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EPSILON-INFLATION WITH CONTRACTIVE INTERVAL FUNCTIONS

Günther Mayer, Rostock

Dedicated to Prof. Dr. Gerhard Heindl on the occasion of his 60th birthday

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Abstract. For contractive interval functions \([g]\) we show that \([g]([x]_k^0) \subseteq \text{int}([x]_k^0)\) results from the iterative process \([x]_k^{k+1} := [g]([x]_k^k)\) after finitely many iterations if one uses the epsilon-inflated vector \([x]_\varepsilon^k\) as input for \([g]\) instead of the original output vector \([x]^k\). Applying Brouwer’s fixed point theorem, zeros of various mathematical problems can be verified in this way.

Keywords: epsilon-inflation, P-contraction, contraction, verification algorithms, interval computation, nonlinear equations, eigenvalues, singular values

MSC 2000: 65F05, 65F10, 65F15, 65G05, 65G10, 65H10, 65H15, 65L05

1. Introduction

If \(G\) denotes a nonempty convex, compact subset of \(\mathbb{R}^n\) and if \(t\) is a continuous self-mapping of \(G\) then Brouwer’s fixed point theorem guarantees that \(t\) has at least one fixed point in \(G\). Often \(G\) is an interval vector and \(t\) is a function which is defined and continuous in an open superset \(D\) of \(G\). Assume that with \(t\) an interval function \([g]\) is associated such that the inclusion property

\[(1) \quad t(x) \in [g]([x])\]

holds for all \(x \in [x]\) and for all \([x] \subseteq D\). If

\[(2) \quad [g]([x]) \subseteq [x] \quad (\text{or, more strongly, } [g]([x]) \subseteq \text{int}([x]))\]
is valid for some interval vector \([x] \subseteq D\) then \(t\) has a fixed point \(x^*\) in \([x]\) by the above mentioned Brouwer's fixed point theorem, since (1) and (2) guarantee the self-mapping property of \(t\).

A simple choice of \([g]\) is the interval arithmetic evaluation of \(t\) (cf. [2]) which guarantees (1). But often \([g]\) is chosen in a more sophisticated way. In order to find a vector \([x]\) which satisfies (2) one usually starts with an approximation \(\tilde{x}\) of a fixed point \(x^*\) of \(t\) and one iterates by

\[
[x]^0 := [\tilde{x}, \tilde{x}], \quad [x]^{k+1} := [g]([x]_\varepsilon^k), \quad k = 0, 1, \ldots
\]

until (2) holds for some \([x] = [x]_\varepsilon^k\) with \(k \leq k_{\max}\). Here \(k_{\max}\) is a given bound for the number of iterates and \([x]_\varepsilon^k\) is any interval vector which contains \([x]^k\) in its interior. Usually, \([x]_\varepsilon^k\) is called the \(\varepsilon\)-inflation of \([x]_\varepsilon^k\). This name stems from the fact that the construction of \([x]_\varepsilon^k\) normally depends on a parameter \(\varepsilon > 0\). A simple example is \([x]_\varepsilon := [x] + \varepsilon[-1,1](1, \ldots, 1)^T\), further possibilities can be found e.g. in [9]. The iteration (3) does not always end up with (2) as the example \([g]([x]) := 2[x], \tilde{x} := 1\) shows for an arbitrary \(\varepsilon\)-inflation. But often it helps as in the case \(g([x]) := \frac{1}{2}[x], \tilde{x} := 1\) if one chooses the \(\varepsilon\)-inflation from above with \(\varepsilon := 0.1\) whence \([x]^4 \subseteq [x]^3\).

It is an open question in which situations (3) ends up with (2) for some \([x] = [x]_\varepsilon^k\) in at most \(k_{\max}\) steps. For contractive interval functions \([g]\), in particular for functions \([g]\) of the form

\[
[g]([x]) := t(\tilde{x}) + \{t'(\tilde{x}) + [H]([x])\}([x] - \tilde{x}),
\]

we will at least be able to show that (3) results in (2) after finitely many steps of iterations. In (4) the vector \(\tilde{x}\) is a fixed vector from \(D\); \([H]\) is an interval matrix function for which we require the Lipschitz condition

\[
\|q([H]([x]), [H]([y]))\| \leq \kappa\|q([x], [y])\|
\]

and the value

\[
[H](\tilde{x}) = O;
\]

\(q\) denotes the Hausdorff distance; \(\kappa\) is a positive constant which is independent of \([x]\) but which may depend on \(\tilde{x}\); \(\|\cdot\|\) denotes any monotone vector norm and the corresponding operator norm for matrices, respectively. Functions \([g]\) as in (4) occur, when involving second derivatives in order to compute zeros of a function \(f\); in particular, they arise when verifying eigenpairs, singular values, and solutions of quadratic systems (cf. Section 4). For example, when verifying and enclosing zeros
of functions $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f = (f_i) \in C^2(D)$, one often transforms the problem $f(x) = 0$ into the fixed point problem

$$x = t(x) := x - Cf(x), \quad C \in \mathbb{R}^{n \times n} \text{ nonsingular.}$$

