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Zdeněk Skalák

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A CONTINUITY PROPERTY FOR THE INVERSE  
OF MAÑÉ'S PROJECTION

ZDENĚK SKALÁK, Praha

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*Abstract.* Let  $X$  be a compact subset of a separable Hilbert space  $H$  with finite fractal dimension  $d_F(X)$ , and  $P_0$  an orthogonal projection in  $H$  of rank greater than or equal to  $2d_F(X) + 1$ . For every  $\delta > 0$ , there exists an orthogonal projection  $P$  in  $H$  of the same rank as  $P_0$ , which is injective when restricted to  $X$  and such that  $\|P - P_0\| < \delta$ . This result follows from Mañé's paper. Thus the inverse  $(P|_X)^{-1}$  of the restricted mapping  $P|_X: X \rightarrow PX$  is well defined. It is natural to ask whether there exists a universal modulus of continuity for the inverse of Mañé's projection  $(P|_X)^{-1}$ . It is known that when  $H$  is finite dimensional then  $(P|_X)^{-1}$  is Hölder continuous. In this paper we shall prove that if  $X$  is a global attractor of an infinite dimensional dissipative evolutionary equation then in some cases (e.g. two-dimensional Navier-Stokes equations with homogeneous Dirichlet boundary conditions)  $\|x - y\| \cdot \ln \ln \frac{1}{\gamma\|Px - Py\|} \leq 1$  for every  $x, y \in X$  such that  $\|Px - Py\| \leq \frac{1}{\gamma e^e}$ , where  $\gamma$  is a positive constant.

*Keywords:* dissipative evolutionary equations, Navier-Stokes equations, attractors, Mañé's projection, fractal dimension

*MSC 2000:* 35Q10, 76D05, 76F99

INTRODUCTION

A large class of infinite dimensional dissipative evolution equations has a global attractor (see [26]) which describes the long-time behaviour of the solutions of these equations. The attractor is usually a subset of a suitable infinite dimensional Hilbert space  $H$ . It is supposed to have a complex geometric structure (see [6, 17]), perhaps a fractal one, which reflects the complicated dynamics of the corresponding system.

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It was proved in many cases of the infinite dimensional dissipative evolution equations that the global attractor is a subset of the graph of a mapping  $\varphi$  defined from a finite dimensional subspace of  $H$  to  $H$ . If  $\varphi$  is sufficiently smooth then the dynamics on the attractor can be described by a system of finite dimensional ordinary differential equations. This fact has often been used in numerical computations (see [21, 23]). The existence of the (smooth) mapping  $\varphi$  gives us also better understanding of the geometrical structure of the attractor.

In those cases of the evolution dissipative equations where the existence of inertial manifolds was proved (see [4, 5, 10, 15, 16]), the mapping  $\varphi$  is Lipschitz and is defined on the finite dimensional space which is spanned by the first  $n$  eigenvectors of the linear part of the corresponding equation. In these cases, the dynamics on the global attractor is described by the finite dimensional system of ordinary differential equations, referred to as an inertial form. This leads to considerable simplification in the study of the dynamics. The existence of inertial manifolds was proved for the systems satisfying the spectral gap condition in the spectrum of the linear part of the equation, for example for the modified Navier-Stokes equations and some Reaction-Diffusion equations (see [15]) or the Kuramoto-Shivasinski equation (see [14]). A different method of showing the existence of inertial manifolds was used in [20] for the two-dimensional Navier-Stokes equations with periodic boundary conditions or in [27] for the Navier-Stokes equations on a two-dimensional sphere.

However, for some dissipative evolution equations the existence of inertial manifolds has not been proved yet. Here, in many cases, only the existence of approximate inertial manifolds was established (see [7, 11, 12, 13, 19, 25, 28]), which means that the global attractor is not a subset of a smooth finite dimensional manifold but only a subset of its open neighborhood. Fortunately, it was shown in some cases of these systems (e.g. the two-dimensional Navier-Stokes equations with homogeneous Dirichlet boundary conditions) that the fractal dimension of their global attractor is finite (see [2, 3, 9, 18]). Therefore, as was proved in [22], any orthogonal projection in the underlying Hilbert space  $H$  of sufficiently high finite rank can be perturbed so that its restriction to the global attractor is injective. It means that in the case of these systems the global attractor is a subset of the graph of a continuous mapping defined on a finite dimensional subspace of  $H$ . It was proved in [1] that the mapping is Hölder continuous as long as the underlying Hilbert space  $H$  is finite dimensional. It is natural to ask whether the same theorem also holds for the infinite dimensional space  $H$ . In this paper we show the following weaker result:

**Theorem R.** *Let us consider a system of infinite dimensional dissipative evolution equations which has a global attractor  $X$  of finite fractal dimension in the phase space  $H$  (the precise definition of the system together with the necessary conditions*

[A1], [A2] and [A3] will be stated in Section 1). Let  $P_0$  be an orthogonal projection in  $H$  of sufficiently high finite rank  $N$ . Then for any  $\delta > 0$  there exists an orthogonal projection  $P$  in  $H$  of rank  $N$  and a positive number  $\gamma = \gamma(P_0, \delta)$  such that

- (1)  $\|P - P_0\| \leq \delta$ ,
- (2)  $\|x - y\| \cdot \ln \ln \frac{1}{\gamma \|Px - Py\|} \leq 1$  for all  $x, y \in X$  such that  $\|Px - Py\| \leq \frac{1}{\gamma e}$ .

## 1. PRELIMINARIES, NOTATION AND STATEMENT OF MAIN THEOREM

Assume we are given an infinite dimensional Hilbert space  $H$  with a scalar product  $(\cdot, \cdot)$  and a norm  $\|\cdot\|$ . In this paper we consider a class of nonlinear evolutionary equations of the type

$$(1.1) \quad \frac{du}{dt} + Au + R(u) = f,$$

where  $A$  is a linear unbounded self-adjoint positive operator in  $H$  with domain  $D(A)$  dense in  $H$ . We assume that  $A^{-1}$  is compact and, accordingly,  $H$  has an orthonormal basis  $\{w_j\}_{j=1}^{\infty}$  consisting of eigenfunctions of the operator  $A$  with the corresponding eigenvalues  $\lambda_j$  (i.e.,  $Aw_j = \lambda_j w_j$ ,  $j = 1, 2, \dots$ ) satisfying  $0 < \lambda_1 \leq \lambda_2 \leq \dots$ ,  $\lambda_j \rightarrow +\infty$  as  $j \rightarrow +\infty$ .  $R$  is an arbitrary (nonlinear) operator from  $D(A)$  into  $H$  and  $f \in H$ .

For  $n = 1, 2, \dots$  we denote by  $P_n$  the orthogonal projection of  $H$  onto  $\text{Span}\{w_1, w_2, \dots, w_n\}$ .

In the paper, we use the concept of the fractal dimension (for more details see e.g. [1] or [26]):

**Definition 1.1.** Let  $Y$  be a compact subset of a Hilbert space  $H$  and  $\varepsilon > 0$ . Let  $N_\varepsilon(Y)$  be the minimum number of  $\varepsilon$ -balls necessary to cover  $Y$ . The fractal dimension of  $Y$ , denoted by  $d_F(Y)$ , is then defined as

$$d_F(Y) = \limsup_{\varepsilon \rightarrow 0_+} \frac{\ln N_\varepsilon(Y)}{\ln(1/\varepsilon)}.$$

Moreover, let the following assumptions be satisfied:

[A1] For every  $u_0 \in H$  there is a unique global solution  $u = u(t)$  to (1.1) such that

$$(1.2) \quad u(0) = u_0$$

and  $u \in C((0, \infty), H)$ .

- [A2] There exists a global attractor  $X \subset H$  for (1.1). The fractal dimension of  $X$ , denoted by  $d_F(X)$ , is finite.
- [A3] There exist a real number  $a \in (0, 1)$  and a natural number  $M$  such that for all natural numbers  $n \geq M$  and for all  $x, y \in X$  we have:  $\|x - y\| \geq a^n \implies \|P_n x - P_n y\| \geq \frac{1}{2}\|x - y\|$ .

**Remark 1.2.** The assumptions [A1] and [A2] have been proved for a large class of dissipative systems defined in (1.1), see e.g. [26]. As far as [A3] is concerned, let us now point out that this condition holds for global attractors of some dissipative systems defined in (1.1). This follows from the theory of inertial and approximate inertial manifolds developed in [4, 5, 7, 10, 11, 12, 13, 15, 16, 19, 25, 28], where it has been shown that the global attractors are included in a thin neighbourhood of these manifolds. In the article, we do not work directly with the systems (1.1) (and therefore we do not specify the terms of (1.1) in detail), but we consider the global attractors of those particular systems (1.1) where [A3] can be proved and use only this key condition. For example, the following proposition has been proved in the case of the two-dimensional Navier-Stokes equations:

**Proposition.** *Let  $X$  be the global attractor of two-dimensional Navier-Stokes equations with periodic or homogeneous Dirichlet boundary conditions. Then there exist a natural number  $M$  and a real number  $a \in (0, 1)$  such that for all natural numbers  $n \geq M$  there exists a Lipschitz mapping (approximate inertial manifold)  $\Sigma_n: P_n H \rightarrow (I - P_n)H$  with a Lipschitz constant 1 such that*

$$\text{dist}(x, \Sigma_n) \leq a^n, \quad \forall x \in X.$$

**Proof.** The proof follows immediately from Theorem 3.1 in [13]. □

Clearly, [A3] is now a simple consequence of the proposition.

