Milan Práger

Eigenvalues and eigenfunctions of the Laplace operator on an equilateral triangle

Applications of Mathematics, Vol. 43 (1998), No. 4, 311--320

Persistent URL: http://dml.cz/dmlcz/134391

EIGENVALUES AND EIGENFUNCTIONS OF THE LAPLACE OPERATOR ON AN EQUILATERAL TRIANGLE

Milan Práger, Praha

(Received January 5, 1998)

Abstract. A boundary value problem for the Laplace equation with Dirichlet and Neumann boundary conditions on an equilateral triangle is transformed to a problem of the same type on a rectangle. This enables us to use, e.g., the cyclic reduction method for computing the numerical solution of the problem. By the same transformation, explicit formulae for all eigenvalues and all eigenfunctions of the corresponding operator are obtained.

Keywords: Laplace operator, boundary value problem, eigenvalues, eigenfunctions

MSC 2000: 35P10, 35J05

0. Introduction

Fast solvers for simple boundary value problems on simple domains are very useful tools for the solution of complex problems. In this paper we show that problems for the Laplace operator on an equilateral triangle can be transformed into problems on a rectangle. Fast solvers for the rectangle can be thus applied to problems on the triangle. The Fourier analysis on a rectangle yields the eigenfunction expansion on the triangle. The corresponding eigenvalues can be used e.g. for the preconditioning of related problems. In the next paper we will describe the eigenproblem for the discrete Laplace operator on a triangle mesh.

Let T be an equilateral triangle with vertices $\left(\frac{-1}{\sqrt{3}},0\right)$, $\left(\frac{1}{\sqrt{3}},0\right)$, $(0,1)$. Its altitude is equal to one and its side is equal to $\frac{2}{\sqrt{3}}$.

Let a function $f(x,y)\in L_2$ be given on T. We decompose it into the symmetric part $f^s(x,y)=\frac{1}{2}\big(f(x,y)+f(-x,y)\big)$ and the skew symmetric part $f^a(x,y)=\frac{1}{2}\big(f(x,y)-f(-x,y)\big)$.

Let R be the rectangle $\langle 0, \sqrt{3}\rangle \times \langle 0, 1\rangle$. We define the prolongation of a function $u \in L_2(T_1)$ from the triangle $T_1 = T \cap R$ onto R so that we prolong it successively by skew symmetry with respect to the dotted lines (see Fig. 1).

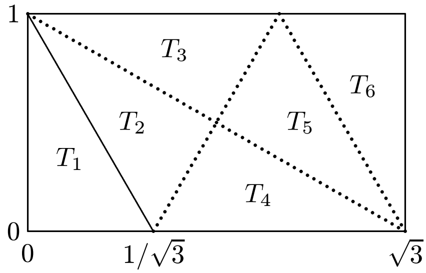

Fig. 1

In order to describe such a prolongation, we first introduce transformations K_i of the open triangle T_1 onto open triangles T_i by the equations

$$
\begin{aligned}
&x_1 = \xi, \quad x_2 = \tfrac{1}{2}\big(-\xi - \sqrt{3}\eta + \sqrt{3}\big), \quad x_3 = \tfrac{1}{2}\big(\xi - \sqrt{3}\eta + \sqrt{3}\big),\\
&y_1 = \eta, \quad y_2 = \tfrac{1}{2}\big(-\sqrt{3}\xi + \eta + 1\big), \qquad y_3 = \tfrac{1}{2}\big(\sqrt{3}\xi + \eta + 1\big),\\
&x_4 = \tfrac{1}{2}\big(-\xi + \sqrt{3}\eta + \sqrt{3}\big), \quad x_5 = \tfrac{1}{2}\big(\xi + \sqrt{3}\eta + \sqrt{3}\big), \quad x_6 = \sqrt{3} - \xi,\\
&y_4 = \tfrac{1}{2}\big(-\sqrt{3}\xi - \eta + 1\big), \qquad y_5 = \tfrac{1}{2}\big(\sqrt{3}\xi - \eta + 1\big), \qquad y_6 = 1 - \eta,
\end{aligned} \tag{1}
$$

where $B_i = (x_i, y_i) \in T_i$ for $B = (\xi, \eta) \in T_1$. We thus have $B_i = K_i B$.

