axes $\alpha = 0, \beta = 0$ (since $\tilde{\varphi} < \vartheta$ in (3.12)—to skech the graphs of functions $y_1 = \gamma \cos 2\varphi$, $y_2 = \varrho \sin(\varphi - \vartheta)$ whose intersection gives $\tilde{\varphi}$—see (3.3)—helps to visualize the situation). If $\gamma_0 \in (-1, 0]$, $\vartheta \in [0, \frac{\pi}{4}]$, this distance, denote it by $d(\vartheta, \gamma_0)$, satisfies $d(\vartheta, \gamma_0) > d(\vartheta, 0) = 1$ and for $\gamma \in (0, 2)$, $\vartheta \in [0, \frac{\pi}{4}]$ we have $d(\vartheta, \gamma) < 1$.

To study the shape of the set M in more detail, the following lemmas play a fundamental role.

Lemma 3.5. *Let* $\tilde{M} = \{[\varrho, \vartheta, \gamma] \in M \colon \gamma \in [-1, 0], \vartheta \in [-\arctan \gamma, \frac{\pi}{4}\}$ *and let* $\tilde{\varphi} = \tilde{\varphi}(\varrho, \vartheta, \gamma)$ *be the argument which realizes the maximum in the definition of* H, *i.e.*

$$H(\varrho, \vartheta, \gamma) = \varrho \cos(\tilde{\varphi} - \vartheta) + \frac{1}{2}\gamma \sin 2\tilde{\varphi}.$$

Then $\tilde{\varphi}$ *is differentiable in the interior of* $\tilde{M}$ *and*

$$\begin{aligned} \tilde{\varphi}'_\varrho &= \frac{-\sin(\tilde{\varphi} - \vartheta)}{1 + \frac{3}{2}\gamma \sin 2\tilde{\varphi}}, \\ \tilde{\varphi}'_\vartheta &= \frac{\varrho \cos(\tilde{\varphi} - \vartheta)}{1 + \frac{3}{2}\gamma \sin 2\tilde{\varphi}}, \\ \tilde{\varphi}'_\gamma &= \frac{\cos 2\tilde{\varphi}}{1 + \frac{3}{2}\gamma \sin 2\tilde{\varphi}}. \end{aligned} \tag{3.13}$$

P r o o f. Observe that $\tilde{M}$ is just the northern part of M in the region $\alpha \geqslant \beta \geqslant 0$, $\gamma \leqslant 0$. Since $\tilde{\varphi}$ realizes the maximum of H and $(\varrho, \vartheta, \gamma) \in M$, we have

$$\begin{aligned} \varrho \cos(\tilde{\varphi} - \vartheta) + \frac{1}{2}\gamma \sin 2\tilde{\varphi} &= 1, \\ -\varrho \sin(\tilde{\varphi} - \vartheta) + \gamma \cos 2\tilde{\varphi} &= 0. \end{aligned} \tag{3.14}$$

We will prove only the relation for $\tilde{\varphi}'_\varrho$, the proof of the remaining equalities is the same. Differentiating the second equation in (3.14) with respect to ϱ and taking into account the former one, we have

$$\begin{aligned} 0 &= \sin(\tilde{\varphi}-\vartheta) - \varrho\tilde{\varphi}'_\varrho \cos(\tilde{\varphi}-\vartheta) - 2\gamma\tilde{\varphi}'_\varrho \sin 2\tilde{\varphi} \\ &= -\tilde{\varphi}_\varrho[\varrho\cos(\tilde{\varphi}-\vartheta) + \frac{1}{2}\sin 2\tilde{\varphi} + \frac{3}{2}\gamma\sin 2\tilde{\varphi}] - \sin(\tilde{\varphi}-\vartheta) \\ &= -\tilde{\varphi}'_\varrho(1+\frac{3}{2}\gamma\sin 2\tilde{\varphi}) - \sin(\tilde{\varphi}-\vartheta). \end{aligned}$$

