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TEMPERATURE-DEPENDENT HYSTERESIS IN ONE-DIMENSIONAL THERMOVISCO-ELASTOPLASTICITY

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Abstract. In this paper, we develop a thermodynamically consistent description of the uniaxial behavior of thermovisco-elastoplastic materials for which the total stress σ contains, in addition to elastic, viscous and thermic contributions, a plastic component σ^p of the form $\sigma^p(x,t) = \mathcal{P}[\varepsilon, \theta(x,t)](x,t)$. Here ε and θ are the fields of strain and absolute temperature, respectively, and $\{\mathcal{P}[\cdot,\theta]\}_{\theta>0}$ denotes a family of (rate-independent) hysteresis operators of Prandtl-Ishlinskii type, parametrized by the absolute temperature. The system of momentum and energy balance equations governing the space-time evolution of the material forms a system of two highly nonlinearly coupled partial differential equations involving partial derivatives of hysteretic nonlinearities at different places. It is shown that an initial-boundary value problem for this system admits a unique global strong solution which depends continuously on the data.

Keywords: thermoplasticity, viscoelasticity, hysteresis, Prandtl-Ishlinskii operator, PDEs with hysteresis, thermodynamical consistency

MSC 2000: 35G25, 73B30, 73E60, 73B05

0. Introduction

For many materials the stress-strain (σ-ε) relations measured in uniaxial load-deformation experiments strongly depend on the absolute (Kelvin) temperature θ and, at the same time, exhibit a strong plastic behavior witnessed by the occurrence of rate-independent hysteresis loops. Figure 1 shows a typical diagram, where the elasticity modulus and the yield limit depend on temperature.

Among the materials exhibiting temperature-dependent, but rate-independent hysteretic effects are shape memory alloys (see, for instance, Chapter 5 in [BS]) and even, although to a smaller extent, quite ordinary steels.

If the σ-ε relation exhibits a hysteresis, it can no longer be expressed in terms of simple single-valued functions since the latter are certainly not able to give a correct account of the inherent memory structures that are responsible for the complicated loopings in the interior of experimentally observed hysteresis loops.

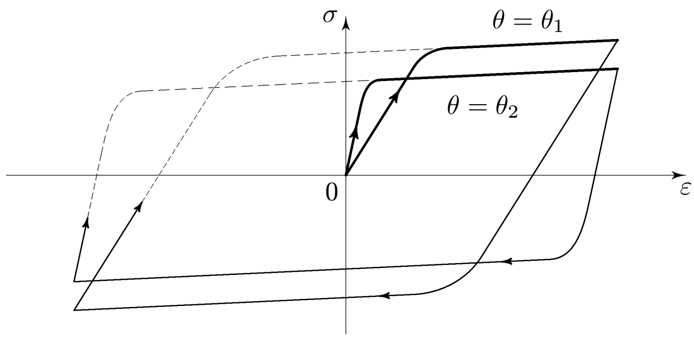

Figure 1. Strain-stress diagrams at constant temperatures $\theta_1 \neq \theta_2$.

To avoid these difficulties, a different approach to thermoelastoplastic hysteresis based on the notion of *hysteresis operators* introduced by the Russian group around M. A. Krasnosel'skii in the seventies (see [KP]) has been proposed by the authors in [KS]. In this approach, the temperature-dependent plastic stress σ^p has been assumed in the form of an *operator equation* with a temperature-dependent hysteretic constitutive operator $\mathcal{P}$ of *Prandtl-Ishlinskii type*, namely

$$\sigma^p = \mathcal{P}[\varepsilon, \theta] := \int_0^\infty \varphi(r, \theta)\mathfrak{s}_r[\varepsilon]\,\mathrm{d}r. \tag{0.1}$$

In this connection, $\mathfrak{s}_r$ denotes the so-called *stop operator* or the *elastic-plastic element* with threshold $r > 0$ (to be defined in the next section), and $\varphi(\cdot, \theta) \geqslant 0$ is a density function with respect to $r > 0$, parametrized by the absolute temperature θ.

The advantage of this approach is that an operator equation like (0.1) is suited much better than a simple functional relation to keep track of the *memory effects* imprinted on the material in the past history; in fact, the output at any time $t \in [0, T]$ may depend on the whole evolution of the input in the time interval $[0, t]$. Observe that the requirement of rate-independence implies that $\mathcal{P}$ cannot be expressed in terms of an integral operator of convolution type, i.e. we are not dealing with a model with fading memory.

For the isothermal case, i.e. if $\mathcal{P}$ is independent of θ, a one-dimensional approach to elastoplasticity using rate-independent hysteresis operators has been carried out earlier by P. Krejčí in a series of papers (cf. e.g. [K1, K2, K]); the (simpler) case of viscoplasticity has been treated in [BS1]. In these cases, the space-time evolution is governed by the equation of motion which takes the form

$$\Big(\varrho\, u_{tt} - \big(\mathcal{P}[u_x]\big)_x - \mu u_{xxt}\Big)(x, t) = f(x, t), \tag{0.2}$$

where ϱ, $\mu \geqslant 0$ and u denote mass density, viscosity coefficient and displacement, respectively.

In the non-isothermal case the equation of motion has to be complemented by a field equation representing the balance law of internal energy, and the second principle of thermodynamics in the form of the Clausius-Duhem inequality must be obeyed. It is, however, not obvious how the correct expressions for thermodynamic state functions like the densities of free energy, internal energy and entropy, should look like for a constitutive law like (0.1). In [KS], a corresponding construction has been carried out. It turned out that in a setting like ours, where the relation between the strain and the plastic stress is given in an *operator* form, it is quite natural to consider the densities of free energy, internal energy and entropy as *operators* rather than as *functions*.

The aim of this paper is to extend the investigations of [KS] to other situations. More precisely, while in [KS] we have studied the case when the total stress σ is composed of a plastic stress σ^p of the form (0.1) and a so-called *couple stress*, we consider here the situation when σ comprises, in addition to the plastic stress (0.1), (nonlinear) elastic, (linear) viscous, and (linear) thermic contributions σ^e, σ^v and σ^d, respectively; that is, we assume a constitutive law of the form

$$\sigma = \sigma^p + \sigma^e + \sigma^v + \sigma^d, \tag{0.3}$$

with σ^p given as in (0.1).

It should be mentioned at this place that hysteretic relations usually can not be described in an explicit form and, as a rule, enjoy only very restricted smoothness properties. Therefore, the classical techniques of one-dimensional thermovisco-elasticity developed for cases in which the stress-strain relation is given through a simple (possibly nonconvex, but differentiable) function (we only refer to the fundamental papers [D, DH]) do not apply, and new techniques tailored to deal with the specific behavior of hysteretic nonlinearities have to be employed.

The paper is organized as follows. In Section 1, the field equations governing the space-time evolution in thermovisco-elastoplastic materials with the constitutive law (0.3) are derived. We obtain a system of nonlinearly coupled partial differential equations involving partial derivatives of hysteretic nonlinearities at different places, even in derivatives of highest order. Section 2 brings the statement of the initial-boundary value problem under investigation, and the general existence and uniqueness result is formulated. In Section 3, we employ space discretization to construct approximations to the solution for which global a priori estimates are shown in Section 4. Section 5 contains the proof of existence using compactness arguments and a passage-to-the-limit procedure. In the final Section 6, stability with respect to the data of the system and uniqueness are established.

The stop operator $\mathfrak{s}_r\colon\ W^{1,1}(0,T) \to W^{1,1}(0,T)$ in the equation (0.1) is defined as the solution operator $\sigma_r = \mathfrak{s}_r[\varepsilon]$ of the variational inequality

$$|\sigma_r(t)| \leqslant r, \quad (\dot\varepsilon - \dot\sigma_r)(\sigma_r - \tilde\sigma) \geqslant 0 \quad \text{a.e.,} \quad \forall \tilde\sigma \in [-r, r], \tag{1.1}$$

with an initial condition

$$\sigma_r(0) = \operatorname{sign}(\varepsilon(0)) \min\{r, |\varepsilon(0)|\} \tag{1.2}$$

which describes the strain-stress law of Prandtl's model for elastic-perfectly plastic materials with a unit elasticity modulus and yield point r.

The density function φ in (0.1) is assumed to be given. It can be identified from the isothermal initial loading curves $\sigma = \Phi(\varepsilon, \theta)$ obtained experimentally by letting ε monotonically increase for each fixed temperature θ starting from the origin. The corresponding formula reads (see [K])

$$\Phi(\varepsilon, \theta) = \int_0^\varepsilon \int_s^\infty \varphi(r, \theta) \, \mathrm{d}r \, \mathrm{d}s. \tag{1.3}$$

We consider here only the case when φ is nonnegative, i.e. the initial loading curves at each constant temperature are concave and nondecreasing as in Figure 1.

The operator $\mathfrak{s}_r$ has the following properties (for a proof, see [BS], [K]).

Proposition 1.1. *Let $r > 0$ be given. Then*

(i) *For every $\varepsilon \in W^{1,1}(0,T)$, we have*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{s}_r[\varepsilon]\right)^2 = \dot\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{s}_r[\varepsilon] \quad \text{a.e. in }]0,T[\,. \tag{1.4}$$

(ii) *For every $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$, we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2]\right)^2 \leqslant (\dot\varepsilon_1 - \dot\varepsilon_2)(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2]) \quad \text{a.e. in }]0,T[\,, \tag{1.5}$$

$$\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2])\right|(t)\,\mathrm{d}t \leqslant |\varepsilon_1(0) - \varepsilon_2(0)| + 2\int_0^T |\dot\varepsilon_1 - \dot\varepsilon_2|(t)\,\mathrm{d}t, \tag{1.6}$$

$$|(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2])(t)| \leqslant 2 \max_{0 \leqslant \tau \leqslant t} |\varepsilon_1(\tau) - \varepsilon_2(\tau)| \quad \forall t \in [0,T]. \tag{1.7}$$

(iii) *For every $r, q > 0$ and $\varepsilon \in W^{1,1}(0,T)$, we have*

$$|(\mathfrak{s}_r[\varepsilon] - \mathfrak{s}_q[\varepsilon])(t)| \leqslant |r - q| \quad \forall t \in [0,T]. \tag{1.8}$$

The inequalities (1.6), (1.7) entail that the stop operator $\mathfrak{s}_r$ is Lipschitz continuous in $W^{1,1}(0,T)$ and admits a Lipschitz continuous extension onto $C([0,T])$. Moreover, we immediately see by definition that $\mathfrak{s}_r$ is a *causal* operator, that is, we have the implication

$$\varepsilon_1(\tau) = \varepsilon_2(\tau) \quad \forall \tau \in [0,t] \qquad \Rightarrow \qquad \mathfrak{s}_r[\varepsilon_1](t) = \mathfrak{s}_r[\varepsilon_2](t) \tag{1.9}$$

for every $t \in [0,T]$, which means that the output values at time t depend only on the past values of the input. This enables us to consider $\mathfrak{s}_r$ as a family of operators acting in the spaces $C([0,t])$ for all $t \in \,]0,T]$.

Inequality (1.5) immediately yields

Corollary 1.2. *For all* $\varepsilon, \varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$ *we have*

$$\mathfrak{s}_r[\varepsilon]\left(\dot{\varepsilon} - \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{s}_r[\varepsilon]\right) \geqslant 0 \quad \text{a.e. in }]0,T[\,, \tag{1.10}$$

$$|(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2])(t)| \leqslant |\varepsilon_1(0) - \varepsilon_2(0)| + \int_0^t |\dot{\varepsilon}_1 - \dot{\varepsilon}_2|(\tau)\,\mathrm{d}\tau \quad \forall t \in [0,T]. \tag{1.11}$$

In this paper we consider the one-dimensional equation of motion

$$\varrho u_{tt} = \sigma_x + f, \tag{1.12}$$

where $\varrho > 0$ is a constant referential density, u is the displacement, σ is the total unaxial stress and f is the volume force density.

We assume that σ can be decomposed into the sum

$$\sigma = \sigma^p + \sigma^e + \sigma^v + \sigma^d, \tag{1.13}$$

where

$$\sigma^e = \gamma(\varepsilon) \tag{1.14}$$

with a given nondecreasing Lipschitz continuous function $\gamma\colon \mathbb{R}^1 \to \mathbb{R}^1$, $\gamma(0) = 0$, is the (nonlinear) kinematic hardening component,

$$\sigma^v = \mu\dot{\varepsilon} \tag{1.15}$$

with a constant $\mu > 0$ is the viscous component,

$$\sigma^d = -\beta\theta \tag{1.16}$$

with a constant $\beta \in \mathbb{R}^1$ is the thermic dilation component and σ^p is the thermoplastic component given by (0.1). Equation (1.13) can be interpreted rheologically as a parallel combination of the above components (see [LC]). The stop operator $\mathfrak{s}_r$ is assumed to act on functions of x and t according to the formula

$$\mathfrak{s}_r[\varepsilon](x,t) := \mathfrak{s}_r[\varepsilon(x,\cdot)](t), \tag{1.17}$$

i.e. x plays the role of a parameter. The equation of motion (1.12) has to be coupled with the energy balance equation

$$U_t = \sigma\varepsilon_t - q_x + g, \tag{1.18}$$

where U is the total internal energy, q is the heat flux and g is the heat source density. The model is thermodynamically consistent provided the temperature θ and the entropy S satisfy the inequalities

$$\theta > 0, \tag{1.19}$$

$$S_t \geqslant \frac{g}{\theta} - \left(\frac{q}{\theta}\right)_x \qquad \text{(Clausius-Duhem)}, \tag{1.20}$$

in an appropriate sense.