The transformations K_i are successive reflections with respect to the dotted lines in Fig. 1 and are therefore compositions of rotations, reflections and translations of the triangle T_1.

The prolongation $\mathscr{P}u$ of a function $u \in L_2(T_1)$ from T_1 onto R is defined by

$$\mathscr{P}u(B_i) = c_i u(B) \quad \text{on} \quad T_i, \tag{2}$$

where $c_i = 1$ for $i = 1, 3, 4, 6$ and $c_i = -1$ for $i = 2, 5$.

Let further $v \in L_2(R)$. We define a transformation $\mathscr{F}$, which we call a folding, from R onto T_1 as follows:

$$\mathscr{F}v(B) = \sum_{i=1}^{6} c_i v(B_i),$$

where $B = K_i^{-1} B_i$.

Let us notice that for a function $u \in H_0^1(T_1)$ we can use the transformation (2) for closed triangles and that we have $\mathscr{P}u \in H_0^1(R)$, see e.g. [1]. Conversely, it is easily seen that for a function $u \in H_0^1(R)$, its transform $\mathscr{F}u$ belongs to $H_0^1(T_1)$.

1. The boundary value problem

We will show that a Dirichlet boundary value problem for the Laplace equation on the triangle T_1 can be transformed into the same problem on R if we take the prolongation of the right-hand side given on T_1 for the right-hand side. The basis for it is the following

Theorem. *Let $f \in L_2(T_1)$ and let $u \in H_0^1(T_1)$ be the solution of the boundary value problem*

$$\int_{T_1} (\operatorname{grad} u, \operatorname{grad} \varphi)\, \mathrm{d}\xi\, \mathrm{d}\eta = \int_{T_1} (f, \varphi)\, \mathrm{d}\xi\, \mathrm{d}\eta \quad \textit{for } \ \forall \varphi \in H_0^1(T_1). \tag{3}$$

Then $\mathscr{P}u$ is the solution of the boundary value problem on R

$$\int_{R} (\operatorname{grad} \mathscr{P}u, \operatorname{grad} \psi)\, \mathrm{d}x\, \mathrm{d}y = \int_{R} (\mathscr{P}f, \psi)\, \mathrm{d}x\, \mathrm{d}y \quad \textit{for } \ \forall \psi \in H_0^1(R). \tag{4}$$

P r o o f. We start with the left-hand side of (4). With the use of the transformations K_i^{-1} we have

$$\int_{R} (\operatorname{grad} \mathscr{P}u, \operatorname{grad} \psi)\, \mathrm{d}x\, \mathrm{d}y = \sum_{i=1}^{6} \int_{T_i} (\operatorname{grad} \mathscr{P}u, \operatorname{grad} \psi)\, \mathrm{d}x_i\, \mathrm{d}y_i =$$

$$\sum_{i=1}^{6} \int_{T_1} c_i (\operatorname{grad} u, \operatorname{grad} \psi)\, \mathrm{d}\xi\, \mathrm{d}\eta = \int_{T_1} (\operatorname{grad} u, \operatorname{grad} \mathscr{F}\psi)\, \mathrm{d}\xi\, \mathrm{d}\eta,$$

where the gradients in the last integral are taken with respect to ξ and η. This is a consequence of the isometry of the transformations (1). The modulus of the Jacobian equals to one for the same reason. Further we obtain from (3) and with the transformations K_i that

$$\int_{T_1} (f, \mathscr{F}\psi)\, \mathrm{d}\xi\, \mathrm{d}\eta = \sum_{i=1}^{6} \int_{T_i} (\mathscr{P}f, \psi)\, \mathrm{d}x_i\, \mathrm{d}y_i = \int_{R} (\mathscr{P}f, \psi)\, \mathrm{d}x\, \mathrm{d}y, \quad \text{q.e.d.}$$

□

From this theorem and from the unicity of the solution it follows that the solution on R with the prolonged right-hand side restricted onto T_1 is the solution sought.