To prove (3.13) we need to show that $1+\frac{3}{2}\gamma\sin 2\tilde{\varphi} \neq 0$ for $[\varrho,\vartheta,\gamma]$ in the interior of $\tilde{M}$. For $\gamma \in (-\frac{2}{3}, 0]$ this is obvious. For $\gamma \in [-1, -\frac{2}{3}]$ we proceed as follows. If $\vartheta = -\arctan\gamma$ and $[\varrho,\vartheta,\gamma] \in \tilde{M}$ then $\tilde{\varphi}(\varrho,\vartheta,\gamma) = 0$ and for $\vartheta = \frac{\pi}{4}$, according to (3.6),

$$\tan\tilde{\varphi}\Big(\varrho, \frac{\pi}{4}, \gamma\Big) = \frac{\varrho - \sqrt{4\gamma^2-\varrho^2}}{\varrho + \sqrt{4\gamma^2-\varrho^2}}$$

and since $\frac{\varrho^2}{2} + (\gamma+1)^2 = 1$ for $[\varrho, \frac{\pi}{4}, \gamma] \in \tilde{M}$, after some computation we get

$$\sin 2\tilde{\varphi} = \frac{2\tan\tilde{\varphi}}{1+\tan^2\tilde{\varphi}} = -\frac{2\gamma+2}{\gamma}. \tag{3.15}$$

Hence, for $\vartheta = -\arctan\gamma$ the denominator in (3.13) is positive and by the second relation in (3.13) $\tilde{\varphi}$ is increasing with respect to ϑ at all points where the denominator remains positive. At $\vartheta = \frac{\pi}{4}$, by (3.15) $1+\frac{3}{2}\gamma\sin 2\tilde{\varphi} = 1-3(\gamma+1) > 0$ except for $\gamma = -\frac{2}{3}$ but for this γ we have $\varphi(\varrho,\vartheta,-\frac{2}{3}) < \frac{\pi}{4}$ if $\tilde{\vartheta} < \frac{\pi}{4}$. Consequently, for a fixed $\gamma \in [-1, 0]$, $\vartheta \in (-\arctan\gamma, \frac{\pi}{4})$ and ϱ such that $[\varrho,\vartheta,\gamma] \in M$, the denominator in (3.15) is positive. This means that $\tilde{\varphi} = \tilde{\varphi}(\varrho,\tilde{\vartheta},\gamma)$ is an increasing function of ϑ which increases from $\tilde{\varphi} = 0$ to $\tilde{\varphi} = \frac{\pi}{4}$ if $\gamma \in [-\frac{2}{3}, 0]$ and from $\tilde{\varphi} = 0$ to $\tilde{\varphi} = -\frac{1}{2}\arcsin\frac{2(\gamma+1)}{\gamma}$ if $\gamma \in [-1, -\frac{2}{3}]$. □

Using the same argument as in the previous proof one may show that for a fixed $\gamma \in [-1, 0]$, $\vartheta \in (\frac{\pi}{4}, \frac{\pi}{2}+\arctan\gamma)$ and ϱ such that $[\varrho,\vartheta,\gamma] \in M$ the function $\tilde{\varphi}$ is increasing with respect to ϑ. This function increases from $\tilde{\varphi} = \frac{\pi}{2} + \frac{1}{2}\arcsin\frac{2(\gamma+1)}{\gamma}$ to $\tilde{\varphi} = \frac{\pi}{2}$ if $\gamma \in [-1, -\frac{2}{3}]$ and from $\tilde{\varphi} = \frac{\pi}{4}$ to $\tilde{\varphi} = \frac{\pi}{2}$ if $\gamma \in [-\frac{2}{3}, 0]$. Particularly, for $\gamma \in [-1, -\frac{2}{3}]$ this function is discontinuous (and hence not differentiable) at $\vartheta = \frac{\pi}{4}$.