Let us also note that [A3] is easily verifiable for those dissipative systems where the existence of Lipschitz inertial manifolds was proved (see [4, 5, 10, 15, 16]).

Let us denote  $A_X = \{x - y; x, y \in X\}$  and  $S_X = \sup\{\|u\|; u \in A_X\}$ . Obviously, one has  $d_F(A_X) \leq 2d_F(X) < \infty$ . Let  $P_0$  be an orthogonal projection in  $H$  of rank  $N > d_F(A_X)$  and  $\delta > 0$ . Let  $D$  and  $\theta$  be numbers such that  $d_F(A_X) < D < N$  and  $0 < \theta < 1 - D/N$ . Since  $D > d_F(A_X)$ , there exists a number  $c \geq 1$  such that  $N_\varepsilon(A_X) \leq c(1/\varepsilon)^D$  for every  $\varepsilon \in (0, 1)$ , where  $N_\varepsilon(A_X)$  denotes the minimum number of  $\varepsilon$ -balls necessary to cover  $A_X$ .

The following theorem is the main result of this paper.

**Theorem 1.3.** *Let  $X \in H$  be the global attractor of the system defined in (1.1) and let the assumptions [A1], [A2] and [A3] be satisfied. Assume we are given the operator  $P_0$  and the numbers  $N, \delta, D, \theta$  and  $c$  as in the preceding paragraph. Then there exists an orthogonal projection  $P$  in  $H$  of rank  $N$  such that  $\|P - P_0\| \leq \delta$  and  $\frac{\|u\|}{f(\|Pu\|)} \leq 1$  for every  $u \in A_X$ . The real function  $f$  is defined on the interval  $(0, 1)$  as*

$$f(x) = \begin{cases} \frac{1}{\ln \ln \frac{1}{\gamma x}} & \text{for every } x \in \left(0, \frac{1}{\gamma e^e}\right), \\ 1 & \text{for every } x \in \left(\frac{1}{\gamma e^e}, 1\right) \end{cases}$$

and  $\gamma = \gamma(a, c, N, D, P_0, \delta, \theta, S_X)$  is a positive constant.

## 2. STATEMENT OF AUXILIARY LEMMAS

In this section we formulate three lemmas which will be used in the proof of Theorem 1.3.

Let us denote by  $G(N, M)$  the set consisting of all  $N$  dimensional subspaces of the Euclidean space  $\mathbb{R}^M$ . There exists a nontrivial measure  $m$  defined on  $G(N, M)$  which is invariant under the action of the orthogonal group. Let us note that all open subsets of  $G(N, M)$  are  $m$ -measurable and their measure is greater than zero. For a subspace  $S$  of  $\mathbb{R}^M$ , we denote by  $S^\perp$  the orthogonal complement of  $S$  in  $\mathbb{R}^M$ . For  $0 \neq a \in \mathbb{R}^M$  and  $\varrho > 0$ , let us define the set

$$W(a, \varrho) = \{S \in G(N, M); S^\perp \cap B(a, 2\varrho) \neq \emptyset\}.$$

For every natural number  $k$  let  $O_k$  denote the surface of the  $k$  dimensional unique sphere of  $\mathbb{R}^{k+1}$  and let  $O(M, N) = O_{M-1}O_{M-2} \dots O_{M-N}$  for  $1 \leq N < M$ . Let us note that the definitions and the propositions of this section are also mentioned in [1] and intensively studied in [24].

**Lemma 2.1.** *If  $0 \neq a \in \mathbb{R}^M$  and  $\varrho > 0$  then for  $W(a, \varrho) \subset G(N, M)$ ,*

$$(2.1) \quad m(W(a, \varrho)) \leq 4^N M^N O(M, N) \left(\frac{\varrho}{\|a\|}\right)^N.$$

**Lemma 2.2.** *Let  $P_0 \in G(N, M)$  and  $\delta \in (0, 1)$ . Then*

$$(2.2) \quad m(\{P \in G(N, M); \|P - P_0\| \leq \delta\}) > \varepsilon^M \delta^{MN} O(M, N),$$

where  $\varepsilon = 1/(4N)^N$ .

