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BAD LUCK IN QUADRATIC IMPROVEMENT OF THE LINEAR ESTIMATOR IN A SPECIAL LINEAR MODEL

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Abstract. The paper concludes our investigations in looking for the locally best linear-quadratic estimators of mean value parameters and of the covariance matrix elements in a special structure of the linear model (2 variables case) where the dispersions of the observed quantities depend on the mean value parameters. Unfortunately there exists no linear-quadratic improvement of the linear estimator of mean value parameters in this model.

Keywords: linear model with dispersions depending on the mean value parameters, locally best linear-quadratic unbiased estimator (LBLQUE) of mean value parameters

MSC 2000: 62J05, 62F10

1. Introduction

In the case of measuring a linear dependence (2 variables case) with a measuring device whose dispersion characteristic is linear-quadratically dependent on the actual measured value we obtain the model

$$(\mathbf{Y}, \mathbf{X}\beta, \mathbf{\Sigma}), \tag{1.1}$$

where $\mathbf{Y}_{n,1}$ is considered to be a normally distributed random vector. Its realization $\mathbf{y}_{n,1}$ is the result of the measurement. The mean value is $\mathscr{E}(\mathbf{Y}) = \mathbf{X}_{n,2}\beta_{2,1}$, where

$$\mathbf{X}_{n,2} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_k \end{pmatrix}$$

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with $\mathbf{X}_i = \begin{pmatrix} 1 & t_i \\ 1 & t_i \\ \vdots & \vdots \\ 1 & t_i \end{pmatrix}$ of order $n_i \times 2,\ n_i \geqslant 1,\ i = 1, 2, \ldots, k,\ k \geqslant 3,\ n = \sum_{i=1}^{k} n_i,$ $t_1 < t_2 < \ldots < t_k,\ \beta \in \mathbb{R}^2$ (two dimensional Euclidean space).

The covariance matrix of the vector $\mathbf{Y}$ is

$$\mathbf{\Sigma} = \sigma^2 \mathbf{\Sigma}(\beta) = \sigma^2 \begin{pmatrix} (a + b|\mathbf{e}_1'\mathbf{X}\beta|)^2 & 0 & \ldots & 0 \\ 0 & (a + b|\mathbf{e}_2'\mathbf{X}\beta|)^2 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & (a + b|\mathbf{e}_n'\mathbf{X}\beta|)^2 \end{pmatrix},$$

where $a,\ b$ and σ^2 are known positive constants (the characteristics of the measuring device, for more details see e.g. [3] p. 456, 914), $\mathbf{e}_i'$ is the transpose of the i-th unit vector.

The paper is based on results obtained in [4], [5], [6], [7] and [8]. In [4], Lemma 3.1 a necessary and sufficient condition for the statistic $\mathbf{p}'\mathbf{Y}$ to be the UBLUE (uniformly best linear estimator, see e.g. [1]) of its mean value was shown. According to this condition it is stated in Section 2 that in model (1.1) the UBLUE of the parametric function (linear functional) $\mathbf{f}'\beta$ does not exist (it exists only for $\mathbf{f} = \mathbf{0}_{2,1}$). That is why (according to further considerations in [4]) only the β_0-LBLUE (locally best linear unbiased estimator, see e.g. [1]) exists. Our effort is to find the β_0-LBLQUE (locally best linear-quadratic unbiased estimator, see e.g. [6], [7]) in model (1.1). In Section 3 it is shown that our effort ended unsuccessfully in the sense that in model (1.1) exists no locally best linear-quadratic unbiased estimator as an improvement of the β_0-LBLUE of $\mathbf{f}'\beta$. This is also the goal of this paper.

We want only to remark for completeness that the β_0-LBLQUE of the covariance matrix elements in model (1.1) (in the case of no or one independently repeated observation) can be found in [8] together with its asymptotic behaviour (some results are stated also in [5]).