Lemma 3.6. *Let $\bar\gamma \in [-1,0]$ be fixed. Then for $r \in (0,1)$ and $\varphi \in [0,\frac{\pi}{4}]$ we have*

(3.16)
$$\sup_{H(\varrho,\vartheta,\bar\gamma)=1}\left[r\varrho\cos(\varphi-\vartheta)+\frac{1}{2}\bar\gamma r^2\sin 2\varphi\right]$$
$$=\begin{cases}\sqrt{2}r\sqrt{-\bar\gamma^2-2\bar\gamma}\cos\left(\varphi-\frac{\pi}{4}\right)+\frac{1}{2}\bar\gamma r^2\sin 2\varphi, & \text{if } \bar\gamma\in[-1,-\frac{2}{3}]\\ & \text{and } \varphi<-\frac{1}{2}\arcsin\frac{2\bar\gamma+2}{\bar\gamma},\\ r-\bar\gamma\frac{r(1-r)}{2}\sin 2\varphi, & \text{if } \bar\gamma\in[-\frac{2}{3},0]\\ & \text{or } \varphi\geqslant-\frac{1}{2}\arcsin\frac{2\bar\gamma+2}{\bar\gamma}.\end{cases}$$

Proof. Recall that in the extremal problem (3.16) it suffices to consider $[\varrho,\vartheta,\bar\gamma]\in\tilde M$ (the northern part of M), i. e., we take $\vartheta\in[-\arctan\gamma,\frac{\pi}{2}+\arctan\gamma]$. Since the term $\frac{1}{2}\bar\gamma r^2\sin 2\varphi$ is independent of ϱ,ϑ, we look for maximum of the term $r\varrho\cos(\varphi-\vartheta)$. Again, similar to the previous lemma, we may consider $\vartheta\in[0,\frac{\pi}{4}]$ only.

Let $\bar\gamma\in[-1,0]$ be fixed and $[\varrho,\vartheta,\bar\gamma]\in M$, i.e., $H(\varrho,\vartheta,\bar\gamma)=1$. This equation determines implicitly the function $\varrho=\varrho(\vartheta)$ and in view of Lemma 3.4

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\varrho(\vartheta)\cos(\varphi-\vartheta)&=-\varrho\tan(\tilde\varphi-\vartheta)\cos(\varphi-\vartheta)+\varrho\sin(\varphi-\vartheta)\\ &=-\frac{\varrho}{\cos(\tilde\varphi-\vartheta)}\left[\sin(\tilde\varphi-\vartheta)\cos(\varphi-\vartheta)-\cos(\tilde\varphi-\vartheta)\sin(\varphi-\vartheta)\right]\\ &=\frac{\varrho}{\cos(\tilde\varphi-\vartheta)}\sin(\varphi-\tilde\varphi).\end{aligned}\tag{3.17}$$

By the previous lemma, for a fixed $\bar\gamma\in[-1,0]$ and $[\varrho,\vartheta,\bar\gamma]\in M$, $\tilde\varphi(\varrho,\vartheta,\bar\gamma)$ is an increasing function of ϑ for $\vartheta\in[-\arctan\bar\gamma,\frac{\pi}{4}]$, $\tilde\varphi(\varrho,-\arctan\bar\gamma,\bar\gamma)=0$ and according to the note below Lemma 3.5, $\tilde\varphi(\varrho,\frac{\pi}{4},\bar\gamma)=\frac{\pi}{4}$ if $\bar\gamma\in[-\frac{2}{3},0]$ and $\tilde\varphi(\varrho,\frac{\pi}{4},\bar\gamma)=-\frac{1}{2}\arctan\frac{2(\bar\gamma+1)}{\bar\gamma}$ if $\bar\gamma\in[-1,-\frac{2}{3}]$. Now, if $\bar\gamma\in[-\frac{2}{3},0]$, there exist ϱ,ϑ such that $[\varrho,\vartheta,\bar\gamma]\in\tilde M$ and $\tilde\varphi(\varrho,\vartheta,\bar\gamma)=\varphi$. For these ϱ,ϑ and $\tilde\varphi$ we have

