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BAD LUCK IN QUADRATIC IMPROVEMENT OF THE LINEAR ESTIMATOR IN A SPECIAL LINEAR MODEL

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Abstract. The paper concludes our investigations in looking for the locally best linear-quadratic estimators of mean value parameters and of the covariance matrix elements in a special structure of the linear model (2 variables case) where the dispersions of the observed quantities depend on the mean value parameters. Unfortunately there exists no linear-quadratic improvement of the linear estimator of mean value parameters in this model.

Keywords: linear model with dispersions depending on the mean value parameters, locally best linear-quadratic unbiased estimator (LBLQUE) of mean value parameters

MSC 2000: 62J05, 62F10

1. Introduction

In the case of measuring a linear dependence (2 variables case) with a measuring device whose dispersion characteristic is linear-quadratically dependent on the actual measured value we obtain the model

\[ (Y, X\beta, \Sigma), \]

where \( Y_{n,1} \) is considered to be a normally distributed random vector. Its realization \( y_{n,1} \) is the result of the measurement. The mean value is \( \mathcal{E}(Y) = X_{n,2}\beta_{2,1} \), where

\[
X_{n,2} = \begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_k
\end{pmatrix}
\]

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with \( X_i = \begin{pmatrix} 1 & t_i \\ 1 & t_i \\ \vdots & \vdots \\ 1 & t_i \end{pmatrix} \) of order \( n_i \times 2 \), \( n_i \geq 1 \), \( i = 1, 2, \ldots, k \), \( k \geq 3 \), \( n = \sum_{i=1}^{k} n_i \),
t_1 < t_2 < \ldots < t_k$, \( \beta \in \mathbb{R}^2 \) (two dimensional Euclidean space).

The covariance matrix of the vector \( Y \) is

\[
\Sigma = \sigma^2 \Sigma(\beta) = \sigma^2 \begin{pmatrix}
(a + b|e'_1 X \beta|)^2 & 0 & \ldots & 0 \\
0 & (a + b|e'_2 X \beta|)^2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & (a + b|e'_n X \beta|)^2
\end{pmatrix},
\]

where \( a \), \( b \) and \( \sigma^2 \) are known positive constants (the characteristics of the measuring device, for more details see e.g. [3] p. 456, 914), \( e'_i \) is the transpose of the \( i \)-th unit vector.

The paper is based on results obtained in [4], [5], [6], [7] and [8]. In [4], Lemma 3.1 a necessary and sufficient condition for the statistic \( p'Y \) to be the UBLUE (uniformly best linear estimator, see e.g. [1]) of its mean value was shown. According to this condition it is stated in Section 2 that in model (1.1) the UBLUE of the parametric function (linear functional) \( f' \beta \) does not exist (it exists only for \( f = 0 \) \( _{2,1} \)). That is why (according to further considerations in [4]) only the \( \beta_0 \)-LBLUE (locally best linear unbiased estimator, see e.g. [1]) exists. Our effort is to find the \( \beta_0 \)-LBLQUE (locally best linear-quadratic unbiased estimator, see e.g. [6], [7]) in model (1.1). In Section 3 it is shown that our effort ended unsuccessfully in the sense that in model (1.1) exists no locally best linear-quadratic unbiased estimator as an improvement of the \( \beta_0 \)-LBLUE of \( f' \beta \). This is also the goal of this paper.

We want only to remark for completeness that the \( \beta_0 \)-LBLQUE of the covariance matrix elements in model (1.1) (in the case of no or one independently repeated observation) can be found in [8] together with its asymptotic behaviour (some results are stated also in [5]).