2. The UBLUE of $\mathbf{f}'\beta$ in model (1.1)

According to Lemma 3.1 and Lemma 3.2 in [4] the statistic $\mathbf{p}'\mathbf{Y}$ is the UBLUE of its mean value in model (1.1) if and only if

$$(2.1) \qquad \forall\{\beta \in \mathbb{R}^2\} \qquad (\mathbf{I} - \mathbf{X}\mathbf{X}^-)\mathbf{\Sigma}(\beta)\mathbf{p} = \mathbf{0}$$

for an arbitrary but fixed g-inverse $\mathbf{X}^-$ (see e.g. [2]). In this case $\mathbf{p} \in \mu(\mathbf{X}) = \{\mathbf{X}\mathbf{u}\colon\ \mathbf{u} \in \mathbb{R}^2\}$.

Let the matrix $\mathbf{C}_1$ (of order $2 \times n_1$) be

$$\begin{pmatrix} -\frac{t_2}{t_1-t_2} & 0 & \dots & 0 \\ \frac{1}{t_1-t_2} & 0 & \dots & 0 \end{pmatrix},$$

while the matrix $\mathbf{C}_2$ (of order $2 \times n_2$) is

$$\begin{pmatrix} \frac{t_1}{t_1-t_2} & 0 & \dots & 0 \\ -\frac{1}{t_1-t_2} & 0 & \dots & 0 \end{pmatrix}.$$

It can be easily shown that the matrix $\mathbf{X}^- = (\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{O})$ (of order $2 \times n$) is a g-inverse of the matrix $\mathbf{X}$. So the matrix $\mathbf{I} - \mathbf{X}\mathbf{X}^-$ (of order $n \times n$) is

$$\mathbf{I} - \mathbf{X}\mathbf{X}^- = \begin{pmatrix} \mathbf{D}_{1,1} & \mathbf{D}_{1,2} & \dots & \mathbf{D}_{1,k} \\ \mathbf{D}_{2,1} & \mathbf{D}_{2,2} & \dots & \mathbf{D}_{2,k} \\ \vdots & & \ddots & \\ \mathbf{D}_{k,1} & \mathbf{D}_{k,2} & \dots & \mathbf{D}_{k,k} \end{pmatrix},$$

where the matrices $\mathbf{D}_{i,j}\ \ i = 1, 2, \dots, k\ ,\ j = 1, 2, \dots, k$ are of orders $n_i \times n_j$, respectively. Here

$$\mathbf{D}_{1,1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}$$

is of order $n_1 \times n_1$,

$$\mathbf{D}_{2,2} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}$$

is of order $n_2 \times n_2$,

$$\mathbf{D}_{i,1} = \begin{pmatrix} \frac{t_2-t_i}{t_1-t_2} & 0 & \dots & 0 \\ \frac{t_2-t_i}{t_1-t_2} & 0 & \dots & 0 \\ \vdots & & & \\ \frac{t_2-t_i}{t_1-t_2} & 0 & \dots & 0 \end{pmatrix}$$

is of order $n_i \times n_1$, $i = 3, 4, \dots, k$,

$$\mathbf{D}_{i,2} = \begin{pmatrix} \frac{t_i-t_1}{t_1-t_2} & 0 & \dots & 0 \\ \frac{t_i-t_1}{t_1-t_2} & 0 & \dots & 0 \\ \vdots & & & \\ \frac{t_i-t_1}{t_1-t_2} & 0 & \dots & 0 \end{pmatrix}$$

is of order $n_i \times n_2$, $i = 3, 4, \ldots, k$,

$$\mathbf{D}_{i,i} = \mathbf{I}$$

is the unit matrix of order $n_i \times n_i$, $i = 3, 4, \ldots, k$ and the other $\mathbf{D}_{i,j}$ are equal to $\mathbf{O}$ of proper orders. As $\mathbf{p}$ in (2.1) belongs to $\mu(\mathbf{X})$, we can write it as $\mathbf{p} = \mathbf{X}\mathbf{w}$, where $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

