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EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS
OF A NONLINEAR EVOLUTION PROBLEM

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Abstract. We prove existence and asymptotic behaviour of a weak solutions of a mixed problem for

$$(*) \quad \begin{cases} u'' + Au - \Delta u' + |v|^{\varrho+2} |u|^{\varrho} u = f_1 \\ v'' + Av - \Delta v' + |u|^{\varrho+2} |v|^{\varrho} v = f_2 \end{cases}$$

where A is the pseudo-Laplacian operator.

Keywords: Nonlinear problem, existence of solutions, Galerkin method, compactness, pseudo-Laplacian, asymptotic behaviour

MSC 2000: 35L70

1. INTRODUCTION

In 1987 Medeiros-Miranda [7] proved the existence and uniqueness of weak solutions of the system

$$(**) \quad \begin{cases} \square u + |v|^{\varrho+2} |u|^{\varrho} u = f_1, \\ \square v + |u|^{\varrho+2} |v|^{\varrho} v = f_2, \quad \varrho > -1, \end{cases}$$

where $\square = \frac{\partial^2}{\partial t^2} - \Delta$ is the d'Alembertian operator. They proved existence of solutions for $n \geq 1$ (n : spatial dimension) and uniqueness for $n = 1, 2, 3$. We have studied the existence of solutions to a system analogous to that in (**), namely

$$(***) \quad \begin{cases} u'' + Au - \Delta u' + |v|^{\varrho+2} |u|^{\varrho} u = f_1, \\ v'' + Av - \Delta v' + |u|^{\varrho+2} |v|^{\varrho} v = f_2, \end{cases}$$

where

$$A\omega = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left| \frac{\partial \omega}{\partial x_i} \right|^{(p-2)} \frac{\partial \omega}{\partial x_i} \right), \quad p > 2.$$

Many significant variations of the (**) and (***)-problems had been studied by many authors. Tsutsumi [8] studied the differential equation

$$u'' + Au + Bu' = f$$

with initial conditions $u(0) = u_0$, $u'(0) = u_1$, where A is a nonlinear operator with some strong properties and B is a bounded linear operator associated with a bounded symmetric bilinear form. Biazutti [1] studied the existence of weak solutions of the initial boundary value problem for the system

$$\begin{aligned} u'' + Au - \Delta u' + G_1(u', v') &= f_1, \\ v'' + Av - \Delta v' + G_2(u', v') &= f_2, \end{aligned}$$

where A is as before, $p \geq 2$ and G_1, G_2 have some properties as functions of u' and v' .

(For other authors see references at the end).

2. NOTATION AND MAIN RESULTS

Let Ω be a regular bounded domain of \mathbb{R}^n . Let $T > 0$ be a real number and $Q = \Omega \times]0, T[$. The norm and inner product in $H_0^1(\Omega)$ and $L^2(\Omega)$ are denoted by $\| \cdot \|$, $((\cdot, \cdot))$ and $| \cdot |$, (\cdot, \cdot) , respectively.

Let X be a Banach space and $1 \leq p \leq \pm\infty$.

Then $L^p(0, T; X)$ is the Banach space of vector X -valued measurable functions $u:]0, T[\rightarrow X$ such that $\|u(t)\|_X \in L^p(0, T)$.

If $1 \leq p < +\infty$, then $L^p(0, T; X)$ is normed by

$$\|u\|_{L^p(0, T; X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}}.$$

In the case $p = +\infty$, we have

$$\|u\|_{L^\infty(0, T; X)} = \operatorname{ess\,sup}_{[0, T]} \|u(t)\|_X.$$

Now we list some results and relations that will be used in the sequel.

2.1 Let $n \in \mathbb{N}$, $p \in \mathbb{R}$, with $n > p$, $p > 2$.

If $-1 < p \leq \frac{4(1-n+p)}{2(n-p-1)+np}$, then $\varrho < \frac{4}{np-2}$.

2.2 If $n \in \mathbb{N}$, $e \frac{4n+2}{4+n} < p < n$, then $\frac{4}{np-2} \leq \frac{1}{n-p}$.

2.3 Let n , p and ϱ be as before and let

$$\theta = \frac{2np(\varrho+2)}{(np-2)(\varrho+2)+2np(\varrho+1)} \quad \text{and} \quad \gamma = \frac{2np(\varrho+2)}{(np+2)(\varrho+2)-2np(\varrho+1)}.$$

Then

$$\text{i) } 1 < \theta < \frac{\varrho+2}{\varrho+1}, \quad \text{ii) } 1 < \gamma \leq \frac{np}{n-p}, \quad \text{iii) } \frac{1}{\theta} + \frac{1}{\gamma} = 1.$$

2.4 Let $\alpha = \frac{\varrho+2}{(\varrho+1)\theta}$, $\beta = \frac{\varrho+2}{(\varrho+2)-(\varrho+1)\theta}$.

Then

$$\text{i) } \alpha > 1, \beta > 1, \quad \text{ii) } \theta\beta = \frac{2np}{np-2}, \quad \text{iii) } \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

2.5 Let $u, v \in W_0^{1,p}(\Omega)$. Then

$$\text{i) } uv \in L^{e+2}(\Omega), \quad \text{ii) } |v|^{e+2}|u|^e u, |u|^{e+2}|v|^e v \in L^\theta(\Omega).$$

The proofs of 2.1–2.5 are straightforward and can be found in Castro [3].

3. AN EXISTENCE THEOREM

Theorem 1. Let n , p and ϱ be as before and suppose that

$$(1) \quad \begin{aligned} f_1, f_2 &\in L^2(0, T; L^2(\Omega)), \\ u_0, v_0 &\in W_0^{1,p}(\Omega), \\ u_1, v_1 &\in L^2(\Omega). \end{aligned}$$

Then there exist functions $u, v: Q \rightarrow \mathbb{R}$ such that

$$\begin{aligned} u, v &\in L^\infty(0, T; W_0^{1,p}(\Omega)), \\ u', v' &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \\ \frac{d}{dt}(u'(t), w) + \langle Au(t), w \rangle + ((u'(t), w)) \\ &+ \langle |v(t)|^{e+2}|u(t)|^e u(t), w \rangle = (f_1(t), w), \quad \forall w \in W_0^{1,p}(\Omega) \end{aligned}$$

in the sense of $D'(0, T)$,

$$\begin{aligned} \frac{d}{dt}(v'(t), w) + \langle Av(t), w \rangle + ((v'(t), w)) \\ + \langle |u(t)|^{e+2}|v(t)|^e v(t), w \rangle = (f_2(t), w), \quad \forall w \in W_0^{1,p}(\Omega) \end{aligned}$$

in the sense of $D'(0, T)$,

$$\begin{aligned} u(0) &= u_0, & u'(0) &= u_1, \\ v(0) &= v_0, & v'(0) &= v_1. \end{aligned}$$

Proof. Let $\{w_j\}_j$ be a spectral basis of $H_0^s(\Omega)$, $s > n(\frac{1}{2} - \frac{1}{p}) + 1$, which is an orthonormal complete system in $L^2(\Omega)$. Let $V_m = [w_1, \dots, w_m]$ be a subspace of $H_0^s(\Omega)$ generated by the first m vectors w_1, \dots, w_m . \square

Approximate problem. We consider the system

$$\begin{aligned} (2) \quad & (u_m''(t), w) + \langle Au_m(t), w \rangle + ((u_m'(t), w)) + \langle |v_m(t)|^{\varrho+2} |u_m(t)|^{\varrho} u_m(t), w \rangle \\ & = (f_1(t), w), \\ & (v_m''(t), w) + \langle Av_m(t), w \rangle + ((v_m'(t), w)) + \langle |u_m(t)|^{\varrho+2} |v_m(t)|^{\varrho} v_m(t), w \rangle \\ & = (f_2(t), w) \quad \forall w \in V_m, \\ & u_m(0) = u_{0m} \rightarrow u_0, \text{ in } W_0^{1,p}(\Omega); \\ & u_m'(0) = u_{1m}(0) \rightarrow u_1, \text{ in } L^2(\Omega), \\ & v_m(0) = v_{0m} \rightarrow v_0, \text{ in } W_0^{1,p}(\Omega); \\ & v_m'(0) = v_{1m}(0) \rightarrow v_1, \text{ in } L^2(\Omega). \end{aligned}$$

The system (2) is in the form required by the Caratheodory existence theorem, so there exists a solution $\{u_m(t), v_m(t)\}$ of (2) defined in $[0, t_m[$, $t_m > 0$. In what follows we will obtain some “a priori” estimates that will enable us to extend the solutions $u_m(t)$, $v_m(t)$ to the interval $[0, T]$.

Estimate I. In the system (2) we replace w by $u_m'(t)$ in equation (2)₁, and by $v_m'(t)$ in equation (2)₂.

Then adding both the expressions we get

$$\begin{aligned} (3) \quad & \frac{d}{dt} \left\{ \frac{1}{2} |u_m'(t)|^2 + \frac{1}{2} |v_m'(t)|^2 + \frac{1}{p} \|u_m(t)\|_0^p + \frac{1}{p} \|v_m(t)\|_0^p \right. \\ & \quad \left. + \frac{1}{\varrho+2} \|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \right\} \\ & + \frac{1}{2} (\|u_m'(t)\|^2 + \|v_m'(t)\|^2) \leq |f_1(t)| |u_m'(t)| + |f_2(t)| |v_m'(t)|. \end{aligned}$$

Now, integration from 0 to $t < t_m$, the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ and the continuous immersion of $H_0^1(\Omega)$ in $L^2(\Omega)$, implies

$$\begin{aligned}
 (4) \quad & \frac{1}{2}|u'_m(t)|^2 + \frac{1}{2}|v'_m(t)|^2 + \frac{1}{p}\|u_m(t)\|_0^p + \frac{1}{p}\|v_m(t)\|_0^p \\
 & + \frac{1}{\varrho + 2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} + \frac{1}{2}\int_0^t (\|u'_m(s)\|^2 + \|v'_m(s)\|^2) \, ds \\
 & \leq c \int_0^t (|f_1(s)|^2 + |f_2(s)|^2) \, ds + \frac{1}{2}|u'_m(0)|^2 + \frac{1}{2}|v'_m(0)|^2 \\
 & + \frac{1}{p}\|u_m(0)\|_0^p + \frac{1}{p}\|v_m(0)\|_0^p + \frac{1}{\varrho + 2}\|x_m(0)v_m(0)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}.
 \end{aligned}$$

(*) We remember that $\|\cdot\|_0$ means the norm in $W_0^{1,p}(\Omega)$.

Taking into account hypothesis (1) on f_1, f_2 , (2)₃–(2)₄ and 2.5, from the above expression we get

$$\begin{aligned}
 (5) \quad & \frac{1}{2}|v'_m(t)|^2 + \frac{1}{2}|v'_m(t)|^2 + \frac{1}{p}\|u_m(t)\|_0^p + \frac{1}{p}\|v_m(t)\|_0^p \\
 & + \frac{1}{\varrho + 2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} + \frac{1}{2}\int_0^t (\|u'_m(s)\|^2 + \|v'_m(s)\|^2) \, ds \leq C,
 \end{aligned}$$

where C is a constant independent of t and m .

So, we have:

$u_m(t), v_m(t)$ may be extended to the interval $[0, T]$,

$$(6) \quad (u_m)_m, (v_m)_m \quad \text{are bounded in } L^\infty(0, T; W_0^{1,p}(\Omega)),$$

$$(7) \quad (u'_m)_m, (v'_m)_m \quad \text{are bounded in } L^\infty(0, T; L^2(\Omega)),$$

$$(8) \quad (u'_m)_m, (v'_m)_m \quad \text{are bounded in } L^2(0, T; H_0^1(\Omega)),$$

$$(9) \quad (u_m v_m)_m \quad \text{is bounded in } L^\infty(0, T; L^{\varrho+2}(\Omega)).$$

Furthermore,

$$(10) \quad (Au_m)_m, (Av_m)_m \quad \text{are bounded in } L^\infty(0, T; W^{-1,p'}(\Omega)), \text{ because } A \text{ is}$$

a “bounded” operator, that is, it takes bounded sets into bounded sets.

Estimate II. Now we will obtain an estimate for u''_m, v''_m .

To this end we consider the projection operator given by

$$\begin{aligned}
 (11) \quad & P_m: L^2(\Omega) \longrightarrow L^2(\Omega), \\
 & h \longmapsto P_m h = \sum_{j=1}^m (h, w_j) w_j
 \end{aligned}$$

and suppose that $L^2(\Omega)$ is identified with its dual, so that we have the following sequence of continuous imbeddings:

$$(12) \quad H_0^s(\Omega) \subset W_0^{1,p}(\Omega) \subset H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) \subset W^{-1,p^1}(\Omega) \subset H^{-s}(\Omega)$$

Also, $W_0^{1,p}(\Omega) \subset L^\gamma(\Omega)$ and $L^\theta(\Omega) \subset W^{-1,p'}(\Omega)$.

By using the imbeddings as in (12), 2.5 and the projection operator, we get from the approximate problem, in a standard way, that

$$(13) \quad (u''_m)_-m, (v''_m)_-m \text{ are bounded in } L^2(0, T; H^{-s}(\Omega)).$$

Passage to the limit. As a consequence of (6)–(9) and (13) there exist subsequences denoted by $(u_\nu)_-v$, $(v_\nu)_-v$ such that

$$(14) \quad \begin{aligned} u_\nu &\overset{*}{\rightharpoonup} u, v_\nu \overset{*}{\rightharpoonup} \nu \text{ in } L^\infty(0, T; W_0^{1,p}(\Omega)), \\ u'_\nu &\overset{*}{\rightharpoonup} u', v'_\nu \overset{*}{\rightharpoonup} v' \text{ in } L^\infty(0, T; L^2(\Omega)), \\ u'_\nu &\rightharpoonup u', v'_\nu \rightharpoonup v' \text{ in } L^\infty(0, T; H_0^1(\Omega)), \\ u_\nu v_\nu &\overset{*}{\rightharpoonup} z \text{ in } L^\infty(0, T; L^{\theta+2}(\Omega)), \\ u''_\nu &\rightharpoonup u'', v''_\nu \rightharpoonup v'' \text{ in } L^2(0, T; H^{-s}(\Omega)), \\ Au_\nu &\overset{*}{\rightharpoonup} \eta, Av_\nu \overset{*}{\rightharpoonup} \xi \text{ in } L^\infty(0, T; W^{-1,p'}(\Omega)), \\ |v_\nu|^{\theta+2} |u_\nu|^\theta u_\nu &\overset{*}{\rightharpoonup} \lambda, |u_\nu|^{\theta+2} |v_\nu|^\theta v_\nu \overset{*}{\rightharpoonup} \mu \text{ in } L^\infty(0, T; L^\theta(\Omega)). \end{aligned}$$

Now, by (14), Aubin-Lions Compactness Theorem and Lion's Lemma 1.3 (see [5], [7]), we get

$$(15) \quad \begin{aligned} u_\nu v_\nu &\overset{*}{\rightharpoonup} uv \text{ in } L^\infty(0, T; L^{\theta+2}(\Omega)), \\ |v_\nu|^{\theta+2} |u_\nu|^\theta u_\nu &\overset{*}{\rightharpoonup} |v|^{\theta+2} |u|^\theta u \text{ in } L^\infty(0, T; L^\theta(\Omega)), \\ |u_\nu|^{\theta+2} |v_\nu|^\theta v_\nu &\overset{*}{\rightharpoonup} |u|^{\theta+2} |v|^\theta v \text{ in } L^\infty(0, T; L^\theta(\Omega)). \end{aligned}$$

From now on we consider the equation $(2)_1$ in the form

$$(16) \quad \begin{aligned} (u''_\nu(t), w) + \langle Au_\nu(t), w \rangle + ((u'_\nu(t), w)) \\ + \langle |v_\nu(t)|^{\theta+2} |u_\nu(t)|^\theta u_\nu(t), w \rangle = (f_1(t), w) \end{aligned}$$

where $w \in V_m$, $\nu \geq m$.

Multiplying (16) by $\varphi \in D(0, T)$, integrating from 0 to t and passing to the limit as $\nu \rightarrow \infty$, we deduce from the convergence in (14) and (15) that

$$(17) \quad \begin{aligned} - \int_0^T (u'(t), w) \varphi' dt + \int_0^T \langle \eta(t), w \rangle \varphi dt + \int_0^T ((u'(t), w)) \varphi dt \\ + \int_0^T \langle |v(t)|^{\theta+2} |u(t)|^\theta u(t), w \rangle \varphi dt = \int_0^T (f_1(t), w) \varphi dt, \quad \forall w \in V_m, \end{aligned}$$

$\forall \varphi \in D(T)$ and, by a density argument, $\forall w \in W_0^{1,p}(\Omega)$, $\forall \varphi \in D(0, T)$.

In a similar way it results from equation (2)₂ that

$$(17') \quad - \int_0^T (v'(t), w) \varphi' dt + \int_0^T \langle \xi(t), w \rangle \varphi dt + \int_0^T ((v'(t), w)) \varphi dt \\ + \int_0^T \langle |u(t)|^{e+2} |v(t)|^e v(t), w \rangle \varphi dt = \int_0^T (f_2(t), w) \varphi dt, \quad \forall w \in W_0^{1,p}(\Omega).$$

In order to establish the theorem we next prove that $Au(t) = \eta(t)$, $Av(t) = \xi(t)$. To this end we suppose the initial conditions $u(0) = u_0$, $u'(0) = u_1$, $v(0) = v_0$ and $v'(0) = v_1$ are already proved.

Here, as it is known, it is essential to have the strong convergence

$$(18) \quad u'_\nu \rightarrow u', \quad v'_\nu \rightarrow v' \quad \text{in} \quad L^2(0, T; L^2(\Omega)) \equiv L^2(Q).$$

So let us multiply the equation (16) by $\varphi \in C^1([0, T])$ and integrate from 0 to T obtaining

$$(19) \quad (u'_\nu(T), w\varphi(T)) - (u'_\nu(0), w\varphi(0)) - \int_0^T (u'_\nu(t), w\varphi(t)) dt \\ + \int_0^T \langle Au_\nu(t), w\varphi(t) \rangle dt + \int_0^T ((u'_\nu(t), w\varphi(t)) dt \\ + \int_0^T \langle |v'_\nu(t)|^{e+2} |u_\nu(t)|^e u_\nu(t), w\varphi(t) \rangle dt = \int_0^T (f_1(t), w\varphi(t)) dt, \quad \forall w \in V_m.$$

Since the set of finite linear combinations of products of the type $w\varphi$, $w \in W_0^{1,p}(\Omega)$, $\varphi \in C^1([0, T])$, is dense in $V = \{v \in L^2(0, T; W_0^{1,p}(\Omega)); v' \in L^2(0, T; L^2(\Omega))\}$ and since $u \in V$, by passing to the limit with $\nu \rightarrow \infty$ in the equation (19) we get

$$(u'(T), u(T)) - (u'(0), u(0)) - \int_0^T (u'(t), u'(t)) dt + \int_0^T \langle \eta(t), u(t) \rangle dt \\ + \int_0^T ((u'(t), u(t))) dt + \int_0^T \|v(t)u(t)\|_{L^{e+2}(\Omega)}^{e+2} = \int_0^T (f_1(t), u(t)) dt.$$

On the other hand, since A is a monotone operator we have

$$0 \leq \int_0^T \langle Au_\nu(t) - Aw, u_\nu(t) - w \rangle dt, \quad \forall w \in W_0^{1,p}(\Omega),$$

and by a straightforward but lengthy calculation, we conclude

$$\begin{aligned}
 (21) \quad 0 &\leq (u'(0), u(0)) - (u'(T), u(T)) + \int_0^T |u'(t)|^2 dt \\
 &+ \frac{1}{2} \|u(0)\|^2 - \frac{1}{2} \|u(T)\|^2 - \int_0^T \|u(t)v(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \\
 &+ \int_0^T (f_1(t), u(t)) dt - \int_0^T \langle \eta(t), w \rangle dt - \int_0^T \langle Aw, u(t) - w \rangle dt, \\
 &\forall w \in W_0^{1,p}(\Omega).
 \end{aligned}$$

Next we substitute (20) into (21) to obtain that

$$(22) \quad 0 \leq \int_0^T \langle \eta(t) - Aw, u(t) - w \rangle dt, \quad \forall w \in W_0^{1,p}(\Omega).$$

From this inequality, as A is a hemicontinuous operator, we have $Au(t) = \eta(t)$.

In an analogous way we show that $Av(t) = \xi(t)$.

The initial conditions are proved in a standard way (see [3]) and so the proof of Theorem 1 is complete.

ASYMPTOTIC BEHAVIOUR

In what follows we will consider $f_1 = f_2 = 0$ and in this case we can extend the solution $\{u, v\}$, obtained in Theorem 1, to the interval $[0, +\infty)$. So in order to study the asymptotic behaviour of the solution of the problem (***) with $f_1 = f_2 = 0$, we first consider the energy of the following approximate problem:

$$\begin{aligned}
 (23) \quad & (u_m''(t), w) + \langle Au_m(t), w \rangle + ((u_m'(t), w)) + \langle |v_m(t)|^{\varrho+2} |u_m(t)|^\varrho u_m(t), w \rangle = 0, \\
 & (v_m''(t), w) + \langle Av_m(t), w \rangle + ((v_m'(t), w)) + \langle |u_m(t)|^{\varrho+2} |v_m(t)|^\varrho v_m(t), w \rangle = 0, \\
 & u_m(0) = u_{0m} \rightarrow u_0, \quad \text{in } W_0^{1,p}(\Omega); \quad u_m'(0) = u_{1m} \rightarrow u_1, \quad \text{in } L^2(\Omega), \\
 & v_m(0) = v_{0m} \rightarrow v_0, \quad \text{in } W_0^{1,p}(\Omega); \quad v_m'(0) = v_{1m} \rightarrow v_1, \quad \text{in } L^2(\Omega).
 \end{aligned}$$

We remember that in this case, $u_m(t), v_m(t)$ may be extended to the whole interval $[0, \infty)$.

We define the energy of the system (23) by

$$\begin{aligned}
 (24) \quad E_m(t) &= \frac{1}{2} |u_m'(t)|^2 + \frac{1}{2} |v_m'(t)|^2 + \frac{1}{p} \|u_m(t)\|_0^p + \frac{1}{p} \|v_m(t)\|_0^p \\
 &+ \frac{1}{\varrho+2} \|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}
 \end{aligned}$$

and it is simple to verify that $E_m(t)$ is a decreasing function for $t \geq 0$, with $0 \leq E_m(t) \leq E_m(0)$, $\forall t \geq 0$.

The study of the behaviour of the energy $E_m(t)$ of the system in (23) in the interval $[t, t + 1]$ leads us after a rather lengthy calculation (see [3], [6] for details) to the inequality

$$(25) \quad E_m^{\frac{2}{p'}}(t) \leq c(E_m(t) - E_m(t + 1)).$$

From (25) and by Nakao's Lemma [6] we obtain

$$(26) \quad E_m(t) \leq c(1 + t)^{-\frac{1}{\beta}}, \quad \forall t \geq 0, \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1, \\ \frac{2}{p'} = 1 + \beta, \quad \beta > 0.$$

The inequality (26) means that the energy of the approximate system (23) has an algebraic decay.

The next step is to take $\liminf (m \rightarrow \infty)$ in the expression of the approximate energy $E_m(t)$ to obtain

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|v'(t)|^2 + \frac{1}{p}\|u(t)\|_0^p + \frac{1}{p}\|v(t)\|_0^p \\ + \frac{1}{\varrho + 2}\|u(t)v(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \leq c(1 + t)^{-\frac{1}{\beta}}, \quad \forall t \geq 0,$$

where β is as defined before and given by

$$\beta = \frac{2}{p'} - 1 > 0.$$

Therefore the energy associated to the system (***) with $f_1 = f_2 = 0$ has an algebraic decay.

References

- [1] *A. Biazutti*: Sobre uma Equação não Linear de Vibrações. Existência de Soluções Fracas e Comportamento Assintótico. Tese de Doutorado – IMUFRJ.
- [2] *F.E. Browder, Bui An Ton.*: Nonlinear functional equations in Banach spaces and elliptic super regularization. *Math. Zeitsch.* 105 (1968), 177–195.
- [3] *N.N. Castro*: Sobre um Problema de Evolução não Linear. Tese de Doutorado, IMUFRJ, 1995.
- [4] *E. Coddington, N. Levinson*: Theory of Ordinary Differential Equations. New York.
- [5] *J.L. Lions*: Quelques Méthodes de Résolutions des Problèmes aux Limites Non Linéaires. Dunod Paris, 1969.

- [6] *M. Nakao*: A difference inequality and its applications to nonlinear evolution equations. J. Math. Soc. Jap. (10)4 (1978), 747–762.
- [7] *L.A. Medeiros, M.M. Miranda*: Weak solutions for a system of nonlinear Klein-Gordon equations. Annali Di Matematica Pura ed Applicata IV (CXLVI), 173–183.
- [8] *M. Tsutsumi*: Some nonlinear evolution equations of second order. Proc. Jap. Acad. 47 (1971), 950–955.

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