The interval function $[g]$ from (4) reads then

$$[g](\lfloor x \rfloor) := \tilde{x} - Cf(\tilde{x}) + \{I - Cf'(\tilde{x}) + [H](\lfloor x \rfloor)\}(\lfloor x \rfloor - \tilde{x})$$

with $[H](x) := f''([\lfloor x \rfloor \cup \tilde{x}])((\lfloor x \rfloor - \tilde{x})$, for example, where $f''(x)y$ is defined by

$$f''(x)y := \left(y^T \left(\frac{\partial^2 f_i(x)}{\partial x_l \partial x_k}\right)\right)_{i=1,...,n} \in \mathbb{R}^{n \times n}$$

with the Hessian $\left(\frac{\partial^2 f_i(x)}{\partial x_l \partial x_k}\right) \in \mathbb{R}^{n \times n}$ of $f_i$ and with the convex hull $\lfloor x \rfloor \cup \tilde{x}$ of $\lfloor x \rfloor$ and $\tilde{x}$.

The technique and the name $\varepsilon$-inflation have been introduced in [13]. Remarks concerning its practical applicability can be found e.g. in [5] and [6]. Theoretical considerations have been done in [8], [9], [11], [15] and [16]. The idea of replacing a starting interval $\lfloor x \rfloor^0$ by another one with a larger diameter, say $\lceil \tilde{x} \rceil^0$, was already used in [4]. But $\lceil \tilde{x} \rceil^0 \supseteq \lfloor x \rfloor^0$ was not required there. Our paper generalizes the results of [8], [9] and [11] where $P$-contractivity was assumed. Note that each $P$-contraction is a contraction but not vice versa; see [9] for a counterexample. Our present paper deals with contractive functions; it uses an access which is different from that in [10], where quantitative aspects played the crucial role.

2. Preliminaries

By $\mathbb{I} \mathbb{R}$, $\mathbb{I} \mathbb{R}^n$, $\mathbb{I} \mathbb{R}^{n \times n}$ we denote the set of intervals, the set of interval vectors with $n$ components and the set of $n \times n$ interval matrices, respectively. By ‘interval’ we always mean a real compact interval. We write interval quantities in brackets with the exception of degenerate interval quantities which we identify with the element which they contain. Examples are the identity matrix $I$, its $i$-th column $e^{(i)}$ and the vector $e = (1, 1, \ldots, 1)^T$. With $[z] \in \mathbb{I} \mathbb{R}^n$ we define the subset $I([z]) := \{\lfloor x \rfloor \mid \lfloor x \rfloor \subseteq [z]\}$ of $\mathbb{I} \mathbb{R}^n$. We apply the notation $\lfloor x \rfloor = ([x], \underline{x}, \overline{x}) = ([x_i, x_i], [\underline{x}, \overline{x}]) \in \mathbb{I} \mathbb{R}^n$ simultaneously without further reference, and we proceed similarly with the elements of $\mathbb{I} \mathbb{R}$ and $\mathbb{I} \mathbb{R}^{n \times n}$. By $\text{int}([a])$ we denote the topological interior of an interval $[a]$ and by $\bar{a}$ we mean its midpoint. We define the absolute value $|[a]|$ by $|[a]| := \max\{a, |\overline{a}|\}$, the diameter $d([a])$ by $d([a]) := \overline{a} - \underline{a}$ and the distance $q([a], [b])$ by $q([a], [b]) :=$
max\{\|a - b\|, |\overline{a} - \overline{b}|\}. For interval vectors and interval matrices these items are applied entrywise. Continuity in $1\mathbb{R}$, $1\mathbb{R}^n$ and $1\mathbb{R}^{n \times n}$ is to be understood with respect to $q$. If $g(x)$ is an expression for some function $g$, we write $g([x])$ for the interval arithmetic evaluation of this expression (cf. [2]), assuming that $g([x])$ exists. Note that we distinguish between $g([x])$ and $[g([x])]$, where $[g]$ means any interval function. For details on interval arithmetic we refer to [2] or [12].

By $\rho(A)$ we denote the spectral radius of $A \in \mathbb{R}^{n \times n}$; $A \succ 0$ means $a_{ij} > 0$ for $i, j = 1, \ldots, n$, and $x > 0$ is used for $x \in \mathbb{R}^n$ if $x_i > 0$, $i = 1, \ldots, n$.

As in [2], we define $[g] : 1\mathbb{R}^n \to 1\mathbb{R}^n$ to be a P-contraction if there is a matrix $P \in 1\mathbb{R}^{n \times n}$ with $P \succ 0$, $\rho(P) < 1$ such that

$$q([g](x), [g](y)) \leq Pq(x, y)$$

for all $x, y \in 1\mathbb{R}^n$. If $[g]$ fulfils (9) only for all $x, y \subseteq [z]$ with a given $[z] \in 1\mathbb{R}^n$, we call $[g]$ a P-contraction on $[z]$. Similarly, we define $[g] : 1\mathbb{R}^n \to 1\mathbb{R}^n$ to be a contraction (with respect to some vector norm $\|\cdot\|$) if there is a real constant $\alpha \in (0, 1)$ such that

$$\|q([g](x), [g](y))\| \leq \alpha\|q(x, y)\|$$

holds for all $x, y \in 1\mathbb{R}^n$. If $[g]$ fulfils (10) only on $I([z])$ for a given $[z] \in 1\mathbb{R}^n$, we call $[g]$ a contraction on $[z]$ (with respect to some vector norm $\|\cdot\|$).

