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ERROR ESTIMATES FOR DISTRIBUTED PARAMETER IDENTIFICATION IN PARABOLIC PROBLEMS WITH OUTPUT LEAST SQUARES AND CRANK-NICOLSON METHOD

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Abstract. The identification problem of a functional coefficient in a parabolic equation is considered. For this purpose an output least squares method is introduced, and estimates of the rate of convergence for the Crank-Nicolson time discretization scheme are proved, the equation being approximated with the finite element Galerkin method with respect to space variables.

Keywords: Parameter identification, parabolic problem, finite element method, Crank-Nicolson scheme

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1. Introduction

In this article we consider the parabolic equation

$$
\begin{aligned}
(1.1)\qquad \frac{\partial u(t,x)}{\partial t} - \nabla\cdot(b(t,x)\,\nabla u(t,x)) &= f(t,x) \quad \text{in } (0,T]\times\Omega,\\
b\frac{\partial u}{\partial n}\Big|_{\partial\Omega} &= g \quad \text{in } (0,T],\\
u(0,x) &= u_0(x) \quad \text{in } \Omega,
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, and $[0,T]$ is a fixed time interval with $T<\infty$. A direct problem in (1.1) consists in finding the unknown solution u when we know both functions b and f, but here we are interested in the inverse problem related to this equation: With some information about the solution u, recover the parameter b.

We assume that we have a distributed observation of the solution u and we use the output least squares method to transform the identification problem of b into a minimization problem. The aim of this paper is to generalize the analysis for elliptic identification problems in [5] to the case of the parabolic equation (1.1). For other works containing estimates of the rate of convergence for parabolic identification problems we refer to [9], [6] and references therein.

The convergence analysis of inverse problems is based on the techniques used for the corresponding direct parabolic problems. This kind of calculations can be found, for example, in the books [10], [4], and in the papers [3], [7] and [11]. In our work we have adopted these techniques especially in the proofs of Lemmas 3.3 and 3.4. However, at least we do not know any work where a method like ours is used in order to obtain estimates for time derivatives in those lemmas.

The paper is organized as follows. In Section 2 we recall some approximation results and inequalities needed in the analysis. In Section 3 we formulate the identification problem as an optimal control problem by introducing a cost functional, which is minimized in the computational procedure. This is followed by estimates of the rate of convergence, when the equation (1.1) is discretized in time with the Crank-Nicolson scheme.

2. Notation and preliminaries

The standard notation for Sobolev spaces and associated norms will be used. We will not include the domain Ω in the spaces and norms, since we assume it always fixed. We use $(\cdot,\cdot)$ to denote the L^2 inner product on Ω and $\langle\cdot,\cdot\rangle$ on $\partial\Omega$. We regard C as a generic constant, which may vary in different contexts, but is always independent of h. By D_t we denote the derivative with respect to the time variable t.

In order to define the finite element spaces let $\mathcal{T}_h$, $0 < h < 1$, be a family of triangulations of $\overline{\Omega}$. If the boundary of Ω is curved, we use triangles with one edge replaced by a curved segment of the boundary (see [8] for details). We assume that the family $\mathcal{T}_h$ is regular and quasi-uniform. For fixed integers $0 \leqslant l \leqslant r$, $r \geqslant 1$, we define a finite element space as

$$S^r_{h,l} = \Big\{ v \mid v \in C^{l-1}(\overline{\Omega}),\ \ v|_T \in P_r\ \forall\ T \in \mathcal{T}_h \Big\}, \tag{2.1}$$

where P_r is the space of polynomials of degree less than or equal to r and C^{-1} is interpreted as L^2. By standard results ([2], [1]) we know that for all $v \in W^{m,p}(\Omega)$ there is $v_h \in S^r_{h,l}$ such that

$$\|v - v_h\|_{k,p} \leqslant C\, h^{m-k} \|v\|_{m,p} \text{ for } \ 0 \leqslant k \leqslant l,\ \ k \leqslant m \leqslant r+1,\ \ 1 \leqslant p \leqslant \infty. \tag{2.2}$$

Also, these spaces satisfy an inverse inequality

$$\|v_h\|_{1,p} \leqslant C\, h^{-1} \|v_h\|_{0,p} \quad \forall v_h \in S^r_{h,l},\ 1 \leqslant p \leqslant \infty. \tag{2.3}$$

We will make use of Young's inequality:

Let $a, b \in \mathbb{R}$. Then, for $\alpha > 0$ we have

$$a\, b \leqslant \frac{1}{4\,\alpha} a^2 + \alpha b^2. \tag{2.4}$$

Using (2.4) with $\alpha = \frac{1}{2}$ it is easy to prove that for $v_1, \ldots, v_m \in X$ we have an estimate

$$\|v_1 + \ldots + v_m\|_X^2 \leqslant m\, (\|v_1\|_X^2 + \ldots + \|v_m\|_X^2). \tag{2.5}$$

We denote by $H^{-1} = H^{-1}(\Omega)$ the dual space $\big(H^1(\Omega)\big)'$ equipped with the natural norm

$$\|v\|_{-1} = \sup_{\substack{\psi \in H^1 \\ \psi \neq 0}} \frac{|(v, \psi)|}{\|\psi\|_1}. \tag{2.6}$$

A direct consequence of this definition for $v \in H^{-1}$ and $\psi \in H^1$ is the inequality

$$|(v, \psi)| \leqslant \|v\|_{-1} \|\psi\|_1. \tag{2.7}$$

If v is a strongly measurable map of $(0, T)$ into the Banach space X with a norm $\|\cdot\|_X$, we set

$$\|v\|^2_{L^2((0,T);X)} = \|v\|^2_{L^2(X)} = \int_0^T \|v(s)\|_X^2 \,\mathrm{d}s. \tag{2.8}$$

Moreover, if v is continuous from $[0, T]$ into X, we put

$$\|v\|_{C^0([0,T];X)} = \|v\|_{C^0(X)} = \max_{t \in [0,T]} \|v(t)\|_X. \tag{2.9}$$

3. Estimates for Crank-Nicolson scheme

First we fix some notation. Let $1 \leqslant n \in \mathbb{N}$ be a positive integer and set $\Delta t = \frac{T}{n}$. We divide the time-axis $[0, T]$ into the subintervals $[t_j, t_{j+1}]$, $j = 0, \ldots, n-1$, where $t_j = j\Delta t$. In the sequel, we use the following notation for functions ψ, $\tilde{\psi}$, which are defined on $[0, T]$ or its division:

$$\begin{gathered}\psi_j = \psi(t_j),\ \partial_t \psi_j = \frac{\psi_{j+1} - \psi_j}{\Delta t},\ \psi_{j+\frac{1}{2}} = \frac{\psi_{j+1} + \psi_j}{2}, \\ \overline{\psi}_{j+\frac{1}{2}} = \psi(t_{j+\frac{1}{2}}),\ (\psi\,\tilde{\psi})_{j+\frac{1}{2}} = \psi_{j+\frac{1}{2}}\,\tilde{\psi}_{j+\frac{1}{2}}.\end{gathered} \tag{3.1}$$

Let $z(t, x) \in C^0(H^1)$ be a distributed observation of the state u and $\varphi(t, x) \in C^0(H^{-1})$ that of $D_t u$ at each time level t. We assume that the observation error with these functions is of the form

$$\begin{gathered}\|u - z\|_1 \leqslant \varepsilon_1, \\ \|D_t u - \varphi\|_{-1} \leqslant \varepsilon_2,\end{gathered} \tag{3.2}$$

for all $t \in (0, T)$. Notice that if we have measurements of u at different time levels, we can get an observation also for $D_t u$ by differentiating in time the obtained interpolation of observations. Later we will discuss possibilities how to remove the observation φ, but in the following analysis it is more convenient to consider z and φ as separate observations.