**Lemma 2.3.** *Let  $P_0$  be an orthogonal projection in  $H$  of rank  $N$  and  $\delta > 0$ . Then there exists an orthogonal projection  $P$  in  $H$  of rank  $N$  and a natural number  $M = M(P_0, \delta)$  such that  $\|P_0 - P\| \leq \delta$  and  $PH \subset \text{Span}\{w_1, w_2, \dots, w_M\}$ .*

**Remark 2.4.** For the methods to prove Lemma 2.1 and Lemma 2.2, see [24]. The proof of Lemma 2.3 is elementary.

### 3. PROOF OF MAIN THEOREM

From now on, we suppose that all assumptions of Theorem 1.3 hold. Let  $\{M_n\}_{n=1}^\infty$  be a sequence of natural numbers and  $0 = M_0 < 4 \leq M_1 < M_2 < \dots$ ,  $M_1 > N$  and  $M_1 \geq M$ , where  $M$  is the number from the assumption [A3]. Using Lemma 2.3 we can assume, without loss of generality, that for  $M_1 = M_1(\delta, P_0)$  sufficiently large,  $P_0H \subset \text{Span}\{w_1, w_2, \dots, w_{M_1}\}$ . We can further assume that  $S_X \leq 1$  and  $\delta \in (0, \frac{1}{2})$ .

In the proof of the following lemma we use a method which was developed in [1].

**Lemma 3.1.** *Let  $k$  be a natural number or zero, let  $\tilde{P}$  be an orthogonal projection in  $P_{M_{k+1}}(H)$  of rank  $N$  and  $\beta > 0$ . Then there exists an orthogonal projection  $P$  in  $P_{M_{k+1}}(H)$  of rank  $N$  such that*

- (1)  $\frac{\|x\|}{\|P_x\|^\theta} \leq \frac{2}{\eta a^\theta}$ ,  $\forall x \in A_k \stackrel{\text{def}}{=} \{x \in A_X; a^{M_{k+1}} < \|x\| \leq a^{M_k}\}$  and
- (2)  $\|P - \tilde{P}\| \leq \frac{1}{2}\beta$ , where

$$(3.1) \quad \eta = \left( \frac{\varepsilon^{M_{k+1}} \beta^{M_{k+1}N}}{B M_{k+1}^N G^{M_k} 2^{M_{k+1}N}} \right)^{\theta/(N-D)},$$

$$(3.2) \quad B = c2^D 4^N H a^{(N-D)(\theta-1)/\theta}, \quad H = G/(1-G) \text{ and } G = a^{(N-D)/\theta-N}.$$

**Proof.** Let us consider the finite dimensional space  $P_{M_{k+1}}(H)$ . Then obviously,  $P_{M_{k+1}}(A_k) \subset P_{M_{k+1}}(H)$ ,  $d_F(P_{M_{k+1}}(A_k)) \leq d_F(A_k) \leq d_F(A_X) < D < N$  and  $N_\varepsilon(P_{M_{k+1}}(A_k)) \leq c(1/\varepsilon)^D$  for every  $\varepsilon \in (0, 1)$ . For all natural numbers  $n$  let  $r_n = a^n$  and let us consider the sets  $Y_k^n = \{y \in P_{M_{k+1}}(A_k); r_n < \|y\| \leq r_{n-1}\}$ . Now we set  $\varrho_n = (a^{\theta-1} \eta r_n)^{1/\theta}$  for every natural number  $n$ , where  $\eta > 0$  is to be specified later. For each  $n = 1, 2, \dots$  we choose a covering of  $P_{M_{k+1}}(A_k)$  with  $\frac{1}{2}\varrho_n$ -balls consisting of at most  $c(\frac{1}{2}\varrho_n)^{-D}$  balls. We disregard the balls not intersecting  $Y_k^n$ . Let  $B(\alpha_j^{nk}, \frac{1}{2}\varrho_n)$ ,  $j = 1, 2, \dots, m_n^k$ , be the remaining balls, where  $m_n^k \leq c(\frac{1}{2}\varrho_n)^{-D}$ .

For each  $j = 1, 2, \dots, m_n^k$  let us choose an arbitrary point  $a_j^{nk} \in Y_k^n \cap B(\alpha_j^{nk}, \frac{1}{2}\varrho_n)$ . Then  $B(\alpha_j^{nk}, \frac{1}{2}\varrho_n) \subset B(a_j^{nk}, \varrho_n)$  and therefore

$$Y_k^n \subset \bigcup_{j=1}^{m_n^k} B(\alpha_j^{nk}, \frac{1}{2}\varrho_n) \subset \bigcup_{j=1}^{m_n^k} B(a_j^{nk}, \varrho_n).$$

If a given projection  $P$  in  $P_{M_{k+1}}(H)$  satisfies

$$(3.3) \quad (I - P)P_{M_{k+1}}(H) \cap B(a_j^{nk}, 2\varrho_n) = \emptyset$$

for every  $n = 1, 2, \dots$  and  $j = 1, 2, \dots, m_n^k$ ,

then it is easy to see that

$$\|y\| \leq \frac{1}{a^\theta \eta} \|Py\|^\theta, \quad \forall y \in P_{M_{k+1}}(A_k).$$