$$\begin{aligned}&r\left[\varrho\cos(\varphi-\vartheta)+\frac{1}{2}r\bar\gamma\sin 2\varphi\right]\\ &=r\left[\varrho\cos(\tilde\varphi-\vartheta)+\frac{1}{2}\bar\gamma\sin 2\tilde\varphi+\frac{1}{2}\bar\gamma(r-1)\sin 2\varphi\right]=r\left[1-\frac{1}{2}\bar\gamma(1-r)\sin 2\varphi\right].\end{aligned}$$

If $\bar\gamma\in[-1,-\frac{2}{3})$, let us distinguish the cases $\varphi<-\frac{1}{2}\arcsin\frac{2(\bar\gamma+1)}{\bar\gamma}=\tilde\varphi(\varrho,\frac{\pi}{4},\bar\gamma)$ and $\varphi\geqslant-\frac{1}{2}\arcsin\frac{2(\bar\gamma+1)}{\bar\gamma}$. In the first case again there exist ϱ,ϑ such that $[\varrho,\vartheta,\bar\gamma]\in M$ and $\tilde\varphi(\varrho,\vartheta,\bar\gamma)=\varphi$. By the same computation as above

$$\sup_{H(\varrho,\vartheta,\bar\gamma)=1}\left[r\varrho\cos(\varphi-\vartheta)+\frac{1}{2}\bar\gamma r^2\sin 2\varphi\right]=r\left[1-\frac{1}{2}\bar\gamma(1-r)\sin 2\varphi\right].$$

Finally, if $\varphi \in [-\frac{1}{2}\arcsin\frac{2(\bar\gamma+1)}{\bar\gamma}, \frac{\pi}{4}]$ then $\varrho\cos(\varphi-\vartheta)$ attains its maximum over $[\varrho,\vartheta,\bar\gamma]$ satisfying $H(\varrho,\vartheta,\bar\gamma)=1$ at $\vartheta=\frac{\pi}{4}$ and $\varrho=\sqrt{-2\bar\gamma^2-4\bar\gamma}$ (see (3.10)). Hence

$$\begin{aligned}&\sup_{(\varrho,\vartheta,\bar\gamma)\in M}\left[r\cos(\varphi-\vartheta)+\frac{1}{2}\bar\gamma r^2\sin 2\varphi\right]\\ &= r\sqrt{-2\bar\gamma^2-4\bar\gamma}\cos\left(\varphi-\frac{\pi}{4}\right)+\frac{1}{2}\bar\gamma r^2\sin 2\varphi.\end{aligned}$$

$\square$

IV. Polyconvex envelope of the function f_∞

In this section we deal with the function $f_\infty\colon\ \mathbb{R}^{2\times 2}$ given by

$$f_\infty(A)=\begin{cases}0 & \text{if } A=0,\\ 1 & \text{if } \|A\|\leqslant 1,\\ \infty & \text{if } \|A\|>1\end{cases}$$

and we will omit the subscript ∞ if no ambiguity may arise.

Theorem 4.1.

$$Pf_\infty(A)=\begin{cases}\sqrt{(\det A)^2+2|\det A|+\|A\|^2}-|\det A| & \text{if } \|A\|\leqslant 1,\\ \infty & \text{if } \|A\|>1.\end{cases}\tag{4.1}$$