The original equation (1.1) in the weak Galerkin form reads as follows: find $u\colon [0, T] \to H^1$ such that

$$\begin{gathered}(D_t u, v) + (b\,\nabla u, \nabla v) = (f, v) + \langle g, v\rangle \quad \forall v \in H^1, \\ u = u_0 \quad \text{for } t = 0.\end{gathered} \tag{3.3}$$

In the semidiscrete finite element approximation of (3.3) we seek for a function $u_h\colon [0, T] \to U_h$ such that

$$\begin{gathered}(D_t u_h, v_h) + (b\,\nabla u_h, \nabla v_h) = (f, v_h) + \langle g, v_h\rangle \quad \forall v_h \in U_h, \\ u_h = u_{0,h} \quad \text{for } t = 0,\end{gathered} \tag{3.4}$$

where $U_h = S^{r+1}_{h,k} \subset H^1$ for $k \geqslant 1$, and $u_{0,h}$ is the interpolant of u_0 in U_h. In order to get a totally discrete scheme we must discretize also the time derivative in (3.4). We introduce the well-known Crank-Nicolson scheme, where the equation is discretized in a symmetric fashion around the points $t_{j+\frac{1}{2}}$. So, let us define a totally discrete solution $U_j(b) = u_h(b)(t_j)$, which corresponds to a given parameter b and can be computed recursively by

$$(3.5)\qquad \begin{cases} (\partial_t U_j, v_h) + ((\bar{b}\,\nabla U)_{j+\frac{1}{2}}, \nabla v_h) = (\bar{f}_{j+\frac{1}{2}}, v_h) + \langle \bar{g}_{j+\frac{1}{2}}, v_h\rangle \quad \forall v_h \in U_h, \\ U_0 = u_{0,h}, \end{cases}$$

for $j = 0, \ldots, n-1$.

For the computational procedure we introduce a cost functional

$$(3.6)\qquad J(b_h) = \sum_{j=0}^{n-1} \|\nabla(\hat{U} - \bar{z})_{j+\frac{1}{2}}\|_0^2 + \|\partial_t \hat{U}_j - \bar{\varphi}_{j+\frac{1}{2}}\|_{-1}^2,$$

where $\hat{U} = U_j(\hat{b}_h)$ is the solution of (3.5) calculated with the parameter $\hat{b}_h$. It can be seen that the cost functional consists of a sum of output least squares fits between the computed discrete function $\hat{U}$ and the given observations. The actual identification problem with the cost functional $J(b_h)$ can be defined as follows:

$$(3.7)\qquad \text{find } b_h \in M_h\colon\ J(b_h) \leqslant J(\tilde{b}_h) \quad \forall\, \tilde{b}_h \in M_h,$$

where

$$M_h = \Big\{\bar{\tilde{b}}_h \in S^r_{h,k-1} \mid \forall 0 \leqslant j \leqslant n-1\colon\ 0 < \lambda_1 \leqslant \bar{\tilde{b}}_{h,j+\frac{1}{2}} \leqslant \lambda_2 < \infty \text{ a.e. in } \Omega\Big\}$$

is the set of admissible parameters with given positive constants $\lambda_1, \lambda_2 \in \mathbb{R}$. From the definition of M_h we see that the discrete parameter $\tilde{b}_h$ is defined as a finite element function with respect to the space variables at time levels $t_{j+\frac{1}{2}}$, $j = 0, \ldots, n-1$. In the sequel, let b_h be the minimizer of (3.7) and $W = U_j(b_h)$ the solution of (3.5) with the minimizing parameter.

Concerning the smoothness of the functions we assume that

$$(3.8)\qquad \begin{gathered} u \in C^0(H^{r+2}),\ D_t u \in L^2(H^r),\ D_{tt} u \in L^2(H^1), \\ D_{ttt} u \in L^2(H^{-1}),\ b \in C^0(W^{1,\infty} \cap H^{r+1}) \end{gathered}$$

for $r \geqslant 1$. Moreover, we require that the true parameter b satisfy

$$(3.9)\qquad \lambda_1 + \delta < b(x) < \lambda_2 - \delta \text{ a.e. in } \Omega\ \forall t \in (0,T)$$

for some $\delta > 0$. Finally, as in [5] we assume that

$$(3.10)\qquad \begin{gathered} \text{there exists a constant unit vector } \vec{\nu} \text{ and a constant } \delta > 0 \\ \text{such that } \nabla u \cdot \vec{\nu} \geqslant \delta > 0\ \forall x \in \Omega,\ \forall t \in (0,T), \end{gathered}$$

$$(3.11)\qquad u \in W^{r+3}(\Omega) \text{ and } \Gamma_1 = \left\{x \in \partial\Omega\colon \frac{\partial u}{\partial n} > 0\right\} \in C^{r+2}\ \forall t \in (0,T).$$

Lemma 3.1. *Assume* (3.10) *is fulfilled and* $u \in W^{2,\infty}$ *for all* $t \in (0,T)$. *Then there is at most one coefficient* $b(t,x) \in H^1(\Omega)$ *satisfying* (3.3) $\forall t \in (0,T)$.

Proof. As in [5], Lemma 1. □

Let us first prove some estimates for the true solution u of (1.1) and its L^2 projection.

Lemma 3.2. *Let* $\bar u_h$ *be the* L^2 *projection of* u *into* U_h. *Then the following estimates for* u *and* $\bar u_h$ *are valid:*

$$\|u-\bar u_h\|_k \leqslant C\,h^{s-k}\,\|u\|_s,\quad 1\leqslant s\leqslant r+2, -1\leqslant k\leqslant 1,$$
$$\|D_t(u-\bar u_h)\|_k \leqslant C\,h^{s-k}\,\|D_t u\|_s,\quad 0\leqslant s\leqslant r+2, -1\leqslant k\leqslant 1.$$

Proof. By the definition, u and $\bar u_h$ satisfy the relation

$$(3.12)\qquad (u-\bar u_h, v_h) = 0\quad \forall v_h \in U_h.$$

A direct calculation using (3.12) shows

$$(3.13)\qquad \|u-\bar u_h\|_0^2 = (u-\bar u_h, u-\bar u_h) = (u-\bar u_h, u-\chi) \leqslant \|u-\bar u_h\|_0\|u-\chi\|_0,$$

where χ is an arbitrary element in U_h. Then, (2.2) yields

$$(3.14)\qquad \|u-\bar u_h\|_0 \leqslant C\,h^s\,\|u\|_s,\quad 0\leqslant s\leqslant r+2.$$