This follows from the fact that if  $y \in P_{M_{k+1}}(A_k)$  and  $y \neq 0$ , then  $y \in Y_k^n \subset \bigcup_{j=1}^{m_n^k} B(a_j^{nk}, \varrho_n)$  for some natural number  $n$ ,  $\|y\| \leq r_{n-1}$ ,  $\|Py\| = \text{dist}(y, (\text{Ker } P)^\perp) \geq \varrho_n$  (see (3.3)), and from the definition of  $r_{n-1}$  and  $\varrho_n$ .

Our goal is to perturb  $\tilde{P}$  just enough to realize (3.3). Since

$$P_{M_{k+1}}(A_k) \subset \bigcup_{n=M_k+1}^{\infty} Y_k^n \subset \bigcup_{n=M_k+1}^{\infty} \bigcup_{j=1}^{m_n^k} B(a_j^{nk}, 2\varrho_n),$$

we first proceed to estimate  $\sum_{n=M_k+1}^{\infty} \sum_{j=1}^{m_n^k} m(W(a_j^{nk}, \varrho_n))$ . It follows from Lemma 2.1, the inequalities  $m_n^k \leq c2^D \varrho_n^{-D}$ ,  $r_n \leq |a_j^{nk}|$  and  $\varrho_n = (a^{\theta-1} \eta r_n)^{1/\theta}$  that

$$(3.4) \quad \begin{aligned} \sum_{n=M_k+1}^{\infty} \sum_{j=1}^{m_n^k} m(W(a_j^{nk}, \varrho_n)) &\leq \sum_{n=M_k+1}^{\infty} m_n^k 4^N M_{k+1}^N O(M_{k+1}, N) \left(\frac{\varrho_n}{r_n}\right)^N \\ &\leq c2^D 4^N M_{k+1}^N O(M_{k+1}, N) \sum_{n=M_k+1}^{\infty} \varrho_n^{N-D} r_n^{-N} \\ &= c2^D 4^N a^{(N-D)(\theta-1)/\theta} M_{k+1}^N \eta^{(N-D)/\theta} O(M_{k+1}, N) \sum_{n=M_k+1}^{\infty} r_n^{(N-D)/\theta - N} \\ &= B \eta^{(N-D)/\theta} O(M_{k+1}, N) M_{k+1}^N G^{M_k}, \end{aligned}$$



where for  $B$  and  $G$  see (3.2). If we now set

$$\eta = \left( \frac{\varepsilon^{M_{k+1}} \beta^{M_{k+1}N}}{B M_{k+1}^N G^{M_k} 2^{M_{k+1}N}} \right)^{\theta/(N-D)},$$

then

$$B\eta^{(N-D)/\theta} O(M_{k+1}, N) M_{k+1}^N G^{M_k} \leq \varepsilon^{M_{k+1}} \left( \frac{\beta}{2} \right)^{M_{k+1}N} O(M_{k+1}, N)$$

and due to (3.3), (3.4) and Lemma 2.2 there exists an orthogonal projection  $P$  in  $P_{M_{k+1}}(H)$  of rank  $N$  so that item (2) of Lemma 3.1 holds, i.e.  $\|P - \tilde{P}\| \leq \frac{1}{2}\beta$ , and

$$\|P_{M_{k+1}}x\| \leq \frac{1}{\eta a^\theta} \|Px\|^\theta, \quad \forall x \in A_k.$$

Since  $\|x\| \geq a^{M_{k+1}}$  for every  $x \in A_k$  it follows from the assumption [A3] that  $\|P_{M_{k+1}}x\| \geq \frac{1}{2}\|x\|$  and therefore

$$\frac{\|x\|}{\|Px\|^\theta} \leq \frac{2}{\eta a^\theta}, \quad \forall x \in A_k,$$

that is item (1) of Lemma 3.1 holds.  $\square$

**Remark 3.2.** Let us show that  $B$  from (3.2) is greater than 1: Since  $a, G \in (0, 1)$  and  $D > 0$ , then  $a^{-D} \geq 1 - G$  and therefore  $a^{(N-D)(\theta-1)/\theta} a^{(N-D)/\theta-N} \geq 1 - G$ . Since  $c \geq 1$ , it means according to (3.2) that  $B > H a^{(N-D)(\theta-1)/\theta} \geq 1$ .