So finally we have

$$(2.2) \qquad (\mathbf{I} - \mathbf{X}\mathbf{X}^-)\boldsymbol{\Sigma}(\beta)\mathbf{p} = (\mathbf{I} - \mathbf{X}\mathbf{X}^-)\boldsymbol{\Sigma}(\beta)\mathbf{X}\mathbf{w} = \mathbf{g} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \\ \vdots \\ \mathbf{g}_k \end{pmatrix},$$

where $\mathbf{g}$ is of order $n \times 1$, $\mathbf{g}_i = \mathbf{0}$ is of order $n_i \times 1$, $i = 1, 2$ and

$$\mathbf{g}_j = \mathbf{1}_j \otimes [(a + b|\beta_1 + t_1\beta_2|)^2 \frac{t_2 - t_j}{t_1 - t_2}(w_1 + t_1 w_2)$$
$$+ (a + b|\beta_1 + t_2\beta_2|)^2 \frac{t_j - t_1}{t_1 - t_2}(w_1 + t_2 w_2) + (a + b|\beta_1 + t_j\beta_2|)^2 (w_1 + t_j w_2)]$$

with $\mathbf{1}_j = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ of order $n_j \times 1$, $j = 3, 4, \ldots, k$.

By $\otimes$ we denote the Kronecker product (see e.g. [2], p. 11).

According to (2.1) we are looking for a statistic $\mathbf{p}'\mathbf{Y}$ to be the UBLUE of its mean value, i.e. we are searching for such a vector $\mathbf{w}$ that $\mathbf{g}$ in (2.2) is equal to $\mathbf{O}$ for all $\beta \in \mathbb{R}^2$. As $k \geqslant 3$, let us take the first element of the vector $\mathbf{g}_3$. (2.1) implies that for all $\beta_1, \beta_2 \in \mathbb{R}$,

$$(2.3) \quad (a + b|\beta_1 + t_1\beta_2|)^2 \frac{t_2 - t_3}{t_1 - t_2}(w_1 + t_1 w_2)$$
$$+ (a + b|\beta_1 + t_2\beta_2|)^2 \frac{t_3 - t_1}{t_1 - t_2}(w_1 + t_2 w_2) + (a + b|\beta_1 + t_3\beta_2|)^2 (w_1 + t_3 w_2) = 0.$$

Let $\beta_1^{(1)} = \frac{t_3}{t_3 - t_1}$, $\beta_2^{(1)} = \frac{-1}{t_3 - t_1}$. So (2.3) is of the form

$$(2.4) \quad \Big[(a + b)^2 \frac{t_2 - t_3}{t_1 - t_2} + \Big(a + b\frac{t_3 - t_2}{t_3 - t_1}\Big)^2 \frac{t_3 - t_1}{t_1 - t_2} + a^2\Big] w_1$$
$$+ \Big[(a + b)^2 \frac{t_2 - t_3}{t_1 - t_2} t_1 + \Big(a + b\frac{t_3 - t_2}{t_3 - t_1}\Big)^2 \frac{t_3 - t_1}{t_1 - t_2} t_2 + a^2 t_3\Big] w_2 = 0.$$

For another choice of β_1, β_2: $\beta_1^{(2)} = \frac{t_3}{t_3-t_2}$, $\beta_2^{(2)} = \frac{-1}{t_3-t_2}$, (2.3) is of the form

$$\text{(2.5)} \quad \Big[\Big(a + b\frac{t_3 - t_1}{t_3 - t_2}\Big)^2 \frac{t_3 - t_2}{t_1 - t_2} + (a+b)^2 \frac{t_3 - t_1}{t_1 - t_2} + a^2\Big] w_1 + \Big[\Big(a + b\frac{t_3 - t_1}{t_3 - t_2}\Big)^2 \frac{t_2 - t_3}{t_1 - t_2} t_1 + (a+b)^2 \frac{t_3 - t_1}{t_1 - t_2} + a^2 t_3\Big] w_2 = 0.$$