P r o o f. According to the statement at the beginning of Section II, we consider only the diagonal matrices $A=\left(\begin{smallmatrix}x&0\\0&y\end{smallmatrix}\right)$. Moreover, by the symmetry relations we prove (4.1) only for those diagonal matrices for which $0\leqslant y\leqslant x$ (since $Pf(x,y)=Pf(y,x)$). As mentioned in the previous section, the fundamental role in computing Pf is played by the extremal problem (2.8) which in the case

$$\tilde f(x,y)=\begin{cases}0 & \text{if } x^2+y^2\leqslant 1,\\ \infty & \text{if } x^2+y^2>1\end{cases}$$

reads

$$\alpha x+\beta y+\gamma xy\to\max,\quad [\alpha,\beta,\gamma]\in M.\tag{4.2}$$

Recall that $M=\{[\alpha,\beta,\gamma]\in\mathbb{R}^3\mid H(\alpha,\beta,\gamma)=1\}$. In this extremal problem, if $x,y\geqslant 0$, $x\geqslant y$, in view of (3.1) it suffices to consider $\alpha,\beta\geqslant 0$ and $\alpha\geqslant\beta$.

Solving (4.2), we pass to the polar coordinates $x = r\cos\varphi, y = r\sin\varphi$, $\alpha = \varrho\cos\vartheta$, $\beta = \varrho\sin\vartheta$. Then (4.2) becomes

$$r\varrho\cos(\varphi-\vartheta)+\frac{1}{2}\gamma r^2\sin 2\varphi \to \max, \quad H(\varrho,\vartheta,\gamma)=1. \tag{4.3}$$

Let $[x,y]=[r\cos\varphi, r\sin\varphi]$, $r\in(0,1)$, $\varphi\in[0,\frac{\pi}{4}]$ be fixed and denote by $[\tilde\varrho,\tilde\vartheta,\tilde\gamma]\in M$ the extremal point of (4.3), i.e.,

$$Pf(x,y)=Pf(r,\varphi)=r\tilde\varrho\cos(\varphi-\tilde\vartheta)+\frac{1}{2}\tilde\gamma r^2\sin 2\varphi.$$

We shall show that $\tilde\gamma\in[-1,-\frac{2}{3}]$, $\vartheta=\frac{\pi}{4}$ and $\tilde\varrho=\sqrt{-2\tilde\gamma^2-4\tilde\gamma}$.

According to the second relation in (2.4), $H(\varrho,\vartheta,\gamma)=1$ implies $|\gamma|\leqslant 2$ and for $\gamma\in[-2,-1]$, $\vartheta\in[0,\frac{\pi}{4}]$, the points $[\varrho,\vartheta,\gamma]$ lie on the southern part of M, so we consider $\gamma\in[-1,2]$ only.

Now we proceed as follows. We keep $\gamma\in[-1,2]$ constant and first we optimize the function with respect to (ϱ,ϑ). If $\gamma\in[-\frac{2}{3},0]$, by Lemma 3.6 we have

$$\begin{aligned}&\sup_{(\varrho,\vartheta,\gamma)\in M}\left[r\varrho\cos(\varphi-\vartheta)+\frac{1}{2}\gamma r^2\sin 2\varphi\right]=1+\frac{1}{2}|\gamma|(1-r)\sin 2\varphi\\ &\leqslant 1+\frac{1}{3}(1-r)\sin 2\varphi=\sup_{(\varrho,\vartheta,-\frac{2}{3})\in M}\left[r\varrho\cos(\tilde\varphi-\vartheta)+\frac{1}{2}\gamma r^2\sin 2\varphi\right],\end{aligned}$$

hence surely $\tilde\gamma\notin(-\frac{2}{3},0]$.

If $\gamma\in[0,2]$, using (3.3) with $r=1$ one may verify that $\tilde\varphi\big(\sqrt{1+\gamma^2},-\arctan\gamma,\gamma\big)=0$ and $[\sqrt{1+\gamma^2},-\arctan\gamma,\gamma]\in M$. By (3.10) we have $\tilde\varphi(1-\frac{\gamma}{2},\frac{\pi}{4},\gamma)=\frac{\pi}{4}$ and the same argument as for $\gamma\in[-\frac{2}{3},0]$ implies that

$$\sup_{(\varrho,\vartheta,\gamma)\in M}\left[r\varrho\cos(\varphi-\vartheta)+\frac{1}{2}\gamma r^2\sin 2\varphi\right]\leqslant 1+\frac{1}{3}(1-r)\sin 2\varphi,$$

i.e., $\tilde\gamma\notin[0,2]$.