**Proof of Theorem 1.3.** For every  $k = 0, 1, \dots$ , let us now define an orthogonal projection  $P^k$  in  $H$  of rank  $N$  and a positive number  $\delta_k < 1$ . The projection  $P$  from Theorem 1.3 will then be defined as  $\lim_{k \rightarrow \infty} P^k$ . We put  $P^0 = P_0$ ,  $\delta_0 = \delta$  and use the principle of mathematical induction. Let us assume that for some  $k \in \{0, 1, \dots\}$  an orthogonal projection  $P^k$  in  $P_{M_{k+1}}(H)$  of rank  $N$  and a positive number  $\delta_k < 1$  have already been defined. Let us define  $P^{k+1}$  and  $\delta_{k+1}$ . By virtue of Lemma 3.1, there exists an orthogonal projection  $P^{k+1}$  in  $P_{M_{k+1}}(H) \subset P_{M_{k+2}}(H)$  of rank  $N$  such that

$$\|P^{k+1} - P^k\| \leq \frac{1}{2}\delta_k$$

and

$$(3.5) \quad \frac{\|x\|}{\|P^{k+1}x\|^\theta} \leq \frac{2}{\eta_{k+1} a^\theta}, \quad \forall x \in A_k,$$

where

$$(3.6) \quad \eta_{k+1} = \left( \frac{\varepsilon^{M_{k+1}} \delta_k^{M_{k+1}N}}{BM_{k+1}^N G^{M_k} 2^{M_{k+1}N}} \right)^{\theta/(N-D)}$$

with constants  $B, H$  and  $G$  defined in (3.2). Let us set  $c_{k+1} = \frac{2}{\eta_{k+1} a^\theta}$  and define

$$(3.7) \quad \delta_{k+1} = \frac{1}{2} \left( \frac{a^{M_{k+1}}}{c_{k+1}} \right)^{1/\theta} a^{-M_{k+1}}.$$

Let us show now that  $\delta_{k+1} \leq \frac{1}{2} \delta_k$ . It follows from the definitions of  $\delta_{k+1}$ ,  $c_{k+1}$  and  $\eta_{k+1}$  that

$$(3.8) \quad \begin{aligned} \delta_{k+1} &= \frac{1}{2} \left( \frac{a^{M_{k+1}}}{c_{k+1}} \right)^{1/\theta} a^{-M_{k+1}} \\ &= \frac{a}{2^{(1+\theta)/\theta}} \left( \frac{\varepsilon^{M_{k+1}} \delta_k^{M_{k+1}N}}{BM_{k+1}^N G^{M_k} 2^{M_{k+1}N}} \right)^{1/(N-D)} a^{M_{k+1}(1-\theta)/\theta} \\ &= \left( \frac{a}{2^{(1+\theta)/\theta} B^{1/(N-D)}} \right) \left( \frac{\varepsilon^{M_{k+1}}}{M_{k+1}^N 2^{M_{k+1}N}} \right)^{1/(N-D)} \\ &\quad \times \left( \frac{a^{M_{k+1}(1-\theta)/\theta}}{G^{M_k/(N-D)}} \right) \delta_k^{M_{k+1}N/(N-D)}. \end{aligned}$$

Using the definition of  $G$  in (3.2) one immediately has  $\left( \frac{a^{M_{k+1}(1-\theta)/\theta}}{G^{M_k/(N-D)}} \right) \leq 1$ . From here and since  $B \geq 1$  and  $\varepsilon \leq 1$ , one obtains from (3.8) that  $\delta_{k+1} \leq \frac{1}{2} \delta_k^{M_{k+1}N/(N-D)}$ . Since  $M_{k+1}N/(N-D) \geq 1$  and  $\delta_k < 1$ , we conclude that  $\delta_{k+1} \leq \frac{1}{2} \delta_k < 1$ .

Let  $Q$  be an orthogonal projection in  $H$  of rank  $N$  and  $\|Q - P^{k+1}\| \leq \delta_{k+1}$  for  $k = 0, 1, \dots$ . Then for every  $x \in A_k$

$$(3.9) \quad \begin{aligned} \|Q - P^{k+1}\| \cdot \|x\| &\leq \frac{1}{2} \left( \frac{a^{M_{k+1}}}{c_{k+1}} \right)^{1/\theta} a^{-M_{k+1}} \|x\| \\ &= \frac{1}{2} \left( \frac{1}{c_{k+1}} \right)^{1/\theta} a^{M_{k+1}(1/\theta-1)} \|x\| \\ &< \frac{1}{2} \left( \frac{1}{c_{k+1}} \right)^{1/\theta} \|x\|^{1/\theta-1} \|x\| = \frac{1}{2} \left( \frac{\|x\|}{c_{k+1}} \right)^{1/\theta} \end{aligned}$$