A vector norm $\|\cdot\|$ on $\mathbb{R}^n$ is termed monotone if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in \mathbb{R}^n$.

If the same symbol $\|\cdot\|$ is used for vectors and matrices then we always assume that the matrix norm is the operator norm generated by the vector norm $\|\cdot\|$. Throughout our paper, $\|\cdot\|_\infty$ denotes the maximum norm when applied to vectors, and the row sum norm when applied to matrices; $\mu, \nu$ denote positive constants such that

$$\mu\|x\|_\infty \leq \|x\| \leq \nu\|x\|_\infty.$$

3. Results

We start our results with a theorem which is well-known for P-contractions (cf. [2] and [8], [9]) and which we formulate now for contractive mappings. In Theorems 3.1–3.4 the function $[g]$ need not necessarily be defined by (4).

**Theorem 3.1.** Let $[g] : 1\mathbb{R}^n \to 1\mathbb{R}^n$ be a contraction with respect to a monotone norm $\|\cdot\|$. Then each sequence of iterates $[x]^{k+1} := [g([x]^k)]$, $k = 0, 1, \ldots$ converges to the same limit $[x]^*$, which is the unique fixed point of $[g]$.
If
\[(12) \quad [g](x) \in \mathbb{R}^n\]
holds for all \(x \in \mathbb{R}^n\), then \([x]^*\) is a degenerate interval vector.

If a function \(t: \mathbb{R}^n \to \mathbb{R}^n\) satisfies the inclusion property (1) for all \(x \in [x]\) and all \([x] \in I \mathbb{R}^n\), then \([x]^*\) contains all the fixed points of \(t\). If, in addition, \(t\) is continuous, then it has at least one fixed point in \([x]^*\).

If (12) and (1) hold, then \(t\) is a contraction. It has a unique fixed point which can be identified with \([x]^*\).

The assertions hold analogously if \(\mathbb{R}^n\) is replaced by \([z]\) and if \(I \mathbb{R}^n\) is replaced by \(I([z])\) for a fixed vector \([z] \in I \mathbb{R}^n\).

**Proof.** Since \((I \mathbb{R}^n, \|q(\cdot, \cdot)\|)\) is a complete metric space, the existence and uniqueness of \([x]^*\) follow from Banach’s fixed point theorem.

Assume now that (1) holds and that \([x]^*\) does not contain some fixed point \(y^*\) of \(t\). Start the iterative process \([x]^{k+1} := [g]([x]^k)\) with \([x]^0 := y^*\). Then \(y^* = t(y^*) \in [g](y^*) = [g]([x]^0) = [x]^1\) and, by induction, \(y^* \in [x]^k\), \(k = 0, 1, \ldots\). Therefore, \(y^* \in \lim_{k \to \infty} [x]^k = [x]^*\), which contradicts our assumption. Hence \([x]^*\) contains all fixed points of \(t\). Since \(t(x) \in [g]([x]^*) = [x]^*\) for all \(x \in [x]^*\), Brouwer’s fixed point theorem guarantees at least one fixed point of \(t\) in \([x]^*\), provided that \(t\) is continuous.

Let now (12) and (1) hold simultaneously. Then, clearly, \([g](x) = t(x)\) for all \(x \in \mathbb{R}^n\), and the contractivity of \([g]\) and the monotonicity of \(\| \cdot \|\) imply

\[
\|t(x) - t(y)\| = \|t(x) - t(y)\| = \|q(t(x), t(y))\| = \|q([g](x), [g](y))\| \\
\leq \alpha \|x - y\| = \alpha \|x - y\|,
\]

where \(\alpha\) is the contraction constant of \([g]\). Hence \(t\) is a contraction. \(\Box\)

**Theorem 3.2.** Let \([z] := \mathbb{R}^n\) be a fixed vector and let \([g]: I([z]) \to I \mathbb{R}^n\) be a contraction on \([z]\) with respect to a monotone vector norm \(\| \cdot \|\). Let \([z]\) be a vector such that \([z] \supseteq [z] + \frac{\|d([z])\|}{\mu(1 - \alpha)}[-1, 1]e\), where \(\alpha\) is the contraction constant and where \(\mu\) is from (11). Choose \([x]^0 \subseteq [z]\) and assume \([x]^1 := [g]([x]^0) \subseteq [z]\). Then the iterates \([x]^{k+1} := [g]([x]^k)\) are defined for \(k = 0, 1, \ldots\), i.e., they are all contained in \([z]\). They converge to a vector \([x]^* \subseteq [z]\) which is independent of \([x]^0\).

