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A STUDY OF BENDING WAVES IN INFINITE AND ANISOTROPIC PLATES

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Abstract. In this paper we present a unified approach to obtain integral representation formulas for describing the propagation of bending waves in infinite plates. The general anisotropic case is included and both new and well-known formulas are obtained in special cases (e.g. the classical Boussinesq formula). The formulas we have derived have been compared with experimental data and the coincidence is very good in all cases.

Keywords: Kirchoff plate equation, bending waves, anisotropic plates, orthotropic plates, isotropic plates, the Fourier transform

MSC 2000: 35C15, 35Q72, 73K10

1. Introduction

This paper is a result of a cooperation between the departments of Experimental Mechanics and Mathematics at Luleå University of Technology. The starting point was the recent Ph.D. Theses [3] and [10] where, in particular, the classical Boussinesq formula (see our Example 4.4) was important. In order to be able achieve better understanding of the existing experimental data in more general cases we have here in particular derived suitable integral representation formulas which cover also these cases.

More precisely we have given in this paper a unified approach to obtain integral representations for describing the propagation of bending waves in infinite plates. We start by handling the most general case with an anisotropic plate with very modest restrictions on the outer force. After that we consider various special cases and get both new and well-known formulas. For example in the simplest case with an isotropic plate when the outer force is a unit impulse in both the plane and the time variable we obtain the classical Boussinesq solution. Our starting point is to use Fourier analysis and some exact formulas for the Fourier transform which are very important for our investigation.

The formulas we have derived have been compared with experimental data and the coincidence is very good in all cases, see Section 5 of this paper and also the recent paper [4], which may be seen as a complement of this paper.

This paper is organized in the following way: In Section 2 a preliminary discussion on the plate equation can be found. In Section 3 we have derived an integral representation formula for a general anisotropic plate. However, this formula contains a Fourier transform as a factor in the integrand. Fortunately, in some cases this formula can be simplified in a very useful way (see our Remark 3.5 and Statement 3.6). In Section 4 we discuss the special case with an isotropic plate. Here we can give a representation formula which does not contain any Fourier transform as a factor in the integrand. In particular, this gives us a much better understanding of the Boussinesq formula. Section 5 is reserved for the announced comparison of our results with experimental data and for some concluding remarks. Finally, in order not to disturb some of our discussions (in Sections 4 and 5) we have collected the proofs of some crucial formulas in a separate Appendix (together with some necessary theory concerning distributions and Fourier analysis).

2. A preliminary discussion of the plate equation

The general form of the standard (Kirchoff's) plate equation for an anisotropic plate is

$$\begin{aligned} D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} \\ + \varrho h \frac{\partial^2 w}{\partial t^2} = q(x,y,t), \end{aligned} \tag{1}$$

where $w = w(x, y, t)$, $(x, y) \in \mathbb{R}^2$ is the deflection of the plate and t represents the time. D_{ij} are the principal stiffnesses, h the thickness of the plate, ϱ the volume density of the plate and $q(x, y, t)$ the perpendicular load per unit area. D_{ij}, ϱ and h are supposed to be constants. For an anisotropic plate there is no elastic symmetry. In this model the resistance of the surrounding medium and the internal friction are disregarded. A more detailed description of the model can be found in [3], [6].

For later purposes it is convenient to simplify equation (1)by making the scaling transformations $x = \frac{\overline{x}}{b}$ and $y = \frac{\overline{y}}{c}$ where $\frac{D_{11}}{b^4} = \frac{D_{22}}{c^4} = D$ (D is a given fixed constant). Then the equation (1) can be written

$$\frac{\partial^4 w}{\partial x^4} + A_1 \frac{\partial^4 w}{\partial x^3 \partial y} + A_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + A_3 \frac{\partial^4 w}{\partial x \partial y^3} + \frac{\partial^4 w}{\partial y^4} + \frac{1}{a^2} \frac{\partial^2 w}{\partial t^2} = \frac{q(x,y,t)}{D}, \quad (x,y) \in \mathbb{R}^2,\ t > 0, \tag{2}$$

where $a = \sqrt{\frac{D}{\varrho h}}$ and

$$\begin{cases} A_1 = \dfrac{4D_{16}}{(D_{11})^{3/4}(D_{22})^{1/4}} \\ A_2 = \dfrac{2(D_{12} + 2D_{66})}{(D_{11}D_{22})^{1/2}} \\ A_3 = \dfrac{4D_{26}}{(D_{11})^{1/4}(D_{22})^{3/4}}. \end{cases}$$

Note that here we have dropped the bars on x and y. A formal solution of equation (2) and, thus, of (1) is derived in our statement 3.1.

We also consider the special case where $D_{16} = D_{26} = 0$, that is when we have a plate with orthogonal anisotropy or simply an orthotropic plate. This special case is of great interest in physics. See e.g. the recent Theses [3] and [10]. In this case $A_1 = A_3 = 0$ and (2) can be written as

$$\Delta^2 w + \varepsilon \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{1}{a^2} \frac{\partial^2 w}{\partial t^2} = \frac{q(x,y,t)}{D}, \qquad (x,y) \in \mathbb{R}^2,\ t > 0, \tag{3}$$

where

$$\begin{cases} \varepsilon = A_2 - 2 = 2\left(\dfrac{D_{12} + 2D_{66}}{\sqrt{D_{11}D_{22}}} - 1\right) \\ a = \sqrt{\dfrac{D}{\varrho h}} \end{cases}$$

and, as usual, $\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$ denotes the biharmonic differential operator.

3. On an integral representation of the solution of (2)

Here we consider the special case when $q(x,y,t)$ can be separated as

$$q(x,y,t) = f(t)p(x,y).$$

Moreover, $\widehat{p}$ denotes the Fourier transform of p, i.e.

$$\widehat{p}(\xi,\eta) = \iint_{\mathbb{R}^2} p(x,y)\mathrm{e}^{-\mathrm{i}(\xi x + \eta y)}\,\mathrm{d}x\,\mathrm{d}y,\ \ (\xi,\eta) \in \mathbb{R}^2.$$

3.1. A formal solution of (2)**.** The solution of the Cauchy-problem

$$(4)\qquad \begin{cases} \dfrac{\partial^4 w}{\partial x^4} + A_1 \dfrac{\partial^4 w}{\partial x^3 \partial y} + A_2 \dfrac{\partial^4 w}{\partial x^2 \partial y^2} + A_3 \dfrac{\partial^4 w}{\partial x \partial y^3} + \dfrac{\partial^4 w}{\partial y^4} + \dfrac{1}{a^2}\dfrac{\partial^2 w}{\partial t^2} \\ \qquad = \dfrac{f(t)p(x,y)}{D}, \ (x,y) \in \mathbb{R}^2, \ t > 0 \\ w\big|_{t=0} = 0, \qquad (x,y) \in \mathbb{R}^2 \\ w_t\big|_{t=0} = 0, \qquad (x,y) \in \mathbb{R}^2 \end{cases}$$

can formally be written as the integral representation

$$w(x,y,t) = \frac{a^2}{4\pi^2 D} \int_0^t f(\tau)\,\mathrm{d}\tau \iint_{\mathbb{R}^2} \widehat{p}(\xi,\eta) \frac{\sin\alpha(t-\tau)}{\alpha} \mathrm{e}^{\mathrm{i}(\xi x + \eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where $\alpha^2 = a^2\left[\xi^4 + A_1\xi^3\eta + A_2\xi^2\eta^2 + A_3\xi\eta^3 + \eta^4\right]$.