As a small example illustrating the situation, let us solve numerically the problem

$$\Delta u = 2\left(x + \sqrt{3}y\right) \quad \text{on} \quad T_1$$

with homogeneous Dirichlet conditions. Its exact solution is $xy\left(\sqrt{3}x + y - 1\right)$.

The numerical solution was obtained by applying the method of cyclic reduction and factorization to the “prolonged problem” on R.

n	max.error$\times n^2$	time (sec)$/n^2\log n$
4	.1205389	4.40276E-7
8	.1263714	7.33793E-8
16	.1275594	6.87931E-8
32	.1277278	8.31250E-8
64	.1277067	1.76760E-7
128	.1276699	2.00646E-7
256	.5252685	2.95553E-7

Table 1. Results of the numerical example

The computation was performed on a rectangular mesh with n panels in the y-direction and with $3n$ panels in the x-direction on R. In Table 1 we briefly show the characteristic behaviour of results for different values of n (in repeated computations, values of the time elapsed were slightly varying). They are in full agreement with what has been expected, i.e., the errors are of order n^{-2} and the time elapsed is proportional to $n^2\log n$.

2. Eigenproblem

Let us consider the Fourier expansion of $\mathscr{P}f^a$ on R. The Fourier coefficients are

$$\begin{aligned}\mathscr{P}f^a_{k,l} &= \int_R \mathscr{P}f^a(x,y)\sin\frac{k\pi x}{\sqrt{3}}\sin l\pi y\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{T_1} f^a(x,y)u_{k,l}(x,y)\,\mathrm{d}x\,\mathrm{d}y,\qquad k,l=1,2,\dots,\end{aligned}$$

where

(5)
$$\begin{aligned}u_{k,l}(x,y) = \sin\frac{k\pi x}{\sqrt{3}}\sin l\pi y &- \sin\frac{k\pi}{2\sqrt{3}}\big(-x-\sqrt{3}y+\sqrt{3}\big)\sin\frac{l\pi}{2}\big(-\sqrt{3}x+y+1\big)\\ &+ \sin\frac{k\pi}{2\sqrt{3}}\big(x-\sqrt{3}y+\sqrt{3}\big)\sin\frac{l\pi}{2}\big(\sqrt{3}x+y+1\big)\\ &+ \sin\frac{k\pi}{2\sqrt{3}}\big(-x+\sqrt{3}y+\sqrt{3}\big)\sin\frac{l\pi}{2}\big(-\sqrt{3}x-y+1\big)\\ - \sin\frac{k\pi}{2\sqrt{3}}\big(x+\sqrt{3}y+\sqrt{3}\big)\sin\frac{l\pi}{2}\big(\sqrt{3}x-y+1\big) &+ \sin\frac{k\pi(\sqrt{3}-x)}{\sqrt{3}}\sin l\pi(1-y).\end{aligned}$$

We will simplify this expression for $u_{k,l}(x,y)$. The manipulations are dependent on the parity of the subscripts k and l. We show it in more detail only for the case when both subscripts are even. We join together the first and last terms, the second and fifth terms, the third and fourth terms in (5), and obtain

$$\begin{aligned} u_{k,l}(x,y) = 2\sin\frac{k\pi x}{\sqrt{3}}\sin l\pi y - 2(-1)^{(k+l)/2}\sin k\pi\frac{x+\sqrt{3}y}{2\sqrt{3}}\sin l\pi\frac{\sqrt{3}x-y}{2} \\ + 2(-1)^{(k+l)/2}\sin k\pi\frac{x-\sqrt{3}y}{2\sqrt{3}}\sin l\pi\frac{\sqrt{3}x+y}{2}, \end{aligned}$$

and by further manipulation with the aid of the identity

$$\sin(a+b)\,\sin(c-d)-\sin(a-b)\,\sin(c+d) = \sin(a+c)\,\sin(b-d)-\sin(a-c)\,\sin(b+d) \tag{6}$$

we have

$$\begin{aligned} u_{k,l}(x,y) = 2\sin\frac{k\pi x}{\sqrt{3}}\sin l\pi y - 2(-1)^{(k+l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k+3l)\sin\frac{\pi y}{2}(k-l) \\ + 2(-1)^{(k+l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k-3l)\sin\frac{\pi y}{2}(k+l). \end{aligned}$$

For $k = l$ or $k = 3l$ the function $u_{k,l}$ is apparently equal to zero.