In [KS] we derived the following expressions for the thermoplastic parts of the internal energy U^p and the entropy S^p in operator form corresponding to the constitutive law (0.1):

$$U^p = \mathcal{V}[\varepsilon,\theta] := \frac{1}{2}\int_0^\infty (\varphi(r,\theta) - \theta\varphi_\theta(r,\theta))\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r, \tag{1.21}$$

$$S^p = \mathcal{S}[\varepsilon,\theta] := -\frac{1}{2}\int_0^\infty \varphi_\theta(r,\theta)\mathfrak{s}_r^2[\varepsilon]\,\mathrm{d}r. \tag{1.22}$$

In accordance with (1.13), (1.21), (1.22) we put

$$U := C_V\theta + \mathcal{V}[\varepsilon,\theta] + \Gamma(\varepsilon) + V_0, \tag{1.23}$$

$$S := C_V\log\theta + \mathcal{S}[\varepsilon,\theta] + \beta\varepsilon, \tag{1.24}$$

where $C_V > 0$, the purely caloric part of the *specific heat*, is a constant, $V_0 > 0$ is a constant which is chosen in order to ensure that $U \geqslant 0$ according to Hypothesis 2.2 below, and $\Gamma(\varepsilon) := \int_0^\varepsilon \gamma(s)\,\mathrm{d}s$. For the heat flux we assume Fourier's law

$$q = -\kappa\theta_x \tag{1.25}$$

with a constant heat conduction coefficient $\kappa > 0$. We complete the system (1.12), (1.18) with the small deformation hypothesis

$$\varepsilon = u_x \tag{1.26}$$

and rewrite it in the form

$$\varrho u_{tt} - (\gamma(u_x) + \mathcal{P}[u_x, \theta] + \mu u_{xt} - \beta\theta)_x = f, \tag{1.27}$$

$$(C_V\theta + \mathcal{V}[u_x, \theta])_t - \kappa\theta_{xx} = (\mathcal{P}[u_x, \theta] + \mu u_{xt} - \beta\theta)\, u_{xt} + g. \tag{1.28}$$

In fact, the model can be interpreted in the framework of classical thermodynamics using a continuous family of internal parameters. In the above setting, the memory state at point x and time t is described by the function

$$r \longmapsto \mathfrak{s}_r[\varepsilon](x, t), \tag{1.29}$$

i.e. the internal parameter function takes values in an infinite-dimensional subset of the metric space

$$\Lambda = \left\{\lambda \in W^{1,\infty}(0, \infty);\ |\lambda'(r)| \leqslant 1 \quad \text{a.e. in } \ ]0, \infty[\right\} \tag{1.30}$$

according to (1.8). The operator notation we introduced in [KS] and use here is much more elegant, indeed.

2. Statement of the problem

We consider a model problem for a system of the form (1.27), (1.28), namely

$$u_{tt} - \gamma(u_x)_x - (\mathcal{P}[u_x, \theta])_x - \mu u_{xxt} + \beta\theta_x = f(\theta, x, t), \tag{2.1}$$

$$(C_V\theta + \mathcal{V}[u_x, \theta])_t - \theta_{xx} = \mathcal{P}[u_x, \theta]u_{xt} + \mu u_{xt}^2 - \beta\theta u_{xt} + g(\theta, x, t), \tag{2.2}$$

for $x \in \,]0, 1[$, $t \in [0, T]$, where $T > 0$, $\mu > 0$, $C_V > 0$, $\beta \in \mathbb{R}^1$ are fixed constants, $\gamma\colon \mathbb{R}^1 \to \mathbb{R}^1$, $f, g\colon \,]0, \infty[\, \times \,]0, 1[\, \times [0, T] \to \mathbb{R}^1$ are given functions, and $\mathcal{P}$, $\mathcal{V}$ are the operators defined by (0.1), (1.21) with a given distribution function $\varphi\colon (]0, \infty[)^2 \to [0, \infty[$ satisfying Hypothesis 2.2 below.

In other words, we assume in (1.27), (1.28) that the volume force and heat source densities are given functions of x and t which may also depend on the instantaneous value of θ, and we rescale the units in such a way that $\varrho \equiv \kappa \equiv 1$. The system

(2.1), (2.2) is coupled with boundary and initial conditions which are chosen in the following simple form:

$$u(0,t) = u(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \tag{2.3}$$

$$u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ \theta(x,0) = \theta^0(x). \tag{2.4}$$

The data are assumed to satisfy the following hypotheses.

Hypothesis 2.1.

(i) *$u^0, u^1 \in W^{2,2}(0,1) \cap \overset{\circ}{W}{}^{1,2}(0,1)$, $\theta^0 \in W^{1,2}(0,1)$, and there exists a constant $\delta > 0$ such that*

$$\theta^0(x) \geqslant \delta \qquad \forall x \in [0,1]. \tag{2.5}$$

(ii) *$\gamma\colon \mathbb{R}^1 \to \mathbb{R}^1$ is an absolutely continuous function, $\gamma(0) = 0$, and there exists a constant $\gamma_0 > 0$ such that*

$$0 \leqslant \frac{d\gamma(\varepsilon)}{d\varepsilon} \leqslant \gamma_0 \qquad \text{a.e. in } \mathbb{R}^1. \tag{2.6}$$

(iii) *The functions f, g are measurable, $f(\cdot,x,t)$, $g(\cdot,x,t)$ are absolutely continuous in $[0,\infty[$ for a.e. $(x,t) \in]0,1[\times]0,T[$. Moreover, there exist a constant $K > 0$ and functions $f_0, g_0 \in L^2(]0,1[\times]0,T[)$ such that*

$$g(0,x,t) = g_0(x,t) \geqslant 0 \qquad \text{a.e.}, \tag{2.7}$$

$$|f(\theta,x,t)| + |f_t(\theta,x,t)| \leqslant f_0(x,t) \quad \text{a.e.}, \tag{2.8}$$

$$|f_\theta(\theta,x,t)| + |g_\theta(\theta,x,t)| \leqslant K \qquad \text{a.e.} \tag{2.9}$$

Hypothesis 2.2. *The function $\varphi\colon (]0,\infty[)^2 \to [0,\infty[$ is measurable, $\varphi(r,\cdot)$, $\varphi_\theta(r,\cdot)$ are absolutely continuous for a.e. $r > 0$, and there exist constants $L > 0$, $V_0 > 0$ such that for a.e. $\theta > 0$ the following inequalities hold:*

$$\int_0^\infty \varphi(r,\theta)\,\mathrm{d}r \leqslant L, \tag{2.10}$$

$$\int_0^\infty |\varphi_\theta(r,\theta)| r\,\mathrm{d}r \leqslant L, \tag{2.11}$$

$$\int_0^\infty \theta|\varphi_{\theta\theta}(r,\theta)| r^2\,\mathrm{d}r \leqslant C_V, \tag{2.12}$$

where C_V *is the constant introduced in* (1.23),

$$\frac{1}{2}\int_0^\infty |\varphi(r,\theta)-\theta\varphi_\theta(r,\theta)|\,(1+r^2)\,\mathrm{d}r \leqslant V_0. \tag{2.13}$$

E x a m p l e 2.3. A typical function φ satisfying Hypothesis 2.2 can be chosen as

$$\varphi(r,\theta) = \overline{E}(\theta)c(r-\bar{r}(\theta)), \tag{2.14}$$

where $c \in \mathcal{D}(]-m,m[)$ is a mollifier such that

$$\int_{-m}^{m} c(s)\,\mathrm{d}s = 1, \qquad c \geqslant 0, \tag{2.15}$$

with a (small) constant $m > 0$, and $\overline{E}, \bar{r}$ are given functions such that $\overline{E}(\theta) \leqslant L$, $m \leqslant \bar{r}(\theta) \leqslant R$, for some constant $R \geqslant m$, with $(1+\theta)\big(|\overline{E}'(\theta)| + |\bar{r}'(\theta)|\big)$ bounded and $\theta\big(|\overline{E}''(\theta)| + |\bar{r}''(\theta)| + \overline{E}'^2(\theta) + \bar{r}'^2(\theta)\big)$ small, uniformly with respect to θ.

We now state the main result of this paper.

Theorem 2.4. *Let Hypotheses* 2.1, 2.2 *hold. Then there exists a unique solution* (u,θ) *to the problem* (2.1)–(2.4) *such that*

$$u_{tt}, u_{xx}, u_{xxt}, \theta_x \in L^\infty(0,T;L^2(0,1)), \tag{2.16}$$

$$u_{xtt}, \theta_t, \theta_{xx} \in L^2(]0,1[\times]0,T[), \tag{2.17}$$

$$\theta, u, u_x, u_t, u_{xt} \in C([0,1]\times[0,T]), \tag{2.18}$$

there exists a constant $c_0 > 0$ *depending only on the given data such that for all* $t \in [0,T]$ *and* $x \in [0,1]$ *we have*

$$\theta(x,t) \geqslant \delta \mathrm{e}^{-c_0 t} > 0, \tag{2.19}$$

and (2.1)–(2.4) *are satisfied almost everywhere.*

We first check that the model is thermodynamically consistent according to (1.19), (1.20).

Corollary 2.5. *The solution from Theorem* 2.4 *satisfies the Clausius-Duhem inequality* (1.20) *with* S *defined by* (1.24), (1.22) *almost everywhere in* $]0,1[\times]0,T[$.

Proof of Corollary 2.5. For a.e. x and t we have

$$
\begin{aligned}
(2.20)\qquad & \theta S_t + \theta\left(\frac{q}{\theta}\right)_x - g \\
&= C_V\theta_t + \theta\left(\mathcal{S}[u_x,\theta]\right)_t + \beta\theta u_{xt} - \theta_{xx} - g + \frac{1}{\theta}\theta_x^2 \\
&= -\left(\mathcal{V}[u_x,\theta]\right)_t + \theta\left(\mathcal{S}[u_x,\theta]\right)_t + \mathcal{P}[u_x,\theta]u_{xt} + \mu u_{xt}^2 + \frac{1}{\theta}\theta_x^2 \\
&= \int_0^\infty \varphi(r,\theta)\mathfrak{s}_r[u_x](u_x - \mathfrak{s}_r[u_x])_t\,\mathrm{d}r + \mu u_{xt}^2 + \frac{1}{\theta}\theta_x^2
\end{aligned}
$$

and the assertion follows from (1.10). □

The existence result in Theorem 2.4 is proved via compactness methods based on a space-discrete approximation scheme. We use a stepwise estimation technique which will be explained in the next two sections. It depends substantially on the following properties of the hysteresis operators $\mathcal{P}$ and $\mathcal{V}$.