Let χ be the interpolant of u in U_h. Using (2.3), (2.2) and (3.14) we deduce

$$(3.15)\qquad \begin{aligned}\|u-\bar u_h\|_1 &\leqslant \|u-\chi\|_1 + C\,h^{-1}\,\|\chi-\bar u_h\|_0\\ &\leqslant C\,h^{s-1}\,\|u\|_s + C\,h^{-1}\,(\|\chi-u\|_0 + \|u-\bar u_h\|_0)\\ &\leqslant C\,h^{s-1}\,\|u\|_s,\quad 1\leqslant s\leqslant r+2.\end{aligned}$$

Moreover, by the definition of the H^{-1} norm it follows from (3.12), (2.2) and (3.14) that

$$(3.16)\qquad \begin{aligned}\|u-\bar u_h\|_{-1} &= \sup_{\psi\in H^1}\frac{|(u-\bar u_h,\psi)|}{\|\psi\|_1} = \sup_{\psi\in H^1}\frac{|(u-\bar u_h,\psi-\psi_h)|}{\|\psi\|_1}\\ &\leqslant \sup_{\psi\in H^1}\frac{\|u-\bar u_h\|_0\|\psi-\psi_h\|_0}{\|\psi\|_1}\\ &\leqslant C\,h\,\|u-\bar u_h\|_0\\ &\leqslant C\,h^{s+1}\,\|u\|_s,\quad 0\leqslant s\leqslant r+2.\end{aligned}$$

By differentiating (3.12) with respect to t we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(u-\bar{u}_h, v_h) = (D_t(u-\bar{u}_h), v_h) = 0 \quad \forall v_h \in U_h, \tag{3.17}$$

which means that the L^2 projection commutes with time differentiation. This implies, as in (3.14) - (3.16), that estimates

$$\begin{aligned} \|D_t(u-\bar{u}_h)\|_0 &\leqslant C\,h^s\,\|D_t u\|_s, \quad 0\leqslant s\leqslant r+2,\\ \|D_t(u-\bar{u}_h)\|_1 &\leqslant C\,h^{s-1}\,\|D_t u\|_s, \quad 1\leqslant s\leqslant r+2,\\ \|D_t(u-\bar{u}_h)\|_{-1} &\leqslant C\,h^{s+1}\,\|D_t u\|_s, \quad 0\leqslant s\leqslant r+2, \end{aligned} \tag{3.18}$$

are also valid. This proves the results. □

Lemma 3.3. *For u and the discrete solution $U_j(b)$ of (3.5) an estimate*

$$\Delta t \sum_{j=0}^{n-1} \Big(\|(U-\bar{u})_{j+\frac12}\|_1^2 + \|\partial_t U_j - D_t \bar{u}_{j+\frac12}\|_{-1}^2 \Big) \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4)$$

holds, for Δt sufficiently small.

Proof. From (3.3) and $U_h \subset H^1$ it follows that the formula

$$(D_t\bar{u}_{j+\frac12}, v_h) + ((\bar{b}\,\nabla\bar{u})_{j+\frac12}, \nabla v_h) = (\bar{f}_{j+\frac12}, v_h) + \langle \bar{g}_{j+\frac12}, v_h\rangle \quad \forall v_h \in U_h \tag{3.19}$$

is valid. Hence, a combination of (3.5) and (3.19) leads to

$$(\partial_t U_j - D_t\bar{u}_{j+\frac12}, v_h) + ((\bar{b}\,\nabla(U-\bar{u}))_{j+\frac12}, \nabla v_h) = 0 \quad \forall v_h \in U_h. \tag{3.20}$$

Let χ be an arbitrary element in U_h. By adding and subtracting some terms to (3.20) we get

$$\begin{aligned} &(\partial_t(U-\chi)_j, v_h) + ((\bar{b}\,\nabla(U-\chi))_{j+\frac12}, \nabla v_h)\\ &\quad = (\partial_t(u-\chi)_j, v_h) + ((\bar{b}\,\nabla(u-\chi))_{j+\frac12}, \nabla v_h)\\ &\quad + (D_t\bar{u}_{j+\frac12} - \partial_t u_j, v_h) + ((\bar{b}\,\nabla(\bar{u}-u))_{j+\frac12}, \nabla v_h) \quad \forall v_h \in U_h. \end{aligned} \tag{3.21}$$

Now we choose $v_h = (U-\chi)_{j+\frac12}$. Using the inequalities (2.7) and (2.4) we obtain from (3.21), when $\lambda_1\|(U-\chi)_{j+\frac12}\|_0^2$ is added to both sides,

$$\begin{aligned} &\frac{1}{2\Delta t}(\|(U-\chi)_{j+1}\|_0^2 - \|(U-\chi)_j\|_0^2) + \lambda_1\|(U-\chi)_{j+\frac12}\|_1^2\\ &\quad \leqslant C\Big(\|\partial_t(u-\chi)_j\|_{-1}^2 + \lambda_2^2\|\nabla(u-\chi)_{j+\frac12}\|_0^2 + \|D_t\bar{u}_{j+\frac12} - \partial_t u_j\|_{-1}^2\\ &\qquad + \lambda_2^2\|\nabla(\bar{u}-u)_{j+\frac12}\|_0^2 + \lambda_1\|(U-\chi)_{j+\frac12}\|_0^2\Big) + \alpha\|(U-\chi)_{j+\frac12}\|_1^2. \end{aligned} \tag{3.22}$$

Here we have used the formula

$$(3.23)\qquad (\partial_t(U-\chi)_j, (U-\chi)_{j+\frac{1}{2}}) = \frac{1}{2\Delta t}(\|(U-\chi)_{j+1}\|_0^2 - \|(U-\chi)_j\|_0^2),$$

and the result (see [4], p. 152)

$$(3.24)\qquad \Delta t\, \|D_t\bar{u}_{j+\frac{1}{2}} - \partial_t u_j\|_{-1}^2 \leqslant C(\Delta t)^4 \int_{t_j}^{t_{j+1}} \|D_{ttt}u\|_{-1}^2 \,\mathrm{d}s,$$

which implies

$$(3.25)\qquad \begin{aligned} \Delta t \sum_{j=0}^{n-1} \|D_t\bar{u}_{j+\frac{1}{2}} - \partial_t u_j\|_{-1}^2 &\leqslant C(\Delta t)^4 \int_0^{t_n} \|D_{ttt}u\|_{-1}^2 \,\mathrm{d}s \\ &\leqslant C(\Delta t)^4 \|D_{ttt}u\|_{L^2(H^{-1})}^2. \end{aligned}$$

A direct calculation shows that the inequality

$$(3.26)\qquad \Delta t \sum_{j=0}^{n-1} \|(\bar{u}-u)_{j+\frac{1}{2}}\|_1^2 \leqslant C(\Delta t)^4\, \|D_{tt}u\|_{L^2(H^1)}^2$$

is valid, too. Moreover, a straightforward calculation gives

$$(3.27)\qquad \partial_t(u-\chi)_j = \frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} D_t(u-\chi)\,\mathrm{d}s,$$

whence we get

$$(3.28)\qquad \Delta t \sum_{j=0}^{n-1} \|\partial_t(u-\chi)_j\|_{-1}^2 \leqslant \int_0^{t_n} \|D_t(u-\chi)\|_{-1}^2 \,\mathrm{d}s \leqslant \|D_t(u-\chi)\|_{L^2(H^{-1})}^2.$$