P r o o f. The Fourier transformation of (4) yields

$$(5)\qquad \begin{cases} \widehat{w}_{tt} + \alpha^2 \widehat{w} = \dfrac{a^2 f(t)\widehat{p}(\xi,\eta)}{D} \\ \widehat{w}\big|_{t=0} = 0 \\ \widehat{w}_t\,\big|_{t=0} = 0 \end{cases}$$

where

$$\widehat{w}(\xi,\eta,t) = \iint_{\mathbb{R}^2} w(x,y,t)\mathrm{e}^{-\mathrm{i}(\xi x + \eta y)}\,\mathrm{d}x\,\mathrm{d}y.$$

It is well-known and easy to see that (5) has the unit impulse ($\delta(t)$) response $\frac{\sin\alpha t}{\alpha}$. Therefore, by solving (5) by Green's method, we find that $\widehat{w}$ can be represented as a convolution:

$$(6)\qquad \widehat{w}(\xi,\eta,t) = \frac{a^2\widehat{p}(\xi,\eta)}{D} \int_0^t f(\tau)\frac{\sin\alpha(t-\tau)}{\alpha}\,\mathrm{d}\tau.$$

By using the inverse Fourier transform and (6) we find that

$$\begin{aligned} w(x,y,t) &= \left(\frac{1}{2\pi}\right)^2 \iint_{\mathbb{R}^2} \widehat{w}(\xi,\eta,t)\mathrm{e}^{\mathrm{i}(\xi x + \eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{a^2}{4\pi^2 D} \int_0^t f(\tau)\,\mathrm{d}\tau \iint_{\mathbb{R}^2} \widehat{p}(\xi,\eta) \frac{\sin\alpha(t-\tau)}{\alpha} \mathrm{e}^{\mathrm{i}(\xi x + \eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned}$$

and the proof is complete. □

3.2. Remark. From the above proof we see that the formal solution is even pointwise correct at each point where the Fourier inversion formula holds (see e.g. [15]).

3.3. Remark. In applications it is sometimes convenient to introduce the polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \text{and} \quad \begin{cases} \xi = \sigma\cos\varphi \\ \eta = \sigma\sin\varphi \end{cases}$$

and rewrite the solution of (4) as

(7)

$$w(r,\theta,t) = \frac{a}{4\pi^2 D}\int_0^t f(\tau)\,\mathrm{d}\tau \int_0^{2\pi}\int_0^{\infty} \widehat{p}(\sigma,\varphi)\frac{\sin\left[a\sigma^2(t-\tau)\cdot g(\varphi)\right]}{\sigma\cdot g(\varphi)}\mathrm{e}^{\mathrm{i}\sigma r\cos(\varphi-\theta)}\,\mathrm{d}\sigma\,\mathrm{d}\varphi,$$

where

$$g(\varphi) = \left[1 + A_1\cos^3(\varphi)\sin(\varphi) + (A_2-2)\cos^2(\varphi)\sin^2(\varphi) + A_3\cos(\varphi)\sin^3(\varphi)\right]^{1/2}.$$

3.4. Example. Let $p(x,y) = \delta(x,y)$ and $f(t) = I\delta(t)$, where δ as usual denotes the Dirac delta function. This means that the outer force $q(x,y,t)$ is a momentary load at $t=0$ which is concentrated at $(0,0)$. Then (4) has the solution

$$w(x,y,t) = \frac{Ia^2}{4\pi^2 D}\iint_{\mathbb{R}^2}\frac{\sin\alpha t}{\alpha}\mathrm{e}^{\mathrm{i}(\xi x+\eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{8}$$

Proof. We note that the Fourier transform of $\delta(x,y)$ is $\widehat{\delta} = \widehat{\delta}(\xi,\eta) \equiv 1$. Thus, by the statement 3.1,

$$\begin{aligned} w(x,y,t) &= \frac{Ia^2}{4\pi^2 D}\int_0^t \delta(\tau)\,\mathrm{d}\tau \iint_{\mathbb{R}^2}\frac{\sin\alpha(t-\tau)}{\alpha}\mathrm{e}^{\mathrm{i}(\xi x+\eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{Ia^2}{4\pi^2 D}\iint_{\mathbb{R}^2}\left(\int_0^t \delta(\tau)\frac{\sin\alpha(t-\tau)}{\alpha}\mathrm{e}^{\mathrm{i}(\xi x+\eta y)}\,\mathrm{d}\tau\right)\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{Ia^2}{4\pi^2 D}\iint_{\mathbb{R}^2}\frac{\sin\alpha t}{\alpha}\mathrm{e}^{\mathrm{i}(\xi x+\eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned}$$

□

3.5. Remark. The solution w in (8) can be written in polar coordinates:

$$w(r,\theta,t) = \frac{Ia}{4\pi^2 D}\int_0^{2\pi}\int_0^{\infty}\frac{\sin\left[a\sigma^2 t\cdot g(\varphi)\right]}{\sigma\cdot g(\varphi)}\mathrm{e}^{\mathrm{i}\sigma r\cos(\varphi-\theta)}\,\mathrm{d}\sigma\,\mathrm{d}\varphi, \tag{9}$$

with

$$g(\varphi) = \left[1 + A_1\cos^3(\varphi)\sin(\varphi) + (A_2-2)\cos^2(\varphi)\sin^2(\varphi) + A_3\cos(\varphi)\sin^3(\varphi)\right]^{1/2}.$$

In Remark A5 in Appendix we formally show that (9) can be rewritten in the form

$$w(r,\theta,t) = \frac{Ia}{8\pi D}\int_0^{2\pi}\left(\frac{1}{2} - \left[S\left(\sqrt{\frac{2q}{\pi}}\right)\right]^2 - \left[C\left(\sqrt{\frac{2q}{\pi}}\right)\right]^2\right)\frac{1}{g(\varphi)}\,\mathrm{d}\varphi, \tag{10}$$

where $q = \frac{r^2\cos^2(\varphi-\theta)}{4at\cdot g(\varphi)}$, $S(x) = \int_0^x \sin\frac{\pi u^2}{2}\,\mathrm{d}u$ and $C(x) = \int_0^x \cos\frac{\pi u^2}{2}\,\mathrm{d}u$ (the Fresnel integrals). This form (10) is much more suitable for numerical calculations, see [4].