Proposition 2.6. *Let Hypothesis* 2.2 *hold. Then the operators* $\mathcal{P},\mathcal{V}$ *are causal and have the following properties.*

(i) *For every* $\varepsilon,\theta \in W^{1,1}(0,T)$, $\theta > 0$, *we have*

$$
(2.21)\qquad |\mathcal{P}[\varepsilon,\theta](t)| \leqslant V_0, \quad |\mathcal{V}[\varepsilon,\theta](t)| \leqslant V_0,
$$

$$
(2.22)\qquad \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}[\varepsilon,\theta](t)\right| \leqslant L\left(|\dot\varepsilon(t)| + |\dot\theta(t)|\right) \quad \text{a.e. in }]0,T[\,.
$$

(ii) *For every* $\varepsilon,\varepsilon_2,\theta_1,\theta_2 \in W^{1,1}(0,T)$, $\theta_1 > 0$, $\theta_2 > 0$ *and for every* $t \in [0,T]$, *we have*

$$
\begin{aligned}
(2.23)\qquad & \left|\mathcal{P}[\varepsilon_1,\theta_1] - \mathcal{P}[\varepsilon_2,\theta_2]\right|(t) \\
&\leqslant L\left(|\theta_1-\theta_2|(t) + |\varepsilon_1-\varepsilon_2|(0) + \int_0^t |\dot\varepsilon_1 - \dot\varepsilon_2|(\tau)\,\mathrm{d}\tau\right),
\end{aligned}
$$

$$
\begin{aligned}
(2.24)\qquad & \left|\mathcal{V}[\varepsilon_1,\theta_1] - \mathcal{V}[\varepsilon_2,\theta_2]\right|(t) \\
&\leqslant \frac{C_V}{2}|\theta_1-\theta_2|(t) + V_0\left(|\varepsilon_1-\varepsilon_2|(0) + \int_0^t |\dot\varepsilon_1 - \dot\varepsilon_2|(\tau)\,\mathrm{d}\tau\right),
\end{aligned}
$$

$$
(2.25)\qquad \left|\mathcal{P}[\varepsilon_1,\theta_1] - \mathcal{P}[\varepsilon_2,\theta_2]\right|(t) \leqslant L\left(|\theta_1-\theta_2|(t) + 2\max_{0\leqslant\tau\leqslant t}|\varepsilon_1-\varepsilon_2|(\tau)\right),
$$

$$
(2.26)\qquad \left|\mathcal{V}[\varepsilon_1,\theta_1] - \mathcal{V}[\varepsilon_2,\theta_2]\right|(t) \leqslant \frac{C_V}{2}|\theta_1-\theta_2|(t) + 2V_0\max_{0\leqslant\tau\leqslant t}|\varepsilon_1-\varepsilon_2|(\tau).
$$

P r o o f. The causality is obvious. To prove part (ii), we just note that

$$(2.27)\quad \begin{aligned}&\big|\mathcal{P}[\varepsilon_1,\theta_1]-\mathcal{P}[\varepsilon_2,\theta_2]\big|\\ &\leqslant \int_0^\infty |\varphi(r,\theta_1)-\varphi(r,\theta_2)||\mathfrak{s}_r[\varepsilon_1]|\,\mathrm{d}r+\int_0^\infty \varphi(r,\theta_2)|\mathfrak{s}_r[\varepsilon_1]-\mathfrak{s}_r[\varepsilon_2]|\,\mathrm{d}r,\end{aligned}$$

$$(2.28)\quad \begin{aligned}&\big|\mathcal{V}[\varepsilon_1,\theta_1]-\mathcal{V}[\varepsilon_2,\theta_2]\big|\\ &\leqslant \frac{1}{2}\int_0^\infty |\varphi(r,\theta_1)-\theta_1\varphi_\theta(r,\theta_1)-\varphi(r,\theta_2)+\theta_2\varphi_\theta(r,\theta_2)|\mathfrak{s}_r^2[\varepsilon_1]\,\mathrm{d}r\\ &\quad+\frac{1}{2}\int_0^\infty |\varphi(r,\theta_2)-\theta_2\varphi_\theta(r,\theta_2)|\,\big|\mathfrak{s}_r^2[\varepsilon_1]-\mathfrak{s}_r^2[\varepsilon_2]\big|\,\mathrm{d}r,\end{aligned}$$

and the inequalities (2.23)–(2.26) follow from the hypotheses (2.10)–(2.13) and the inequalities (1.7), (1.11). In addition, by definition we have

$$(2.29)\qquad |\mathfrak{s}_r[\varepsilon](t)|\leqslant r\qquad \forall\varepsilon,\quad \forall t,$$

and from (1.4) it follows that

$$(2.30)\qquad \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{s}_r[\varepsilon](t)\right|\leqslant|\dot\varepsilon(t)|\qquad \text{a.e.}\quad \forall\varepsilon.$$

A straightforward argument yields (2.22) and the second inequality of (2.21). The proof of Proposition 2.6 will be complete if we check that Hypothesis 2.2 implies

$$(2.31)\qquad \int_0^\infty r\varphi(r,\theta)\,\mathrm{d}r\leqslant V_0\qquad \forall\theta>0.$$

To this end, we introduce the function

$$(2.32)\qquad \psi(r,\theta):=\frac{r}{\theta}\varphi(r,\theta).$$

By (2.10), (2.11), (2.13) we have for all $\theta>0$

$$(2.33)\qquad \int_0^\infty \frac{1}{r}\,\psi(r,\theta)\,\mathrm{d}r\leqslant\frac{L}{\theta},$$

$$(2.34)\qquad \int_0^\infty |\psi(r,\theta)+\theta\psi_\theta(r,\theta)|\,\mathrm{d}r\leqslant L,$$

$$(2.35)\qquad \int_0^\infty |\psi_\theta(r,\theta)|\,\mathrm{d}r\leqslant\frac{V_0}{\theta^2},$$

and the triangle inequality yields that

$$(2.36)\qquad \int_0^\infty \psi(r,\theta)\,\mathrm{d}r\leqslant L+\frac{V_0}{\theta}.$$

The functions $\psi(\cdot,\theta)$ thus belong to $L^1(0,\infty)$ for each value of the parameter $\theta > 0$. Moreover, for $\theta_2 > \theta_1 > 0$ it follows from (2.35) that

$$\int_0^\infty |\psi(r,\theta_1) - \psi(r,\theta_2)|\, \mathrm{d}r \leqslant \int_0^\infty \int_{\theta_1}^{\theta_2} |\psi_\theta(r,\theta)|\, \mathrm{d}\theta\, \mathrm{d}r \leqslant V_0 \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right). \tag{2.37}$$

Since the space $L^1(0,\infty)$ is complete, there exists a function $\psi_\infty \in L^1(0,\infty)$ such that

$$\lim_{\theta\to\infty} \int_0^\infty |\psi(r,\theta) - \psi_\infty(r)|\, \mathrm{d}r = 0. \tag{2.38}$$

Passing to the limit in (2.37) as $\theta_2 \to \infty$ we obtain

$$\int_0^\infty |\psi(r,\theta) - \psi_\infty(r)|\, \mathrm{d}r \leqslant \frac{V_0}{\theta} \qquad \forall \theta > 0. \tag{2.39}$$

On the other hand, for every $R > 0$ and $\theta > 0$ we have

$$\int_0^R \psi_\infty(r)\, \mathrm{d}r \leqslant R \int_0^R \frac{1}{r}\psi(r,\theta)\, \mathrm{d}r + \int_0^R |\psi(r,\theta) - \psi_\infty(r)|\, \mathrm{d}r. \tag{2.40}$$

Hence (2.33), (2.39) yield that $\psi_\infty = 0$ a.e.; inequality (2.31) now follows immediately from (2.39), (2.32). □

3. Space discretization

Let $n > 1$ be a given integer. We replace (2.1)–(2.4) by the following system of ODEs for unknown functions $u_1, \dots, u_{n-1}$, $\theta_1, \dots, \theta_n$:

$$\ddot{u}_k = n(\sigma_{k+1} - \sigma_k) + f_k(\theta_k, t), \qquad k = 1, \dots, n-1, \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(C_V\theta_k + \mathcal{V}[\varepsilon_k, \theta_k]\right) = n^2(\theta_{k+1} - 2\theta_k + \theta_{k-1}) + \left(\mathcal{P}[\varepsilon_k, \theta_k] + \mu\dot{\varepsilon}_k - \beta\theta_k\right)\dot{\varepsilon}_k + g_k(\theta_k, t), \quad k = 1, \dots, n, \tag{3.2}$$

$$\varepsilon_k = n(u_k - u_{k-1}), \qquad k = 1, \dots, n, \tag{3.3}$$

$$\sigma_k = \gamma(\varepsilon_k) + \mathcal{P}[\varepsilon_k, \theta_k] + \mu\dot{\varepsilon}_k - \beta\theta_k, \quad k = 1, \dots, n, \tag{3.4}$$

$$u_0 = u_n = 0, \quad \theta_0 = \theta_1, \quad \theta_{n+1} = \theta_n, \tag{3.5}$$

$$f_k(\theta, t) = n \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(\theta, x, t)\, \mathrm{d}x, \quad g_k(\theta, t) = n \int_{\frac{k-1}{n}}^{\frac{k}{n}} g(\theta, x, t)\, \mathrm{d}x, \quad k = 1, \dots, n, \tag{3.6}$$

$$u_k(0) = u^0\left(\frac{k}{n}\right), \quad \dot{u}_k(0) = u^1\left(\frac{k}{n}\right), \quad \theta_k(0) = \theta^0\left(\frac{k}{n}\right), \quad k = 1, \dots, n. \tag{3.7}$$

It can be proved in a standard way that the system (3.1)–(3.7) admits a unique local solution; indeed, it suffices to put $v_k := \dot{u}_k$ and to rewrite (3.1), (3.2) as integral equations,

$$
\begin{aligned}
&(3.8)\quad v_k(t) = u^1\left(\frac{k}{n}\right) + \int_0^t (n(\sigma_{k+1} - \sigma_k) + f_k(\theta_k, \cdot))(\tau)\,\mathrm{d}\tau,\\
&(3.9)\quad u_k(t) = u^0\left(\frac{k}{n}\right) + \int_0^t v_k(\tau)\,\mathrm{d}\tau,\\
&(3.10)\ \theta_k(t) = \theta^0\left(\frac{k}{n}\right) + \frac{1}{C_V}\left(\mathcal{V}[\varepsilon_k, \theta_k](0) - \mathcal{V}[\varepsilon_k, \theta_k](t)\right)\\
&\qquad + \int_0^t \frac{1}{C_V}\big[n^2(\theta_{k+1} - 2\theta_k + \theta_{k-1})\\
&\qquad + (\mathcal{P}[\varepsilon_k, \theta_k] + \mu n(v_k - v_{k-1}) - \beta\theta_k)\, n(v_k - v_{k-1}) + g_k(\theta_k, \cdot)\big](\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

The system (3.8)–(3.10) is of the form

$$
W(t) = W(0) + A(W)(t) - A(W)(0) + \int_0^t B(W, \cdot)(\tau)\,\mathrm{d}\tau, \tag{3.11}
$$

where W is a vector function with components $\{v_k, u_k, \theta_k\,;\ k = 1, \dots, n\}$, A is an operator in $C([0,t]; \mathbb{R}^{3n})$ for every $t \in\,]0, T[$ with components $\{\underbrace{0, \dots, 0}_{2n}, -\frac{1}{C_V}\mathcal{V}[\varepsilon_k, \theta_k](t);$ $k = 1, \dots, n\}$, and the operator B is given by the expressions under the integral signs in (3.8)–(3.10). We endow the space $\mathbb{R}^{3n}$ with the norm $|||W||| = \sum_{k=1}^{n} (|\theta_k| + \frac{8nV_0}{C_V}|u_k| + |v_k|)$. Then we have, by Proposition 2.6,

$$
|||A(W_1)(t) - A(W_2)(t)||| \leqslant \frac{1}{2} \max_{\tau \in [0,t]} |||W_1(\tau) - W_2(\tau)||| \tag{3.12}
$$

for every W_1, $W_2 \in C([0,t]; \mathbb{R}^{3n})$. The operator B is Lipschitz in $C([0,\tau]; \mathbb{R}^{3n})$ for every $\tau \in [0,t]$ by Proposition 2.6 and Hypothesis 2.1. In a standard way we conclude from the Contraction Mapping Principle that equation (3.11) (and therefore also system (3.1)–(3.7)) admits a unique classical solution in an interval $[0, T_n]$. Taking a smaller $T_n > 0$ if necessary, we may assume that

$$
\theta_k(t) > 0 \quad \text{for all } t \in [0, T_n], \quad k = 1, \dots, n, \tag{3.13}
$$

due to hypothesis (2.5).

In the interval $[0,T_n]$ the solution $u_1,\dots,u_{n-1}$, $\theta_1,\dots,\theta_n$ of (3.1)–(3.7) satisfies the following estimates.

Theorem 3.1. *There exists a constant $\overline{C}$ which depends only on T, on the number*

$$M := \|u^0\|_{W^{2,2}} + \|u^1\|_{W^{2,2}} + \|\theta^0\|_{W^{1,2}} + \|f_0\|_{L^2} + \|g_0\|_{L^2}, \tag{3.14}$$

and on the constants C_V, β, μ, K, L, V_0 and γ_0, such that for all $t\in[0,T_n]$ we have

$$\frac{1}{n}\sum_{k=1}^{n}\left(\dot u_k^2+\ddot u_k^2+\varepsilon_k^2+\dot\varepsilon_k^2+\theta_k^2+n^2(\theta_{k+1}-\theta_k)^2\right)(t)\leqslant\overline{C}, \tag{3.15}$$

$$n\sum_{k=1}^{n-1}\left((\varepsilon_{k+1}-\varepsilon_k)^2+(\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2\right)(t)\leqslant\overline{C}, \tag{3.16}$$

$$\frac{1}{n}\sum_{k=1}^{n}\int_0^t(\ddot\varepsilon_k^2+\dot\theta_k^2)(\tau)\,\mathrm{d}\tau\leqslant\overline{C}, \tag{3.17}$$

$$n^3\sum_{k=1}^{n-1}\int_0^t(\theta_{k+1}-2\theta_k+\theta_{k-1})^2(\tau)\,\mathrm{d}\tau\leqslant\overline{C}. \tag{3.18}$$

We devote the next section to the proof of Theorem 3.1 which requires several consecutive steps (Lemmas 4.1–4.10 below). For this purpose it is convenient to rewrite equation (3.2) in the form

$$\begin{aligned}\dot\theta_k\left(C_V-\frac{1}{2}\int_0^\infty\theta_k\varphi_{\theta\theta}(r,\theta_k)\mathfrak{s}_r^2[\varepsilon_k]\,\mathrm{d}r\right)\\ =n^2\left(\theta_{k+1}-2\theta_k+\theta_{k-1}\right)+\theta_k\left(\int_0^\infty\varphi_\theta(r,\theta_k)\mathfrak{s}_r[\varepsilon_k](\mathfrak{s}_r[\varepsilon_k])_t\,\mathrm{d}r-\beta\dot\varepsilon_k\right)\\ +\int_0^\infty\varphi(r,\theta_k)\mathfrak{s}_r[\varepsilon_k](\varepsilon_k-\mathfrak{s}_r[\varepsilon_k])_t\,\mathrm{d}r+\mu\dot\varepsilon_k^2+g_k(\theta_k,t).\end{aligned} \tag{3.19}$$

Theorem 3.1 has the following consequence.