By the definition it follows that

$$(3.29)\qquad \|(u-\chi)_{j+\frac{1}{2}}\|_1 \leqslant C(\|(u-\chi)_{j+1}\|_1 + \|(u-\chi)_j\|_1),$$

and a repeated application shows that

$$(3.30)\qquad \Delta t \sum_{j=0}^{n-1} \|(u-\chi)_{j+\frac{1}{2}}\|_1^2 \leqslant C\,T\, \|u-\chi\|_{C^0(H^1)}^2.$$

Finally, a direct calculation gives

$$\sum_{j=0}^{n-1} \frac{1}{2\Delta t} \left(\|(U-\chi)_{j+1}\|_0^2 - \|(U-\chi)_j\|_0^2 \right) = \frac{1}{2\Delta t} \left(\|(U-\chi)_n\|_0^2 - \|U_0-\chi_0\|_0^2 \right). \tag{3.31}$$

Then, summing (3.22) for $j = 0, \ldots, n-1$, choosing $\alpha < \lambda_1$ and using the results (3.25)–(3.31) we obtain

$$\begin{aligned} \|(U-\chi)_n\|_0^2 &+ \Delta t \sum_{j=0}^{n-1} \|(U-\chi)_{j+\frac{1}{2}}\|_1^2 \\ &\leqslant C\Delta t \sum_{j=0}^{n-1} \Big(\|\partial_t(u-\chi)_j\|_{-1}^2 + \|\nabla(u-\chi)_{j+\frac{1}{2}}\|_0^2 + \|D_t\bar{u}_{j+\frac{1}{2}} - \partial_t u_j\|_{-1}^2 \\ &\quad + \|\nabla(\bar{u}-u)_{j+\frac{1}{2}}\|_0^2 + \|(U-\chi)_{j+\frac{1}{2}}\|_0^2 \Big) + \|(U-\chi)_0\|_0^2 \\ &\leqslant C\,T \left(\|D_t(u-\chi)\|_{L^2(H^{-1})}^2 + \|\nabla(u-\chi)\|_{C^0(L^2)}^2 \right. \\ &\quad \left. + \|u_{0,h}-u_0\|_0^2 + \|u_0-\chi_0\|_0^2 \right) \\ &\quad + C(\Delta t)^4 (\|D_{ttt}u\|_{L^2(H^{-1})}^2 + \|D_{tt}u\|_{L^2(H_0^1)}^2) + C\Delta t \sum_{j=0}^{n} \|(U-\chi)_j\|_0^2 \\ &\leqslant C\,T\,h^{2(r+1)} (\|D_t u\|_{L^2(H^r)}^2 + \|u\|_{C^0(H^{r+2})}^2 + \|u_0\|_{r+1}^2) \\ &\quad + C(\Delta t)^4 (\|D_{ttt}u\|_{L^2(H^{-1})}^2 + \|D_{tt}u\|_{L^2(H^1)}^2) + C\Delta t \sum_{j=0}^{n} \|(U-\chi)_j\|_0^2 \\ &\leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4) + +C\Delta t \sum_{j=0}^{n} \|(U-\chi)_j\|_0^2. \end{aligned} \tag{3.32}$$

Here we have chosen χ to be the L^2 projection of u and used the results from Lemma 3.2. Using the discrete Gronwall's inequality ([4], Lemma 4.7) we deduce from (3.32), for Δt sufficiently small,

$$\Delta t \sum_{j=0}^{n-1} \|(U-\chi)_{j+\frac{1}{2}}\|_1^2 \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4). \tag{3.33}$$

From this we obtain, using again Lemma 3.2, (3.30) and (3.26)

$$\Delta t \sum_{j=0}^{n-1} \|(U-\bar{u})_{j+\frac{1}{2}}\|_1^2 \leqslant \Delta t \sum_{j=0}^{n-1} \Big(\|(U-\chi)_{j+\frac{1}{2}}\|_1^2 + \|(\chi-u)_{j+\frac{1}{2}}\|_1^2 + \|(u-\bar{u})_{j+\frac{1}{2}}\|_1^2 \Big) \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4). \tag{3.34}$$

Let $\psi_h \in U_h$ be the L^2 projection and $\chi \in U_h$ the H^1 projection of a given $\psi \in H^1$. Then it follows from (2.3) and (2.2) that

$$\begin{aligned} \|\psi_h\|_1 &\leqslant C\,h^{-1}\,\|\psi_h-\chi\|_0 + \|\chi-\psi\|_1 + \|\psi\|_1 \\ &\leqslant C\,h^{-1}\,(\|\psi_h-\psi\|_0 + \|\psi-\chi\|_0) + C\,\|\psi\|_1 \\ &\leqslant C\,\|\psi\|_1. \end{aligned} \tag{3.35}$$

Hence, using the equation (3.21) and the definitions of the H^{-1} norm and the L^2 projection we get for $\partial_t(U-\chi)_j \in U_h$

(3.36)

$$\begin{aligned} \|\partial_t(U-\chi)_j\|_{-1} &= \sup_{\psi\in H^1} \frac{(\partial_t(U-\chi)_j,\psi)}{\|\psi\|_1} = \sup_{\psi\in H^1} \frac{(\partial_t(U-\chi)_j,\psi_h)}{\|\psi\|_1} \\ &\leqslant \sup_{\psi\in H^1} \Bigg\{ \frac{|((\bar{b}\,\nabla(U-\chi))_{j+\frac{1}{2}},\nabla\psi_h) + (\partial_t(u-\chi)_j,\psi_h) + ((\bar{b}\,\nabla(u-\chi))_{j+\frac{1}{2}},\nabla\psi_h)|}{\|\psi\|_1} \\ &\qquad + \frac{|(D_t\bar{u}_{j+\frac{1}{2}} - \partial_t u_j,\psi_h) + ((\bar{b}\,\nabla(\bar{u}-u))_{j+\frac{1}{2}},\nabla\psi_h)|}{\|\psi\|_1} \Bigg\} \\ &\leqslant C\Big(\|\nabla(U-\chi)_{j+\frac{1}{2}}\|_0 + \|\partial_t(u-\chi)_j\|_{-1} + \|\nabla(u-\chi)_{j+\frac{1}{2}}\|_0 \\ &\qquad + \|D_t\bar{u}_{j+\frac{1}{2}} - \partial_t u_j\|_{-1} + \|\nabla(\bar{u}-u)_{j+\frac{1}{2}}\|_0 \Big) \end{aligned}$$

$\forall 0 \leqslant j \leqslant n-1$. This together with (3.33), (3.28), (3.30), (3.25) and (3.26) yields

$$\Delta t \sum_{j=0}^{n-1} \|\partial_t U_j - D_t\bar{u}_{j+\frac{1}{2}}\|_{-1}^2 \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4), \tag{3.37}$$

which proves the result. □

Lemma 3.4. *Let θ_h be the interpolant of b in B_h for all $t \in [0,T]$ and let $\widetilde{U}_j = u_h(\theta_h)(t_j)$ be the corresponding discrete solution of* (3.5). *Then, for h small enough,*