and since by virtue of the definition of  $c_{k+1}$  and (3.5) we have  $\|P^{k+1}x\| \geq \left( \frac{\|x\|}{c_{k+1}} \right)^{1/\theta}$  for every  $x \in A_k$ , it follows from (3.9) that  $\|Q - P^{k+1}\| \cdot \|x\| \leq \frac{1}{2} \|P^{k+1}x\|$  for every  $x \in A_k$ . Therefore we have  $\|Qx\| \geq \|P^{k+1}x\| - \|Q - P^{k+1}\| \cdot \|x\| \geq \frac{1}{2} \|P^{k+1}x\|$  for every  $x \in A_k$  and finally

$$(3.10) \quad \frac{\|x\|}{\|Qx\|^\theta} \leq \frac{\|x\|}{\left(\frac{1}{2}\|P^{k+1}x\|\right)^\theta} \leq 2^\theta c_{k+1}, \quad \forall x \in A_k.$$

Let us sum up the properties of the sequences  $\{\delta_k\}_{k=0}^\infty$  and  $\{P^k\}_{k=0}^\infty$ :

- (1)  $\delta_0 = \delta, \delta_k > 0$  and  $\delta_{k+1} \leq \frac{1}{2}\delta_k, \quad \forall k = 0, 1, \dots$
- (2)  $P^0 = P_0$  and  $P^k$  is an orthogonal projection in  $P_{M_{k+1}}(H), \quad \forall k = 0, 1, \dots$
- (3)  $\|P^{k+1} - P^k\| \leq \frac{1}{2}\delta_k, \quad \forall k = 0, 1, \dots$
- (4)  $\frac{\|x\|}{\|P^{k+1}x\|^\theta} \leq c_{k+1}, \quad \forall x \in A_k,$

where  $c_{k+1} = \frac{2}{\eta_{k+1}a^\theta}$  and  $\eta_{k+1}$  was defined in (3.6). It follows easily from conditions (1) and (3) that  $\lim_{k \rightarrow \infty} P^k = P$ , where  $P$  is an orthogonal projection in  $H$  of rank  $N$ . For  $n = 0, 1, \dots$  one has

$$(3.11) \quad \|P - P^n\| \leq \sum_{k=0}^{\infty} \|P^{n+k} - P^{n+k+1}\| \leq \sum_{k=0}^{\infty} \frac{1}{2}\delta_{n+k} \leq \frac{1}{2} \sum_{k=0}^{\infty} \delta_n \frac{1}{2^k} = \delta_n$$

and due to (3.10)

$$(3.12) \quad \frac{\|x\|}{\|P^kx\|^\theta} \leq 2^\theta c_{k+1}, \quad \forall x \in A_k, \forall k = 0, 1, \dots$$

Using (3.6), (3.7) and the definition of  $c_{k+1}$  we shall now express  $c_{k+1}$  as a function of  $c_k$ :

$$(3.13) \quad c_{k+1} = \frac{2}{a^\theta \eta_{k+1}} = \underbrace{\left(\frac{2}{a^\theta} B^{\theta/(N-D)}\right)}_{\geq 1} \underbrace{\left(G^{M_k \theta/(N-D)}\right)}_{\leq 1} \\ \times \left(M_{k+1}^{N\theta/(N-D)}\right) \underbrace{\left(\frac{4^N}{\varepsilon}\right)^{M_{k+1}\theta/(N-D)}}_{\geq 1} \\ \times \left(a^{(\theta-1)NM_k M_{k+1}/(N-D)}\right) \left(c_k^{M_{k+1}N/(N-D)}\right).$$

Moreover, the maximum of the real function  $x \rightarrow x^{1/x} \forall x > 0$  is  $e^{1/e}$ . Therefore  $M_{k+1}^{1/M_{k+1}} \leq 16N$  and using the definition of  $\varepsilon$  from Lemma 2.2 one has  $\left(M_{k+1}^{N\theta/(N-D)}\right) \leq \left(\frac{4^N}{\varepsilon}\right)^{M_{k+1}\theta/(N-D)}$ . If we now denote

$$b = b(a, \theta, N, D, c) = \frac{2}{a^\theta} B^{\theta/(N-D)} \left(\frac{16^N}{\varepsilon^2}\right)^{\theta/(N-D)} a^{(\theta-1)N/(N-D)}$$