Corollary 3.2. *The solution $(u_1,\dots,u_{n-1},\theta_1,\dots,\theta_n)$ of (3.1)–(3.7) can be extended to $[0,T]$, the estimates (3.15)–(3.18) hold for all $t\in[0,T]$, and there exists a constant $c_0>0$, independent of γ and n, such that*

$$\theta_k(t)\geqslant\delta\mathrm{e}^{-c_0t}\quad\textit{for}\quad k=1,\dots,n,\quad t\in[0,T]. \tag{3.20}$$

The proof of Corollary 3.2 is based on the following "discrete maximum principle".

Lemma 3.3. *Let $w_1, \dots, w_n$ be absolutely continuous functions satisfying the system*

$$b_k(t)\dot{w}_k(t) - A(w_{k+1} - 2w_k + w_{k-1})(t) + a_k(t)w_k(t) = r_k(t) \quad \text{for a.e. } t \in \,]0,T[\,, \tag{3.21}$$

$$w_0 = w_1, \quad w_{n+1} = w_n, \tag{3.22}$$

$$b_k(t) \geqslant B, \quad |a_k(t)| \leqslant C, \quad r_k(t) \geqslant 0 \qquad \text{a.e. in }]0,T[\,, \tag{3.23}$$

$$w_k(0) \geqslant \delta, \tag{3.24}$$

for all $k = 1, \dots, n$, where $A \geqslant 0$, $B > 0$, $C > 0$, $\delta > 0$ are given constants and a_k, b_k, r_k are measurable functions. Then

$$w_k(t) \geqslant \delta \mathrm{e}^{-\frac{C}{B}t} \quad \text{for all} \quad t \in [0,T], \ k = 1, \dots, n. \tag{3.25}$$

Proof of Lemma 3.3. For a fixed $C^* > \frac{C}{B}$ put $p_k(t) := w_k(t)\mathrm{e}^{C^*t}$. Then, a.e. in $]0,T[$, the functions p_k for $k = 1, \dots, n$ solve the system

$$b_k(t)\dot{p}_k(t) - A(p_{k+1} - 2p_k + p_{k-1})(t) = (C^*b_k(t) - a_k(t))p_k(t) + r_k(t)\mathrm{e}^{C^*t}. \tag{3.26}$$

Assume that there exist $\eta \in \,]0,\delta[$, $k \in \{1, \dots, n\}$ and $t \in [0,T]$ such that $p_k(t) < \delta - \eta$. Moreover, put

$$\bar{t} = \sup\{t \in [0,T]; \quad p_j(\tau) \geqslant \delta - \eta \quad \forall j \in \{1, \dots, n\}, \quad \forall \tau \in [0,t]\}. \tag{3.27}$$

We fix some j such that $p_j(\bar{t}) = \delta - \eta$ and $\varrho > 0$ such that

$$A\,|p_k(\bar{t}) - p_k(t)| \leqslant \frac{B}{8}\Big(C^* - \frac{C}{B}\Big)(\delta - \eta), \quad k = 1, \dots, n, \ \forall t \in [\bar{t} - \varrho, \bar{t}]. \tag{3.28}$$

Then we have

$$\begin{aligned}
0 &\leqslant \left(\frac{A}{\varrho}\int_{\bar{t}-\varrho}^{\bar{t}} \frac{\mathrm{d}t}{b_j(t)}\right)(p_{j+1}(\bar{t}) - 2p_j(\bar{t}) + p_{j-1}(\bar{t})) \\
&\leqslant \frac{4A}{\varrho B}\int_{\bar{t}-\varrho}^{\bar{t}} \max_k |p_k(\bar{t}) - p_k(t)| \,\mathrm{d}t + \frac{A}{\varrho}\int_{\bar{t}-\varrho}^{\bar{t}} \frac{1}{b_j(t)}\,(p_{j+1}(t) - 2p_j(t) + p_{j-1}(t))\,\mathrm{d}t \\
&\leqslant \frac{1}{2}\left(C^* - \frac{C}{B}\right)(\delta - \eta) + \frac{1}{\varrho}\int_{\bar{t}-\varrho}^{\bar{t}} \left(\dot{p}_j(t) - \left(C^* - \frac{a_j(t)}{b_j(t)}\right)p_j(t) - \frac{r_j(t)}{b_j(t)}\mathrm{e}^{C^*t}\right)\mathrm{d}t \\
&\leqslant -\frac{1}{2}\left(C^* - \frac{C}{B}\right)(\delta - \eta) < 0,
\end{aligned}$$

which is a contradiction. We therefore have $w_k(t) \geqslant \delta\mathrm{e}^{-C^*t}$ for all $C^* > \frac{C}{B}$ and $t \in [0,T]$, and the assertion easily follows. $\square$

Proof of Corollary 3.2. Equation (3.19) is of the form (3.21) with $A = n^2$,

$$r_k(t) = \mu\dot{\varepsilon}_k^2 + \int_0^\infty \varphi(r,\theta_k)\mathfrak{s}_r[\varepsilon_k](\varepsilon_k - \mathfrak{s}_r[\varepsilon_k])_t \,\mathrm{d}r + g_k(\theta_k,t) + K\theta_k,$$
$$a_k(t) = K + \beta\dot{\varepsilon}_k - \int_0^\infty \varphi_\theta(r,\theta_k)\mathfrak{s}_r[\varepsilon_k]\left(\mathfrak{s}_r[\varepsilon_k]\right)_t \,\mathrm{d}r,$$
$$b_k(t) = C_V - \frac{1}{2}\int_0^\infty \theta_k\varphi_{\theta\theta}(r,\theta_k)\mathfrak{s}_r^2[\varepsilon_k]\,\mathrm{d}r.$$

By hypothesis (2.12) we have $b_k(t) \geqslant \frac{1}{2}C_V > 0$. Using the elementary inequality

$$|\dot{\varepsilon}_j(t)|^2 \leqslant \left(|\dot{\varepsilon}_k(t)| + \sum_{i=1}^{n-1}|\dot{\varepsilon}_{i+1} - \dot{\varepsilon}_i|(t)\right)^2$$

for $j,k \in \{1,\dots,n\}$, we obtain from (3.15), (3.16) that

$$\max_{1\leqslant j\leqslant n}|\dot{\varepsilon}_j(t)|^2 \leqslant \frac{2}{n}\sum_{k=1}^n \dot{\varepsilon}_k^2(t) + 2n\sum_{k=1}^{n-1}(\dot{\varepsilon}_{k+1} - \dot{\varepsilon}_k)^2(t) \leqslant 4\overline{C},$$

hence $|a_k(t)| \leqslant K + 4\overline{C}(\beta + L)$ a.e. for all k by Hypothesis 2.2. We further have

$$g_k(\theta_k,t) + K\theta_k = n\int_{\frac{k-1}{n}}^{\frac{k}{n}}(g(\theta_k,x,t) + K\theta_k)\,\mathrm{d}x \geqslant 0$$

by hypotheses (2.8), (2.9) provided $\theta_k > 0$, and from (1.10) it follows that $r_k(t) \geqslant 0$ a.e. for all k. By Lemma 3.3, for all $t \in [0,T_n]$ and $k = 1,\dots,n$ we have $\theta_k(t) \geqslant \delta\mathrm{e}^{-c_0 t}$ for some c_0. This and the estimates (3.15)–(3.18) imply that the solution ε_k, θ_k of (3.1)–(3.7) can be extended onto the whole interval $[0,T]$, and Corollary 3.2 is proved.

□

4. Estimates

In a series of lemmas below we derive the estimates (3.15)–(3.18). Throughout this section we denote by C, C_i positive constants that depend only on C_V, β, γ_0, μ, K, L, V_0, T and the constant M defined by (3.14). We start with two discrete versions of Nirenberg's inequality.

Lemma 4.1. *For each $\alpha \in\ ]0,1[$ there exists a constant C_α such that for every $n \in \mathbb{N}$ and every sequence $z_1,\dots,z_n$ of positive numbers we have*

(4.1)
$$\max_{1\leqslant j\leqslant n} z_j \leqslant C_\alpha\left[\frac{1}{n}\sum_{k=1}^n z_k + \left(n\sum_{k=1}^n (z_{k+1} - z_k)(z_k^{-\alpha} - z_{k+1}^{-\alpha})\right)^{\frac{1}{2-\alpha}}\left(\frac{1}{n}\sum_{k=1}^n z_k\right)^{\frac{1}{2-\alpha}}\right].$$

Proof. Let a sequence $z_1, \dots, z_n$ be given, and let j be such that $z_j \geqslant z_k$ for all $k = 1, \dots, n$. Then we have for all k

$$z_j^{1-\frac{\alpha}{2}} \leqslant z_k^{1-\frac{\alpha}{2}} + \sum_{i=1}^{n-1} \left| z_{i+1}^{1-\frac{\alpha}{2}} - z_i^{1-\frac{\alpha}{2}} \right|, \tag{4.2}$$

whence

$$z_j \leqslant 2^{\frac{\alpha}{2-\alpha}} \left(z_k + \left(\sum_{i=1}^{n-1} \left| z_{i+1}^{1-\frac{\alpha}{2}} - z_i^{1-\frac{\alpha}{2}} \right| \right)^{\frac{2}{2-\alpha}} \right). \tag{4.3}$$

Using the elementary inequality

$$\left(a^{1-\frac{\alpha}{2}} - b^{1-\frac{\alpha}{2}}\right)^2 \leqslant K_\alpha (a+b)(a-b)(b^{-\alpha} - a^{-\alpha}) \quad \text{for every } \ a, b > 0, \tag{4.4}$$

where

$$K_\alpha := \sup_{s>0} \frac{(1+s)^\alpha ((1+s)^{1-\frac{\alpha}{2}} - 1)^2}{s(2+s)((1+s)^\alpha - 1)} < \infty, \tag{4.5}$$

we obtain from (4.3), after summing over k, that

$$z_j \leqslant 2^{\frac{\alpha}{2-\alpha}} \left(\frac{1}{n} \sum_{k=1}^{n} z_k + \left(K_\alpha^{1/2} \sum_{i=1}^{n-1} (z_{i+1} - z_i)^{\frac{1}{2}} (z_i^{-\alpha} - z_{i+1}^{-\alpha})^{\frac{1}{2}} (z_i + z_{i+1})^{\frac{1}{2}} \right)^{\frac{2}{2-\alpha}} \right), \tag{4.6}$$

and (4.1) follows from the discrete Hölder inequality. □

Lemma 4.2. *For every sequence $z_1, \dots, z_n$ of real numbers we have*

$$\max_{1 \leqslant j \leqslant n} z_j^2 \leqslant \frac{1}{n} \sum_{k=1}^{n} z_k^2 + 2 \left(\frac{1}{n} \sum_{k=1}^{n} z_k^2 \right)^{\frac{1}{2}} \left(n \sum_{k=1}^{n-1} (z_{k+1} - z_k)^2 \right)^{\frac{1}{2}}. \tag{4.7}$$

Proof. We proceed as in Lemma 4.1, where (4.2) is replaced by $z_j^2 \leqslant z_k^2 + \sum_{i=1}^{n-1} |z_{i+1}^2 - z_i^2|$. □

In the following Lemmas 4.3–4.10 we derive upper bounds for the solution $(u_1, \dots, u_{n-1}, \theta_1, \dots, \theta_n)$ of the system (3.1)–(3.7).