$\theta_h \in M_h$, *and the following estimate holds for* $\widetilde{U}$ *and* u:

$$\Delta t \sum_{j=0}^{n-1} \Big(\|\nabla(\widetilde{U} - \bar{u})_{j+\frac12}\|_0^2 + \|\partial_t \widetilde{U}_j - D_t \bar{u}_{j+\frac12}\|_{-1}^2 \Big) \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4)$$

for Δt *sufficiently small.*

Proof. By (2.2) and the regularity of b the following estimates hold for b and θ_h:

$$\|b - \theta_h\|_0 \leqslant C\,h^{r+1}\,\|b\|_{r+1}, \qquad \|b - \theta_h\|_\infty \leqslant C\,h\,\|b\|_{1,\infty}, \tag{3.38}$$

which gives, for h small enough,

$$\lambda_1 \leqslant \theta_h \leqslant \lambda_2 \text{ a.e. in } \Omega \tag{3.39}$$

$\forall t \in (0,T)$, so $\bar{\theta}_h \in M_h$ for h small enough. $\widetilde{U}$ satisfies the equation

$$(\partial_t \widetilde{U}_j, v_h) + ((\bar{\theta}_h \nabla \widetilde{U})_{j+\frac12}, \nabla v_h) = (\bar{f}_{j+\frac12}, v_h) + \langle \bar{g}_{j+\frac12}, v_h \rangle \quad \forall v_h \in U_h. \tag{3.40}$$

Hence, a combination of (3.5) and (3.40) gives

$$\begin{aligned}(\partial_t(\widetilde{U} - U)_j, v_h) &+ ((\bar{\theta}_h \nabla(\widetilde{U} - U))_{j+\frac12}, \nabla v_h)\\ &= (((\bar{b} - \bar{\theta}_h)\nabla U)_{j+\frac12}, \nabla v_h)\\ &= (((\bar{b} - \bar{\theta}_h)\nabla(U - \bar{u}))_{j+\frac12}, \nabla v_h) + (((\bar{b} - \bar{\theta}_h)\nabla \bar{u})_{j+\frac12}, \nabla v_h) \quad \forall v_h \in U_h.\end{aligned} \tag{3.41}$$

As before we take $v_h = (\widetilde{U} - U)_{j+\frac12}$. Then (3.41) implies, as in (3.22), by virtue of the fact $\bar{u}_{j+\frac12} \in H^{r+2} \subset W^{1,\infty}$ for $r \geqslant 1$ that

$$\begin{aligned}\frac{1}{2\Delta t}(\|(\widetilde{U} - U)_{j+1}\|_0^2 &- \|(\widetilde{U} - U)_j\|_0^2) + \lambda_1 \|\nabla(\widetilde{U} - U)_{j+\frac12}\|_0^2\\ &\leqslant C\Big(\|\nabla(U - \bar{u})_{j+\frac12}\|_0^2 + \|(\bar{b} - \bar{\theta}_h)_{j+\frac12}\|_0^2\Big) + \alpha \|\nabla(\widetilde{U} - U)_{j+\frac12}\|_0^2.\end{aligned} \tag{3.42}$$

Thus, we choose again $\alpha < \lambda_1$ and sum (3.42) from 0 to $n-1$:

$$\begin{aligned}\|(\widetilde{U} - U)_n\|_0^2 &+ \Delta t \sum_{j=0}^{n-1} \|\nabla(\widetilde{U} - U)_{j+\frac12}\|_0^2\\ &\leqslant C\Delta t \sum_{j=0}^{n-1}(\|\nabla(U - \bar{u})_{j+\frac12}\|_0^2 + \|(\bar{b} - \bar{\theta}_h)_{j+\frac12}\|_0^2)\\ &\leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4 + \|b - \theta_h\|_{C^0(L^2)}^2)\\ &\leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4),\end{aligned} \tag{3.43}$$

having used the results from (3.38) and Lemma 3.3. So, again as in Lemma 3.3, it follows from (3.43) with the triangle inequality, for Δt small enough,

$$\Delta t \sum_{j=0}^{n-1} \|\nabla(\widetilde{U} - \bar{u})_{j+\frac{1}{2}}\|_0^2 \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4). \tag{3.44}$$

The estimate in H^{-1} follows exactly as in (3.36)–(3.37) combined with (3.41). This completes the proof. □

Lemma 3.5. *The following estimate for b, u and b_h, $W = \hat{U}(b_h)$ is valid, for Δt and h small enough:*

$$\Delta t \sum_{j=0}^{n-1} \Big(\|\nabla(W - \bar{u})_{j+\frac{1}{2}}\|_0^2 + \|\partial_t W_j - D_t \bar{u}_{j+\frac{1}{2}}\|_{-1}^2 \Big) \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4 + \varepsilon_1^2 + \varepsilon_2^2).$$

Proof. Because b_h is the minimizer of the cost functional (3.6), we have, for h small enough, $\Delta t\, J(b_h) \leqslant \Delta t\, J(\theta_h)$. This means

$$\begin{aligned} \Delta t \sum_{j=0}^{n-1} & \Big(\|\nabla(W - \bar{z})_{j+\frac{1}{2}}\|_0^2 + \|\partial_t W_j - \bar{\varphi}_{j+\frac{1}{2}}\|_{-1}^2 \Big) \\ & \leqslant \Delta t \sum_{j=0}^{n-1} \Big(\|\nabla(\widetilde{U} - \bar{z})_{j+\frac{1}{2}}\|_0^2 + \|\partial_t \widetilde{U}_j - \bar{\varphi}_{j+\frac{1}{2}}\|_0^2 \Big) \\ & = I_1 + I_2. \end{aligned} \tag{3.45}$$

For I_1 we have, using Lemma 3.4 and (3.2),

$$\begin{aligned} I_1 & \leqslant \Delta t \sum_{j=0}^{n-1} 2\,(\|\nabla(\widetilde{U} - \bar{u})_{j+\frac{1}{2}}\|_0^2 + \|\nabla(\bar{u} - \bar{z})_{j+\frac{1}{2}}\|_0^2 \\ & \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4 + \varepsilon_1^2). \end{aligned} \tag{3.46}$$

For I_2 we get, using again Lemma 3.4 and (3.2),

$$\begin{aligned} I_2 & \leqslant \Delta t \sum_{j=0}^{n-1} 2\,(\|\partial_t \widetilde{U}_j - D_t \bar{u}_{j+\frac{1}{2}}\|_{-1}^2 + \|(D_t \bar{u} - \bar{\varphi})_{j+\frac{1}{2}}\|_{-1}^2 \\ & \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4 + \varepsilon_2^2). \end{aligned} \tag{3.47}$$

Hence, from (3.45)–(3.47) we conclude

$$\begin{aligned} \Delta t \sum_{j=0}^{n-1} & \Big(\|\nabla(W - \bar{z})_{j+\frac{1}{2}}\|_0^2 + \|\partial_t W_j - \bar{\varphi}_{j+\frac{1}{2}}\|_{-1}^2 \Big) \\ & \leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4 + \varepsilon_1^2 + \varepsilon_2^2). \end{aligned} \tag{3.48}$$