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Proof. Since $\| \cdot \|$ is a monotone norm we get

$$
\mu \|q([x]^{k+1}, [x]^0)\|_{\infty} \leq \|q([x]^{k+1}, [x]^0)\| \leq \left\| \sum_{i=0}^{k} q([x]^{i+1}, [x]^i) \right\|
$$

$$
\leq \sum_{i=1}^{k} \|q([g]([x]^i), [g]([x]^{i-1}))\| + \|q([x]^{1}, [x]^0)\| \leq \alpha \sum_{i=1}^{k} \|q([x]^i, [x]^{i-1})\| + \|q([x]^{1}, [x]^0)\| \leq \ldots \leq \left( \sum_{i=0}^{\infty} \alpha^i \right) \|q([x]^{1}, [x]^0)\|
$$

$$
= \frac{1}{1-\alpha} \|q([x]^1, [x]^0)\| \leq \frac{1}{1-\alpha} \|z - \bar{z}\| = \frac{1}{1-\alpha} \|d([z])\|.
$$

Therefore,

$$
(x)^{k+1} \subseteq [x]^0 + \frac{\|d([z])\|}{\mu(1-\alpha)} [-1, 1] e \subseteq [z]^c,
$$

in particular, $[x]^k$ exists for all $k \in \mathbb{N}$. Since

$$
\mu \|\overline{x}^{k+m} - \overline{x}^m\| \leq \mu \|q([x]^{k+m}, [x]^k)\|_{\infty} \leq \|q([g]([x]^{k-1+m}), [g]([x]^{k-1}))\|
$$

$$
\leq \alpha \|q([x]^{k-1+m}, [x]^{k-1})\| \leq \ldots \leq \alpha^k \|q([x]^m, [x]^0)\| \leq \frac{\alpha^k}{1-\alpha} \|d([z])\|
$$

for all $m = 0, 1, \ldots$, and since an analogous inequality holds for the upper bounds, the sequences $\{\overline{x}^k\}, \{\overline{x}^k\}$ converge to limits $\overline{x}^*$ and $\overline{x}^*$, respectively, with $\overline{x}^* \subseteq \overline{x}^*$. Therefore, $\lim_{k \to \infty} [x]^k = [\overline{x}^*, \overline{x}^*] =: [x]^* \subseteq [z]^c$ by (13). Uniqueness follows from $\|q([x]^*, [y]^*)\| = \|q([g]([x]^*), [g]([y]^*)))\| \leq \alpha \|q([x]^*), [y]^*)\|$ for two different fixed points $[x]^*, [y]^*$ of $[g]$. $\square$

Theorem 3.3. Let $[g] : \mathbb{I} \mathbb{R}^n \to \mathbb{I} \mathbb{R}^n$ be a contraction with respect to a monotone norm $\| \cdot \|$ and with a contraction constant $\alpha$. Iterate by inflation according to

$$
\begin{cases}
[x]^0 := \overline{x}, \\
[x]^k_e := [x]^k + [\delta]^k \\
[x]^{k+1} := [g]([x]^k_e)
\end{cases}
$$

$k = 0, 1, \ldots,$

where $[\delta]^k \in \mathbb{I} \mathbb{R}^n$ are given vectors which converge to some limit $[\delta]$. If $[\delta]$ contains 0 in its interior then there is an integer $k_0 = k_0([x]^0_e)$ such that

$$
[g]([x]^{k_0}_e) \subseteq \text{int}([x]^{k_0}_e)
$$

holds.

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Proof. Let \([s][x] := [g][x] + [\delta].\) Then
\[
\|q([s][x]), [s][y])\| = \|q([g][x]), [g][y])\| \leq \alpha \|q([x], [y])\|,
\]
hence \([s]\) is a contraction. By Theorem 3.1 it has a unique fixed point \([x]^*\) which satisfies
\[
[x]^* = [g]([x]^*) + [\delta].
\]
Assume for the moment that
\[
\lim_{k \to \infty} [x]^k = [x]^*.
\]
holds for the sequence in (14). By the continuity of \([g]\) we have
\[
\lim_{k \to \infty} [g]([x]^k) = [g]([x]^*) .
\]
Since \(0 \in \text{int}([\delta]),\) equation (17) implies \([g]([x]^*) \subseteq \text{int}([x]^*).\) Together with (18) and (19) this yields (15) for all sufficiently large integers \(k_0.\)