3.6. The orthotropic case. A formal solution of the Cauchy-problem

$$\begin{cases} \Delta^2 w + \varepsilon\dfrac{\partial^4 w}{\partial x^2\partial y^2} + \dfrac{1}{a^2}\dfrac{\partial^2 w}{\partial t^2} = \dfrac{f(t)p(x,y)}{D} & (x,y)\in\mathbb{R}^2,\ t>0 \\ w\big|_{t=0} = 0, \qquad (x,y)\in\mathbb{R}^2 \\ w_t\big|_{t=0} = 0, \qquad (x,y)\in\mathbb{R}^2 \end{cases} \tag{11}$$

$\left(\varepsilon = A_2 - 2 = 2\left(\frac{D_{12}+2D_{66}}{\sqrt{D_{11}D_{22}}} - 1\right)\right.$ and $\left. a = \sqrt{\frac{D}{\varrho h}}\right)$ is

$$w(x,y,t) = \frac{a^2}{4\pi^2 D}\int_0^t f(\tau)\,\mathrm{d}\tau\iint_{\mathbb{R}^2}\widehat{p}(\xi,\eta)\frac{\sin\alpha(t-\tau)}{\alpha}\mathrm{e}^{i(\xi x+\eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where

$$\alpha^2 = a^2\left[\left(\xi^2+\eta^2\right)^2 + \varepsilon\xi^2\eta^2\right].$$

Proof. Apply the statement 3.1 with $A_1 = A_3 = 0$. □

3.7. Remark. The solution w for the orthotropic case can be written in polar coordinates:

(12)

$$w(r,\theta,t) = \frac{Ia}{8\pi D}\int_0^{2\pi}\left(\frac{1}{2} - \left[S\left(\sqrt{\frac{2q}{\pi}}\right)\right]^2 - \left[C\left(\sqrt{\frac{2q}{\pi}}\right)\right]^2\right)\frac{1}{\sqrt{1+\varepsilon\left(\frac{\sin 2\varphi}{2}\right)^2}}\,\mathrm{d}\varphi,$$

with $q = \frac{r^2\cos^2(\varphi-\theta)}{4at\cdot\sqrt{1+\varepsilon\left(\frac{\sin 2\varphi}{2}\right)^2}}$.

4. The isotropic case ($\varepsilon = 0$)

In this important special case (see e.g. Theses [3] and [10]) it is even possible to obtain an integral representation of the solution of (11) which only contains $p(u,v)$ (instead of its Fourier transform). One key argument in our investigation is to use some exact formulas for the Fourier transform. These formulas are most easily understood and derived in the context of distributions, but in order not to disturb our discussion here we present the details and proofs in a separate Appendix. We are now ready to formulate our main result in this section.

4.1. A formal solution of the isotropic case. The solution of the Cauchy-problem

$$\begin{cases} \Delta^2 w + \dfrac{1}{a^2}\dfrac{\partial^2 w}{\partial t^2} = \dfrac{f(t)p(x,y)}{D} & (x,y) \in \mathbb{R}^2,\ t > 0 \\ w\big|_{t=0} = 0, & (x,y) \in \mathbb{R}^2 \\ w_t\big|_{t=0} = 0, & (x,y) \in \mathbb{R}^2 \end{cases} \tag{13}$$

can formally be written as

$$w(x,y,t) = \frac{a}{4\pi D}\int_0^t \frac{F(\tau)}{t-\tau}\,\mathrm{d}\tau \iint_{\mathbb{R}^2} p(u,v)\sin\frac{(x-u)^2+(y-v)^2}{4a(t-\tau)}\,\mathrm{d}u\,\mathrm{d}v, \tag{14}$$

where $F(t) = \int_0^t f(u)\,\mathrm{d}u$.

Proof. Integration by parts gives

$$\int_0^t \frac{\sin\alpha\,(t-\tau)}{\alpha} f(\tau)\,\mathrm{d}\tau = \int_0^t \cos\alpha\,(t-\tau)\,F(\tau)\,\mathrm{d}\tau.$$

Therefore, by using statement 3.6 with $\varepsilon = 0$, we find that

$$w(x,y,t) = \frac{a^2}{4\pi^2 D}\int_0^t F(\tau)\,\mathrm{d}\tau \iint_{\mathbb{R}^2} \widehat{p}(\xi,\eta)\cos\alpha(t-\tau)\mathrm{e}^{\mathrm{i}(\xi x+\eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{15}$$

where $\alpha = a(\xi^2+\eta^2)$. Our key observation now is that

$$\iint_{\mathbb{R}^2} \sin k(u^2+v^2)\mathrm{e}^{-\mathrm{i}(\xi u+\eta v)}\,\mathrm{d}u\,\mathrm{d}v = \frac{\pi}{k}\cos\frac{\xi^2+\eta^2}{4k}$$

for every $k > 0$ and all $(\xi, \eta) \in \mathbb{R}^2$, see Lemma A2 in Appendix. By using this formula with $k = \frac{1}{4a(t-\tau)}$ we find that if $g(u,v) = \frac{1}{4a(t-\tau)\pi} \sin \frac{u^2+v^2}{4a(t-\tau)}$ then

$$\widehat{g}(\xi, \eta) = \iint\limits_{\mathbb{R}^2} \frac{1}{4a(t-\tau)\pi} \sin \frac{u^2+v^2}{4a(t-\tau)} \, \mathrm{e}^{-\mathrm{i}(\xi u + \eta v)} \, \mathrm{d}u \, \mathrm{d}v = \cos \alpha(t-\tau).$$

Moreover, according to the convolution theorem,

$$\frac{1}{4\pi^2} \iint\limits_{\mathbb{R}^2} \widehat{p}(\xi, \eta) \cos \alpha(t-\tau) \mathrm{e}^{\mathrm{i}(\xi x + \eta y)} \, \mathrm{d}\xi \, \mathrm{d}\eta = \mathscr{F}^{-1} \{\widehat{p}(\xi, \eta) \cos \alpha(t-\tau)\}$$

$$= \mathscr{F}^{-1} \{\widehat{p}(\xi, \eta) \widehat{g}(\xi, \eta)\} = \frac{1}{4a(t-\tau)\pi} \iint\limits_{\mathbb{R}^2} p(u,v) \sin \frac{(x-u)^2 + (y-v)^2}{4a(t-\tau)} \, \mathrm{d}u \, \mathrm{d}v.$$

($\mathscr{F}^{-1}$ denotes the Fourier inverse transform). We insert this expression into (15) and obtain (14). The proof is complete. □

4.2. Example. If a point load is applied to an isotropic (infinite) plate at $(0,0)$ i.e. if $p(u,v) = \delta(u,v)$, then, according to the statement 4.1, the deflection w is equal to

$$w(x,y,t) = \frac{a}{4\pi D} \int_0^t \frac{F(\tau)}{t-\tau} \sin \frac{x^2+y^2}{4a(t-\tau)} \, \mathrm{d}\tau. \tag{16}$$