By the triangle inequality and the above results we then get

$$\begin{aligned}\Delta t \sum_{j=0}^{n-1} \|\nabla(W-\bar u)_{j+\frac12}\|_0^2 &\leqslant \Delta t \sum_{j=0}^{n-1} 2\left(\|\nabla(W-\bar z)_{j+\frac12}\|_0^2 + \|\nabla(\bar z-\bar u)_{j+\frac12}\|_0^2\right)\\ &\leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4 + \varepsilon_1^2 + \varepsilon_2^2)\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}\Delta t \sum_{j=0}^{n-1} \|\partial_t W_j - D_t\bar u_{j+\frac12}\|_{-1}^2 &\leqslant \Delta t \sum_{j=0}^{n-1} 2\left(\|\partial_t W_j - \bar\varphi_{j+\frac12}\|_{-1}^2 + \|(\bar\varphi - D_t\bar u)_{j+\frac12}\|_{-1}^2\right)\\ &\leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4 + \varepsilon_1^2 + \varepsilon_2^2).\end{aligned} \tag{3.50}$$

This completes the proof. □

Now we are ready to present our main theorem:

Theorem 3.1. *Let b_h be the minimizing parameter of the cost functional* (3.6) *and let assumptions* (3.10) *and* (3.11) *be valid. Then, for Δt and h sufficiently small, the estimate*

$$\left(\Delta t \sum_{j=0}^{n-1} \|(\bar b - \bar b_h)_{j+\frac12}\|_0^2\right)^{\frac12} \leqslant C\,T\,h^{-1}\left(h^{r+1} + (\Delta t)^2 + \varepsilon_1 + \varepsilon_2\right)$$

holds for b and b_h.

Proof. From (3.5) we know that the equation

$$\begin{aligned}&\left(((\bar\theta_h - \bar b_h)\,\nabla\widetilde U)_{j+\frac12}, \nabla v_h\right)\\ &\quad= \left(\partial_t(W-\widetilde U)_j, v_h\right) + \left((\bar b_h\,\nabla(W-\widetilde U))_{j+\frac12}, \nabla v_h\right)\end{aligned} \tag{3.51}$$

is valid for all $0 \leqslant j \leqslant n-1$. By adding and subtracting some terms we then get

$$\begin{aligned}\left(((\bar\theta_h - \bar b_h)\,\nabla\bar u)_{j+\frac12}, \nabla v\right) &= \left(((\bar\theta_h - \bar b_h)\,\nabla\bar u)_{j+\frac12}, \nabla(v-v_h)\right)\\ &+ \left(((\bar\theta_h - \bar b_h)\,\nabla(\bar u - \widetilde U))_{j+\frac12}, \nabla v_h\right) + \left(\partial_t(W-\widetilde U)_j, v_h\right)\\ &+ \left((\bar b_h\,\nabla(W-\widetilde U))_{j+\frac12}, \nabla v_h\right).\end{aligned} \tag{3.52}$$

We now choose $v = \varrho \exp^{-2k\vec{x}\cdot\vec{\nu}}[(\bar{\theta}_h - \bar{b}_h)_{j+\frac{1}{2}}]$. For the details of this choice and the subsequent results we refer to [5]. Then we can show, using assumptions (3.10) and (3.11), that

$$\begin{aligned} &\Big(((\bar{\theta}_h - \bar{b}_h)\,\nabla \bar{u})_{j+\frac{1}{2}}, \nabla v\Big) \geqslant \tau \|(\bar{\theta}_h - \bar{b}_h)_{j+\frac{1}{2}}\|_0^2 \ \text{ for } \tau > 0, \\ &\|v - v_h\|_1 \leqslant Ch\|(\bar{\theta}_h - \bar{b}_h)_{j+\frac{1}{2}}\|_0, \\ &\|v\|_1 \leqslant Ch^{-1}\|(\bar{\theta}_h - \bar{b}_h)_{j+\frac{1}{2}}\|_0, \\ &\|v_h\|_1 \leqslant \|v_h - v\|_1 + \|v\|_1 \leqslant Ch^{-1}\|(\bar{\theta}_h - \bar{b}_h)_{j+\frac{1}{2}}\|_0. \end{aligned} \tag{3.53}$$

Applying this choice to (3.52) and using (3.53), (2.7) and (2.4) we obtain

$$\begin{aligned} &\tau\,\|(\bar{\theta}_h - \bar{b}_h)_{j+\frac{1}{2}}\|_0^2 \\ &\leqslant Ch\|(\bar{\theta}_h - \bar{b}_h)_{j+\frac{1}{2}}\|_0^2 + Ch^{-1}\Big(\|\nabla(\bar{u} - \widetilde{U})_{j+\frac{1}{2}}\|_0 \\ &\quad + \|\partial_t(W - \widetilde{U})_j\|_{-1} + \|\nabla(W - \widetilde{U})_{j+\frac{1}{2}}\|_0\Big)\,\|(\bar{\theta}_h - \bar{b}_h)_{j+\frac{1}{2}}\|_0 \\ &\leqslant (Ch + \alpha)\|(\bar{\theta}_h - \bar{b}_h)_{j+\frac{1}{2}}\|_0^2 + Ch^{-2}\Big(\|\nabla(\bar{u} - \widetilde{U})_{j+\frac{1}{2}}\|_0^2 \\ &\quad \|\partial_t(W - \widetilde{U})_j\|_{-1}^2 + \|\nabla(W - \widetilde{U})_{j+\frac{1}{2}}\|_0^2\Big). \end{aligned} \tag{3.54}$$

Hence, for h and α small enough,

$$\begin{aligned} &\|(\bar{b} - \bar{b}_h)_{j+\frac{1}{2}}\|_0^2 \\ &\leqslant \|(\bar{b} - \bar{\theta}_h)_{j+\frac{1}{2}}\|_0^2 + Ch^{-2}\Big(\|\nabla(\bar{u} - \widetilde{U})_{j+\frac{1}{2}}\|_0^2 \\ &\quad + \|\partial_t(W - \widetilde{U})_j\|_{-1}^2 + \|\nabla(W - \widetilde{U})_{j+\frac{1}{2}}\|_0^2\Big) \ \forall 0 \leqslant j \leqslant n-1. \end{aligned} \tag{3.55}$$

Thus, by summing (3.55) from $j = 0, \ldots, n-1$ and multiplying with Δt we have

$$\begin{aligned} &\Delta t \sum_{j=0}^{n-1} \|(\bar{b} - \bar{b}_h)_{j+\frac{1}{2}}\|_0^2 \\ &\leqslant C\Delta t \sum_{j=0}^{n-1} \Big(\|(\bar{b} - \bar{\theta}_h)_{j+\frac{1}{2}}\|_0^2 \\ &\qquad + h^{-2}(\|\nabla(\bar{u} - \widetilde{U})_{j+\frac{1}{2}}\|_0^2 + \|\partial_t(W - \widetilde{U})_j\|_{-1}^2 + \|\nabla(W - \widetilde{U})_{j+\frac{1}{2}}\|_0^2)\Big) \\ &\leqslant CT\,\|b - \theta_h\|_{C^0(L^2)}^2 + Ch^{-2}\Delta t \sum_{j=0}^{n-1} \Big(\|\nabla(\bar{u} - \widetilde{U})_{j+\frac{1}{2}}\|_0^2 + \|\nabla(W - \bar{u})_{j+\frac{1}{2}}\|_0^2 \\ &\quad + \|\partial_t W_j - D_t\bar{u}_{j+\frac{1}{2}}\|_{-1}^2 + \|D_t\bar{u}_{j+\frac{1}{2}} - \partial_t\widetilde{U}_j\|_{-1}^2\Big). \end{aligned} \tag{3.56}$$

This combined with (3.38) and the results in Lemmas 3.4 and 3.5 yields the result. □

R e m a r k 3.1. The estimate in Theorem 3.1 is of the same form as the estimates obtained for the Crank-Nicolson scheme in direct parabolic problems, for example, in [3] and [4].