In an analogous manner we find out that for the subscripts of different parity, the function $u_{k,l}$ equals zero, too. For both subscripts odd we use an identity derived from (6) by differentiation with respect to a and c, and we have in the same way

$$\begin{aligned} u_{k,l}(x,y) = 2\sin\frac{k\pi x}{\sqrt{3}}\sin l\pi y - 2(-1)^{(k+l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k+3l)\sin\frac{\pi y}{2}(k-l) \\ - 2(-1)^{(k+l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k-3l)\sin\frac{\pi y}{2}(k+l). \end{aligned}$$

We can write both cases as

$$\begin{aligned} u_{k,l}(x,y) = 2\sin\frac{k\pi x}{\sqrt{3}}&\sin l\pi y \\ &- 2(-1)^{(k+l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k+3l)\sin\frac{\pi y}{2}(k-l) \\ &+ 2(-1)^{(k-l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k-3l)\sin\frac{\pi y}{2}(k+l). \end{aligned} \tag{7}$$

We thus obtain nonzero functions $u_{k,l}$ only for subscripts of the same parity and such that $k \neq l$ or $k \neq 3l$. We decompose this set of subscripts into three disjoint subsets defined by the inequalities $0 < k < l$, $l < k < 3l$ and $3l < k$.

For $l' < k' < 3l'$ we set

(8) $$k = \tfrac{1}{2}(3l' - k'),\ l = \tfrac{1}{2}(k' + l').$$

Then $0 < k < l$ and the pair (k, l) belongs to the first subset of subscripts. We obtain

$$\begin{aligned}
u_{k',l'}(x,y) &= 2\sin\frac{k'\pi x}{\sqrt{3}}\sin l'\pi y \\
&\quad - 2(-1)^{(k'+l')/2}\sin\frac{\pi x}{2\sqrt{3}}(k'+3l')\sin\frac{\pi y}{2}(k'-l') \\
&\quad + 2(-1)^{(k'-l')/2}\sin\frac{\pi x}{2\sqrt{3}}(k'-3l')\sin\frac{\pi y}{2}(k'+l') \\
&= 2(-1)^{(k-l)/2}\Bigg[(-1)^{(k-l)/2}\sin\frac{\pi x}{2\sqrt{3}}(3l-k)\sin\frac{\pi y}{2}(k+l) \\
&\quad - (-1)^{(k+l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k+3l)\sin\frac{\pi y}{2}(l-k) \\
&\quad + \sin\frac{-k\pi x}{\sqrt{3}}\sin l\pi y\Bigg] \\
&= -(-1)^{(k-l)/2}u_{k,l}.
\end{aligned}$$

For $3l'' < k''$ we also set

(9) $$k = \tfrac{1}{2}(k'' - 3l''),\ l = \tfrac{1}{2}(k'' + l'').$$

Then $0 < k < l$ and the pair (k, l) belongs to the first subset of subscripts and

$$u_{k'',l''}(x,y) = (-1)^{(k+l)/2}u_{k,l}.$$

For the symmetric part f^s of the function f we define the prolongation $\mathscr{P}f^s$ by (2) only with $c_i = 1$ for $i = 1, 4, 5$ and $c_i = -1$ for $i = 2, 3, 6$, and the folding $\mathscr{F}$ in an obvious way. For a function u vanishing on the horizontal sides of R, the folded function $\mathscr{F}u$ vanishes on ∂T_1 except for the vertical side.