We can also prove the following useful result:

4.3. A generalized Boussinesq formula. The solution of the Cauchy-problem

$$\begin{cases} \Delta^2 w + \dfrac{1}{a^2} \dfrac{\partial^2 w}{\partial t^2} = \dfrac{f(t)\delta(x,y)}{D} & (x,y) \in \mathbb{R}^2, \ t > 0 \\ w\big|_{t=0} = 0, & (x,y) \in \mathbb{R}^2 \\ w_t\big|_{t=0} = 0, & (x,y) \in \mathbb{R}^2 \end{cases} \tag{17}$$

is

$$w(x,y,t) = \frac{a}{4\pi D} \left(\int_0^t \left(\frac{\pi}{2} - \mathrm{Si} \left(\frac{r^2}{4a(t-\tau)} \right) \right) f(\tau) \, \mathrm{d}\tau \right),$$

where $\mathrm{Si}(s) = \int_0^s \frac{\sin u}{u} \, \mathrm{d}u$ and $r = \sqrt{x^2+y^2}$.

P r o o f. From Example 4.2 we know that

$$w(x,y,t)=\frac{a}{4\pi D}\int_0^t \frac{F(\tau)}{t-\tau}\sin\frac{x^2+y^2}{4a(t-\tau)}\,\mathrm{d}\tau. \tag{18}$$

Now we put $s=\frac{x^2+y^2}{4a(t-\tau)}$ in (18) and find that

$$\begin{aligned}
w(x,y,t)&=\frac{a}{4\pi D}\int_{\frac{x^2+y^2}{4at}}^{\infty} F\Bigl(t-\frac{x^2+y^2}{4as}\Bigr)\frac{\sin s}{s}\,\mathrm{d}s=\bigl[r^2=x^2+y^2\bigr]\\
&=\frac{a}{4\pi D}\int_{\frac{r^2}{4at}}^{\infty} F\Bigl(t-\frac{r^2}{4as}\Bigr)\frac{\sin s}{s}\,\mathrm{d}s=[\text{Integration by parts}]\\
&=\frac{a}{4\pi D}\biggl(F(t)\frac{\pi}{2}-\int_{\frac{r^2}{4at}}^{\infty} f\Bigl(t-\frac{r^2}{4as}\Bigr)\frac{r^2}{4as^2}\,\mathrm{Si}(s)\,\mathrm{d}s\biggr)=\biggl[\tau=t-\frac{r^2}{4as}\biggr]\\
&=\frac{a}{4\pi D}\left(\frac{\pi}{2}\int_0^t f(\tau)\,\mathrm{d}\tau-\int_0^t f(\tau)\,\mathrm{Si}\Bigl(\frac{r^2}{4a(t-\tau)}\Bigr)\,\mathrm{d}\tau\right)\\
&=\frac{a}{4\pi D}\left(\int_0^t\Bigl(\frac{\pi}{2}-\mathrm{Si}\Bigl(\frac{r^2}{4a(t-\tau)}\Bigr)\Bigr)f(\tau)\,\mathrm{d}\tau\right).
\end{aligned}$$

The proof is complete. □

4.4. E x a m p l e. Let $f(\tau)=I\,\delta(\tau)$ in the statement 4.3. Then we obtain the classical Boussinesq solution

$$w(x,y,t)=\frac{aI}{4\pi D}\left(\frac{\pi}{2}-\mathrm{Si}\Bigl(\frac{r^2}{4at}\Bigr)\right). \tag{19}$$

We remark that this formula has been rediscovered and/or applied in many connections, see e.g. [1], [3], [8], [10] and [13].

5. Numerical calculations, experimental results and further remarks

First we present some examples. The numerical experiments agree very well with experimental results (see Example 5.3).

5.1. Example. (The isotropic case.)

In Figure 1 the deflection $w(r,\theta,t)$ is illustrated for $a = 765\,\mathrm{ms}^{-2}$, $t = 40\,\mu\mathrm{s}$. We have used both our formula (12, normalized)

$$w(r,\theta,t) = \frac{\frac{Ia}{8\pi D}\int_0^{2\pi}\left(\frac{1}{2} - \left[S\left(\sqrt{\frac{2q}{\pi}}\right)\right]^2 - \left[C\left(\sqrt{\frac{2q}{\pi}}\right)\right]^2\right)\frac{1}{\sqrt{1+\varepsilon\left(\frac{\sin 2\varphi}{2}\right)^2}}\,\mathrm{d}\varphi}{\frac{Ia}{8\pi D}\int_0^{2\pi}\frac{1}{\sqrt{1+\varepsilon\left(\frac{\sin 2\varphi}{2}\right)^2}}\,\mathrm{d}\varphi} \tag{20}$$

with $\varepsilon = 0$, $q = \frac{r^2\cos^2(\varphi-\theta)}{4at\sqrt{1+\varepsilon\left(\frac{\sin 2\varphi}{2}\right)^2}}$, $\theta = 0$ (θ can be any number) and the Boussinesq formula (19, normalized)

$$w(r,t) = \frac{\frac{aI}{4\pi D}\left(\frac{\pi}{2} - \mathrm{Si}\left(\frac{r^2}{4at}\right)\right)}{\frac{aI}{4\pi D}\frac{\pi}{2}} = 1 - \frac{2}{\pi}\,\mathrm{Si}\left(\frac{r^2}{4at}\right)$$

and, as expected, the results are identical.

5.2. Example. (An orthotropic case.)

In Figure 2 the deflection $w(r,\theta,t)$ in formula (20) is illustrated for $a = 765\,\mathrm{ms}^{-2}$, $t = 40\mu s$, $\varepsilon = -0.8$, $\theta = 0$ resp. $\theta = \frac{\pi}{4}$. In this case we of course get different curves for different values of θ.

5.3. Example. (An anisotropic case.)

Figure 3 shows a comparison between the theoretical formula (10) and experimental results. The plate in this case is a composite anisotropic plate with a thickness of 4.1 mm. The size is $0.298 \times 0.192\,\mathrm{m}$. Figures 3a and 3b show respectively a 3-D map and a contour map 78 μs after the impact start. Figure 3c shows the interferogram of the same plate after 78 μs after the impact start. The bending waves are created with a laserpulse as short as 25 ns. The focused spot is about 0.4 mm in diameter. That is, we can approximate the "impact" with a Dirac delta function in time and space. The stiffnesses for the plate are $D_{11} = 171\,\mathrm{Nm}$, $D_{22} = 76\,\mathrm{Nm}$, $D_{12} + 2D_{66} = 127\,\mathrm{Nm}$, $D_{16} = -16.9\,\mathrm{Nm}$ and $D_{26} = -5.0\,\mathrm{Nm}$. Furthermore, the scaling factors are $b = 1$ and $c = \left(\frac{D_{22}}{D_{11}}\right)^{1/4}$ and the density $\varrho = 1957\,\mathrm{kg/m}^3$.