2. THE UBLUE OF \( f' \beta \) IN MODEL (1.1)

According to Lemma 3.1 and Lemma 3.2 in [4] the statistic \( p'Y \) is the UBLUE of its mean value in model (1.1) if and only if

\[
\forall \{\beta \in \mathbb{R}^2\} \quad (I - XX^-) \Sigma(\beta) p = 0
\]

for an arbitrary but fixed g-inverse \( X^- \) (see e.g. [2]). In this case \( p \in \mu(X) = \{Xu: u \in \mathbb{R}^2\} \).
Let the matrix $C_1$ (of order $2 \times n_1$) be
\[
\begin{pmatrix}
-\frac{t_2}{t_1-t_2} & 0 & \ldots & 0 \\
\frac{1}{t_1-t_2} & 0 & \ldots & 0
\end{pmatrix},
\]
while the matrix $C_2$ (of order $2 \times n_2$) is
\[
\begin{pmatrix}
\frac{t_1}{t_1-t_2} & 0 & \ldots & 0 \\
-\frac{1}{t_1-t_2} & 0 & \ldots & 0
\end{pmatrix}.
\]
It can be easily shown that the matrix $X^-=(C_1\ C_2\ O)$ (of order $2 \times n$) is a g-inverse of the matrix $X$. So the matrix $I-XX^-$ (of order $n \times n$) is
\[
I-XX^- = \begin{pmatrix}
D_{1,1} & D_{1,2} & \ldots & D_{1,k} \\
D_{2,1} & D_{2,2} & \ldots & D_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
D_{k,1} & D_{k,2} & \ldots & D_{k,k}
\end{pmatrix},
\]
where the matrices $D_{i,j} \ i = 1, 2, \ldots, k \ j = 1, 2, \ldots, k$ are of orders $n_i \times n_j$, respectively. Here
\[
D_{1,1} = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
-1 & 1 & 0 & \ldots & 0 \\
-1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-1 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]
is of order $n_1 \times n_1$,
\[
D_{2,2} = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
-1 & 1 & 0 & \ldots & 0 \\
-1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-1 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]
is of order $n_2 \times n_2$,
\[
D_{i,1} = \begin{pmatrix}
\frac{t_2-t_i}{t_1-t_2} & 0 & \ldots & 0 \\
\frac{t_2-t_i}{t_1-t_2} & 0 & \ldots & 0 \\
\frac{t_2-t_i}{t_1-t_2} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{t_2-t_i}{t_1-t_2} & 0 & \ldots & 0
\end{pmatrix}
\]
is of order $n_i \times n_1$, $i = 3, 4, \ldots, k$,
\[
D_{i,2} = \begin{pmatrix}
\frac{t_i-t_1}{t_1-t_2} & 0 & \ldots & 0 \\
\frac{t_i-t_1}{t_1-t_2} & 0 & \ldots & 0 \\
\frac{t_i-t_1}{t_1-t_2} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{t_i-t_1}{t_1-t_2} & 0 & \ldots & 0
\end{pmatrix}
\]
is of order \( n_i \times n_2, i = 3, 4, \ldots, k, \)
\[
D_{i,i} = I
\]
is the unit matrix of order \( n_i \times n_i, i = 3, 4, \ldots, k \) and the other \( D_{i,j} \) are equal to \( O \) of proper orders. As \( p \) in (2.1) belongs to \( \mu(X) \), we can write it as \( p = Xw \), where \( w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \).

So finally we have

\begin{equation}
(I - XX^{-})\Sigma(\beta)p = (I - XX^{-})\Sigma(\beta)Xw = g = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_k \end{pmatrix},
\end{equation}

where \( g \) is of order \( n \times 1 \), \( g_i = 0 \) is of order \( n_i \times 1, i = 1, 2 \) and

\[
g_j = 1_j \otimes [(a + b|\beta_1 + t_1\beta_2|)^2 \frac{t_2 - t_j}{t_1 - t_2}(w_1 + t_1w_2) + (a + b|\beta_1 + t_2\beta_2|)^2 \frac{t_j - t_1}{t_1 - t_2}(w_1 + t_2w_2) + (a + b|\beta_1 + t_3\beta_2|)^2 (w_1 + t_3w_2)]
\]

with \( 1_j = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \) of order \( n_j \times 1, j = 3, 4, \ldots, k. \)

By \( \otimes \) we denote the Kronecker product (see e.g. [2], p. 11).

According to (2.1) we are looking for a statistic \( p'Y \) to be the UBLUE of its mean value, i.e. we are searching for such a vector \( w \) that \( g \) in (2.2) is equal to \( O \) for all \( \beta \in \mathbb{R}^2 \). As \( k \geq 3 \), let us take the first element of the vector \( g_3 \). (2.1) implies that for all \( \beta_1, \beta_2 \in \mathbb{R}, \)

\begin{equation}
(a + b|\beta_1 + t_1\beta_2|)^2 \frac{t_2 - t_3}{t_1 - t_2}(w_1 + t_1w_2) + (a + b|\beta_1 + t_2\beta_2|)^2 \frac{t_3 - t_1}{t_1 - t_2}(w_1 + t_2w_2) + (a + b|\beta_1 + t_3\beta_2|)^2 (w_1 + t_3w_2) = 0.
\end{equation}

Let \( \beta_1^{(1)} = \frac{t_3}{t_3 - t_1}, \beta_2^{(1)} = \frac{-1}{t_3 - t_1} \). So (2.3) is of the form

\begin{equation}
[(a + b)^2 \frac{t_2 - t_3}{t_1 - t_2} + (a + b \frac{t_3 - t_2}{t_3 - t_1})^2 \frac{t_3 - t_1}{t_1 - t_2} + a^2]w_1
+ [(a + b)^2 \frac{t_2 - t_3}{t_1 - t_2} t_1 + (a + b \frac{t_3 - t_2}{t_3 - t_1})^2 \frac{t_3 - t_1}{t_1 - t_2} t_2 + a^2 t_3]w_2 = 0.
\end{equation}
For another choice of \( \beta_1, \beta_2: \beta_1^{(2)} = \frac{t_3}{t_3-t_2}, \beta_2^{(2)} = \frac{-1}{t_3-t_2}, \) (2.3) is of the form

\[
(2.5) \left[ \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_3-t_2}{t_1-t_2} + \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_3-t_1}{t_1-t_2} + a^2 \right] w_1 \\
+ \left[ \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_2-t_3}{t_1-t_2} + \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_3-t_1}{t_1-t_2} + a^2 t_3 \right] w_2 = 0.
\]

Because of

\[
\left[ (a+b)^2 \frac{t_2-t_3}{t_1-t_2} + \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_3-t_1}{t_1-t_2} + a^2 \right] \\
\times \left[ \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_2-t_3}{t_1-t_2} + \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_3-t_1}{t_1-t_2} + a^2 t_3 \right] \\
- \left[ (a+b)^2 \frac{t_2-t_3}{t_1-t_2} + \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_2-t_3}{t_1-t_2} + a^2 t_3 \right] \\
\times \left[ \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_3-t_2}{t_1-t_2} + \left( a + \frac{t_3-t_1}{t_3-t_2} \right)^2 \frac{t_3-t_1}{t_1-t_2} + a^2 \right] \\
= 2ab^3(t_1-t_2) \neq 0
\]

(which can be obtained after a straightforward calculation), equations (2.4) and (2.5) imply \( w_1 = w_2 = O. \)

So \( p'Y \) (with \( p = Xw \)) is the UBLUE of its mean value if and only if \( p = O. \)

The only linear functional \( f'\beta \) having the UBLUE in model (1.1) is equal to 0 for all \( \beta \in \mathbb{R}^2. \) There exists no UBLUE of a (nonzero) linear functional \( f'\beta \) in model (1.1). That is why we can only look for the \( \beta_0 \)-UBLUE of the functional \( f'\beta. \) According to Lemma 2.4 and Remark 2.5 in [4] the \( \beta_0 \)-UBLUE of \( f'\beta \) (\( f \in \mu(X') \)) is

\[
(2.6) \quad f'(X'\Sigma^{-1}(\beta_0)X)^{-1}X'\Sigma^{-1}(\beta_0)Y
\]

where \( (X'\Sigma^{-1}(\beta_0)X)^{-1} \) is an arbitrary but fixed g-inverse of the matrix \( X'\Sigma^{-1}(\beta_0)X. \)

We only note that the \( \beta_0 \)-UBLUE of \( f'\beta \) exists if and only if \( f \in \mu(X') \) and is unique.