As in [5] we can prove a better estimate in the one-dimensional case, if we have a condition

$$u' \geqslant \delta > 0 \ \text{ a. e. in } \Omega \ \forall t \in (0,T), \tag{3.57}$$

where $'$ means the differentiation with respect to x.

Theorem 3.2. *If u' satisfies* (3.57), *the estimate*

$$\Bigl(\Delta t \sum_{j=0}^{n-1} \|(\bar{b} - \bar{b}_h)_{j+\frac12}\|_0^2\Bigr)^{\frac12} \leqslant C\,T\,\bigl(h^{r+1} + (\Delta t)^2 + \varepsilon_1 + \varepsilon_2\bigr)$$

is, for Δt and h small enough, valid in $1d$ *for b and b_h.*

P r o o f. From (3.51) we obtain the equation

$$\begin{aligned} &\bigl(((\bar{\theta}_h - \bar{b}_h)\,\bar{u}')_{j+\frac12}, v_h'\bigr) \\ &\quad = \bigl(((\bar{\theta}_h - \bar{b}_h)\,(\bar{u} - \widetilde{U})')_{j+\frac12}, v_h'\bigr) + \bigl(\partial_t (W - \widetilde{U})_j, v_h\bigr) \\ &\qquad + \bigl((\bar{b}_h\,(W - \widetilde{U})')_{j+\frac12}, v_h'\bigr). \end{aligned} \tag{3.58}$$

In $1d$ the domain Ω reduces to an interval $I = (a,b)$. Let v_h be the solution of the differential equation

$$\begin{cases} \ \ v_h' = (\bar{\theta}_h - \bar{b}_h)_{j+\frac12} & \text{in } I, \\ v_h(a) = 0. \end{cases} \tag{3.59}$$

Then, as in [5], by the definition $v_h' \in B_h = S^r_{h,k-1}$, and therefore $v_h \in S^{r+1}_{h,k} = U_h$. Moreover, because $v_h(a) = 0$, we get from the Poincaré inequality $\|v_h\|_0 \leqslant C\|v_h'\|_0 \leqslant \|(\bar{\theta}_h - \bar{b}_h)_{j+\frac12}\|_0$, which means $\|v_h\|_1 \leqslant C\|(\bar{\theta}_h - \bar{b}_h)_{j+\frac12}\|_0$. Applying this v_h to (3.58) and using (3.57) and (2.7) we get

$$\begin{aligned} &\delta \|(\bar{\theta}_h - \bar{b}_h)_{j+\frac12}\|_0^2 \\ &\quad \leqslant C\Bigl(\|(\bar{u} - \widetilde{U})'_{j+\frac12}\|_0 + \|\partial_t (W - \widetilde{U})_j\|_{-1} + \|(W - \widetilde{U})'_{j+\frac12}\|_0\Bigr)\,\|v_h\|_1 \\ &\quad \leqslant C\Bigl(\|(\bar{u} - \widetilde{U})'_{j+\frac12}\|_0 + \|\partial_t (W - \widetilde{U})_j\|_{-1} + \|(W - \widetilde{U})'_{j+\frac12}\|_0\Bigr) \\ &\qquad \|(\bar{\theta}_h - \bar{b}_h)_{j+\frac12}\|_0 \quad \forall 0 \leqslant j \leqslant n-1. \end{aligned} \tag{3.60}$$

Again, multiplying (3.60) with Δt, summing from 0 to $n-1$, applying the triangle inequality and using the results from Lemmas 3.4 and 3.5 we obtain the result. □

Next we consider one possibility of removing the observation φ for $D_t u$. Because this could be done in many ways by using some interpolation strategy of the observation z in time direction, we have so far assumed the existence of φ with an observation error ε_2. So, assume that we have only an observation z of u with observation errors

$$\|u - z\|_1 \leqslant \varepsilon_1, \qquad \|u - z\|_{-1} \leqslant \varepsilon_2, \tag{3.61}$$

for all $t \in [0, T]$. Let us define a cost functional by

$$J(b_h) = \sum_{j=0}^{n-1} \|\nabla(\hat{U} - z)_{j+\frac{1}{2}}\|_0^2 + (\Delta t)^2 \|\partial_t(\hat{U} - z)_j\|_{-1}^2, \tag{3.62}$$

where $(\Delta t)^2$ in front of the second term is for balancing the two terms in the cost functional with respect to the observation errors in (3.61).

Theorem 3.3. *Let b_h be the minimizing parameter of the cost functional* (3.62) *and let assumptions* (3.10) *and* (3.11) *be valid. Then, for Δt and h sufficiently small, the estimate*

$$\left(\Delta t \sum_{j=0}^{n-1} \|(\bar{b} - \bar{b}_h)_{j+\frac{1}{2}}\|_0^2\right)^{\frac{1}{2}} \leqslant C\, T\, (h\, \Delta t)^{-1} \left(h^{r+1} + (\Delta t)^2 + \varepsilon_1 + \varepsilon_2\right),$$

holds for b and b_h.

The one-dimensional error estimate corresponding to Theorem 3.2 reads as follows:

Theorem 3.4. *If u' satisfies* (3.57), *the estimate*

$$\left(\Delta t \sum_{j=0}^{n-1} \|(\bar{b} - \bar{b}_h)_{j+\frac{1}{2}}\|_0^2\right)^{\frac{1}{2}} \leqslant C\, T\, (\Delta t)^{-1} \left(h^{r+1} + (\Delta t)^2 + \varepsilon_1 + \varepsilon_2\right)$$

is, for Δt and h small enough, valid in $1d$ for b and the minimizer b_h of (3.62).

Because the proof of Theorem 3.4 follows from the proof of Theorem 3.3 as in Theorem 3.2, we will only prove Theorem 3.3.

Proof (of Theorem 3.3). As in Lemma 3.5 we have

$$\begin{aligned} &\Delta t \sum_{j=0}^{n-1} \left(\|\nabla(W - z)_{j+\frac{1}{2}}\|_0^2 + (\Delta t)^2 \|\partial_t(W - z)_j\|_{-1}^2\right) \\ &\leqslant \Delta t \sum_{j=0}^{n-1} \left(\|\nabla(\widetilde{U} - z)_{j+\frac{1}{2}}\|_0^2 + (\Delta t)^2 \|\partial_t(\widetilde{U} - z)_j\|_0^2\right) \\ &= I_1 + (\Delta t)^2 I_2. \end{aligned} \tag{3.63}$$

By using the previous results and (3.61) we have

$$
\begin{aligned}
I_1 &\leqslant \Delta t \sum_{j=0}^{n-1} 3\,(\|\nabla(\widetilde{U}-\bar{u})_{j+\frac12}\|_0^2 + \|\nabla(\bar{u}-u)_{j+\frac12}\|_0^2 + \|\nabla(u-z)_{j+\frac12}\|_0^2 \\
&\leqslant CT\left(h^{2(r+1)} + (\Delta t)^4 + \|u-z\|^2_{C^0(H_0^1)}\right) \\
&\leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4 + \varepsilon_1^2).
\end{aligned} \tag{3.64}
$$