If $\gamma\in[-1,-\frac{2}{3}]$ is such that

$$\varphi<\tilde\varphi\Big(\varrho,\frac{\pi}{4},\gamma\Big)=-\frac{1}{2}\arcsin\frac{2(\gamma+1)}{\gamma} \tag{4.4}$$

(see (3.16)), then again the second possibility in (3.14) holds. Since the function $-\frac{1}{2}\arcsin\frac{2(\gamma+1)}{\gamma}$ is continuous and increasing, there exists $\gamma_1<\gamma$, $\gamma_1\in[-1,-\frac{2}{3}]$, such that $\varphi<-\frac{1}{2}\arcsin\frac{2(\gamma_1+1)}{\gamma_1}$ still holds, hence

$$\sup_{(\varrho,\vartheta,\gamma)\in M}\left[r\varrho\cos(\varphi-\vartheta)+\frac{1}{2}\gamma r^2\sin 2\varphi\right]<\sup_{(\varrho,\vartheta,\gamma_1)\in M}\left[r\varrho\cos(\varphi-\vartheta)+\frac{1}{2}\gamma_1 r^2\sin 2\varphi\right],$$

i.e. in this case the maximum of (4.3) is not attained, either. Consequently, maximum in (4.3) is realized for $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\gamma}]$ such that $\varphi \geqslant -\frac{1}{2} \arctan \frac{2(\tilde{\gamma}+1)}{\tilde{\gamma}} = \tilde{\varphi}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\gamma})$. By (3.17) the function $r\varrho\cos(\varphi - \vartheta)$ is increasing with respect to ϑ for $\vartheta \in [0, \frac{\pi}{4}]$ and $(\varrho, \vartheta, \gamma) \in M$, hence if $\varphi \geqslant -\frac{1}{2} \arctan \frac{2(\tilde{\gamma}+1)}{\tilde{\gamma}}$, we have $\tilde{\vartheta} = \frac{\pi}{4}$ and by (3.10) $\tilde{\varrho} = \sqrt{-2\tilde{\gamma}^2 - 4\tilde{\gamma}}$.

This leads us to the extremal problem

$$\alpha(x+y) + \gamma xy \to \max, \quad H(\alpha, \alpha, \gamma) = 1, \ \gamma \in \Big[-1, -\frac{2}{3}\Big], \tag{4.5}$$

and by (3.10) for $\gamma \in [-1, -\frac{2}{3}]$

$$H(\alpha, \alpha, \gamma) = \alpha^2 + (\gamma+1)^2.$$

Note that we obtain the same result if we take the extremal problem

$$r\sqrt{-2\gamma^2 - 4\gamma} \cos\Big(\varphi - \frac{\pi}{4}\Big) + \frac{1}{2}\gamma r^2 \sin 2\varphi \to \max, \quad \gamma \in \Big[-1, -\frac{2}{3}\Big]$$

which is a "polar coordinate" modification of (4.5). Solving (4.5), we get the system of equations

$$\begin{aligned} x + y &= 2\lambda\alpha, \\ xy &= 2\lambda(\gamma+1), \\ \alpha^2 + (\gamma+1)^2 - 1 &= 0 \end{aligned}$$

(λ is a Lagrange multiplier) and the solution of this system is

$$\gamma = -1 \pm \frac{xy}{\sqrt{x^2y^2 + (x+y)^2}}, \quad \alpha = \pm \frac{x+y}{\sqrt{x^2y^2 + (x+y)^2}}. \tag{4.6}$$

Since for the solution of (4.3) only the northern part of M is important, we take $+$ in (4.6) and the extremal value of (4.5) is

$$\begin{aligned} h(x,y) &= \frac{(x+y)^2}{\sqrt{x^2y^2 + (x+y)^2}} + \left(-1 + \frac{xy}{\sqrt{x^2y^2 + (x+y)^2}}\right) xy \\ &= \sqrt{x^2 + y^2 + (x+y)^2} - xy. \end{aligned}$$

Consequently, we have (for $x, y \geqslant 0$, $x^2 + y^2 \leqslant 1$)

$$Pf(x,y) = \sqrt{x^2y^2 + (x+y)^2} - xy$$

and similarly we get the relations for the other three quadrants. Now, replacing $x^2 + y^2$ by $\|A\|$ and xy by $|\det A|$ as mentioned at the beginning of Section II, we get (4.1). $\square$

i) Let $r \in (0,1)$ and $\varphi \in [0, \frac{\pi}{4}]$. We have

$$Pf(r,\varphi) = r\sqrt{\frac{1}{4}r^2 \sin^2 2\varphi + 1 + \sin 2\varphi} - \frac{1}{2}r^2 \sin 2\varphi$$

and one may directly verify that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} P(r,\varphi) = 0 \qquad \Longleftrightarrow \qquad \varphi = \frac{\pi}{4}.$$

Comparing $Pf(r, \frac{\pi}{4})$ with $Pf(r,0)$ and $P(r, \frac{\pi}{2})$ we have the inequality

$$Pf\Big(r, \frac{\pi}{4}\Big) \geqslant Pf(r,\varphi), \quad \varphi \in \Big[0, \frac{\pi}{2}\Big]. \tag{5.1}$$

A similar statement may be formulated for $\varphi \in [\frac{\pi}{2}, 2\pi]$.

ii) For $\det A = 0$ we have $Pf_\infty(A) = Cf_\infty(A) = \|A\|$. It may be proved that the same equality holds also for the function $f_p\colon\ \mathbb{R}^{2\times 2} \to \mathbb{R}$, $p \geqslant 2$ given by

$$f(A) = \begin{cases} 1 + \frac{\|A\|^p}{p}, & \text{if } A \neq 0, \\ 0 & \text{if } A = 0. \end{cases} \tag{5.3}$$

This follows from the fact that for $\tilde{f}(x,y) = \frac{(x^2+y^2)^{p/2}}{p}$ the surface

$$M_p = \{[\alpha,\beta,\gamma] \mid \tilde{f}^P(\alpha,\beta,\gamma) = 1\}$$

has essentially the same shape as M.

iii) It is known that the function (1.4_2) also equals the rank one envelope of (1.3), see [4]. This implies the equality $Pf = Qf = Rf$ in this case. Concerning the function f given by (1.5) we have not been successful in computing Rf. B. Dacorogna computed that

$$R_2 f(A) = \begin{cases} y + x\sqrt{\frac{1-y}{1+y}} & \text{if } x^2 + y^2 = \|A\|^2 \leqslant 1, \\ \infty & \text{otherwise,} \end{cases}$$

where $0 \leqslant y \leqslant x$ are the singular values of A given by (2.2). Unfortunately, this function is not rank one convex (since it is not separately convex), hence $Rf < R_2 f$ (in computing $R_3 f$ we have met difficult technical problems which seem to be hard to overcome). The formula for computing $R_n f$ may be found e.g. in [4] and generally

$Rf(A) = \lim_{n\to\infty} R_n f(A)$. In contrast to our example, $Rf = R_2 f$ for the function of Kohn and Strang (1.3).

iv) The previous remarks suggest two immediate lines of extension of our results.

a) To carry out all computations for $n \times n$ matrices.

b) To investigate polyconvex envelopes of the general "Kohn-Strang-like" functions (1.6).

Concerning the first problem, this extension does not seem to be trivial, since the proofs of the fundamental Lemmas 3.5 and 3.6 are rather technical and depend substantially on dimension two. As for the second problem, we have made some preliminary computations for the function (5.3) and the results obtained suggest that a general statement concerning the polyconvex envelope of (1.6) may be formulated which naturally generalizes the results of this paper and of [4].

Some of the above problems we hope to investigate in a subsequent paper.

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