A comparison between Figure 3b and 3c shows that the agreement is very good.

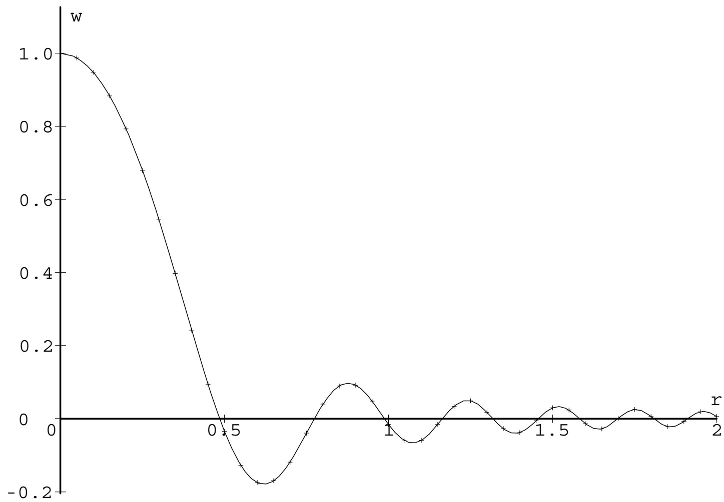

Figure 1

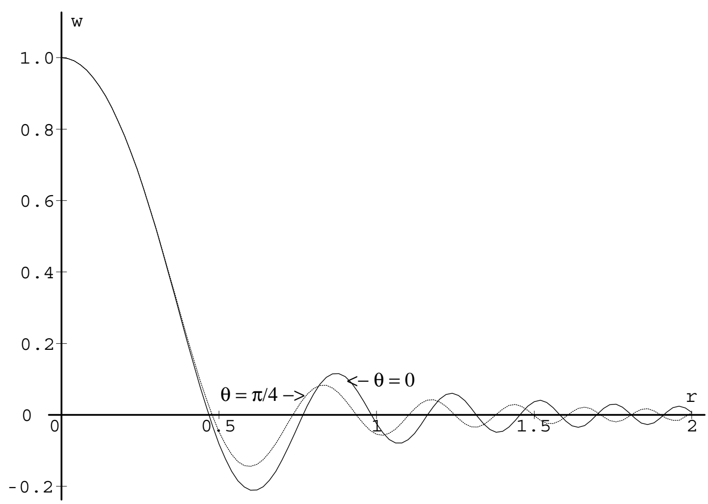

Figure 2

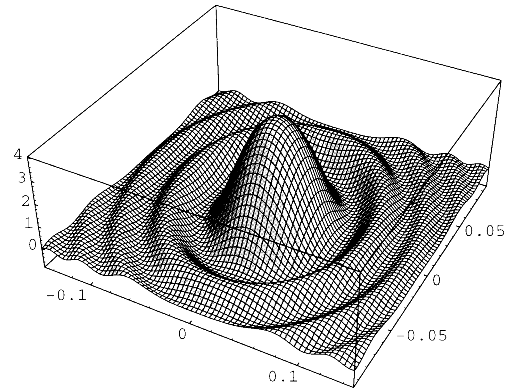

Figure 3a

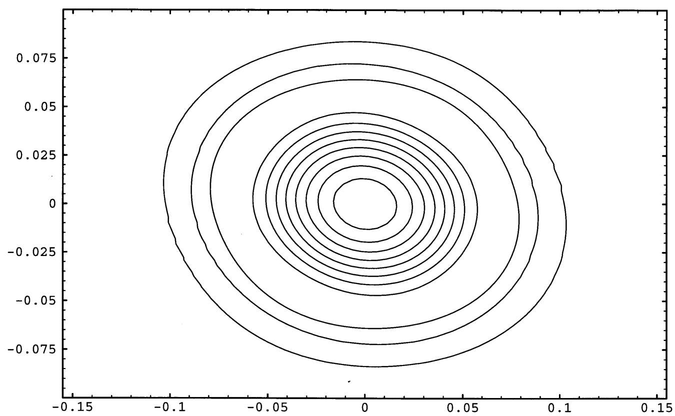

Figure 3b

5.4. R e m a r k . The Boussinesq formula (19) obviously has the following properties:

$$\text{(a)}\ \lim_{r\to\infty} w(r,t) = 0$$

$$\text{(b)}\ \lim_{t\to 0} w(r,t) = 0$$

$$\text{(c)}\ \lim_{t\to\infty} w(r,t) = \frac{aI}{8D}$$

$$\text{(d)}\ w(0,t) = \frac{aI}{8D}$$

$$\text{(e)}\ w_t(r,t) = \frac{aI\sin(\frac{r^2}{4at})}{4\pi Dt},\ r^2 = x^2 + y^2,\ t > 0.$$

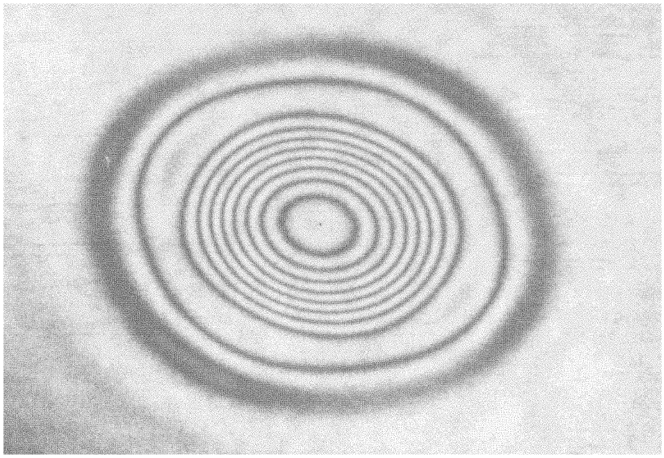

Figure 3c

The results (a)–(c) are expected for physical reasons. In (d) we ought to get $w(0,t) = 0$ but, on the other hand, in our derivation of the Boussinesq formula we have e.g. used the inversion formula so we cannot expect to get agreement at *each* point.

Another formulation of the Cauchy problem of the isotropic case is

$$\tag{21} \begin{cases} \Delta^2 w + \dfrac{1}{a^2}\dfrac{\partial^2 w}{\partial t^2} = 0 & (x,y) \in \mathbb{R}^2,\ t > 0 \\ w\big|_{t=0} = 0 & (x,y) \in \mathbb{R}^2 \\ w_t\big|_{t=0} = \dfrac{a^2 I}{D}\delta(x,y) & (x,y) \in \mathbb{R}^2 \end{cases}$$

where $a = \sqrt{\frac{D}{\varrho h}}$. This is consistent with the law of impulse. In fact we have (mass $\times$ change of velocity):

$$\iint\limits_{\mathbb{R}^2} \varrho h(w_t\big|_{t=0} - 0)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\mathbb{R}^2} \varrho h \frac{a^2 I}{D}\delta(x,y)\,\mathrm{d}x\,\mathrm{d}y = I \iint\limits_{\mathbb{R}^2} \delta(x,y)\,\mathrm{d}x\,\mathrm{d}y = I.$$

On the other hand, if an impulse is given to the plate by the force $F(t)$ and the time of impact is τ we have

$$\int\limits_{\tau} F(t)\,\mathrm{d}t = \int\limits_{\tau}\left(\iint\limits_{\mathbb{R}^2} I\delta(t)\delta(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}t = \int\limits_{\tau} I\delta(t)\,\mathrm{d}t = I.$$

Moreover, it is easy (with Maple or Mathematica) to verify that

$$w(x,y,t) = \frac{aI}{4\pi D}\left(\frac{\pi}{2} - \mathrm{Si}\left(\frac{r^2}{4at}\right)\right)$$

satisfies

$$\begin{aligned}
&\Delta^2 w + \frac{1}{a^2}\frac{\partial^2 w}{\partial t^2} = 0,\ 0 < t,\ 0 < r,\\
&w\big|_{t=0} = 0 \text{ and}\\
&w_t = \frac{aI\sin(\frac{r^2}{4at})}{4\pi Dt}.
\end{aligned}$$

In fact we have now proved that (18) is a solution of (21) and (17) with $f(t) = \delta(t)$.

In addition, according to Lemma A4 in Appendix, we have that

$$\frac{aI\sin(\frac{r^2}{4at})}{4\pi Dt} \to \frac{a^2 I}{D}\delta(x,y) \text{ as } t \to 0,$$

and we have just proved that the Boussinesq solution (19) is a solution of (21) and, thus, of (17) with $f(t) = I\delta(t)$. Therefore also the formula (e) is reasonable and the mathematical modelling seems to be very consistent.

5.5. Remark. In this paper we have considered an infinite plate. For some new results concerning the case with finite plates (more exactly, concerning the biharmonic equation $\Delta^2 u = 0$) we refer to the recent papers [11] and [2] and the references given there. The results presented in this paper cannot be derived by using the methods developed in [11] and [2].

A. Appendix

Here we let $\mathbb{S}$ denote *the space of testfunctions* in $\mathbb{R}^2$, i.e. the space of all infinite differentiable functions φ in $\mathbb{R}^2$ such that

$$\sup_{(x,y)\in\mathbb{R}^2}\left| x^{\alpha_1} y^{\alpha_2} \frac{\partial^{\beta_1+\beta_2}\varphi}{\partial^{\beta_1}x\partial^{\beta_2}y}\right| < \infty$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}_+$. Moreover, the collection $\mathbb{S}'$ of all continuous linear functionals on $\mathbb{S}$ is called *the space of temperated distributions.*

A1. Example. Let $f \in L^p(\mathbb{R}^2), 1 \leqslant p < \infty$, i.e. let

$$\|f\|_{L^p} = \left(\iint_{\mathbb{R}^2} |f(x,y)|\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} < \infty.$$

Then $L = L_f$, defined by

$$L(\varphi) = L_f(\varphi) = \iint_{\mathbb{R}^2} f(x,y)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

is a corresponding ("unique") temperated distribution. Sometimes $L(\varphi)$ is written as $L(\varphi) = \langle L, \varphi \rangle$ (this way of writing is equipped with the natural duality between the spaces $\mathbb{S}$ and $\mathbb{S}'$).

A2. Lemma. *Let* $g(x,y) = \sin k(x^2+y^2)$, $k > 0$. *Then*

$$\widehat{g}(\xi,\eta) = \frac{\pi}{k}\cos\frac{\xi^2+\eta^2}{4k} \quad \textit{in } S'.$$

We have not found a proof of this formula in literature so for the reader's convenience we include such a proof.

Proof of Lemma A2. First of all we will prove that if $f(x,y) = \mathrm{e}^{\mathrm{i}(x^2+y^2)}$, then

$$\widehat{f}(\xi,\eta) = \mathrm{i}\pi\mathrm{e}^{-\mathrm{i}\frac{\xi^2+\eta^2}{4}} \quad \text{in } \mathbb{S}'. \tag{22}$$

In order to prove this fact we first note that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 2\mathrm{i}xf, \ \frac{\mathrm{d}f}{\mathrm{d}y} = 2\mathrm{i}yf \text{ and } f,\ xf,\ yf \in \mathbb{S}'.$$

For the Fourier transform $\widehat{f}$ of f we have

$$\widehat{\frac{\mathrm{d}f}{\mathrm{d}x}} = \mathrm{i}\xi\widehat{f} \quad \text{and} \quad \widehat{\frac{\mathrm{d}f}{\mathrm{d}y}} = \mathrm{i}\eta\widehat{f}$$

$$\widehat{xf} = \mathrm{i}\frac{\mathrm{d}\widehat{f}}{\mathrm{d}\xi} \quad \text{and} \quad \widehat{yf} = \mathrm{i}\frac{\mathrm{d}\widehat{f}}{\mathrm{d}\eta}$$

Hence $-2\frac{\mathrm{d}\widehat{f}}{\mathrm{d}\xi} = \mathrm{i}\xi\widehat{f}$, which has the solution $\widehat{f}(\xi,\eta) = C_1(\eta)\mathrm{e}^{-\mathrm{i}\frac{\xi^2}{4}}$. In the same way we find that $\widehat{f}(\xi,\eta) = C_2(\xi)\mathrm{e}^{-\mathrm{i}\frac{\eta^2}{4}}$ and therefore $C_1(\eta)\mathrm{e}^{-\mathrm{i}\frac{\xi^2}{4}} = C_2(\xi)\mathrm{e}^{-\mathrm{i}\frac{\eta^2}{4}}$. Thus $C_1(\eta)\mathrm{e}^{\mathrm{i}\frac{\eta^2}{4}} = C_2(\xi)\mathrm{e}^{\mathrm{i}\frac{\xi^2}{4}} = C$ and it follows that $\widehat{f}(\xi,\eta) = C\mathrm{e}^{-\mathrm{i}\frac{\xi^2+\eta^2}{4}}$. To determine the constant C take as a testfunction $\varphi(x,y) = \frac{1}{4\pi}\mathrm{e}^{-\frac{x^2+y^2}{4}}$ which has the Fourier transform $\widehat{\varphi}(\xi,\eta) = \mathrm{e}^{-(\xi^2+\eta^2)}$. Since, by definition, $\langle\widehat{f},\varphi\rangle = \langle f,\widehat{\varphi}\rangle$ we must have

$$\frac{C}{4\pi}\iint_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\frac{\xi^2+\eta^2}{4}}\mathrm{e}^{-\frac{\xi^2+\eta^2}{4}}\,\mathrm{d}\xi\,\mathrm{d}\eta = \iint_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}(x^2+y^2)}\mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y.$$

Both integrals are absolute convergent. The integral on the left hand side is equal to

$$\iint\limits_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\frac{\xi^2+\eta^2}{4}} \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} \,\mathrm{d}\xi \,\mathrm{d}\eta = \iint\limits_{\mathbb{R}^2} \mathrm{e}^{-\xi^2 \frac{1+\mathrm{i}}{4}} \mathrm{e}^{-\eta^2 \frac{1+\mathrm{i}}{4}} \,\mathrm{d}\xi \,\mathrm{d}\eta$$

$$= \Bigg(\int\limits_{-\infty}^{\infty} \mathrm{e}^{-\xi^2 \frac{1+\mathrm{i}}{4}} \,\mathrm{d}\xi \Bigg)^2 = \frac{4\pi}{1+\mathrm{i}}$$

(see [16, p. 147]) and similarly the integral on the right hand side is equal to

$$\iint\limits_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}(x^2+y^2)} \mathrm{e}^{-(x^2+y^2)} \,\mathrm{d}x \,\mathrm{d}y = \frac{\pi}{1-\mathrm{i}}.$$

Thus $\frac{C}{4\pi}\frac{4\pi}{1+\mathrm{i}} = \frac{\pi}{1-\mathrm{i}}$ so that $C = \mathrm{i}\pi$ which means that (22) is proved. Now by using (22) and the following property of the Fourier transform:

$$\mathscr{F}\{f(ax, by)\} = \frac{1}{|ab|}\widehat{f}\Big(\frac{\xi}{a}, \frac{\eta}{b}\Big)$$

with $a = b = \sqrt{k}$ and $f(x,y) = \mathrm{e}^{\mathrm{i}(x^2+y^2)}$, we find that if $h(x,y) = \mathrm{e}^{\mathrm{i}k(x^2+y^2)}$, then

$$\widehat{h}(\xi,\eta) = \frac{\mathrm{i}\pi}{k}\mathrm{e}^{-\mathrm{i}\frac{\xi^2+\eta^2}{4k}} \quad \text{in } \mathbb{S}'.$$

Therefore, by comparing the imaginary parts in this relation, we obtain that

$$\widehat{g}(\xi,\eta) = \mathscr{F}\widehat{h}(\xi,\eta) = \frac{\pi}{k}\cos\frac{\xi^2+\eta^2}{4k} \quad \text{in } \mathbb{S}'$$

and the proof is complete. □

A3. Remark. As a reward from the above proof we obtain also that if $g(x,y) = \cos k\left(x^2+y^2\right)$, $k > 0$, then

$$\widehat{g}(\xi,\eta) = \frac{\pi}{k}\sin\frac{\xi^2+\eta^2}{4k} \quad \text{in } \mathbb{S}'.$$

In our discussion we have also used the following fact of independent interest:

A4. Lemma. *Consider* $f_k(x,y) = \frac{k}{\pi}\sin k(x^2+y^2)$, $k > 0$. *Then*

$$f_k(x,y) \to \delta(x,y) \ \textit{in } \mathbb{S}' \textit{ as } k \to \infty.$$

P r o o f. The key is to use the property that if

$$f_k \to f \text{ in } \mathbb{S}', \text{ then } \widehat{f_k} \to \widehat{f} \text{ in } \mathbb{S}' \text{ as } k \to \infty, \tag{23}$$

and the fact that, for all $(x,y) \in \mathbb{R}^2$,

$$\cos\frac{x^2+y^2}{4k} \to 1 \text{ in } \mathbb{S}' \text{ as } k \to \infty. \tag{24}$$

The proof of (24) follows at once by using the dominated convergence theorem and the obvious fact that

$$\cos\frac{x^2+y^2}{4k} \to 1 \text{ when } k \to \infty \text{ pointwise.}$$

Therefore, according to Lemma A2,

$$\mathscr{F}\left\{\frac{k}{\pi}\sin k(x^2+y^2)\right\} = \cos\frac{\xi^2+\eta^2}{4k} \to 1 \text{ in } \mathbb{S}' \text{ as } k \to \infty.$$

Moreover, since

$$\widehat{1} \equiv (2\pi)^2\delta(\xi,\eta)$$

we conclude, by using the inverse Fourier transform, and (24)

$$(2\pi)^2\frac{k}{\pi}\sin k(\xi^2+\eta^2) \to (2\pi)^2\delta(\xi,\eta) \text{ in } \mathbb{S}' \text{ as } k \to \infty,$$

and the proof follows. □

A5. A derivation of formula (10) **in Remark 3.6.** Partial integration yields that

$$I_1 =: \int_0^\infty \frac{\sin\lambda x^2\cos\omega x}{x}\,\mathrm{d}x = -\frac{1}{\omega}\int_0^\infty \sin\omega x\left(2\lambda\cos\lambda x^2 - \frac{\sin\lambda x^2}{x^2}\right)\mathrm{d}x,\ \lambda > 0, \omega > 0.$$

Moreover, according to [5, p. 395] we have

$$\begin{aligned} I_2 &= -\frac{2\lambda}{\omega}\int_0^\infty \sin\omega x\cos\lambda x^2\,\mathrm{d}x \\ &= -\frac{2\lambda}{\omega}\sqrt{\frac{\pi}{2\lambda}}\left[\sin\left(\frac{\omega^2}{4\lambda}\right)C\left(\frac{\omega}{2\sqrt{\lambda}}\sqrt{\frac{2}{\pi}}\right) - \cos\left(\frac{\omega^2}{4\lambda}\right)S\left(\frac{\omega}{2\sqrt{\lambda}}\sqrt{\frac{2}{\pi}}\right)\right] \\ &= -\frac{1}{\sqrt{q}}\sqrt{\frac{\pi}{2}}\left[\sin(q)C\left(\sqrt{\frac{2q}{\pi}}\right) - \cos(q)S\left(\sqrt{\frac{2q}{\pi}}\right)\right] \end{aligned}$$

where $q = \frac{\omega^2}{4\lambda}$, $S(x) = \int\limits_0^x \sin \frac{\pi u^2}{2}\,\mathrm{d}u$ and $C(x) = \int\limits_0^x \cos \frac{\pi u^2}{2}\,\mathrm{d}u$.