Similarly, for I_2 we get

$$
\begin{aligned}
(\Delta t)^2 I_2 &\leqslant \Delta t \sum_{j=0}^{n-1} 3\,(\Delta t)^2\,\big(\|\partial_t \widetilde{U}_j - D_t\bar{u}_{j+\frac12}\|_{-1}^2 \\
&\qquad + \|D_t\bar{u}_{j+\frac12} - \partial_t u_j\|_{-1}^2 + \|\partial_t(u-z)_j\|_{-1}^2\big) \\
&\leqslant C\,T\,(\Delta t)^2\big(h^{2(r+1)} + (\Delta t)^4\big) + C\big(\|(u-z)_j\|_{-1}^2 + \|(u-z)_{j+1}\|_{-1}^2\big) \\
&\leqslant C\,T\,(\Delta t)^2\big(h^{2(r+1)} + (\Delta t)^4\big) + C\|u-z\|^2_{C^0(H^{-1})} \\
&\leqslant C\,T\,(\Delta t)^2\big(h^{2(r+1)} + (\Delta t)^4\big) + \varepsilon_2^2.
\end{aligned} \tag{3.65}
$$

Hence, we conclude

$$
\begin{aligned}
\Delta t \sum_{j=0}^{n-1} \|\nabla(W-\bar{u})_{j+\frac12}\|_0^2 &\leqslant \Delta t \sum_{j=0}^{n-1} 3\,\Big(\|\nabla(W-z)_{j+\frac12}\|_0^2 + \|\nabla(z-u)_{j+\frac12}\|_0^2 \\
&\qquad + \|\nabla(u-\bar{u})_{j+\frac12}\|_0^2\Big) \\
&\leqslant C\,T\,(h^{2(r+1)} + (\Delta t)^4 + \varepsilon_1^2 + \varepsilon_2^2),
\end{aligned} \tag{3.66}
$$

and

$$
\begin{aligned}
\Delta t \sum_{j=0}^{n-1} \|\partial_t W_j - D_t\bar{u}_{j+\frac12}\|_{-1}^2 &\leqslant \Delta t \sum_{j=0}^{n-1} 3\,\big(\|\partial_t(W-z)_j\|_{-1}^2 + \|\partial_t(z-u)_j\|_{-1}^2 \\
&\qquad + \|\partial_t u_j - D_t\bar{u}_{j+\frac12}\|_{-1}^2\big) \\
&\leqslant C\,T\,(\Delta t)^{-2}\,\Big(h^{2(r+1)} + (\Delta t)^4 + \varepsilon_1^2 + \varepsilon_2^2\Big).
\end{aligned} \tag{3.67}
$$

The rest of the proof proceeds as in Theorem 3.1. □

Remark 3.2. It is clear from the previous analysis that if $D_{ttt}z \in L^2(H^{-1})$ and the observation error between the time derivatives of u and z is $\|D_t(u-z)\|_{-1} \leqslant \varepsilon_2$ $\forall t \in (0,T)$, a minimization of the cost functional

$$
J(b_h) = \sum_{j=0}^{n-1} \|\nabla(\hat{U}-z)_{j+\frac12}\|_0^2 + \|\partial_t(\hat{U}-z)_j\|_{-1}^2 \tag{3.68}
$$

will give the same estimates as in Theorems 3.1 and 3.2.

R e m a r k 3.3. In fact, from the calculations we notice that for the validity of our results it would be enough to assume that z is an observation of u in H_0^1 with an observation error $\|\nabla(u-z)\|_0 \leqslant \varepsilon_1 \ \forall t \in [0,T]$.

R e m a r k 3.4. The relation for k in the discrete spaces $B_h = S^r_{h,k-1}$ and $U_h = S^{r+1}_{h,k}$ was only needed for the one-dimensional estimates. Thus, nothing in the general case prevents the use of the same k in both spaces B_h and U_h.

R e m a r k 3.5. If z is the L^2 projection or the interpolant of u in U_h, both observation errors ε_1 and ε_2 can be removed from all error estimates. This follows from Lemma 3.2 and the standard interpolation theory. Moreover, the choice $\partial_t z_j$ in the cost functional (3.62) is the simplest one to approximate the time derivative. Other interpolation strategies in time direction can be used in order to obtain higher order approximations for $D_t u$.

R e m a r k 3.6. Throughout this section we could replace $\overline{f}_{j+\frac{1}{2}}$ with $f_{j+\frac{1}{2}}$ if $D_{tt} f \in L^2(H^{-1})$, and $\bar{g}_{j+\frac{1}{2}}$ with $g_{j+\frac{1}{2}}$ if $D_{tt} g \in L^2(L^2(\partial\Omega))$.

R e m a r k 3.7. In (3.2) and (3.61) we have assumed that the observation z is given in H^1. However, if $z \in C^0(H^{r+2})$ is an L^2 observation of u with an observation error $\|u-z\|_0 \leqslant \tilde{\varepsilon}_1$, we can show, by using the inverse inequality (2.3) and the approximation result (2.2) that $\|u-z\|_1 \leqslant C\,(h^{r+1} + h^{-1}\,\tilde{\varepsilon}_1)$. Hence, in this case the term ε_1 should be replaced with $h^{-1}\,\tilde{\varepsilon}_1$ in all error estimates. Then we can observe that our estimates in this case are exactly of the same order (and coincide if $D_t u = 0$) as those in [5] for elliptic problems.

Lemma 3.6. *Calculation of the dual norm. In our cost functionals we need to compute the H^{-1} norm of a given function g (with the methods proposed this computation is done separately at each time level). This can be done as follows: let φ be the weak Galerkin solution of*

$$\begin{cases} -\Delta\varphi + \varphi = g \text{ in } \Omega, \\ \left.\dfrac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0. \end{cases} \tag{3.69}$$

Then $\|g\|_{-1}$ is equal to $\|\varphi\|_1$.

P r o o f. The equation (3.69) in the weak Galerkin form reads

$$(\nabla\varphi, \nabla v) + (\varphi, v) = (g, v) \quad \forall v \in H^1. \tag{3.70}$$

By choosing $v = \varphi$ in (3.70) and using the inequalities (2.7) and (2.4) with $\alpha = \frac{1}{2}$ we obtain

$$\|\varphi\|_1^2 = (\nabla\varphi, \nabla\varphi) + (\varphi, \varphi) = (g, \varphi) \leqslant \frac{1}{2}\,\|g\|_{-1}^2 + \frac{1}{2}\,\|\varphi\|_1^2, \tag{3.71}$$

which shows that

$$\|\varphi\|_1 \leqslant \|g\|_{-1}. \tag{3.72}$$

By the definition of the H^{-1} norm we get, using again the equation (3.70),

$$\|g\|_{-1} = \sup_{\psi\in H^1} \frac{|(g,\psi)|}{\|\psi\|_1} = \sup_{\psi\in H^1} \frac{|(\nabla\varphi,\nabla\psi)+(\varphi,\psi)|}{\|\psi\|_1} \leqslant \|\varphi\|_1, \tag{3.73}$$

which combined with (3.72) proves the result. □

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