Because of

$$\begin{aligned}
&\Big[(a+b)^2 \frac{t_2 - t_3}{t_1 - t_2} + \Big(a + b\frac{t_3 - t_2}{t_3 - t_1}\Big)^2 \frac{t_3 - t_1}{t_1 - t_2} + a^2\Big] \\
&\qquad \times \Big[\Big(a + b\frac{t_3 - t_1}{t_3 - t_2}\Big)^2 \frac{t_2 - t_3}{t_1 - t_2} t_1 + (a+b)^2 \frac{t_3 - t_1}{t_1 - t_2} + a^2 t_3\Big] \\
&\qquad - \Big[(a+b)^2 \frac{t_2 - t_3}{t_1 - t_2} t_1 + (a + b\frac{t_3 - t_2}{t_3 - t_1})^2 \frac{t_3 - t_1}{t_1 - t_2} t_2 + a^2 t_3\Big] \\
&\qquad \times \Big[\Big(a + b\frac{t_3 - t_1}{t_3 - t_2}\Big)^2 \frac{t_3 - t_2}{t_1 - t_2} + (a+b)^2 \frac{t_3 - t_1}{t_1 - t_2} + a^2\Big] \\
&\quad = 2ab^3(t_1 - t_2) \neq 0
\end{aligned}$$

(which can be obtained after a straightforward calculation), equations (2.4) and (2.5) imply $w_1 = w_2 = O$.

So $\mathbf{p}'\mathbf{Y}$ (with $\mathbf{p} = \mathbf{X}\mathbf{w}$) is the UBLUE of its mean value if and only if $\mathbf{p} = \mathbf{O}$. The only linear functional $\mathbf{f}'\beta$ having the UBLUE in model (1.1) is equal to 0 for all $\beta \in \mathbb{R}^2$. There exists no UBLUE of a (nonzero) linear functional $\mathbf{f}'\beta$ in model (1.1). That is why we can only look for the β_0-LBLUE of the functional $\mathbf{f}'\beta$. According to Lemma 2.4 and Remark 2.5 in [4] the β_0-LBLUE of $\mathbf{f}'\beta$ ($\mathbf{f} \in \mu(\mathbf{X}')$) is

$$\text{(2.6)} \qquad \mathbf{f}'(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta_0)\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta_0)\mathbf{Y}$$

where $(\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta_0)\mathbf{X})^{-}$ is an arbitrary but fixed g-inverse of the matrix $\mathbf{X}'\mathbf{\Sigma}^{-1}(\beta_0)\mathbf{X}$. We only note that the β_0-LBLUE of $\mathbf{f}'\beta$ exists if and only if $\mathbf{f} \in \mu(\mathbf{X}')$ and is unique.

3. The β_0-LBLQUE of $\mathbf{f}'\beta$ in model (1.1)

Based on Lemma 1.8 in [7] we will show that in model (1.1) the β_0-LBLQUE as an improvement of the β_0-LBLUE of any $\mathbf{f}'\beta$ does not exist. Also we have

$$\{\mathbf{f}\colon \exists\ \beta_\circ - \text{LBLQUE for } \mathbf{f}'\beta\} = \{\mathbf{f}\colon \exists\ \beta_\circ - \text{LBLUE for } \mathbf{f}'\beta\} = \mu(\mathbf{X}')\ .$$

Let us denote by $\mathscr{D}$ the class of matrices $\mathbf{D}_{n,n}$ satisfying the following three conditions:

$$(3.1)\qquad \forall\{\beta \in \mathbb{R}^2\} \qquad \operatorname{Tr}\mathbf{D}\begin{pmatrix} |\mathbf{e}_1'\mathbf{X}\beta| & 0 & \dots & 0 \\ 0 & |\mathbf{e}_2'\mathbf{X}\beta| & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & & |\mathbf{e}_n'\mathbf{X}\beta| \end{pmatrix} = 0,$$

$$(3.2)\qquad \operatorname{Tr}\mathbf{D} = 0,$$

$$(3.3)\qquad \mathbf{X}'(\mathbf{D} + \sigma^2 b^2 \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i'\mathbf{D}\mathbf{e}_i\mathbf{e}_i')\mathbf{X} = \mathbf{O}.$$

($\operatorname{Tr}\mathbf{D}$ is the trace of $\mathbf{D}$ i.e. $\sum_{i=1}^{n} \mathbf{e}_i'\mathbf{D}\mathbf{e}_i$.)