We prove now the assumption (18). With the usual rules for \(q\) we obtain
\[
\|q([x]^k, [x]^*)\| = \|q([g]([x]^k-1) + [\delta]^k, [g][x]^* + [\delta])\|
\leq \alpha \|q([x]^k-1, [x]^*)\| + \|q([\delta]^k, [\delta])\|
\leq \alpha^2 \|q([x]^k-2, [x]^*)\| + \alpha \|q([\delta]^k-1, [\delta])\| + \|q([\delta]^k, [\delta])\|
\leq \ldots \leq \alpha^k \|q([x]^0, [x]^*)\| + \sum_{i=0}^{k-1} \alpha^i \|q([\delta]^{k-i}, [\delta])\| .
\]
Fix \(\theta > 0\) and choose the integer \(m\) such that \(\alpha^i \leq \theta\) for all \(i \geq m.\) Since \(\lim_{k \to \infty} [\delta]^k = [\delta],\) there are a constant \(\gamma > 0\) and an integer \(k' > m\) with \(\|q([\delta]^i, [\delta])\| \leq \gamma, \ i = 0, 1, \ldots, \) and \(\|q([\delta]^{k-i}, [\delta])\| \leq \theta, \ k \geq k', \ i = 0, 1, \ldots, m - 1.\) For \(k \geq k'\) we thus get with (20)
\[
\|q([x]^k, [x]^*)\| \leq \theta \|q([x]^0, [x]^*)\| + \sum_{i=0}^{m-1} \alpha^i \theta + \alpha^m \sum_{i=m}^{k-1} \alpha^{i-m} \gamma
\leq \theta \left\{ \|q([x]^0, [x]^*)\| + \frac{1}{1 - \alpha} + \frac{\gamma}{1 - \alpha} \right\} .
\]
Since the expression in braces is independent of \(\theta, \ m\) and \(k,\) and since \(\theta\) can be chosen arbitrarily small, (18) holds. \(\square\)
Relying on Theorem 3.2 one can also formulate a local version of Theorem 3.3. For simplicity, we restrict ourselves to the case \([\delta]^k = [\delta], k = 0, 1, \ldots\).

**Theorem 3.4.** Let \([z]^0 \in \mathbb{I}^{\mathbb{R}^n}\) be a fixed vector and let \([g]: I([z]^0) \to \mathbb{I}^{\mathbb{R}^n}\). Assume that \([z], [z]^c \subseteq [z]^0\) and \([\delta] \in \mathbb{I}^{\mathbb{R}^n}\) possess the following properties:

i) \(0 \in \text{int}([\delta])\),

ii) \([g]\) is contractive with respect to a monotone norm \(\| \cdot \|\) on

\[
[z]^c \supseteq [z] + \frac{\|d([z])\|}{\mu(1 - \alpha)}[-1, 1],
\]

where \(\alpha\) is the contraction constant and \(\mu\) is the constant from (11). If \([x]^0 \subseteq [z]\) and \([x]^1_{\epsilon} \subseteq [z]\) hold for the iterates from (14) with \([\delta]^k := [\delta]\), then there is an integer \(k_0 = k_0([x]^0_{\epsilon})\) such that (15) is true. In particular, \(t\) from (1) has a fixed point in \([x]^k_{\epsilon}\).

**Proof.** Since \([s](x) := [g](x) + [\delta]\) fulfils (16) for all \([x], [y] \subseteq [z]^c\), the function \([s]\) is a contraction on \([z]^c\). By Theorem 3.2 there is a vector \([x]^* \subseteq [z]^c\) which satisfies

\[
\lim_{k \to \infty} [x]^k_{\epsilon} = [x]^* = [s](x)^* = [g](x)^* + [\delta].
\]

Since \(0 \in \text{int}([\delta])\), this yields

\[
[g](x)^* \subseteq \text{int}([x]^*),
\]

and the assertion follows from (19), (22) and from the first equality in (21). \(\Box\)

We want to apply now Theorem 3.4 to the function \([g]\) from (4) when \([H]\) satisfies (5) and (6) with \(\| \cdot \| := \| \cdot \|_{\infty}\). (The choice of the maximum norm is not a severe restriction since by the norm equivalence in \(\mathbb{R}^n\) the norm in (5) can be replaced by any norm, if the constant \(\kappa\) is changed appropriately.) To this end let \([z]^0 \in \mathbb{I}^{\mathbb{R}^n}\) denote a fixed interval vector for which \([g]\) is defined and which contains \(\bar{x}\) in its interior. Following the lines in [11], p. 101, one can show that \([g]\) satisfies the Lipschitz condition

\[
\|q([g](x)), [g](y))\|_{\infty} \leq \beta \|q(x, y)\|_{\infty}, \quad [x], [y] \subseteq [z]
\]

for each fixed \([z] \subseteq [z]^0\) with

\[
\beta := \|t'(\bar{x})\|_{\infty} + 2\kappa\|\bar{z} - \bar{x}\|_{\infty}.
\]

(This even holds for any monotone norm.)
For the remaining part of this section we assume that \( \| t'(\tilde{x}) \|_\infty \) is sufficiently small, \( \tilde{x} \) is a sufficiently good approximation of a fixed point \( x^* \) of \( t \), \( [\delta] \in I \mathbb{R}^n \) is a given vector of sufficiently small diameter which contains 0 in its interior, and \( [x]^k, k = 0, 1, \ldots \), is defined by (14) with \([\delta]^k := [\delta]\).