Lemma 4.3. *There exists a constant $C_1 > 0$ such that for every $t \in [0, T_n]$ we have*

$$\frac{1}{n} \sum_{k=1}^{n} \left(\theta_k + \dot{u}_k^2 + \Gamma(\varepsilon_k) \right)(t) \leqslant C_1. \tag{4.8}$$

P r o o f. Multiply (3.1) by $\dot u_k$ and sum over $k = 1,\dots,n-1$. This yields, for $t \in \,]0,T_n[$,

$$\frac{1}{n}\sum_{k=1}^{n}\left(\ddot u_k \dot u_k + \sigma\dot\varepsilon_k - f_k(\theta_k,\cdot)\dot u_k\right)(t) = 0. \tag{4.9}$$

Summing (3.2) over $k = 1,\dots,n$ and adding the result to (4.9), we obtain

$$\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{n}\left(C_V\theta_k + \mathcal{V}[\varepsilon_k,\theta_k] + \frac{1}{2}\dot u_k^2 + \Gamma(\varepsilon_k)\right)(t) = \frac{1}{n}\sum_{k=1}^{n}\left(g_k(\theta_k,\cdot) + f_k(\theta_k,\cdot)\dot u_k\right)(t), \tag{4.10}$$

where

$$|g_k(\theta_k,t)| \leqslant n\int_{\frac{k-1}{n}}^{\frac{k}{n}} |g(\theta_k,x,t)|\,\mathrm{d}x \leqslant K\theta_k(t) + n\int_{\frac{k-1}{n}}^{\frac{k}{n}} g_0(x,t)\,\mathrm{d}x, \tag{4.11}$$

$$|f_k(\theta_k,t)| \leqslant n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f_0(x,t)\,\mathrm{d}x. \tag{4.12}$$

Furthermore,

$$\frac{1}{n}\sum_{k=1}^{n}\theta_k(0) \leqslant \int_0^1\left(\theta^0(x) + \frac{1}{n}|\theta_x^0(x)|\right)\mathrm{d}x, \tag{4.13}$$

$$\frac{1}{n}\sum_{k=1}^{n}\dot u_k^2(0) \leqslant \int_0^1\left(|u^1(x)|^2 + \frac{2}{n}|u^1(x)||u_x^1(x)|\right)\mathrm{d}x, \tag{4.14}$$

$$\frac{1}{n}\sum_{k=1}^{n}\Gamma(\varepsilon_k)(0) \leqslant \frac{\gamma_0}{2n}\sum_{k=1}^{n}\varepsilon_k^2(0) \leqslant \frac{\gamma_0}{2}\int_0^1 |u_x^0(x)|^2\,\mathrm{d}x, \tag{4.15}$$

and we obtain (4.8) from (4.10)–(4.15) and Gronwall's lemma. □

The following estimate which goes back to Dafermos [D, DH] is crucial for the proof of Theorem 3.1. We fix an auxiliary parameter α and assume

$$\alpha \in \left]0,\frac{1}{3}\right]. \tag{4.16}$$

Lemma 4.4. *There exists a constant $C_2 > 0$ such that for all $t \in [0,T_n]$ we have*

$$\frac{1}{n}\sum_{k=1}^{n}\int_0^t\left(n^2(\theta_{k+1}-\theta_k)\left(\theta_k^{-\alpha} - \theta_{k+1}^{-\alpha}\right) + \theta_k^{-\alpha}\dot\varepsilon_k^2\right)(\tau)\,\mathrm{d}\tau \leqslant C_2, \tag{4.17}$$

$$\frac{1}{n}\sum_{k=1}^{n}\int_0^t \theta_k^{3-\alpha}(\tau)\,\mathrm{d}\tau \leqslant C_2. \tag{4.18}$$

P r o o f. Multiply the equation (3.19) by $-\theta_k^{-\alpha}$. Introducing a function

$$\psi_{-\alpha}(r,\bar\theta) = \int_0^{\bar\theta} \theta^{1-\alpha}\varphi_{\theta\theta}(r,\theta)\,\mathrm{d}\theta \tag{4.19}$$

for $r,\bar\theta > 0$, we obtain, using (1.10), (2.30), (2.7), (2.9) and (2.11),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(-\frac{C_V}{1-\alpha}\theta_k^{1-\alpha} + \frac{1}{2}\int_0^\infty \psi_{-\alpha}(r,\theta_k)\mathfrak{s}_r^2[\varepsilon_k]\,\mathrm{d}r\Big)\\
&\quad + \mu\dot\varepsilon_k^2\theta_k^{-\alpha} + n^2\left(\theta_{k+1} - 2\theta_k + \theta_{k-1}\right)\theta_k^{-\alpha}\\
&= -\,\theta_k^{-\alpha}\Big(g_k(\theta_k,t) + \int_0^\infty \varphi(r,\theta_k)\mathfrak{s}_r[\varepsilon_k]\left(\varepsilon_k - \mathfrak{s}_r[\varepsilon_k]\right)_t\,\mathrm{d}r\Big)\\
&\quad + \beta\theta_k^{1-\alpha}\dot\varepsilon_k - (1-\alpha)\int_0^{\theta_k}\int_0^\infty \theta^{-\alpha}\varphi_\theta(r,\theta)\mathfrak{s}_r[\varepsilon_k](\mathfrak{s}_r[\varepsilon_k])_t\,\mathrm{d}r\,\mathrm{d}\theta\\
&\leqslant \left(|\beta| + L\right)\theta_k^{1-\alpha}|\dot\varepsilon_k| + K\theta_k^{1-\alpha}.
\end{aligned} \tag{4.20}$$

By hypothesis (2.12), we have

$$\Big|\int_0^\infty \psi_{-\alpha}(r,\theta_k)\mathfrak{s}_r^2[\varepsilon_k]\,\mathrm{d}r\Big| \leqslant \int_0^{\theta_k}\theta^{-\alpha}\int_0^\infty \theta\,|\varphi_{\theta\theta}(r,\theta)|\,r^2\,\mathrm{d}r\,\mathrm{d}\theta \leqslant \frac{C_V}{1-\alpha}\theta_k^{1-\alpha} \tag{4.21}$$

and, by Lemma 4.3 and Hölder's inequality,

$$\frac{1}{n}\sum_{k=1}^n \theta_k^{1-\alpha}(t) \leqslant C_1^{1-\alpha} \qquad \forall t \in [0,T_n]. \tag{4.22}$$

Summing and integrating (4.20), we obtain, using (4.21), (4.22),

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^n \int_0^t \left(\mu\theta_k^{-\alpha}\dot\varepsilon_k^2 + n^2(\theta_{k+1} - \theta_k)(\theta_k^{-\alpha} - \theta_{k+1}^{-\alpha})\right)(\tau)\,\mathrm{d}\tau\\
&\qquad\qquad \leqslant C\Big(1 + \frac{1}{n}\sum_{k=1}^n\int_0^t \left(\theta_k^{1-\alpha}|\dot\varepsilon_k|\right)(\tau)\,\mathrm{d}\tau\Big).
\end{aligned} \tag{4.23}$$

From Hölder's inequality it follows that

$$\frac{1}{n}\sum_{k=1}^n\int_0^t \left(\theta_k^{1-\alpha}|\dot\varepsilon_k|\right)(\tau)\,\mathrm{d}\tau \leqslant \Big(\frac{1}{n}\sum_{k=1}^n\int_0^t \theta_k^{2-\alpha}(\tau)\,\mathrm{d}\tau\Big)^{\frac12}\Big(\frac{1}{n}\sum_{k=1}^n\int_0^t \left(\theta_k^{-\alpha}\dot\varepsilon_k^2\right)(\tau)\,\mathrm{d}\tau\Big)^{\frac12}. \tag{4.24}$$

On the other hand, for an arbitrary $p \in \,]0, 2-\alpha]$ we estimate, using Lemmas 4.1 and 4.3,

$$
\begin{aligned}
(4.25) \qquad \frac{1}{n}\sum_{k=1}^{n}\int_0^t \theta_k^{p+1}(\tau)\,\mathrm{d}\tau &\leqslant \max_{0\leqslant\tau\leqslant t}\Big(\frac{1}{n}\sum_{k=1}^{n}\theta_k(\tau)\,\mathrm{d}\tau\Big)\int_0^t \max_j \theta_j^p(\tau)\,\mathrm{d}\tau \\
&\leqslant C\Big(1 + n\sum_{k=1}^{n-1}\int_0^t (\theta_{k+1}-\theta_k)(\theta_k^{-\alpha}-\theta_{k+1}^{-\alpha})(\tau)\,\mathrm{d}\tau\Big)^{\frac{p}{2-\alpha}}.
\end{aligned}
$$

Inequality (4.17) now follows from (4.23), (4.24), (4.25) for $p = 1-\alpha$, and from Young's inequality. The estimate (4.18) is then obtained from (4.25) for $p = 2-\alpha$. □

Lemma 4.5. *There exists a constant $C_3 > 0$ such that for all $t \in [0, T_n]$ we have*

$$
(4.26) \qquad \frac{1}{n}\sum_{k=1}^{n}\Big(\dot u_k^2(t) + \Gamma(\varepsilon_k(t)) + \int_0^t \dot\varepsilon_k^2(\tau)\,\mathrm{d}\tau\Big) \leqslant C_3.
$$

P r o o f. Integrating (4.9) from 0 to t and using (2.21), (4.8) and (4.12)–(4.15), we find

$$
\begin{aligned}
(4.27) \qquad \frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{2}\dot u_k^2(t) + \Gamma(\varepsilon_k(t)) + \mu\int_0^t \dot\varepsilon_k^2(\tau)\,\mathrm{d}\tau\Big) &\\
\leqslant C\Big(1 + \frac{1}{n}\sum_{k=1}^{n}\int_0^t (1+\theta_k)|\dot\varepsilon_k|(\tau)\,\mathrm{d}\tau\Big) &\\
\leqslant C\Big(1 + \frac{1}{n}\sum_{k=1}^{n}\int_0^t \Big(|\dot\varepsilon_k| + \theta_k^{1+\frac{\alpha}{2}}(\theta_k^{-\frac{\alpha}{2}}|\dot\varepsilon_k|)\Big)(\tau)\,\mathrm{d}\tau\Big)&,
\end{aligned}
$$

and (4.26) follows from Hölder's inequality, (4.17) and (4.18). □

Lemma 4.6. *There exists a constant $C_4 > 0$ such that for every $t \in [0, T_n]$ we have*

$$
\begin{aligned}
(4.28) \qquad \frac{1}{n}\sum_{k=1}^{n}\dot\varepsilon_k^2(t) + n\sum_{k=1}^{n-1}\int_0^t (\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2(\tau)\,\mathrm{d}\tau &\\
\leqslant C_4\Big(1 + n\sum_{k=1}^{n-1}\int_0^t |\theta_{k+1}-\theta_k|^2(\tau)\,\mathrm{d}\tau\Big)&.
\end{aligned}
$$

P r o o f. Multiply (3.1) by $\dot\varepsilon_k - \dot\varepsilon_{k+1}$ and sum over $k = 1, \dots, n-1$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\ddot\varepsilon_k\dot\varepsilon_k + n\mu\sum_{k=1}^{n-1}(\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2 = n\sum_{k=1}^{n-1}\Big(\gamma(\varepsilon_{k+1})-\gamma(\varepsilon_k)+\mathcal{P}[\varepsilon_{k+1},\theta_{k+1}] - \mathcal{P}[\varepsilon_k,\theta_k]-\beta(\theta_{k+1}-\theta_k)+\frac{1}{n}f_k(\theta_k,t)\Big)(\dot\varepsilon_k-\dot\varepsilon_{k+1}). \tag{4.29}$$

We have

$$\frac{1}{n}\sum_{k=1}^{n}\dot\varepsilon_k^2(0) = n\sum_{k=1}^{n}\Big(\int_{\frac{k-1}{n}}^{\frac{k}{n}} u_x^1(x)\,\mathrm{d}x\Big)^2 \leqslant \int_0^1 |u_x^1|^2\,\mathrm{d}x, \tag{4.30}$$

$$n\sum_{k=1}^{n-1}(\varepsilon_{k+1}-\varepsilon_k)^2(0) = n^3\sum_{k=1}^{n-1}\Big(\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_x^{x+\frac{1}{n}} u_{xx}^0(\xi)\,\mathrm{d}\xi\,\mathrm{d}x\Big)^2 \leqslant n\int_0^{1-\frac{1}{n}}\int_x^{x+\frac{1}{n}}|u_{xx}^0(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}x \leqslant \int_0^1|u_{xx}^0(\xi)|^2\,\mathrm{d}\xi, \tag{4.31}$$

where the last inequality follows from Fubini's theorem. Furthermore,

$$\Big[n\sum_{k=1}^{n-1}(\varepsilon_{k+1}-\varepsilon_k)^2(t)\Big]^{\frac{1}{2}} \leqslant \Big[n\sum_{k=1}^{n-1}\Big(|\varepsilon_{k+1}-\varepsilon_k|(0)+\int_0^t|\dot\varepsilon_{k+1}-\dot\varepsilon_k|(\tau)\,\mathrm{d}\tau\Big)^2\Big]^{\frac{1}{2}} \leqslant \Big(n\sum_{k=1}^{n-1}(\varepsilon_{k+1}-\varepsilon_k)^2(0)\Big)^{\frac{1}{2}} + \Big(tn\sum_{k=1}^{n-1}\int_0^t(\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2(\tau)\,\mathrm{d}\tau\Big)^{\frac{1}{2}}. \tag{4.32}$$

Integrating (4.29) from 0 to t and using (2.6), (4.30)–(4.32), (2.23), (3.6), (2.8) and Hölder's inequality, we obtain

$$\frac{1}{n}\sum_{k=1}^{n}\dot\varepsilon_k^2(t) + n\sum_{k=1}^{n-1}\int_0^t(\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2(\tau)\,\mathrm{d}\tau \leqslant C\Big(1+n\sum_{k=1}^{n-1}\int_0^t\Big[(\theta_{k+1}-\theta_k)^2(\tau)+\int_0^\tau(\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2(s)\,\mathrm{d}s\Big]\,\mathrm{d}\tau\Big). \tag{4.33}$$

The functions $w(t) := n\sum\limits_{k=1}^{n-1}\int_0^t(\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2(\tau)\,\mathrm{d}\tau$ and $A(t) := 1+n\sum\limits_{k=1}^{n-1}\int_0^t(\theta_{k+1}-\theta_k)^2(\tau)\,\mathrm{d}\tau$ are nonnegative, nondecreasing, and satisfy the inequality

$$w(t) \leqslant C\Big(A(t)+\int_0^t w(\tau)\,\mathrm{d}\tau\Big), \qquad t\in[0,T_n], \tag{4.34}$$

which implies

$$\int_0^t w(\tau)\,\mathrm{d}\tau \leqslant CA(t)\left(\mathrm{e}^{Ct}-1\right) \leqslant CA(t)\mathrm{e}^{CT}. \tag{4.35}$$

The assertion now follows from (4.33) and (4.35). □

Lemma 4.7. *There exists a constant* $C_5 > 0$ *such that for every* $t \in [0, T_n]$ *we have*

$$\frac{1}{n}\sum_{k=1}^{n}\theta_k^2(t) + n\sum_{k=1}^{n-1}\int_0^t (\theta_{k+1}-\theta_k)^2(\tau)\,\mathrm{d}\tau \leqslant C_5, \tag{4.36}$$

$$\frac{1}{n}\sum_{k=1}^{n}\left(\varepsilon_k^2(t)+\dot\varepsilon_k^2(t)\right) + n\sum_{k=1}^{n-1}\left[(\varepsilon_{k+1}-\varepsilon_k)^2(t) + \int_0^t (\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2(\tau)\,\mathrm{d}\tau\right] \leqslant C_5, \tag{4.37}$$

$$\frac{1}{n}\sum_{k=1}^{n}\int_0^t (\dot\varepsilon_k^4+\theta_k^4)(\tau)\,\mathrm{d}\tau \leqslant C_5. \tag{4.38}$$

P r o o f. Multiply (3.19) by θ_k and put

$$\psi_1(r,\bar\theta) := \int_0^{\bar\theta} \theta^2\varphi_{\theta\theta}(r,\theta)\,\mathrm{d}\theta \quad \text{for } r, \bar\theta > 0 \tag{4.39}$$

according to (4.19). Then

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(C_V\theta_k^2 - \int_0^\infty \psi_1(r,\theta_k)\mathfrak{s}_r^2[\varepsilon_k]\,\mathrm{d}r\right) - n^2(\theta_{k+1}-2\theta_k+\theta_{k-1})\theta_k \\
&\quad = \theta_k\left(\mu\dot\varepsilon_k^2 + \int_0^\infty \varphi(r,\theta_k)\mathfrak{s}_r[\varepsilon_k](\varepsilon_k-\mathfrak{s}_r[\varepsilon_k])_t\,\mathrm{d}r + g_k(\theta_k,t)\right) \\
&\quad - \beta\theta_k^2\dot\varepsilon_k + \int_0^\infty (\theta_k^2\varphi_\theta(r,\theta_k)-\psi_1(r,\theta_k))\mathfrak{s}_r[\varepsilon_k](\mathfrak{s}_r[\varepsilon_k]_t)\,\mathrm{d}r,
\end{aligned} \tag{4.40}$$

where Hypothesis 2.2 yields that

$$\left|\theta_k^2\varphi_\theta(r,\theta_k)-\psi_1(r,\theta_k)\right| = 2\left|\int_0^{\theta_k}\theta\varphi_\theta(r,\theta)\,\mathrm{d}\theta\right| \leqslant \theta_k^2\max_\theta|\varphi_\theta(r,\theta)|, \tag{4.41}$$

$$\left|\int_0^\infty \psi_1(r,\theta_k)\mathfrak{s}_r^2[\varepsilon_k]\,\mathrm{d}r\right| \leqslant \int_0^{\theta_k}\int_0^\infty \theta^2|\varphi_{\theta\theta}(r,\theta)|\,r^2\,\mathrm{d}r\,\mathrm{d}\theta \leqslant \frac{C_V}{2}\theta_k^2. \tag{4.42}$$

Similarly as in (4.14) we have

$$\frac{1}{n}\sum_{k=1}^{n}\theta_k^2(0) \leqslant \int_0^1 \left(|\theta^0(x)|^2 + \frac{2}{n}|\theta^0(x)||\theta_x^0(x)|\right)\mathrm{d}x. \tag{4.43}$$

Summing (4.40) over $k = 1, \ldots, n$, and integrating from 0 to t, we obtain from (4.41), (4.42), (2.11), (2.31), (2.7), (2.9), (3.6) and (2.30) that

$$\begin{aligned}(4.44)\quad \frac{1}{n}\sum_{k=1}^{n}\theta_k^2(t) + n\sum_{k=1}^{n-1}\int_0^t(\theta_{k+1}-\theta_k)^2(\tau)\,\mathrm{d}\tau \\ \leqslant C\biggl(1+\frac{1}{n}\sum_{k=1}^{n}\int_0^t\bigl((1+\theta_k^2)(1+|\dot\varepsilon_k|)+\theta_k\dot\varepsilon_k^2\bigr)(\tau)\,\mathrm{d}\tau\biggr).\end{aligned}$$

We now apply Hölder's inequality to the right-hand side of (4.44). We have

$$(4.45)\quad \frac{1}{n}\sum_{k=1}^{n}\int_0^t(\theta_k\dot\varepsilon_k^2)(\tau)\,\mathrm{d}\tau \leqslant \biggl(\frac{1}{n}\sum_{k=1}^{n}\int_0^t\theta_k^2(\tau)\,\mathrm{d}\tau\biggr)^{\frac12}\biggl(\frac{1}{n}\sum_{k=1}^{n}\int_0^t\dot\varepsilon_k^4(\tau)\,\mathrm{d}\tau\biggr)^{\frac12},$$

$$(4.46)\quad \frac{1}{n}\sum_{k=1}^{n}\int_0^t(\theta_k^2|\dot\varepsilon_k|)(\tau)\,\mathrm{d}\tau \leqslant \biggl(\frac{1}{n}\sum_{k=1}^{n}\int_0^t\theta_k^{\frac83}(\tau)\,\mathrm{d}\tau\biggr)^{\frac34}\biggl(\frac{1}{n}\sum_{k=1}^{n}\int_0^t\dot\varepsilon_k^4(\tau)\,\mathrm{d}\tau\biggr)^{\frac14},$$

whence, by Lemma 4.4,

$$(4.47)\quad \frac{1}{n}\sum_{k=1}^{n}\int_0^t\theta_k^2(t) + n\sum_{k=1}^{n-1}\int_0^t(\theta_{k+1}-\theta_k)^2(\tau)\,\mathrm{d}\tau \leqslant C\biggl(1+\frac{1}{n}\sum_{k=1}^{n}\int_0^t\dot\varepsilon_k^4(\tau)\,\mathrm{d}\tau\biggr)^{\frac12}.$$

Moreover,

$$(4.48)\qquad \frac{1}{n}\sum_{k=1}^{n}\int_0^t\dot\varepsilon_k^4(\tau)\,\mathrm{d}\tau \leqslant \max_\tau\biggl(\frac{1}{n}\sum_{k=1}^{n}\dot\varepsilon_k^2(\tau)\biggr)\cdot\int_0^t\max_j\dot\varepsilon_j^2(\tau)\,\mathrm{d}\tau,$$

where, by Lemmas 4.2 and 4.5,

$$\begin{aligned}(4.49)\quad \int_0^t\max_j\dot\varepsilon_j^2(\tau)\,\mathrm{d}\tau &\leqslant \frac{1}{n}\sum_{k=1}^{n}\int_0^t\dot\varepsilon_k^2(\tau)\,\mathrm{d}\tau \\ &\quad + 2\biggl(\frac{1}{n}\sum_{k=1}^{n}\int_0^t\dot\varepsilon_k^2(\tau)\,\mathrm{d}\tau\biggr)^{\frac12}\biggl(n\sum_{k=1}^{n-1}\int_0^t(\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2(\tau)\,\mathrm{d}\tau\biggr)^{\frac12} \\ &\leqslant C\biggl(1+n\sum_{k=1}^{n-1}\int_0^t(\dot\varepsilon_{k+1}-\dot\varepsilon_k)^2(\tau)\,\mathrm{d}\tau\biggr)^{\frac12}.\end{aligned}$$

Combining (4.47)–(4.49) with (4.27), we obtain for all $t \in [0, T_n]$ that

(4.50)

$$\frac{1}{n}\sum_{k=1}^{n}\theta_k^2(t) + n\sum_{k=1}^{n-1}\int_0^t(\theta_{k+1}-\theta_k)^2(\tau)\,\mathrm{d}\tau \leqslant C\biggl(1+n\sum_{k=1}^{n-1}\int_0^t(\theta_{k+1}-\theta_k)^2(\tau)\,\mathrm{d}\tau\biggr)^{\frac34}.$$

Thus, (4.36) follows from Young's inequality, (4.37) is then a consequence of (4.28), (4.36), (4.32) and of the obvious inequality $\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k^2(t) \leqslant \int_0^1 |u_x^0|^2\,\mathrm{d}x + \frac{C}{n}\max_{0\leqslant\tau\leqslant t}\sum_{k=1}^{n}\dot\varepsilon_k^2(\tau)$ analogous to (4.32). Estimate (4.38) is obtained using the argument of (4.48). □

Lemma 4.8. *There exists a constant $C_6 > 0$ such that for all $t\in[0,T_n]$ we have*

$$\frac{1}{n}\sum_{k=1}^{n}\int_0^t \dot\theta_k^2(\tau)\,\mathrm{d}\tau + n\sum_{k=1}^{n-1}(\theta_{k+1}-\theta_k)^2(t) \leqslant C_6, \tag{4.51}$$

$$n^3\sum_{k=1}^{n-1}\int_0^t (\theta_{k+1}-2\theta_k+\theta_{k-1})^2(\tau)\,\mathrm{d}\tau \leqslant C_6. \tag{4.52}$$

Proof. Multiplying (3.19) by $\dot\theta_k$, we infer from Hypotheses 2.1 and 2.2 that

$$\begin{aligned}\frac{C_V}{2n}\sum_{k=1}^{n}\dot\theta_k^2(t) &+ \frac{n}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{n-1}(\theta_{k+1}-\theta_k)^2(t)\\ &\leqslant \frac{C}{n}\sum_{k=1}^{n}|\dot\theta_k(t)|\Big(1+\theta_k^2(t)+\dot\varepsilon_k^2(t)+n\int_{\frac{k-1}{n}}^{\frac{k}{n}} g_0(x,t)\,\mathrm{d}x\Big).\end{aligned} \tag{4.53}$$

Integrating (4.53) with respect to t and using the inequality

$$n\sum_{k=1}^{n-1}(\theta_{k+1}-\theta_k)^2(0) \leqslant \int_0^1 |\theta_x^0(x)|^2\,\mathrm{d}x, \tag{4.54}$$

we obtain (4.51) from (4.38). Inequality (4.52) is an immediate consequence of (4.51) and equation (3.19). □

Lemma 4.9. *There exists a constant $C_7 > 0$ such that for all $t\in[0,T_n]$ we have*

$$\frac{1}{n}\sum_{k=1}^{n}\Big(\ddot u_k^2(t) + \int_0^t \ddot\varepsilon_k^2(\tau)\,\mathrm{d}\tau\Big) \leqslant C_7. \tag{4.55}$$

Proof. The right-hand side of (3.1) is absolutely continuous. Differentiating with respect to t and multiplying by $\ddot u_k(t)$, we obtain for a.e. t that

$$\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ddot u_k^2 + \dot\sigma\ddot\varepsilon\Big)(t) = \frac{1}{n}\sum_{k=1}^{n}\ddot u_k\Big(\frac{\partial f_k}{\partial t} + \dot\theta_k\frac{\partial f_k}{\partial\theta}\Big)(\theta_k(t),t). \tag{4.56}$$

Hence, by hypotheses (2.8) and (2.9) and by inequality (2.22), we find that for all t

$$\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{n}\ddot{u}_k^2(t)+\mu n\sum_{k=1}^{n}\ddot{\varepsilon}_k^2(t) \leqslant \frac{C}{n}\sum_{k=1}^{n}\left[|\ddot{\varepsilon}_k|(1+|\dot{\theta}_k|+|\dot{\varepsilon}_k|)+|\ddot{u}_k|\left(|\dot{\theta}_k|+\left|\frac{\partial f_k}{\partial t}\right|\right)\right](t), \tag{4.57}$$

where

$$\frac{1}{n}\sum_{k=1}^{n}\int_0^t\left|\frac{\partial f_k}{\partial t}(\theta_k(\tau),\tau)\right|^2\mathrm{d}\tau\leqslant\int_0^t\int_0^1 f_0^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.58}$$

For the initial value $\ddot{u}_k(0)$ we obtain from equation (3.1) and inequality (2.23) the estimate

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot{u}_k^2(0)\leqslant Cn\sum_{k=1}^{n-1}\Big((\varepsilon_{k+1}-\varepsilon_k)^2(0)+(\dot{\varepsilon}_{k+1}-\dot{\varepsilon}_k)^2(0) + (\theta_{k+1}-\theta_k)^2(0)\Big)+\frac{2}{n}\sum_{k=1}^{n-1}f_k^2(\theta_k(0),0), \tag{4.59}$$

where

$$\frac{1}{n}\sum_{k=1}^{n-1}f_k^2(\theta_k(0),0)\leqslant\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}f^2(\theta_k(0),x,0)\,\mathrm{d}x. \tag{4.60}$$

For a.e. $x\in\,]0,1[$, $t,s\in[0,T]$ and $\theta>0$, we infer from hypothesis (2.8) that

$$f^2(\theta,x,t)\leqslant f_0^2(x,s)+2\int_s^t f_0^2(x,\tau)\,\mathrm{d}\tau, \tag{4.61}$$

whence

$$\max_{t,\theta}f^2(\theta,x,t)\leqslant C\int_0^T f_0^2(x,\tau)\,\mathrm{d}\tau \quad \text{a.e. in }]0,1[\,. \tag{4.62}$$

The estimate

$$n\sum_{k=1}^{n-1}(\dot{\varepsilon}_{k+1}-\dot{\varepsilon}_k)^2(0)\leqslant\int_0^1|u^1_{xx}(x)|^2\,\mathrm{d}x, \tag{4.63}$$

which is similar to (4.31), now yields that

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot{u}_k^2(0)\leqslant C. \tag{4.64}$$

Integrating (4.57) from 0 to t, we easily obtain (4.55) from (4.58), (4.64) and Lemmas 4.7 and 4.8. □

Lemma 4.10. *There exists a constant $C_8 > 0$ such that for all $t \in [0, T_n]$ we have*

$$n \sum_{k=1}^{n-1} (\dot{\varepsilon}_{k+1} - \dot{\varepsilon}_k)^2 (t) \leqslant C_8. \tag{4.65}$$

P r o o f. Using again equation (3.1) and inequalities (3.23), (4.62), we obtain for $t \in [0, T_n]$ that

$$n \sum_{k=1}^{n-1} (\dot{\varepsilon}_{k+1} - \dot{\varepsilon}_k)^2 (t) \leqslant C\biggl(1 + \frac{1}{n} \sum_{k=1}^{n-1} \Bigl(\ddot{u}_k^2(t) + n^2(\varepsilon_{k+1} - \varepsilon_k)^2(t) + n^2(\theta_{k+1} - \theta_k)^2(t) + \int_0^t (\dot{\varepsilon}_{k+1} - \dot{\varepsilon}_k)^2(\tau)\,\mathrm{d}\tau\Bigr)\biggr). \tag{4.66}$$

The assertion now follows from Lemmas 4.7 to 4.9 and a Gronwall-type argument. □

To conclude this section, we just notice that Theorem 3.1 is proved by Lemmas 4.5–4.10.

5. Existence

In this section we will construct a sequence $\{u^{(n)}, \theta^{(n)}\}$ of approximate solutions to the system (2.1)–(2.4) and use the compactness method to prove that a limit point of this sequence solves (2.1)–(2.4) in the sense of Theorem 2.4.

Let $n \in \mathbb{N}$ be given, and let $u_1, \ldots, u_{n-1}$, $\theta_1, \ldots, \theta_n$ satisfy the system (3.1)–(3.7). For $t \in [0, T]$, $x \in \bigl[\frac{k-1}{n}, \frac{k}{n}\bigr[$, $k = 1, \ldots, n$, we define functions (continuously extended to $x = 1$)

$$\theta^{(n)}(x,t) = \frac{1}{2}(\theta_k + \theta_{k-1}) + n\left(x - \frac{k-1}{n}\right)(\theta_k - \theta_{k-1}) + \frac{n^2}{2}\left(x - \frac{k-1}{n}\right)^2 (\theta_{k+1} - 2\theta_k + \theta_{k-1}), \tag{5.1}$$

$$\tilde{\theta}^{(n)}(x,t) = \theta_k, \tag{5.2}$$

$$u^{(n)}(x,t) = u_{k-1} + n\left(x - \frac{k-1}{n}\right)(u_k - u_{k-1}), \tag{5.3}$$

$$\tilde{u}^{(n)}(x,t) = u_k, \tag{5.4}$$

$$\varepsilon^{(n)}(x,t) = \varepsilon_k + n\left(x - \frac{k-1}{n}\right)(\varepsilon_{k+1} - \varepsilon_k), \tag{5.5}$$

$$\tilde{\varepsilon}^{(n)}(x,t) = \varepsilon_k, \tag{5.6}$$

$$\sigma^{(n)}(x,t) = \sigma_k, \tag{5.7}$$

where we have put $u_{n+1} := -u_{n-1}$ so that $\varepsilon_{n+1} = \varepsilon_n$.

By Theorem 3.1 and Corollary 3.2 there exists a constant $C > 0$, independent of n, such that ($\|\cdot\|$ denotes the norm of $L^2(0,1)$)

$$\begin{aligned}(5.8)\qquad &\|\theta^{(n)}(t)\|^2 + \|\theta_x^{(n)}(t)\|^2 + \|\varepsilon_t^{(n)}(t)\|^2 + \|\varepsilon_{xt}^{(n)}(t)\|^2 + \|u_t^{(n)}(t)\|^2\\ &+ \|u_{tt}^{(n)}(t)\|^2 + \|\varepsilon^{(n)}(t)\|^2 + \|\varepsilon_x^{(n)}(t)\|^2 \leqslant C \qquad \forall t \in [0,T],\end{aligned}$$

$$(5.9)\qquad \int_0^T \left(\|\varepsilon_{tt}^{(n)}\|^2 + \|\theta_t^{(n)}\|^2 + \|\theta_{xx}^{(n)}\|^2\right)(t)\,\mathrm{d}t \leqslant C.$$

We further have for every x and t that

$$(5.10)\qquad \left|\tilde{u}^{(n)}(x,t) - u^{(n)}(x,t)\right|^2 \leqslant \frac{1}{n^2}\sum_{k=1}^{n} \varepsilon_k^2(t) \leqslant \frac{C}{n},$$

$$(5.11)\qquad \left|\tilde{\varepsilon}^{(n)}(x,t) - \varepsilon^{(n)}(x,t)\right|^2 \leqslant \sum_{k=1}^{n-1}(\varepsilon_{k+1} - \varepsilon_k)^2(t) = \frac{1}{n}\|\varepsilon_x^{(n)}(t)\|^2 \leqslant \frac{C}{n},$$

$$(5.12)\qquad \left|\tilde{\theta}^{(n)}(x,t) - \theta^{(n)}(x,t)\right|^2 \leqslant \sum_{k=1}^{n}(\theta_{k+1} - \theta_k)^2(t) \leqslant \frac{C}{n},$$

$$(5.13)\qquad \left|\tilde{\varepsilon}_t^{(n)}(x,t) - \varepsilon_t^{(n)}(x,t)\right|^2 \leqslant \sum_{k=1}^{n-1}(\dot{\varepsilon}_{k+1} - \dot{\varepsilon}_k)^2(t) \leqslant \frac{C}{n},$$

$$(5.14)\qquad \int_0^T \left|\tilde{u}_{tt}^{(n)}(x,t) - u_{tt}^{(n)}(x,t)\right|^2 \mathrm{d}t \leqslant \frac{1}{n^2}\sum_{k=1}^{n}\int_0^T \ddot{\varepsilon}_k^2(t)\,\mathrm{d}t \leqslant \frac{C}{n},$$

$$(5.15)\qquad u_x^{(n)} = \tilde{\varepsilon}^{(n)}.$$

From the estimates (5.8)–(5.9) we conclude that there exist subsequences (still indexed by (n), for the sake of simplicity) and functions u, ε, θ such that

$$(5.16)\quad \theta_{xx}^{(n)} \to \theta_{xx},\ \theta_t^{(n)} \to \theta_t,\ \varepsilon_{tt}^{(n)} \to \varepsilon_{tt}, \quad \text{all weakly in } L^2(]0,1[\times]0,T[),$$

$$(5.17)\quad \varepsilon_{xt}^{(n)} \to \varepsilon_{xt},\ \varepsilon_x^{(n)} \to \varepsilon_x,\ \varepsilon_t^{(n)} \to \varepsilon_t,\ u_{tt}^{(n)} \to u_{tt},\ \theta_x^{(n)} \to \theta_x, \quad \text{all weakly* in } L^\infty(0,T;L^2(0,1)),$$

$$(5.18)\quad u_x = \varepsilon,$$

and, by compact embedding,

$$(5.19)\qquad \varepsilon_t^{(n)} \to \varepsilon_t,\ \varepsilon^{(n)} \to \varepsilon,\ u_t^{(n)} \to u_t,\ u^{(n)} \to u,\ \theta^{(n)} \to \theta, \quad \text{all in } C([0,1]\times[0,T]) \text{ uniformly.}$$

The functions $u^{(n)}$, $\theta^{(n)}$ fulfil the boundary conditions (2.3). The convergence (5.16), (5.19) implies that conditions (2.3), (2.4) holds also for the limit functions.

To prove the existence result, it remains to check that the system (2.1), (2.2) is satisfied almost everywhere.

Let $w \in \overset{\circ}{W}{}^{1,2}(0,1)$, $z \in L^2(0,1)$ and $\eta \in \mathcal{D}(]0,T[)$ be arbitrary test functions. Then the system (3.1)–(3.4) can be rewritten in the form

$$
\begin{aligned}
(5.20)\quad & \int_0^T \eta(t) \int_0^1 \Big[\Big(\tilde{u}_{tt}^{(n)}(x,t) - f(\tilde{\theta}^{(n)}(x,t),x,t)\Big) w(x) + \sigma^{(n)}(x,t) w'(x)\Big] \,\mathrm{d}x\,\mathrm{d}t \\
&= A_n := \int_0^T \eta(t) \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(w\left(\frac{k}{n}\right) - w(x)\right) \left(f(\theta^k,x,t) - \ddot{u}_k\right) \mathrm{d}x\,\mathrm{d}t, \\
(5.21)\quad & \int_0^1 z(x) \int_0^T \bigg[\Big(C_V \tilde{\theta}^{(n)} + \mathcal{V}[u_x^{(n)},\tilde{\theta}^{(n)}]\Big)\eta'(t) + \bigg(\kappa\theta_{xx}^{(n)} + \mu(u_{xt}^{(n)})^2 \\
&\quad + \mathcal{P}[u_x^{(n)},\tilde{\theta}^{(n)}]u_{xt}^{(n)} - \beta\tilde{\theta}^{(n)}u_{xt}^{(n)} + g\left(\tilde{\theta}^{(n)}(x,t),x,t\right)\bigg)\eta(t)\bigg] \,\mathrm{d}t\,\mathrm{d}x \\
&= B_n := \int_0^T \eta(t) n \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} g(\theta_k,x,t) \int_{\frac{k-1}{n}}^{\frac{k}{n}} (z(x) - z(\xi))\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t, \\
(5.22)\quad & \sigma^n = \gamma(u_x^{(n)}) + \mathcal{P}[u_x^{(n)},\tilde{\theta}^{(n)}] + \mu u_{xt}^{(n)} - \beta\tilde{\theta}^{(n)}.
\end{aligned}
$$

The right-hand sides of (5.20), (5.21), respectively, can be estimated as follows.