and

$$T = T(N, D) = \frac{N}{N-D},$$

it follows immediately from (3.13) that

$$c_{k+1} \leq b^{M_k M_{k+1}} c_k^{M_{k+1} T}, \quad \forall k = 1, 2, \dots$$

Let us define a sequence  $\{\Gamma_k\}_{k=1}^\infty$  in the following way:  $\Gamma_1 = \max(c_1, b^{M_1}, a^{-1})$  and  $\Gamma_{k+1} = b^{M_k M_{k+1}} \Gamma_k^{M_{k+1} T}$ ,  $\forall k = 1, 2, \dots$ . Obviously,  $\Gamma_k \geq c_k$ ,  $\forall k = 1, 2, \dots$ . Since  $\Gamma_{k+1} = b^{M_k M_{k+1}} \Gamma_k^{M_{k+1} T} \geq b^{M_{k+1}}$ , one has  $\Gamma_{k+1} = (b^{M_k})^{M_{k+1}} \Gamma_k^{M_{k+1} T} \leq \Gamma_k^{M_{k+1}} \Gamma_k^{M_{k+1} T} = \Gamma_k^{(T+1)M_{k+1}}$ . Therefore one has  $\Gamma_{k+1} \leq \Gamma_k^{(T+1)M_{k+1}}$ ,  $\forall k = 1, 2, \dots$  and subsequently

$$(3.14) \quad \Gamma_k \leq \Gamma_1^{M_2 M_3 \dots M_k (T+1)^{k-1}}, \quad \forall k = 2, 3, \dots,$$

where  $\Gamma_1 = \Gamma_1(a, c, N, D, \theta, \delta_0, M_1)$ . Since  $c_k \leq \Gamma_k$  for every natural  $k$ , it follows from (3.12) that

$$(3.15) \quad \frac{\|x\|}{\|Px\|^\theta} \leq 2^\theta \Gamma_{k+1}, \quad \forall x \in A_k, \forall k = 0, 1, \dots$$

We shall now consider the real function  $f$  which was defined in Theorem 1.3, where we put

$$(3.16) \quad \gamma = \frac{2\Gamma_1^{1/\theta}}{a^{M_1/\theta} e^e}.$$

Clearly,  $f$  is a nondecreasing function on the interval  $(0, 1)$  and  $\lim_{x \rightarrow 0^+} f(x) = 0$ . Let  $x \in A_k$ . It follows from (3.15) and the definition of  $A_k$  that

$$\|Px\| \geq \frac{1}{2} \left( \frac{\|x\|}{\Gamma_{k+1}} \right)^{1/\theta} \geq \frac{1}{2} \frac{a^{M_{k+1}/\theta}}{\Gamma_{k+1}^{1/\theta}}.$$

If we have

$$(3.17) \quad f\left(\frac{1}{2} \frac{a^{M_{k+1}/\theta}}{\Gamma_{k+1}^{1/\theta}}\right) \geq a^{M_k}$$

then  $f(\|Px\|) \geq a^{M_k} \geq \|x\|$ . Since  $A_X = \bigcup_{k=0,1,\dots} A_k$ , one has

$$(3.18) \quad f(\|Px\|) \geq \|x\|, \quad \forall x \in A_X.$$

Let us prove (3.17). By means of (3.16) and the definition of  $f$ , (3.17) is equivalent to

$$(3.19) \quad \ln \ln \left( \frac{a^{M_1/\theta} e^e}{\Gamma_1^{1/\theta}} \frac{\Gamma_{k+1}^{1/\theta}}{a^{M_{k+1}/\theta}} \right) \leq a^{-M_k}.$$

Obviously, (3.19) holds for  $k = 0$ . Let now  $k > 0$ . It is possible to choose  $M_1$  so large that  $\frac{a^{M_1/\theta} e^c}{\Gamma_1^{1/\theta}} \leq 1$ . Since also  $M_1 \geq 4$  and  $\Gamma_1 \geq a^{-1}$ , it is sufficient to prove (see also (3.14)) that  $\ln \ln(\Gamma_1^{M_1 M_2 M_3 \dots M_{k+1} (T+1)^k / \theta}) \leq a^{-M_k}$  or

$$(3.20) \quad (\ln M_1 + \ln M_2 + \dots + \ln M_{k+1}) + (k \ln(T+1)) + (\ln(1/\theta)) + (\ln \ln \Gamma_1) \leq a^{-M_k}.$$

It is elementary to show that (3.20) holds if  $M_1 = M_1(a, N, D, \theta, c, \delta_0)$  is a sufficiently large number. Therefore (3.17) holds and this together with (3.11) concludes the proof of Theorem 1.3.  $\square$

*Remark 3.3.* By setting for example  $D = \frac{1}{2}(N + d_F(A_X))$  and  $\theta = 1 - D/2N$  in Theorem 1.3 we can have that  $\gamma = \gamma(a, c, N, P_0, \delta, S_X)$ .

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*Author's address*: Institute of Hydrodynamics, The Academy of Sciences of the Czech Republic, Pod Patankou 5/30, 166 12 Praha 6, Czech Republic.