Then \([g] \) is a contraction on
\[
[z] := \tilde{x} + [\delta][-1, 1] + \{ \| \tilde{x} - t(\tilde{x}) \|_\infty + \left( \| t'(\tilde{x}) \|_\infty + \kappa \| [\delta] \|_\infty \right) \| [\delta] \|_\infty \} [-1, 1]e,
\]
and \([x]^0 \subseteq [z] \). From
\[
\| [H](x) [z] \| = \| [H](x) - [H](\tilde{x}) \| = \| q([H](x), [H](\tilde{x})) \|
\leq \kappa \| q(x, \tilde{x}) \| = \kappa \| [x] - \tilde{x} \|.
\]
we get
\[
[x]^1 := [g]([x]^0) = t(\tilde{x}) + \{ t'(\tilde{x}) + [H](\tilde{x} + [\delta]) \} [\delta]
\subseteq \tilde{x} + \{ t(\tilde{x}) - \tilde{x} \} \| [t'(\tilde{x})] \|_\infty - [1, 1]e + \{ \| t'(\tilde{x}) \|_\infty + \| [H](\tilde{x} + [\delta]) \|_\infty \} \| [\delta] \|_\infty + [1, 1]e
\subseteq \tilde{x} + \{ t(\tilde{x}) - \tilde{x} \} \| [t'(\tilde{x})] \|_\infty - [1, 1]e + \| t'(\tilde{x}) \|_\infty + \| [H](\tilde{x} + [\delta]) \|_\infty \} \| [\delta] \|_\infty + [1, 1]e.
\]

Hence \([x]^1 \) and \([x]^1 \) are also contained in \([z] \). By our assumptions we can assume that \( \beta < 0.1 \) and that \( \| d([z]) \|_\infty < \frac{0.1}{4\kappa} \). Let \( \alpha := \frac{1}{2} \). By virtue of \([z]^c := [z] + \frac{\| d([z]) \|_\infty}{\alpha} [-1, 1]e = [z] + 2\| d([z]) \|_\infty - [1, 1]e \) we obtain \( \| [z]^c - \tilde{x} \|_\infty \leq \| [z] - \tilde{x} \|_\infty + 2\| d([z]) \|_\infty \). Hence
\[
\tilde{\beta} := \| t'(\tilde{x}) \|_\infty + 2\kappa \| [z]^c - \tilde{x} \|_\infty \leq \| t'(\tilde{x}) \|_\infty + 2\kappa \| [z] - \tilde{x} \|_\infty + 4\kappa \| d([z]) \|_\infty = \beta + 4\kappa \| d([z]) \|_\infty \leq 0.1 + 0.1 \leq 0.5 = \alpha,
\]
and \([g] \) is a contraction on \([z]^c \) with contraction constant \( \tilde{\beta} \) and therefore also with the contraction constant \( \alpha \). Now Theorem 3.4 applies with \( \mu = 1 \).

In order to use this result for the particular situations of Section 4 we assume now that \( t \) is given by (7) with \([g] \) from (8). If \( C \) from (7) approximates \( f'(\tilde{x})^{-1} \) sufficiently well then \( \| t'(\tilde{x}) \|_\infty = \| I - Cf'(\tilde{x}) \|_\infty \) is certainly small. If, in addition, \( \tilde{x} \)

is a sufficiently good approximation of a zero of \( f \) then \( t(\tilde{x}) \approx \tilde{x} \). Hence the ‘essential’ assumptions above are fulfilled and Theorem 3.4 can be applied. We state this result as a separate corollary:

**Corollary 3.1.** Let \([g] \) be defined as in (4) with \( t(x) := x - Cf(x) \) and with \([H] \) satisfying (5) and (6) with respect to \( \| \cdot \|_\infty \). Assume that \( f'(\tilde{x})^{-1} \) exists and that

\[
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\]
C is nonsingular and approximates $f'(\tilde{x})^{-1}$ sufficiently well. If $\tilde{x}$ is a sufficiently good approximation of a zero $x^*$ of $f$ and if the inflation $[\delta]$ is sufficiently small and contains 0 in its interior, then the inflation procedure (14) with $[\delta]^k := [\delta]$ stops with $[x]^{k+1} \subseteq \text{int}([x]^k_{\varepsilon})$ after finitely many steps.

Note that Corollary 3.1 guarantees success in $\varepsilon$-inflation only if some input parameters are sufficiently good. Unfortunately it neither predicts the minimal number $k_0$ of iterates which are necessary to fulfill (2), nor specifies by a measure what ‘sufficiently’ really means. In this respect further work has to be done.