We construct its Fourier expansion, this time in cosines in x on R. The Fourier coefficients are

$$\mathscr{P}f^s_{k,l} = \int_R \mathscr{P}f^s(x,y)\cos\frac{k\pi x}{\sqrt{3}}\sin l\pi y\,\mathrm{d}x\,\mathrm{d}y = \int_{T_1} f^s(x,y)v_{k,l}(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$
$$k = 0, 1, \ldots,\qquad l = 1, 2, \ldots,$$

where

$$
v_{k,l}(x,y) = \cos\frac{k\pi x}{\sqrt{3}}\sin l\pi y
- \cos\frac{k\pi}{2\sqrt{3}}\big(-x-\sqrt{3}y+\sqrt{3}\big)\sin\frac{l\pi}{2}\big(-\sqrt{3}x+y+1\big)
- \cos\frac{k\pi}{2\sqrt{3}}\big(x-\sqrt{3}y+\sqrt{3}\big)\sin\frac{l\pi}{2}\big(\sqrt{3}x+y+1\big)
+ \cos\frac{k\pi}{2\sqrt{3}}\big(-x+\sqrt{3}y+\sqrt{3}\big)\sin\frac{l\pi}{2}\big(-\sqrt{3}x-y+1\big)
+ \cos\frac{k\pi}{2\sqrt{3}}\big(x+\sqrt{3}y+\sqrt{3}\big)\sin\frac{l\pi}{2}\big(\sqrt{3}x-y+1\big)
- \cos\frac{k\pi\big(\sqrt{3}-x\big)}{\sqrt{3}}\sin l\pi(1-y). \tag{10}
$$

Similarly as in the skew symmetric case, the function $v_{k,l}$ is nonzero only for both the subscripts k and l of the same parity and $k \neq l$. The cases $k=0$ and $k=3l$ are now nonzero. The sets of subscripts are defined as follows: $0 \leqslant k < l$, $l < k \leqslant 3l$, $3l < k$. Proceeding in the same way as before, we find the expression

$$
v_{k,l}(x,y) = 2\cos\frac{k\pi x}{\sqrt{3}}\sin l\pi y
+ 2(-1)^{(k+l)/2}\cos\frac{\pi x}{2\sqrt{3}}(k+3l)\sin\frac{\pi y}{2}(k-l)
- 2(-1)^{(k-l)/2}\cos\frac{\pi x}{2\sqrt{3}}(3l-k)\sin\frac{\pi y}{2}(k+l). \tag{11}
$$

We find out as before that for the subscript pairs (k',l') and (k'',l'') from the second and third subset of subscripts, there exists, according to (7) and (8), subscript pair (k,l) in the first subset of subscripts such that

$$
v_{k',l'}(x,y) = -(-1)^{(k-l)/2}v_{k,l} \quad\text{and}\quad v_{k'',l''}(x,y) = -(-1)^{(k+l)/2}v_{k,l},
$$

respectively.

We see immediately that the functions $u_{k,l}$ and $v_{k,l}$ are eigenfunctions of the Laplace operator, the corresponding eigenvalue being $\pi^2\left(\frac{k^2}{3}+l^2\right)$. It is clear that these functions vanish for $y=0$. Since $u_{k,l}$ and $v_{k,l}$ are results of the folding $\mathscr{F}$ of the functions $\sin\frac{k\pi x}{\sqrt{3}}\sin l\pi y$ and $\cos\frac{k\pi x}{\sqrt{3}}\sin l\pi y$ they vanish on the side $y=-\sqrt{3}x+1$ of the triangle T and thus on the side $y=\sqrt{3}x+1$, too.

Theorem. *The functions*

$$
u_{k,l}, \quad k,l = 1,2,\dots, \quad k \equiv l \mod 2, \quad 0 < k < l,
$$

and

$$v_{k,l}, \quad k = 0, 1, 2, \ldots, l = 1, 2, \ldots, \quad k \equiv l \mod 2, \quad 0 \leqslant k < l,$$

form a complete orthogonal system on T.