3. The \( \beta_0 \)-UBLUE of \( f'\beta \) in model (1.1)

Based on Lemma 1.8 in [7] we will show that in model (1.1) the \( \beta_0 \)-UBLUE as an improvement of the \( \beta_0 \)-UBLUE of any \( f'\beta \) does not exist. Also we have

\[
\{ f: \exists \beta_0 - \text{UBLUE for } f'\beta \} = \{ f: \exists \beta_0 - \text{UBLUE for } f'\beta \} = \mu(X').
\]
Let us denote by $\mathcal{D}$ the class of matrices $D_{n,n}$ satisfying the following three conditions:

\begin{align}
\forall \{ \beta \in \mathbb{R}^2 \} & \quad \text{Tr} \, D \begin{pmatrix}
|e'_1 X \beta| & 0 & \cdots & 0 \\
0 & |e'_2 X \beta| & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & |e'_n X \beta|
\end{pmatrix} = 0, \\
(3.1) & \quad \text{Tr} \, D = 0, \\
(3.2) & \quad X'(D + \sigma^2 b^2 \sum_{i=1}^n e'_i D e_i) X = 0.
\end{align}

(Tr $D$ is the trace of $D$ i.e. $\sum_{i=1}^n e'_i D e_i$.)

Let $D \in \mathcal{D}$ have the $(i, j)$-th element $d_{i,j}$, $i, j = 1, 2, \ldots, n$. From (3.1) it follows that for all $\beta_1, \beta_2 \in \mathbb{R}$ we have

\begin{equation}
(3.4) \quad \sum_{i=1}^{n_1} d_{i,i} |\beta_1 + \sum_{j=1}^{n_1} d_{i,j} |\beta_2| + \sum_{i=n_1+1}^{n_1+n_2} d_{i,i} |\beta_1 + \beta_2| + \cdots + \sum_{i=n_1+\ldots+n_k-1}^{n_1+\ldots+n_k} d_{i,i} |\beta_1 + \beta_2| = 0.
\end{equation}

If we denote

\begin{equation}
\sum_{i=1}^{n_1} d_{i,i} = d_1, \quad \sum_{i=n_1+1}^{n_1+n_2} d_{i,i} = d_2, \ldots, \quad \sum_{i=n_1+\ldots+n_k-1}^{n_1+\ldots+n_k} d_{i,i} = d_k
\end{equation}

then following the same way as in [8], Lemma 9.1, we obtain that condition (3.4) is equivalent to

\begin{equation}
(3.5) \quad d_1 = d_2 = \ldots = d_k = 0.
\end{equation}

So (3.2) follows from (3.1). Taking it into account, for all $D \in \mathcal{D}$ from (3.3) we obtain

\begin{equation}
X' DX = -\sigma^2 b^2 X' \sum_{i=1}^n e'_i D e_i X = -\sigma^2 b^2 \left( \sum_{i=1}^k t_i d_i \sum_{i=1}^k t_i^2 d_i \right) = 0.
\end{equation}

Now we apply Lemma 1.8 in [7] and obtain the desired result

\begin{equation}
\{ f : \exists \beta_0 - \text{LBLUQE for } f' \beta \} = \mu(X')
\end{equation}

i.e. the class of linear functionals $f' \beta$ having the $\beta_0$-LBLUQE is the same as the class of linear functionals having the $\beta_0$-LBLUQE. Further the $\beta_0$-LBLUQE is the same as the $\beta_0$-LBLUE of $f' \beta$ for all $f \in \mu(X')$. 6
Final conclusions for the model (1.1).

(i) The uniformly best linear unbiased estimator (UBLUE) of any nonzero linear functional $f'\beta$ does not exist.

(ii) The $\beta_0$-locally best linear unbiased estimator ($\beta_0$-LBLUE) of $f'\beta$ exists if and only if $f \in \mu(X')$, it is unique and of the form (2.6).

(iii) The $\beta_0$-locally best linear-quadratic unbiased estimator ($\beta_0$-LBLQUE) of $f'\beta$ exists for all $f \in \mu(X')$ and is the same as the $\beta_0$-LBLUE of $f'\beta$ (i.e., there exists no quadratic improvement of the $\beta_0$-LBLUE of any $f'\beta$, $f \in \mu(X')$).

□

References


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