Let

$$I_3 = \frac{1}{\omega}\int\limits_0^\infty \sin \omega x \frac{\sin \lambda x^2}{x^2}\,\mathrm{d}x. \tag{25}$$

I_3 is dominated convergent and therefore we have

$$\begin{aligned}\frac{\mathrm{d}I_3}{\mathrm{d}\lambda} &= \frac{1}{\omega}\int\limits_0^\infty \sin \omega x \cos \lambda x^2\,\mathrm{d}x = \{\text{see e.g. [5]}\}\\ &= \frac{1}{\omega}\sqrt{\frac{\pi}{2\lambda}}\left[\sin\left(\frac{\omega^2}{4\lambda}\right) C\left(\frac{\omega}{2\sqrt{\lambda}}\sqrt{\frac{2}{\pi}}\right) - \cos\left(\frac{\omega^2}{4\lambda}\right) S\left(\frac{\omega}{2\sqrt{\lambda}}\sqrt{\frac{2}{\pi}}\right)\right].\end{aligned}$$

Next we put $q = \frac{\omega^2}{4\lambda}$ and note that

$$\begin{aligned}\frac{\mathrm{d}I_3}{\mathrm{d}q} &= \frac{\mathrm{d}I_3}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}q}\\ &= \frac{1}{\omega}\sqrt{\frac{\pi}{2\lambda}}\left[\sin\left(\frac{\omega^2}{4\lambda}\right) C\left(\frac{\omega}{2\sqrt{\lambda}}\sqrt{\frac{2}{\pi}}\right) - \cos\left(\frac{\omega^2}{4\lambda}\right) S\left(\frac{\omega}{2\sqrt{\lambda}}\sqrt{\frac{2}{\pi}}\right)\right]\left(-\frac{\omega^2}{4q^2}\right)\\ &= -\frac{1}{2q^{3/2}}\sqrt{\frac{\pi}{2}}\left[\sin(q) C\left(\sqrt{\frac{2q}{\pi}}\right) - \cos(q) S\left(\sqrt{\frac{2q}{\pi}}\right)\right].\end{aligned}$$

Thus

$$\begin{aligned}I_3(q) &= \int\limits_0^q -\frac{1}{2s^{3/2}}\sqrt{\frac{\pi}{2}}\left[\sin(s) C\left(\sqrt{\frac{2s}{\pi}}\right) - \cos(s) S\left(\sqrt{\frac{2s}{\pi}}\right)\right]\mathrm{d}s + C_0\\ &= -\frac{\pi}{2}\left[S^2\left(\sqrt{\frac{2q}{\pi}}\right) + C^2\left(\sqrt{\frac{2q}{\pi}}\right)\right]\\ &\quad + \frac{1}{\sqrt{q}}\sqrt{\frac{\pi}{2}}\left[\sin(q) C\left(\sqrt{\frac{2q}{\pi}}\right) - \cos(q) S\left(\sqrt{\frac{2q}{\pi}}\right)\right] + C_0.\end{aligned} \tag{26}$$

To determine the constant C_0 we let $q \to \infty$ ($\Leftrightarrow \lambda \to 0+$). According to (26) and (25) we have

$$0 = -\frac{\pi}{2}\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] + C_0 \Longleftrightarrow C_0 = \frac{\pi}{4}.$$

Now $I_1 = I_2 + I_3$, which yields

$$I_1 = \frac{\pi}{2}\left[\frac{1}{2} - S^2\left(\sqrt{\frac{2q}{\pi}}\right) - C^2\left(\sqrt{\frac{2q}{\pi}}\right)\right]. \tag{27}$$

This formula also holds for $\omega \leqslant 0$, since $I_1(-\omega) = I_1(\omega)$ and $\lim_{\omega\to 0} I_1 = \frac{\pi}{4} = \int_0^\infty \frac{\sin\lambda x^2}{x}\,dx$. In order to prove (10) we only have to consider the real part of the integral on the right hand side of (9), that is

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^\infty \frac{\sin\left[a\sigma^2 t\cdot g(\varphi)\right]}{\sigma g(\varphi)}\cos(\sigma r\cos(\varphi-\theta))\,\mathrm{d}\sigma\,\mathrm{d}\varphi \\ &= \int_0^{2\pi}\frac{1}{g(\varphi)}J\,\mathrm{d}\varphi, \end{aligned}$$

where

$$J = \int_0^\infty \frac{\sin\left[\lambda\sigma^2\right]}{\sigma}\cos(\omega\sigma)\,\mathrm{d}\sigma, \text{ with}$$
$$\lambda = at\cdot g(\varphi) \text{ and } \omega = r\cos(\varphi-\theta).$$

According to (27) we have

$$I = \frac{\pi}{2}\int_0^{2\pi}\frac{1}{g(\varphi)}\left[\frac{1}{2} - S^2\left(\sqrt{\frac{2q}{\pi}}\right) - C^2\left(\sqrt{\frac{2q}{\pi}}\right)\right]\mathrm{d}\varphi,$$

and the proof is complete.

References

[1] *A. Das, K. Roy*: Note on the forced vibration of an orthotropic plate on an elastic foundation. Journal of Sound and Vibration *66(4)* (1979), 521–525.

[2] *M. Engliš, J. Peetre*: A Green's function for the annulus. Research report 95/7002. Department of Mathematics, Lund University (ISRN LUTFD2/TFMA) (53 pages), submitted (1995).

[3] *K.-E. Fällström*: Material parameters and defects in anisotropic plates determined by holografic Interferometry. Ph.D. Thesis 1990: 86 D, Department of Physics, Luleå University of Technology, 1990.

[4] *K.-E. Fällström, O. Lindblom*: Material parameters in anisotropic plates determined by using transient bending waves. Research report. Luleå University of Technology (10 pages), submitted 1996.

[5] *I.S. Gradshteyn, I.M. Ryzhik*: Table of Integrals, Series and Products. Academic Press, 1965.

[6] *S.G. Lekhnitskii*: Anisotropic Plates, translated by S.W. Tsai and T. Cheron. Gordon and Breach Science Publishers, 1944.

[7] *J. Malmquist, V. Stenström, S. Danielsson*: Mathematical Analysis, III. Almqvist & Wiksell, 1954, pp. 584–594. (In Swedish.)

[8] *L. Meirovitch*: Analytical Methods in Vibration. The MacMillan Company, New York, 1967.

[9] *S.V. More*: The symmetrical free vibrations of a thin elastic plate with initial conditions as a generalized function. Indian J. Pure Appl. Math. *10(4)* (1979), 431–436.

[10] *Kenneth Olofsson*: Pulsed holographic interferometry for the study of bending wave propagation in paper and in tubes. Ph.D. Thesis 1994:144 D, Department of Physics, Luleå University of Technology, 1994.

[11] *L.-E. Persson, T. Strömberg*: Green's method applied to the plate equation in mechanics. Comment. Math. Prace Mat *33* (1993), 119–133.

[12] *I.N. Sneddon*: The Use of Integral Transforms. McGraw-Hill Company, Inc., New York, 1972.

[13] *I.N. Sneddon*: The Symmetrical Vibrations of a Thin Elastic Plate. Proceedings of the Cambridge Philosophical Society, Volume 41. Cambridge University Press, 27–42, 1945.

[14] *I.N. Sneddon*: The Fourier Transform Solution of an Elastic Wave Equation. Proceedings of the Cambridge Philosophical Society, Volume 41. Cambridge University Press, 1945, pp. 239–243.

[15] *E.M. Stein, G. Weiss*: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, New Yersey, 1971.

[16] *E.C. Titchmarsch*: The Theory of Functions, Second Edition. Oxford University Press, 1939.

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