Proof. Indeed, let f be a function defined on T and let all its coefficients in the expansion with respect to the functions $u_{k,l}$ and $v_{k,l}$ be zero. Then all Fourier coefficients of the prolonged functions $\mathscr{P}f^a$ and $\mathscr{P}f^s$ vanish on the rectangle R. It is, in fact, the assumption for the subscripts in the first subset of subscripts. The functions with subscripts in the other two subsets are equal, apart from the sign, to the corresponding functions of the first subset and the cases $k \equiv l \mod 2, k = l, k = 3l$ are trivial. Both prolonged functions are thus zero and the function f is zero.

It is obvious that functions corresponding to different eigenvalues are orthogonal. But it is possible that the same eigenvalue belongs to two different pairs (k_1, l_1) and (k_2, l_2), e.g. for $k_1 = 8$, $l_1 = 10$ and $k_2 = 1$, $l_2 = 11$.

In order to prove the orthogonality also for such a case, we prove first that the functions $u_{k,l}$ and $v_{k,l}$ are their own prolongations. Since the formulae (7) and (11) make sense for arbitrary x and y it means that $\mathscr{P}u_{k,l} = u_{k,l}$ and $\mathscr{P}v_{k,l} = v_{k,l}$. It is obvious that $u_{k,l}$ is skew symmetric with respect to both axes and that $v_{k,l}$ is symmetric with respect to the axis x and skew symmetric with respect to the axis y. In an elementary, even if a little cumbersome way it can be proved that both the functions are skew symmetric with respect to the straight line $-\sqrt{3}x - y + 1 = 0$. By this skew symmetry the function is transformed from the rhombus $\bigl(1/\sqrt{3}, 0\bigr)$, $(0, 1)$, $\bigl(-1/\sqrt{3}, 0\bigr)$, $(0, -1)$ onto the union of triangles T_2, T_3, T_4, T_5. Both functions are skew symmetric also with respect to the line $-\sqrt{3}x - y + 3 = 0$ and from this we obtain the skew symmetry of the triangle T_6 to the triangle T_5. Therefore, we have

$$\int_{T_1} u_{k,l}(x,y) u_{m,n}(x,y) \,\mathrm{d}x\,\mathrm{d}y = \int_R u_{k,l}(x,y) \sin\frac{m\pi x}{\sqrt{3}} \sin n\pi y \,\mathrm{d}x\,\mathrm{d}y. \tag{12}$$

This integral is, however, equal to zero for $(k,l) \neq (m,n)$. This is easily seen from (7) and the fact that (m,n) is from the first subset of subscripts. The pairs $(k+3l)/2, (l-k)/2$ and $(3l-k)/2, (k+l)/2$ belong to the other subsets of subscripts. □

From (12) it is easily seen that the norms of $u_{k,l}$ and $v_{k,l}$ for $k = 1, 2, \ldots; l = 1, 2, \ldots$ are equal to $3^{\frac{1}{4}}$ and the norms of $v_{0,l}$ are equal to $\sqrt{2\sqrt{3}}$.

We make yet a useful observation. It is obvious that functions $u^+_{k,l}$ and $u^-_{k,l}$ obtained from $u_{k,l}$ by rotation through $\frac{2\pi}{3}$ or $-\frac{2\pi}{3}$ about the center of gravity of the triangle T are eigenfunctions as well.

We find easily that the function $u_{k,l}$ on the union of the triangles T_4 and T_5 is the function $u_{k,l}$ on T shifted and rotated through $-\frac{2\pi}{3}$. We have, therefore,

$$u^-_{k,l}(x,y) = u_{k,l}\left(x + \frac{2}{\sqrt{3}}, y\right)$$

and similarly

$$u^+_{k,l}(x,y) = u_{k,l}\left(x - \frac{2}{\sqrt{3}}, y\right).$$

Each of the functions $u^+_{k,l}(x,y)$ and $u^-_{k,l}(x,y)$ is, however, linearly dependent on $u_{k,l}(x,y)$ and $v_{k,l}(x,y)$. This is immediately seen for $u^+_{k,l}(x,y)$. We have