If one computes $C$ as an approximate inverse of $f'(\tilde{x})$ one normally does not know exactly whether $f'(\tilde{x})$ or $C$ are nonsingular. This can be guaranteed, however, a posteriori, if one assumes $[H]$ to be inclusion monotone, i.e., $[H](|[x]|) \subseteq [H](|[y]|)$ for $|[x]| \subseteq |[y]|$, and if (2) can be checked for some $k_0$ for which $\tilde{x} \in [x]^{k_0}$ still holds—for example for $k_0 = 0$. The proof is based on the following argument:

Since $t'(\tilde{x}) = I - Cf'(\tilde{x})$ in the situation of Corollary 3.1, one gets by standard rules for the diameter (cf. [2] or [12])

$$d([x]^{k_0}) > d([g](|[x]|_{\varepsilon}^{k_0})) \geq d([g](|[x]|^{k_0})) \geq |t'(\tilde{x}) + [H](|[x]|^{k_0})|d([x]^{k_0})$$

$$\geq |t'(\tilde{x}) + [H](|[\tilde{x}]|)|d([x]^{k_0}) = |t'(\tilde{x})|d([x]^{k_0}) = |I - Cf'(\tilde{x})|d([x]^{k_0}).$$

Therefore, $d([x]^{k_0}) > 0$ and $\varrho(I - Cf'(\tilde{x})) < 1$ by Corollary 3.2.3 and Proposition 3.2.4 in [12], for example. If $C$ or $f'(\tilde{x})$ are singular then 1 would be an eigenvalue of $I - Cf'(\tilde{x})$, which contradicts $\varrho(I - Cf'(\tilde{x})) < 1$.

4. Examples

In this section we will apply Corollary 3.1 to various algorithms for verifying and enclosing solutions of mathematical problems.

Example 4.1. (The algebraic eigenproblem for a simple real eigenvalue)

We consider first the algebraic eigenproblem $Av = \lambda v$. Apparently, each real eigenpair $(v^*, \lambda^*)$ of $A \in \mathbb{R}^{n \times n}$ can be viewed as a zero of the function $f(x) := \left(\begin{array}{c} Av - \lambda v \\ v_{i_0} - \zeta \end{array}\right)$ if the eigenvector $v^*$ is normalized by $v^*_{i_0} = \zeta \neq 0$ in a component $i_0$ and if $x := (v^T, \lambda^*)^T$. Let $(\tilde{v}, \tilde{\lambda})$ be an approximation of $(v^*, \lambda^*)$, where $\lambda^*$ is an algebraic simple eigenvalue of $A$. In [14] the interval function

$$[g](|[x]|) := \tilde{x} - Cf(\tilde{x}) + \left\{ I_{n+1} - C \left( \begin{array}{cc} A - \tilde{\lambda} I_n \\ (e^{i\theta})^T \end{array}\right) \right\} (|[x]| - \tilde{x})$$

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was applied with \( [x] := ([v]^T, [\lambda])^T \in \mathbb{I}^{n+1} \) in order to verify and to enclose \( x^* := ((v^*)^T, \lambda^*)^T \). With \( t(x) = x - C f(x) \) as in Corollary 3.1 one gets

\[
t'(\bar{x}) = I_{n+1} - C \begin{pmatrix} A - \bar{\lambda} I_n & -\bar{v} \\ e^{(i_0)^T} & 0 \end{pmatrix}.
\]

In [7] it was mentioned that for degenerate interval vectors \([x] \equiv x\) the expression \([g](x)\) from (23) is the complete Taylor expansion of \( t(x) \) at \( \bar{x} := (\bar{v}^T, \bar{\lambda})^T \) even if \( \bar{x} \not\in [x] \). Therefore, \( t(x) \in [g]([x]) \) holds trivially for all \( x \in [x] \). With

\[
[H]([x]) := C \begin{pmatrix} O & [v] - \bar{v} \\ O & 0 \end{pmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}
\]

the function \([g]\) has the form (4). The property (6) can be seen at once, the Lipschitz condition (5) follows from

\[
\|q([H]([x]), [H]([y]))\| \leq \|C|q\left(\begin{pmatrix} O & [v] - \bar{v} \\ O & 0 \end{pmatrix}, \begin{pmatrix} O & [w] - \bar{v} \\ O & 0 \end{pmatrix}\right)\|_{\infty}
\]

where \([x] = ([v]^T, [\lambda])^T \in \mathbb{I}^{n+1} \) and \([y] = ([w]^T, [\mu])^T \in \mathbb{I}^{n+1} \). Therefore, Corollary 3.1 applies with \( \kappa := \|C\|_{\infty} \). It shows that under appropriate circumstances concerning the approximations \( C, \bar{x} \) and the inflation \( [\delta] \), the iteration (14) ends up with the subset property (2), which guarantees an eigenpair of \( A \) in the final iterate \([x]^{k_0}\).

□

The arguments in Example 4.1 apply without difficulties also to the generalized algebraic eigenproblem \( Av = \lambda Bv \), where \( A, B \) are matrices from \( \mathbb{R}^{n \times n} \). We leave the details to the reader.

Example 4.2. (Two-dimensional invariant subspaces)

In order to verify and to enclose a basis of a two-dimensional subspace of \( \mathbb{R}^n \) which is invariant with respect to a linear mapping given by a matrix \( A \in \mathbb{R}^{n \times n} \), Alefeld and Spreuer start in [3] with the function

\[
f(x) := \begin{pmatrix} Au - m_{11} u - m_{21} v \\ u_{i1} - \varepsilon \\ u_{i2} - \zeta \\ Av - m_{12} u - m_{22} v \\ v_{i1} - \eta \\ v_{i2} - \theta \end{pmatrix} \in \mathbb{R}^{2n+4}
\]
where \( x = (u^T, m_{11}, m_{21}, v^T, m_{12}, m_{22})^T \in \mathbb{R}^{2n+4}, i_1 \neq i_2 \in \{1, \ldots, n\} \) and \( \varepsilon \theta - \zeta \eta \neq 0 \). Taking into account the normalizations, it is obvious that the vectors \( u^\ast, v^\ast \), which are part of a zero \( x^\ast = ((u^\ast)^T, m_{11}^*, m_{21}^*, (v^\ast)^T, m_{12}^*, m_{22}^*)^T \) of \( f \), form a basis of such an invariant subspace. Again we set \( t(x) := x - Cf(x) \) with a nonsingular matrix \( C \in \mathbb{R}^{(2n+4) \times (2n+4)} \), and we choose \( \tilde{x} = (\tilde{u}^T, \tilde{m}_{11}, \tilde{m}_{21}, \tilde{v}^T, \tilde{m}_{12}, \tilde{m}_{22})^T \) as an approximation of \( x^\ast \). Then

\[
t'(\tilde{x}) = I_{2n+4} - C \begin{pmatrix}
    A - \tilde{m}_{11} I_n & -\tilde{u} & -\tilde{v} & -\tilde{m}_{21} I_n & 0 & 0 \\
    (e^{(i_1)})^T & 0 & 0 & 0 & 0 & 0 \\
    (e^{(i_2)})^T & 0 & 0 & 0 & 0 & 0 \\
    -\tilde{m}_{12} I_n & 0 & 0 & A - \tilde{m}_{22} I_n & -\tilde{u} & -\tilde{v} \\
    0 & 0 & 0 & (e^{(i_1)})^T & 0 & 0 \\
    0 & 0 & 0 & (e^{(i_2)})^T & 0 & 0 
\end{pmatrix}
\]

and

\[
[H](\lfloor x \rfloor) = C \begin{pmatrix}
    O & [u] - \tilde{u} & [v] - \tilde{v} & O & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & O & [u] - \tilde{u} & [v] - \tilde{v} \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \in \mathbb{R}^{(2n+4) \times (2n+4)}
\]

for \( [g] \) from (4). Using the usual rules for \( q \) one again easily verifies (5) and (6) which are the crucial assumptions for Corollary 3.1. □

Example 4.3. (The singular value problem)

Let \( ((u^i)^T, (v^i)^T, \sigma_i)^T \in \mathbb{R}^{m+n+1} \) be a vector which gathers a singular value \( \sigma_i \) of a rectangular matrix \( A \in \mathbb{R}^{m \times n} \) and the corresponding \( i \)-th columns \( u^i, v^i \) of the orthogonal matrices \( U \in \mathbb{R}^{n \times n}, V \in \mathbb{R}^{m \times m} \) of the singular value decomposition

\[
A = V \Sigma U^T = V \text{diag}(\sigma_1, \ldots, \sigma_{\text{min}(m,n)}) U^T.
\]

Then this vector is obviously a zero of the function

\[
f(x) := \begin{pmatrix}
    Au - \sigma v \\
    A^T v - \sigma u \\
    u^T u - 1
\end{pmatrix},
\]

where \( x := (u^T, v^T, \sigma)^T \). If \( x^\ast = ((u^\ast)^T, (v^\ast)^T, \sigma^\ast)^T \) is such a zero of \( f \) with \( \sigma^\ast \neq 0 \) then

\[
(v^\ast)^T v^\ast = (v^\ast)^T \frac{1}{\sigma^\ast} A u^\ast = \frac{1}{\sigma^\ast} (A^T v^\ast)^T u^\ast = (u^\ast)^T u^\ast = 1.
\]
Let $\tilde{x} = (\tilde{u}^T, \tilde{v}^T, \tilde{\sigma})^T$ and let $C \in \mathbb{R}^{(m+n+1) \times (m+n+1)}$ be nonsingular. Similar to the development in [7] (cf. also [1]) we use $[g]$ from (4) with $t(x) := x - Cf(x)$,

$$t'(\tilde{x}) = I - C \begin{pmatrix} A & -\tilde{\sigma}I_m & -\tilde{v} \\ -\tilde{\sigma}I_n & A^T & -\tilde{u} \\ 2\tilde{u}^T & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(m+n+1) \times (m+n+1)}$$

and

$$[H](x) := C \begin{pmatrix} O & O & [v] - \tilde{v} \\ O & O & [u] - \tilde{u} \\ ([u] - \tilde{u})^T & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(m+n+1) \times (m+n+1)},$$

in order to verify $x^r$. Again, $[g](x)$ is the complete Taylor expansion of $t(x) := x - Cf(x)$ at $x = \tilde{x}$. As in Example 4.1 one easily checks that (5) and (6) hold for $[H]$. Thus Corollary 3.1 applies. □

We finally mention that Corollary 3.1 also applies to quadratic systems of the form $t(x) := b + Ax + T(x, x) = x$, where $b, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ and where $T: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bilinear operator. The details are left to the reader.

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**References**


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