$$
\begin{aligned}
(5.23)\quad |A_n| &\leqslant \int_0^T |\eta(t)| \sum_{k=1}^n \left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} |w'(x)|\,\mathrm{d}x \int_{\frac{k-1}{n}}^{\frac{k}{n}} (f_0(x,t) + |\ddot{u}_k|)\,\mathrm{d}x\right) \mathrm{d}t \\
&\leqslant \int_0^T |\eta(t)| \left(\sum_{k=1}^n \left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} |w'(x)|\,\mathrm{d}x\right)^2\right)^{1/2} \cdot \\
&\quad \cdot \left(\sum_{k=1}^n \left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} (f_0(x,t) + |\ddot{u}_k|)\,\mathrm{d}x\right)^2\right)^{1/2} \mathrm{d}t \\
&\leqslant \frac{1}{n}\|w'\| \int_0^T |\eta(t)| \left(\sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} (f_0(x,t) + |\ddot{u}_k|)^2\,\mathrm{d}x\right)^{1/2} \mathrm{d}t \\
&\leqslant \frac{1}{n}\|w'\| \int_0^T |\eta(t)| \left(\left(\frac{1}{n}\sum_{k=1}^n \ddot{u}_k^2\right)^{1/2} + \left(\int_0^1 f_0^2(x,t)\,\mathrm{d}x\right)^{1/2}\right) \mathrm{d}t \\
&\leqslant \frac{C}{n}\|w'\|\|\eta\|_{L^2(0,T)},
\end{aligned}
$$

$$(5.24)\quad \begin{aligned}|B_n| &\leqslant n\int_0^T |\eta(t)| \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} (\theta_k + g_0(x,t)) \int_{\frac{k-1}{n}}^{\frac{k}{n}} |z(x)-z(\zeta)|\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant n\int_0^T |\eta(t)| \Big(\sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} (\theta_k + g_0(x,t))^2\,\mathrm{d}x\Big)^{1/2} \cdot \\ &\cdot \Big(\sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \Big(\int_{\frac{k-1}{n}}^{\frac{k}{n}} |z(x)-z(\zeta)|\,\mathrm{d}\zeta\Big)^2\,\mathrm{d}x\Big)^{1/2}\,\mathrm{d}t \\ &\leqslant Z_n^{1/2}\int_0^T |\eta(t)| \Big(\Big(\frac{1}{n}\sum_{k=1}^n \theta_k^2\Big)^{1/2} + \Big(\int_0^1 g_0^2(x,t)\,\mathrm{d}x\Big)^{1/2}\Big) \\ &\leqslant C Z_n^{1/2} \|\eta\|_{L^2(0,T)},\end{aligned}$$

where

$$(5.25)\qquad Z_n := n\sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |z(x)-z(\zeta)|^2\,\mathrm{d}\zeta\,\mathrm{d}x.$$

Let us extend the function z by zero outside the interval $[0,1]$. Then

$$(5.26)\qquad Z_n \leqslant n\sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} |z(x)-z(\zeta)|^2\,\mathrm{d}\zeta\,\mathrm{d}x = n\int_{-\frac{1}{n}}^{\frac{1}{n}} \int_0^1 |z(x)-z(x+s)|^2\,\mathrm{d}x\,\mathrm{d}s.$$

By the Mean Continuity Theorem we have $\lim\limits_{s\to 0} \int_0^1 |z(x)-z(x+s)|^2\,\mathrm{d}x = 0$, hence

$$(5.27)\qquad \lim_{n\to\infty} Z_n = \lim_{n\to\infty} B_n = 0.$$

Using the convergence results (5.10)–(5.19), (5.23), (5.27) and Proposition 2.6 (ii), we can pass to the limit as $n\to\infty$ in (5.20)–(5.22) obtaining

$$(5.28)\qquad u_{tt} - \sigma_x - f(\theta,x,t) = 0 \qquad \text{a.e.},$$

$$(5.29)\qquad (C_V\theta + \mathcal{V}[u_x,\theta])_t - \kappa\theta_{xx} = \mu u_{xt}^2 + \mathcal{P}[u_x,\theta]u_{xt} - \beta\theta u_{xt} + g(\theta,x,t) \qquad \text{a.e.},$$

$$(5.30)\qquad \sigma = \gamma(u_x) + \mathcal{P}[u_x,\theta] + \mu u_{xt} - \beta\theta.$$

Hence (u,θ) is a solution to (2.1), (2.2) satisfying the assertions of Theorem 2.4. Indeed, inequality (2.19) follows from Corollary 3.2 and the uniform convergence $\tilde{\theta}^{(n)} \longrightarrow \theta$.

The proof of Theorem 2.4 will be complete if we prove that the problem (2.1)–(2.4) admits at most one solution. In fact, we can prove more, namely

Theorem 6.1. *Let Hypotheses* 2.1(ii), 2.2 *hold, let* $(u^0, u^1, \theta^0, f, g)$, $(u'^0, u'^1, \theta'^0, f', g')$ *be two sets of given functions satisfying Hypothesis* 2.1, *and let* (u, θ), (u', θ') *be solutions of* (2.1)–(2.4) *corresponding to these data, respectively, which satisfy* (2.16)–(2.19). *Assume moreover that there exist a constant* $\tilde{K} > 0$ *and functions* $d_f, d_g \in L^2(]0,1[\times]0,T[)$ *such that*

$$|f(\theta_1, x, t) - f'(\theta_2, x, t)| \leqslant \tilde{K}|\theta_1 - \theta_2| + d_f(x,t), \tag{6.1}$$

$$|g(\theta_1, x, t) - g'(\theta_2, x, t)| \leqslant \tilde{K}|\theta_1 - \theta_2| + d_g(x,t), \tag{6.2}$$

holds for all $\theta_1, \theta_2 \in \mathbb{R}^+$ *and a.e.* $(x,t) \in]0,1[\times]0,T[$.

Then there exists a constant C *depending only on the constant* $\overline{C}$ *in Theorem* 3.1 *(i.e. on the size of the data in their respective spaces) such that for all* $t \in [0,T]$ *the differences* $\bar{u} = u - u'$, $\bar{\theta} = \theta - \theta'$, *satisfy*

$$\begin{aligned} &\|\bar{u}_t(t)\|^2 + \int_0^t \left(\|\bar{\theta}\|^2 + \|\bar{u}_{xt}\|^2\right)(\tau)\,\mathrm{d}\tau \\ &\quad \leqslant C\left(\|\bar{u}_t(0)\|^2 + \|\bar{u}_x(0)\|^2 + \|\bar{\theta}(0)\|^2 + \int_0^t\int_0^1 (d_f^2 + d_g^2)\,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned} \tag{6.3}$$

P r o o f. It follows from equation (2.1) that

$$\bar{u}_{tt} - \mu\bar{u}_{xxt} = \beta\bar{\theta}_x + (\mathcal{P}[u_x,\theta] - \mathcal{P}[u'_x,\theta'])_x + (\gamma(u_x) - \gamma(u'_x))_x + f(\theta,x,t) - f'(\theta',x,t) \tag{6.4}$$

a.e. in $]0,1[\times]0,T[$. Multiplying (6.4) by $\bar{u}_t$ and integrating over $[0,1]$, we obtain, using (6.1) and (2.19),

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \bar{u}_t^2\,\mathrm{d}x + \mu\int_0^1 \bar{u}_{xt}^2\,\mathrm{d}x \\ &\leqslant K_1\int_0^1 \Big[|\bar{\theta}||\bar{u}_{xt}| + (|\bar{\theta}| + d_f)|\bar{u}_t| \\ &\quad + \left(|\bar{u}_x(0)| + \int_0^t |\bar{u}_{xt}|(\tau)\,\mathrm{d}\tau\right)|\bar{u}_{xt}|\Big]\,\mathrm{d}x \quad \text{a.e.,} \end{aligned} \tag{6.5}$$

where $K_1 > 0$ is a constant. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \bar{u}_t^2\,\mathrm{d}x + \int_0^1 \bar{u}_{xt}^2\,\mathrm{d}x \leqslant K_2\int_0^1 \left(\bar{\theta}^2 + |\bar{u}_x(0)|^2 + d_f^2 + \left(\int_0^t |\bar{u}_{xt}(\tau)|\,\mathrm{d}\tau\right)^2\right)\mathrm{d}x \tag{6.6}$$

for a.e. t with a constant $K_2 > 0$. Similarly, integrating (2.2) over $[0,t]$, we obtain

$$\begin{aligned}(6.7)\qquad & \left(C_V\bar\theta + \mathcal{V}[u_x,\theta] - \mathcal{V}[u'_x,\theta']\right)(x,t) - \kappa\int_0^t \bar\theta_{xx}\,\mathrm{d}\tau \\ &= \left(C_V\bar\theta + \mathcal{V}[u_x,\theta] - \mathcal{V}[u'_x,\theta']\right)(x,0) \\ &\quad + \int_0^t \Big[\mu(u_{xt}^2 - u'^2_{xt}) + \mathcal{P}[u_x,\theta]u_{xt} - \mathcal{P}[u'_x,\theta']u'_{xt} \\ &\quad - \beta(\theta u_{xt} - \theta' u'_{xt}) + g(\theta,x,\tau) - g'(\theta',x,\tau)\Big]\,\mathrm{d}\tau.\end{aligned}$$

The functions u_{xt}, u'_{xt}, θ, θ', $\mathcal{P}[u_x,\theta]$, $\mathcal{P}[u'_x,\theta']$, are uniformly bounded by a constant depending only on the constant $\overline{C}$ from Theorem 3.1. Moreover, using (2.24), we can estimate

$$(6.8)\quad |\mathcal{V}[u_x,\theta] - \mathcal{V}[u'_x,\theta']|\,(x,t) \leqslant \frac{C_V}{2}|\bar\theta(x,t)| + V_0\left(|\bar u_x(x,0)| + \int_0^t |\bar u_{xt}(x,\tau)|\,\mathrm{d}\tau\right).$$

Multiplying (6.7) by $\bar\theta(x,t)$ and integrating over $[0,1]$, we therefore obtain, using (2.23),

$$\begin{aligned}(6.9)\qquad & C_V\int_0^1 \bar\theta^2\,\mathrm{d}x + \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\int_0^t \bar\theta_x\,\mathrm{d}\tau\right)^2\mathrm{d}x \\ &\leqslant \frac{C_V}{2}\int_0^1 \bar\theta^2\,\mathrm{d}x \\ &\quad + K_3\int_0^1 |\bar\theta|\left[|\bar\theta(x,0)| + |\bar u(x,0)| + \int_0^t (|\bar u_{xt}| + |\bar\theta| + d_g)\,\mathrm{d}\tau\right]\mathrm{d}x\end{aligned}$$

with a constant $K_3 > 0$ depending on $\overline{C}$. Moreover, from Schwarz's inequality it follows that

$$\begin{aligned}(6.10)\quad & C_V\int_0^1 \bar\theta^2\,\mathrm{d}x + \kappa\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\int_0^t \bar\theta_x\,\mathrm{d}\tau\right)^2\mathrm{d}x \\ &\leqslant K_4\int_0^1\left(|\bar\theta(x,0)|^2 + |\bar u(x,0)|^2 + \left(\int_0^t (|\bar u_{xt}| + |\bar\theta| + d_g)\,\mathrm{d}\tau\right)^2\right)\mathrm{d}x,\end{aligned}$$

for a suitable constant $K_4 > 0$. An appropriate linear combination of (6.6) and (6.10) then yields

$$\begin{aligned}(6.11)\qquad & \|\bar\theta(t)\|^2 + \|\bar u_{xt}(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\bar u_t(t)\|^2 + \left\|\int_0^t \bar\theta_x\,\mathrm{d}\tau\right\|^2\right) \\ &\leqslant K_5\Bigg(\|\bar\theta(0)\|^2 + \|\bar u_x(0)\|^2 + \int_0^1 d_f^2(x,t)\,\mathrm{d}x \\ &\quad + t\int_0^t\left(\|\bar u_{xt}\|^2 + \|\theta\|^2 + \int_0^1 d_g^2(x,\tau)\,\mathrm{d}x\right)\mathrm{d}\tau\Bigg)\end{aligned}$$

for a constant $K_5 > 0$. Inequality (6.11) is of the form

$$\dot{w}(t) \leqslant a(t) + b(t)w(t), \tag{6.12}$$

where

$$w(t) = \|\bar{u}_t(t)\|^2 + \|\int_0^t \bar{\theta}_x \,\mathrm{d}\tau\| + \int_0^t \left(\|\bar{\theta}\|^2 + \|\bar{u}_{xt}\|^2\right) \mathrm{d}\tau, \tag{6.13}$$

$$a(t) = K_5\biggl(\|\bar{\theta}(0)\|^2 + \|\bar{u}_{xt}(0)\|^2 + \int_0^1 d_f^2(x,t)\,\mathrm{d}x + t\int_0^t\int_0^1 d_g^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\biggr), \tag{6.14}$$

$$b(t) = K_5 t, \tag{6.15}$$

which entails

$$w(t) \leqslant \mathrm{e}^{B(t)}w(0) + \int_0^t \mathrm{e}^{B(t)-B(\tau)}a(\tau)\,\mathrm{d}\tau, \tag{6.16}$$

where $B(t) = \int_0^t b(\tau)\,\mathrm{d}\tau = \frac{1}{2}K_5 t^2$.

Inequality (6.3) then immediately follows with a constant C depending on K_5 and T. □

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