Let $\mathbf{D} \in \mathscr{D}$ have the (i,j)-th element $d_{i,j},\ i,j = 1,2,\dots,n$. From (3.1) it follows that for all $\beta_1, \beta_2 \in \mathbb{R}$ we have

$$(3.4)\ \sum_{i=1}^{n_1} d_{i,i}|\beta_1 + t_1\beta_2| + \sum_{i=n_1+1}^{n_1+n_2} d_{i,i}|\beta_1 + t_2\beta_2| + \ldots + \sum_{i=n_1+\ldots+n_k-1}^{n_1+\ldots+n_k} d_{i,i}|\beta_1 + t_k\beta_2| = 0.$$

If we denote

$$\sum_{i=1}^{n_1} d_{i,i} = d_1,\ \sum_{i=n_1+1}^{n_1+n_2} d_{i,i} = d_2,\ \dots,\ \sum_{i=n_1+\ldots+n_k-1}^{n_1+\ldots+n_k} d_{i,i} = d_k$$

then following the same way as in [8], Lemma 9.1, we obtain that condition (3.4) is equivalent to

$$(3.5)\qquad d_1 = d_2 = \ldots = d_k = 0.$$

So (3.2) follows from (3.1). Taking it into account, for all $\mathbf{D} \in \mathscr{D}$ from (3.3) we obtain

$$\mathbf{X}'\mathbf{D}\mathbf{X} = -\sigma^2 b^2 \mathbf{X}' \sum_{i=1}^{n} \mathbf{e}_i\mathbf{e}_i'\mathbf{D}\mathbf{e}_i\mathbf{e}_i'\mathbf{X} = -\sigma^2 b^2 \begin{pmatrix} \sum_{i=1}^{k} d_i & \sum_{i=1}^{k} t_i d_i \\ \sum_{i=1}^{k} t_i d_i & \sum_{i=1}^{k} t_i^2 d_i \end{pmatrix} = \mathbf{O}.$$

Now we apply Lemma 1.8 in [7] and obtain the desired result

$$\{\mathbf{f} \colon \exists\ \beta_\circ - \text{LBLQUE for } \mathbf{f}'\beta\} = \mu(\mathbf{X}')$$

i.e. the class of linear functionals $\mathbf{f}'\beta$ having the β_0-LBLUE is the same as the class of linear functionals having the β_0-LBLUQE. Further the β_0-LBLUQE is the same as the β_0-LBLUE of $\mathbf{f}'\beta$ for all $\mathbf{f} \in \mu(\mathbf{X}')$.

Final conclusions for the model (1.1).

(i) The uniformly best linear unbiased estimator (UBLUE) of any nonzero linear functional $\mathbf{f}'\beta$ does not exist.

(ii) The β_0-locally best linear unbiased estimator (β_0-LBLUE) of $\mathbf{f}'\beta$ exists if and only if $\mathbf{f} \in \mu(\mathbf{X}')$, it is unique and of the form (2.6).

(iii) The β_0-locally best linear-quadratic unbiased estimator (β_0-LBLQUE) of $\mathbf{f}'\beta$ exists for all $\mathbf{f} \in \mu(\mathbf{X}')$ and is the same as the β_0-LBLUE of $\mathbf{f}'\beta$ (i.e there exists no quadratic improvement of the β_0-LBLUE of any $\mathbf{f}'\beta$, $\mathbf{f} \in \mu(\mathbf{X}')$). □

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