$$\begin{aligned} u^+_{k,l}(x,y) = u_{k,l}\left(x - \frac{2}{\sqrt{3}}, y\right) &= 2\sin k\pi\left(\frac{x}{\sqrt{3}} - \frac{2}{3}\right)\sin l\pi y \\ &\quad - 2(-1)^{(k+l)/2}\sin\pi\left(\frac{x}{\sqrt{3}} - \frac{2}{3}\right)\frac{k+3l}{2}\sin\pi y\frac{k-l}{2} \\ &\quad + 2(-1)^{(k-l)/2}\sin\pi\left(\frac{x}{\sqrt{3}} - \frac{2}{3}\right)\frac{k-3l}{2}\sin\pi y\frac{k+l}{2}. \end{aligned}$$

If k is a multiple of 3, then $u^+_{k,l}(x,y) = u^-_{k,l}(x,y) = u_{k,l}(x,y)$. If this is not the case we obtain, with the aid of elementary trigonometry,

$$u^+_{k,l}(x,y) = -\tfrac{1}{2}u_{k,l} + (-1)^z\tfrac{\sqrt{3}}{2}v_{k,l},$$

where $k \equiv z \mod 3$, $z = 1$ or 2, and we then have $u_{k,l} + u^+_{k,l} + u^-_{k,l} = 0$. Similar formulae are valid for the rotated functions $v_{k,l}$.

4. Neumann boundary condition

The above approach can be used also for the Laplace operator with the Neumann boundary condition on all three sides of the triangle T. Boundary conditions of different types on different sides of the triangle are not suitable for our approach.

We now show the formulae for eigenfunctions for the case of Neumann conditions. The prolongation of the skew symmetric part of the function is now defined by (2) with $c_i = 1$ for $i = 1,2,4$ and $c_i = -1$ for $i = 3,5,6$, and the prolongation of the symmetric part by (2) with all $c_i = 1$.

We proceed as above and with the use of an identity derived from (6) by differentiation with respect to d we finally have

$$
\begin{aligned}
u_{k,l}(x,y) = 2\sin\frac{k\pi x}{\sqrt{3}}\cos l\pi y \\
&- 2(-1)^{(k+l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k+3l)\cos\frac{\pi y}{2}(k-l) \\
&- 2(-1)^{(k-l)/2}\sin\frac{\pi x}{2\sqrt{3}}(k-3l)\cos\frac{\pi y}{2}(k+l), \\
&k = 1,2,\dots,\ l = 0,1,2,\dots, \quad k \equiv l \mod 2, \quad 0 < k \leqslant l.
\end{aligned} \tag{13}
$$

The identity (6) differentiated with respect to a and d yields

$$
\begin{aligned}
v_{k,l}(x,y) = 2\cos\frac{k\pi x}{\sqrt{3}}\cos l\pi y \\
&+ 2(-1)^{(k+l)/2}\cos\frac{\pi x}{2\sqrt{3}}(k+3l)\cos\frac{\pi y}{2}(k-l) \\
&+ 2(-1)^{(k-l)/2}\cos\frac{\pi x}{2\sqrt{3}}(k-3l)\cos\frac{\pi y}{2}(k+l), \\
&k,l = 0,1,2,\dots, \quad k \equiv l \bmod 2, \quad 0 \leqslant k \leqslant l.
\end{aligned} \tag{14}
$$

The system of functions (13) and (14) is a complete orthogonal system of eigenfunctions of the Laplace operator with Neumann boundary conditions. The eigenvalues are, as before, $\pi^2\left(\frac{k^2}{3} + l^2\right)$. Since we have now a singular problem, we obtain for $k = l = 0$ the zero eigenvalue. The proof is essentialy the same as for the case of Dirichlet boundary conditions. The solution of the corresponding boundary-value problem on T_1 can be transformed into a problem on R similarly, too.

Acknowledgement. This research was supported by Grant 201/97/0217 of the Grant Agency of the Czech Republic.

References

[1] *Křížek, M., Neittaanmäki, P.*: Finite Element Approximaton of Variational Problems and Applications. Longman Scientific & Technical, Harlow, 1990.

Author's address: *Milan Práger*, Matematický ústav AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic.