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LOCAL LIPSCHITZ CONTINUITY OF THE STOP OPERATOR

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Abstract. On a closed convex set $Z$ in $\mathbb{R}^N$ with sufficiently smooth ($W^{2,\infty}$) boundary, the stop operator is locally Lipschitz continuous from $W^{1,1}([0,T],\mathbb{R}^N) \times Z$ into $W^{1,1}([0,T],\mathbb{R}^N)$. The smoothness of the boundary is essential: A counterexample shows that $C^1$-smoothness is not sufficient.

Keywords: hysteresis, stop operator, differential inclusion, Lipschitz continuity

MSC 2000: 34A60, Secondary 49J40

1. Introduction and main result

Throughout the paper we will use the following notation: For $1 \leq p < \infty$, an interval $[0,T]$, and a set $Z \subset \mathbb{R}^N$, the space $W^{1,p}([0,T],Z)$ denotes the space of absolutely continuous functions $f: [0,T] \to Z$ whose derivative is in $L^p$. We use the norm

$$\|f\|_{W^{1,p}}^p = \int_0^T |f(t)|^p \, dt + \int_0^T |f'(t)|^p \, dt.$$ 

If $\Omega \subset \mathbb{R}^M$ is a domain, $W^{k,\infty}(\Omega,Z)$ is the space of functions $f: \Omega \to Z$ whose partial derivatives up to order $k-1$ are Lipschitz continuous. By $B(x,r)$ we mean the closed ball with center $x$ and radius $r$.

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Let $Z \subset \mathbb{R}^N$ be a closed convex set. Given $x_0 \in Z$ and a function $u \in W^{1,1}([0,T], \mathbb{R}^N)$, we seek a function $x \in W^{1,1}([0,T], \mathbb{R}^N)$ such that

- $x(0) = x_0$.
- $x(t) \in Z$ for all $t \in [0,T]$.
- For almost all $t$, $x'(t)$ is as close as possible to $u'(t)$.

Then $x$ is characterized by the variational inequality

$$x(0) = x_0,$$

$$x(t) \in Z,$$

$$\forall y \in Z \langle u'(t) - x'(t), y - x(t) \rangle \leq 0.$$  

We denote by $\partial Z$ the boundary and by $Z^\circ$ the interior of $Z$. By $N_Z(x)$ we denote the normal cone of $Z$ at the point $x$. We can rewrite the variational inequality as a differential inclusion

$$x(0) = x_0,$$

$$x(t) \in Z,$$

$$u'(t) - x'(t) \in N_Z(x(t)).$$

If $Z$ is the closure of an open domain $Z^\circ$ with $C^1$-boundary, so that for each point $x \in \partial Z$ the outward unit normal vector $n(x)$ is defined and depends continuously on $x$, then the differential inclusion is in fact a differential equation

$$x'(t) = \begin{cases} 
  u'(t) & \text{if } x(t) \in Z^\circ, \\
  u'(t) & \text{if } x(t) \in \partial Z \text{ and } \langle n(x(t)), u'(t) \rangle < 0, \\
  u'(t) - \langle n(x(t)), u'(t) \rangle n(x(t)) & \text{if } x(t) \in \partial Z \text{ and } \langle n(x(t)), u'(t) \rangle \geq 0.
\end{cases}$$

Given any closed convex set $Z$, it is shown in [6], that for any $x_0 \in Z$ and any $u \in W^{1,1}([0,T], \mathbb{R}^N)$ there exists a unique function $x \in W^{1,1}([0,T], Z)$ solving (1.1). (See also [7, Proposition 2.2], [8].) The operator

$$S: \begin{cases} 
  W^{1,1}([0,T], \mathbb{R}^N) \times Z \rightarrow W^{1,1}([0,T], Z), \\
  (u, x_0) \mapsto x
\end{cases}$$

is called the stop operator with characteristic $Z$. This operator plays a fundamental role in the theory of elastoplastic materials (see, e.g., the monographs [3], [6], [8], [11]).
According to [7, Proposition 3.1 and Corollary 3.4], the stop operator maps $W^{1,p}([0,T], \mathbb{R}^N) \times Z$ continuously into $W^{1,p}([0,T], \mathbb{R}^N)$ for $1 \leq p < \infty$. Moreover, global Lipschitz continuity has been proved on $W^{1,1} \times Z$ into $W^{1,1}$, if $Z \subset \mathbb{R}$ is an interval [10], and, more generally, if $Z \subset \mathbb{R}^N$ is a (bounded or unbounded) polyhedron [4]. If $p > 1$, the stop operator is not Lipschitz continuous from $W^{1,p} \times Z$ into $W^{1,p}$ [10]. The unit ball in $\mathbb{R}^2$ provides a counterexample to global Lipschitz continuity in $W^{1,1}$ for general convex sets, however, if $Z$ is a ball in $\mathbb{R}^N$, the stop operator satisfies a local Lipschitz condition

$$|x(t) - y(t)| + \int_0^T |x'(t) - y'(t)| \, dt \leq M(u) \left[ |x_0 - y_0| + \int_0^T |u'(t) - v'(t)| \, dt \right]$$

if $x = S(u, x_0)$, $y = S(v, y_0)$, and $M(u)$ is a Lipschitz constant depending on $\int_0^T |u'(t)| \, dt$ [2, Corollary A.4 and Example A.6].

It is announced without proof in [6, Chapter 4, Theorem 20.1] that a similar local Lipschitz condition holds on domains with smooth boundaries. In this paper we give a proof for the local Lipschitz continuity of the stop operator if the domain $Z$ is smooth enough so that there exists a unique outward unit normal vector $n(x)$ to $\partial Z$ at every boundary point $x \in \partial Z$ and $n(x)$ depends Lipschitz continuously on $x$.

**Hypothesis 1.1.** Let $Z \subset \mathbb{R}^N$ be a closed convex set with $W^{2,\infty}$-boundary, i.e., for all $z \in \partial Z$ there exists an orthonormal system $(v_1, \ldots, v_N)$, some $\varepsilon > 0$ and a map $a \in W^{2,\infty}([-\varepsilon, \varepsilon]^{N-1}, \mathbb{R})$ such that $a(0, \ldots, 0) = 0$ and for all $\xi \in [-\varepsilon, \varepsilon]$ the following holds:

$$z + \sum_{j=1}^{N-1} \xi_j v_j + (a(\xi_1, \ldots, \xi_{N-1}) + \xi_N) v_N \in Z \quad \text{iff} \quad \xi_N \geq 0.$$

By $n(z)$ we will denote the outward unit normal vector at $z$:

$$n(z) = \frac{1}{\sqrt{1 + \sum_{j=1}^{N-1} \frac{\partial a}{\partial \xi_j}(0)^2 \left( \sum_{j=1}^{N-1} \frac{\partial a}{\partial \xi_j}(0) v_j - v_N \right)}}.$$

With this assumption we prove the following theorem:
Theorem 1.1. Let \( Z \subset \mathbb{R}^N \) satisfy Hypothesis 1.1, and let \( K \) be a compact subset of \( Z \). Let \( R > 0 \) be fixed. Then there exists a constant \( L > 0 \) (depending on \( K \) and \( R \)) such that the following local Lipschitz estimate holds:

If \( x_0, y_0 \in K \) and \( u, v \in W^{1,1}([0, T], \mathbb{R}^N) \) for some \( T > 0 \) with

\[
\int_0^T (|u'(t)| + |v'(t)|) \, dt \leq R,
\]

then \( x = S(u, x_0) \) and \( y = S(v, y_0) \) satisfy

\[
\int_0^T |x'(t) - y'(t)| \, dt \leq L \left[ |x_0 - y_0| + \int_0^T |u'(t) - v'(t)| \, dt \right].
\]

We will give the proof in Section 2. The smoothness assumption on \( \partial Z \) is essential: In Section 3 we present a cone in \( \mathbb{R}^3 \) as a counterexample to local Lipschitz continuity of the stop operator in general convex sets. Moreover, in Example 3.2 we show that Hölder continuous dependence of the normal vector \( n(x) \) on \( x \) is not sufficient to imply that the stop operator is locally Lipschitz.

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2. Proof of the main result

For the proof of the main theorem, we will require some simple facts from differential geometry. Let \( V \) be a relatively compact subset of \( \partial Z \). The tubular neighborhood of radius \( \delta > 0 \) around \( V \) is defined by

\[
\text{Tub}^\delta V = \{ x + \lambda n(x) \mid \lambda \in (-\delta, \delta) \}.
\]

If \( \partial Z \) is a \( C^2 \)-manifold, the implicit function theorem can be used to show that for sufficiently small \( \delta > 0 \), the map

\[
\text{can} \begin{cases}
V \times (-\delta, \delta) & \to \text{Tub}^\delta V \\
(x, \lambda) & \mapsto x + \lambda n(x)
\end{cases}
\]
is a $C^1$-diffeomorphism. (This is, e.g., a special case of the situation treated in [1, Section 2.7].) Since we have required less smoothness than $C^2$, the map can will in general not be contained in $C^1$, and the standard versions of the implicit function theorem do not work. We will therefore relax the smoothness assumption a little and give a different proof:

**Lemma 2.1.** Let $Z$ be as in Hypothesis 1.1, and $z \in \partial Z$. For $x \in \mathbb{R}^N$ we define

$$d(x) = \begin{cases} \text{dist}(x, \partial Z) & \text{if } x \in Z, \\ -\text{dist}(x, \partial Z) & \text{if } x \notin Z. \end{cases}$$

For $\delta > 0$ let $U_\delta$ be the tubular neighborhood of radius $\delta$ around $\partial Z \cap B(z, \delta)$. Then $\delta > 0$ may be chosen sufficiently small, such that the following assertions hold:

(i) $\forall \delta > 0$ let $U_\delta$ be the tubular neighborhood of radius $\delta$ around $\partial Z \cap B(z, \delta)$. Then $\delta > 0$ may be chosen sufficiently small, such that the following assertions hold:

(ii) $d$ is differentiable on $U_\delta$, and its gradient $\nabla d(x)$ depends Lipschitz continuously on $x$. Namely, if $x = \text{can}(y, \lambda)$, then $\nabla d(x) = -n(y)$.

**Proof.** Let $z \in \partial Z$. We utilize the chart generated by $v_1, \ldots, v_N$, $\varepsilon > 0$, and the function $a$ as in Hypothesis 1.1. Without loss of generality (by rotation of the coordinate system, if necessary) we may assume that $n(z) = -v_N$. We define

$$T: \left\{ \begin{array}{l} (-\varepsilon, \varepsilon)^{N-1} \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^N, \\ (\xi, \lambda) \mapsto z + \sum_{j=1}^{N-1} \xi_j v_j + (a(\xi) + \lambda)v_N, \end{array} \right.$$\n
and write the map can and the normal vector $n$ in local coordinates:

$$\tilde{n}: \left\{ \begin{array}{l} (-\varepsilon, \varepsilon)^{N-1} \rightarrow \mathbb{R}^N, \\ \xi \mapsto n\left(z + \sum_{j=1}^{N-1} \xi_j v_j + a(\xi_1, \ldots, \xi_{N-1})v_N\right), \end{array} \right.$$\n
$$\tilde{\text{can}}: \left\{ \begin{array}{l} (-\varepsilon, \varepsilon)^{N-1} \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^N, \\ (\xi, \lambda) \mapsto z + \sum_{j=1}^{N-1} \xi_j v_j + a(\xi)v_N + \lambda \tilde{n}(\xi). \end{array} \right.$$\n
We have to prove that $\tilde{\text{can}}$ has a Lipschitz continuous inverse on a suitable sufficiently small neighborhood of $z$. It is easy to prove that $T^{-1}$ exists and is Lipschitz continuous on a suitable neighborhood of $z$. Let $M$ be a Lipschitz constant for $T^{-1}$. Notice that

$$(T - \tilde{\text{can}})(\xi, \lambda) = \lambda(v_N - \tilde{n}(\xi)).$$
Therefore, if $\eta \in (0, \varepsilon)$ is sufficiently small, $(T - \tilde{\text{can}})$ is Lipschitz on $(-\eta, \eta)^{N-1} \times (-\eta, \eta)$ with a Lipschitz constant $L < 1/(2M)$. From the contraction principle [5, 10.1.3] we infer that for $y$ sufficiently close to $z$, there exists a unique solution to

$$(\xi, \lambda) = T^{-1}[y + T(\xi, \lambda) - \tilde{\text{can}}(\xi, \lambda)],$$

which is equivalent to

$$y = \tilde{\text{can}}(\xi, \lambda).$$

The proof of the contraction principle shows that this solution depends Lipschitz continuously on $y$. Therefore can possesses a Lipschitz continuous inverse on a sufficiently small neighborhood of $W$ of $z$.

Now choose a neighborhood $V$ of $z$ and $\delta > 0$ sufficiently small, such that $U = \text{Tub}^\delta V \subset W$ and for any $x \in U$ the closest point $\Pi(x)$ to $x$ on $\partial Z$ is contained in $W$. For $x \in U$, elementary geometry shows that

$$\text{can}^{-1}(x) = (\Pi(x), -d(x)).$$

The proof above implies therefore that $d$ is Lipschitz continuous. However, we can improve the result and obtain continuous differentiability of $d$. Let $x \in U$ and $\Delta x$ be sufficiently small. We define

$$\Delta \Pi = \Pi(x + \Delta x) - \Pi(x),$$

$$\Delta d = d(x + \Delta x) - d(x),$$

$$\Delta n = n(\Pi(x + \Delta x)) - n(\Pi(x)).$$

Notice that by the Lipschitz continuity of $n$ and $\text{can}^{-1}$, all of the following terms, $\Delta \Pi$, $\Delta d$, and $\Delta n$ are of order $O(\Delta x)$. Thus

$$\Delta x = \left[\Pi(x + \Delta x) - d(x + \Delta x)n(\Pi(x + \Delta x))\right] - \left[\Pi(x) - d(x)n(\Pi(x))\right]$$

$$= \Pi(x) + \Delta \Pi - (d(x) + \Delta d)[n(\Pi(x)) + \Delta n] - \Pi(x) + d(x)n(\Pi(x))$$

$$= \Delta \Pi - d(x)\Delta n - (\Delta d)n(\Pi(x)) + o(\Delta x).$$

Since $n$ is normalized, we infer that $\langle n(\Pi(x)), \Delta n \rangle = o(\Delta x)$, and since $n$ is orthogonal to $\partial Z$, we infer that $\langle n(\Pi(x)), \Delta \Pi \rangle = o(\Delta x)$. We obtain therefore

$$\langle n(\Pi(x)), \Delta x \rangle = -\Delta d + o(\Delta x).$$

This says that $\nabla d(x) = -n(\Pi(x)).$
The following lemma is the core of the proof of Theorem 1.1.

**Lemma 2.2.** Let $Z$ be as in Hypothesis 1.1, and $z \in Z$. Then there exists a neighborhood $V$ of $z$, a constant $R > 0$ and a constant $M > 0$ such that the stop operator satisfies the following local Lipschitz condition:

If $T > 0$, $x_0, y_0 \in V$, $u, v \in W^{1,1}([0, T], \mathbb{R}^N)$ with

$$
\int_0^T (|u'(t)| + |v'(t)|) \, dt \leq R,
$$

and $x = S(u, x_0)$, $y = S(v, y_0)$, then

$$
\int_0^T |x'(t) - y'(t)| \, dt \leq M \left[ |x_0 - y_0| + \int_0^T |u'(t) - v'(t)| \, dt \right].
$$

**Proof.** If $z \in Z^\circ$, then choose a neighborhood $V$ and a constant $R > 0$ such that $V + B(0, R)$ is entirely contained in $Z^\circ$. Since $|x'(t)| \leq |u'(t)|$, $|y'(t)| \leq |v'(t)|$ (e.g., [4, Proposition 1.2]), we infer that $x(t)$ and $y(t)$ remain in $Z^\circ$ for $t \leq T$, so that $x' = u'$ and $y' = v'$. In this case, the assertion is trivial.

Assume now that $z \in \partial Z$. According to Lemma 2.1 we choose a neighborhood $U = U_\delta$ of $Z$ such that $d$ is differentiable with Lipschitz continuous derivative on $U$. For shorthand we denote $n(x_0) = -\nabla d(x_0)$. This notation is consistent with the fact that $n(x_0)$ is the outward unit normal vector to $Z$ at $x_0$, if $x_0 \in \partial Z$. Let $L$ be a Lipschitz constant for $n$ on $U$. Notice also that $|n(x_0)| \leq 1$ for any $x_0 \in U$, since $n$ is the negative gradient of a distance. Again we choose a constant $R > 0$ and a neighborhood $V$ of $z$ such that $V + B(0, R) \subset U$, therefore $x(t)$ and $y(t)$ remain in $U$ for $t \leq T$.

We keep track of the functions $|x'(t) - y'(t)|$, $|x(t) - y(t)|$ and $\beta(t) = |d(x(t)) - d(y(t))|$. Let $t$ be a Lebesgue point of all of the following functions, $x'$, $y'$, $[d(x')]'$, $[d(y')]'$, and $|x(t) - y(t)|'$, and such that (1.2) holds. From [7, (2.6)] we infer easily that

$$
\frac{d}{dt} |x(t) - y(t)| \leq |u'(t) - v'(t)|.
$$

Thus

$$
|x(t) - y(t)| \leq |x_0 - y_0| + \int_0^t |u'(s) - v'(s)| \, ds. \tag{2.1}
$$

To handle the other two functions, we will prove the inequality

$$
|x'(t) - y'(t)| + \beta'(t) \leq 2|u'(t) - v'(t)| + 2L(|u'(t)| + |v'(t)|) |x(t) - y(t)|. \tag{2.2}
$$
Once this equation is proved, we may integrate and obtain
\[
\int_0^T |x'(t) - y'(t)| \, dt \\
\leq \beta(0) - \beta(T) + 2 \int_0^T |u'(t) - v'(t)| \, dt \\
+ 2L \int_0^T (|u'(t)| + |v'(t)|) |x(t) - y(t)| \, dt \\
\leq |x_0 - y_0| - 0 + 2 \int_0^T |u'(t) - v'(t)| \, dt \\
+ 2L \left( |x_0 - y_0| + \int_0^T |u'(s) - v'(s)| \, ds \right) \int_0^T |u'(t) + v'(t)| \, dt \\
\leq (2LR + 1)|x_0 - y_0| + (2LR + 2) \int_0^T |u'(t) - v'(t)| \, dt.
\]

Therefore, Lemma 2.2 is proved, if we can show (2.2). For this purpose we distinguish the following cases:

**Case 1:** \(x(t) \in Z^o\), \(y(t) \in Z^o\):

In this case, \(x' = u'\) and \(y' = v'\). For shorthand we will omit the argument \(t\) in the following computations. Thus

\[
\frac{d}{dt} \beta \leq \left| \frac{d}{dt} d(x) - \frac{d}{dt} d(y) \right| = | - \langle n(x), u' \rangle + \langle n(y), v' \rangle | \\
\leq |\langle n(x), u' - v' \rangle| + |\langle n(x) - n(y), v' \rangle| \leq |u' - v'| + L|x - y||v'|.
\]

Equation (2.2) follows easily.

**Case 2:** \(x(t) \in \partial Z\) and \(y(t) \in \partial Z\).

Since \(x\) is differentiable at the point \(t\) and \(x(t) \in \partial Z\) while \(x(s) \in Z\) for all \(s\), the derivative \(x'(t)\) is necessarily in the tangent space of \(Z\) at \(x(t)\). This is only possible if \(u'(t)\) does not point strictly inward, i.e. \(\langle n(x), u' \rangle \geq 0\). The same argument holds for \(y'\). We have therefore

\[
x' = u' - \langle n(x), u' \rangle n(x), \quad y' = v' - \langle n(y), v' \rangle n(y).
\]

We infer that

\[
|x' - y'| = |u' - \langle n(x), u' \rangle n(x) - v' + \langle n(y), v' \rangle n(y)| \\
\leq |u' - v' - \langle n(x), u' - v' \rangle n(x)| + |\langle n(x) - n(y), v' \rangle n(x)| \\
+ |\langle n(y), v' \rangle (n(x) - n(y))| \\
\leq |u' - v'| + 2L|x - y||v'|.
\]
Since $x'$ and $y'$ are tangential to $\partial Z$, we infer that
\[
\frac{d}{dt} \beta \leq \left| \frac{d}{dt} d(x(t)) \right| + \left| \frac{d}{dt} d(y(t)) \right| = 0.
\]

Summing up these estimates, we infer again (2.2).

**Case 3:** $x(t) \in \partial Z$ and $y(t) \in Z^\circ$, or vice versa.

Again $\langle n(x), u' \rangle \geq 0$ and $x'$ is tangential to $\partial Z$. Then
\[
|x' - y'| = |u' - \langle n(x), u' \rangle n(x) - v'| \leq |u' - v'| + \langle n(x), u' \rangle.
\]

Notice that in this case $d(x) = 0$, $d(y) > 0$, and again $\frac{d}{dt} d(x) = 0$. Therefore
\[
\frac{d}{dt} \beta = \frac{d}{dt} (d(y) - d(x)) = \langle -n(y), v' \rangle - 0 \\
\leq |\langle n(y) - n(x), v' \rangle| + |\langle n(x), v' - u' \rangle| - \langle n(x), u' \rangle \\
\leq L |x - y| |v'| + |u' - v'| - \langle n(x), u' \rangle.
\]

This implies again the estimate (2.2). □

**Proof of Theorem 1.1.** For each $z \in Z$, choose a neighborhood $V(z)$ and constants $M(z), R(z)$ according to Lemma 2.2. Let $W(z)$ be a neighborhood of $z$ and let $\delta(z)$ be sufficiently small, such that $W(z) + B(0, \delta(z)) \subset V(z)$. We cover $K + B(0, R)$ by a finite union of neighborhoods $W(z_i) \ (i = 1, \ldots, m)$. Put $M = \max\{M(z_i) \mid i = 1, \ldots, m\}$, $S = \min\{R, R(z_1), \ldots, R(z_m)\}$ and $\delta = \min\{\delta(z_i) \mid i = 1, \ldots, m\}$. We start proving Equation (1.4) with $R$ replaced by $S$ in (1.3), and with the assumption that
\[
(2.3) \quad x_0, y_0 \in K + B(0, R) \text{ with } |x_0 - y_0| < \delta.
\]

Choose $i$ such that $x_0 \in W(z_i) \subset V(z_i)$. Assumption (2.3) implies $y_0 \in V(z_i)$. Therefore we may apply Lemma 2.2 on the set $V(z_i)$ and obtain exactly Equation (1.4) with $L = M$.

Next we remove the condition (2.3). Let $x_0, y_0 \in K + B(0, R)$ with $|x_0 - y_0| \leq k\delta$, and let $u, v \in W^{1,1}([0, T], \mathbb{R}^N)$ satisfy (1.3) with $S$ instead of $R$. For $j = 0, \ldots, k$ we define functions $z_j = S(u_j, x_j)$ with $u_j = u + \frac{j}{k}(v - u)$ and $x_j = x_0 + \frac{j}{k}(y_0 - x_0)$. Notice that $x = z_0$ and $y = z_k$, and the initial data satisfy $|x_j - x_{j-1}| \leq \delta$. Therefore
(1.4) holds for each of the differences $z_j - z_{j-1}$ and we obtain

$$\int_0^T |x'(t) - y'(t)| \, dt \leq \sum_{j=1}^k \int_0^T |z'_{j-1}(t) - z'_j(t)| \, dt$$

$$\leq M \sum_{j=1}^k \left[ |x_{j-1} - x_j| + \int_0^T |u'_{j-1}(t) - u'_j(t)| \, dt \right]$$

$$= M \left[ |x_0 - y_0| + \int_0^T |u'(t) - v'(t)| \, dt \right].$$

Finally we got rid of the assumption that $R$ is replaced by $S$ in (1.3). Assume that $R \leq kS$ with fixed $k$. Let $x_0, y_0 \in K$ and let $u, v \in W^{1,1}([0, T], \mathbb{R}^N)$ satisfy (1.3). Since $|x'(t)| \leq |u'(t)|$, we infer that $x(t) \in K + B(0, R)$ for all $t \in [0, T]$. The same holds for $y(t)$. Choose $0 = t_0 < t_1 < \ldots < t_k = T$ such that

$$\int_{t_k}^{t_{k+1}} (|u'(t)| + |v'(t)|) \, dt \leq S.$$

The estimate (1.4) holds on the intervals $[t_{j-1}, t_j]$. Utilizing Equation (2.1), we obtain

$$\int_{t_{j-1}}^{t_j} |x'(t) - y'(t)| \, dt$$

$$\leq M \left[ |x(t_{j-1}) - y(t_{j-1})| + \int_{t_{j-1}}^{t_j} |u'(t) - v'(t)| \, dt \right]$$

$$\leq M \left[ |x_0 - y_0| + \int_0^{t_{j-1}} |u'(t) - v'(t)| \, dt + \int_{t_{j-1}}^{t_j} |u'(t) - v'(t)| \, dt \right]$$

$$\leq M \left[ |x_0 - y_0| + \int_0^T |u'(t) - v'(t)| \, dt \right].$$

Summing up all intervals we obtain

$$\int_0^T |x'(t) - y'(t)| \, dt \leq kM \left[ |x_0 - y_0| + \int_0^T |u'(t) - v'(t)| \, dt \right].$$

Therefore (1.4) holds with $L = kM$. □
We show that the local Lipschitz condition proved in Theorem 1.1 for smooth domains is not valid in general convex sets. Our first counterexample is a cone of revolution in $\mathbb{R}^3$. For preparation we show that a local Lipschitz condition in a cone in fact implies a global condition.

**Lemma 3.1.** Let $Z \subset \mathbb{R}^N$ be a closed convex cone with vertex $0$. Suppose that there exist $R > 0$, $M > 0$ and $T > 0$ such that for all $u, v \in W_1^1([0, T], \mathbb{R}^N)$ with

$$\int_0^T (|u'(t)| + |v'(t)|) \, dt \leq R,$$

the solutions $x = S(u, 0)$ and $y = S(v, 0)$ satisfy the estimate

$$\int_0^T |x'(t) - y'(t)| \, dt \leq M \int_0^T |u'(t) - v'(t)| \, dt.$$

Then for all $x_0, y_0 \in Z$ and all $w \in W_{\text{loc}}^{1,1}([0, \infty), \mathbb{R}^N)$ the solutions $x = S(w, x_0)$, $y = S(w, y_0)$ satisfy

$$\int_0^\infty |x'(t) - y'(t)| \, dt \leq M|x_0 - y_0|.$$

**Proof.** Let $w \in W_1^{1,1}([0, \infty), \mathbb{R}^N)$, let $x_0, y_0 \in K$ and $x = S(w, x_0)$, $y = S(w, y_0)$. For $\eta > 0$ define $x_\eta$, $y_\eta$, $u_\eta$, $v_\eta$ by $u_\eta(0) = v_\eta(0) = 0$ and

$$x_\eta(t) = \begin{cases} \frac{t}{\eta} x_0 & \text{if } t \in [0, \eta], \\ \eta x \left( \frac{t}{\eta} - 1 \right) & \text{if } t \geq \eta, \end{cases} \quad y_\eta(t) = \begin{cases} \frac{t}{\eta} y_0 & \text{if } t \in [0, \eta], \\ \eta y \left( \frac{t}{\eta} - 1 \right) & \text{if } t \geq \eta, \end{cases}$$

$$u_\eta'(t) = \begin{cases} x_0 & \text{if } t \in [0, \eta], \\ w' \left( \frac{t}{\eta} - 1 \right) & \text{if } t \geq \eta, \end{cases} \quad v_\eta'(t) = \begin{cases} y_0 & \text{if } t \in [0, \eta], \\ w' \left( \frac{t}{\eta} - 1 \right) & \text{if } t \geq \eta. \end{cases}$$

For $t \leq \eta$ we have $x_\eta'(t) = x_0 = u_\eta(t)$. For $t \geq \eta$ we obtain

$$u_\eta'(t) - x_\eta'(t) = w' \left( \frac{t}{\eta} - 1 \right) - x' \left( \frac{t}{\eta} - 1 \right) \in N_Z \left( x \left( \frac{t}{\eta} - 1 \right) \right) = N_Z (x_\eta(t)).$$

Here we have used that $Z$ is a cone. Thus $x_\eta = S(u_\eta, 0)$. Similarly, $y_\eta = S(v_\eta, 0)$.
Now we fix some $S > 0$. Notice that for any $\eta > 0$,
\[
\int_0^{\eta (S+1)} (|u'_\eta(t)| + |v'_\eta(t)|) dt = \int_0^\eta (|x_0| + |y_0|) dt + 2 \int_\eta^{\eta (S+1)} |w'(\frac{t}{\eta} - 1)| dt = \eta (|x_0| + |y_0|) + 2\eta \int_0^S |w'(s)| ds.
\]
Therefore we can pick $\eta$ sufficiently small such that $\eta (S + 1) \leq T$ and
\[
\int_0^{\eta (S+1)} (|u'(t)| + |v'(t)|) dt < R.
\]
Then by assumption we have
\[
\int_0^T |x'(s) - y'(s)| ds = \frac{1}{\eta} \int_\eta^{\eta (S+1)} |x'_\eta(t) - y'_\eta(t)| dt 
\leq \frac{M}{\eta} \int_0^T |u'_\eta(t) - v'_\eta(t)| dt = \frac{M}{\eta} \int_\eta^0 |x_0 - y_0| dt = M|x_0 - y_0|.
\]
As $S \to \infty$, we obtain (3.1). \hfill \Box

Now we give our counterexample.

Example 3.1. Consider the cone
\[
Z = \left\{ \left( \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \end{array} \right) \in \mathbb{R}^3 \mid \xi_3 \geq \sqrt{\xi_1^2 + \xi_2^2} \right\}.
\]
Then for any $R > 0$, $M > 0$, and any $T > 0$, there are functions $u, v \in W^{1,1}([0, T], \mathbb{R}^3)$ and $x = S(u, 0)$, $y = S(v, 0)$, with
\[
\int_0^T (|u'(t)| + |v'(t)|) dt \leq R
\]
and
\[
\int_0^T |x'(t) - y'(t)| dt > M \int_0^T |u'(t) - v'(t)| dt.
\]
Proof. Assume the contrary. Then the assumptions for Lemma 3.1 are satisfied. We construct \( w, x \) and \( y \) in order to arrive at a contradiction to (3.1). We put

\[
    x(t) = \begin{pmatrix}
        (t + 1)^{-1} \cos(t) \\
        (t + 1)^{-1} \sin(t) \\
        (t + 1)^{-1}
    \end{pmatrix}, \quad y(t) = 0,
\]

\[
    w'(t) = \begin{pmatrix}
        (1 - (t + 1)^{-2}) \cos(t) - (t + 1)^{-1} \sin(t) \\
        (1 - (t + 1)^{-2}) \sin(t) + (t + 1)^{-1} \cos(t) \\
        -1 - (t + 1)^{-2}
    \end{pmatrix}, \quad w(0) = 0.
\]

Thus

\[
    x'(t) = \begin{pmatrix}
        -(t + 1)^{-2} \cos(t) - (t + 1)^{-1} \sin(t) \\
        -(t + 1)^{-2} \sin(t) + (t + 1)^{-1} \cos(t) \\
        -(t + 1)^{-2}
    \end{pmatrix}.
\]

The normal cone at zero is given by

\[
    N_Z(0) = \left\{ \begin{pmatrix}
        \xi_1 \\
        \xi_2 \\
        \xi_3
    \end{pmatrix} \in \mathbb{R}^3 \mid -\xi_3 \geq \sqrt{\xi_1^2 + \xi_2^2} \right\}.
\]

A straightforward computation shows that \( w'(t) \in N_Z(0) \) for all \( t \), thus \( S(w, 0) = 0 = y \). At the other points of \( \partial Z \), the normal cone is given by

\[
    N_Z \left( \begin{pmatrix}
        \gamma \cos(t) \\
        \gamma \sin(t) \\
        \gamma
    \end{pmatrix} \right) = \left\{ \lambda \begin{pmatrix}
        \cos t \\
        \sin t \\
        -1
    \end{pmatrix} \mid \lambda \geq 0 \right\}.
\]

Thus

\[
    w'(t) - x'(t) = \begin{pmatrix}
        \cos(t) \\
        \sin(t) \\
        -1
    \end{pmatrix} \in N_Z(x(t)).
\]

Thus \( x = S(w, x(0)) \). From (3.1) one infers

\[
    \int_0^\infty |x'(t)| \, dt = \int_0^\infty |x'(t) - y'(t)| \, dt \leq M|x(0)|.
\]

However,

\[
    |x'(t)| = \sqrt{2(t + 1)^{-4} + (t + 1)^{-2}} \geq (t + 1)^{-1},
\]

so that \( x' \) is not integrable on \([0, \infty)\). \( \square \)
Remark 3.1. Although Example 3.1 shows an unbounded convex set, a careful analysis of the proof shows that also a truncated cone provides a counterexample.

The following example shows that the stop operator is not necessarily locally Lipschitz continuous if the characteristic is a domain of type $C^1$, i.e., the normal vector $n(x)$ in each boundary point $x \in \partial Z$ is unique and depends continuously on $x$. In fact, the normal vector in the following counterexample depends Hölder continuously on $x$.

Example 3.2. Let

$$Z = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2 \mid \xi_2 \geq \beta(|\xi_1|) \right\}$$

with

$$\beta(\xi) = \int_0^x \gamma(\tau) d\tau, \quad \gamma(\tau) = \sqrt{\frac{\tau}{\tau + 2}}.$$

Then for all $R > 0$ and $M > 0$ there exist $x_0, y_0 \in Z$, $T > 0$, $u \in W^{1,1}([0,T], \mathbb{R}^2)$, $x = S(x_0, u)$, $y = S(y_0, u)$ with $|x_0| \leq R$, $|y_0| \leq R$,

$$\int_0^T |u'(t)| dt \leq R \quad \text{and} \quad \int_0^T |x'(t) - y'(t)| dt \geq M|x_0 - y_0|.$$

Proof. Notice that Hypothesis 1.1 holds everywhere except at the origin. To exploit the singularity at the origin we will construct a forcing function $u$ and solutions

$$x(t) = \begin{pmatrix} \xi(t) \\ \beta(\xi(t)) \end{pmatrix} \in \partial Z, \quad y(t) = \begin{pmatrix} \eta(t) \\ \beta(\eta(t)) \end{pmatrix} \in \partial Z,$$

such that $\xi \leq 0$ and $\eta \geq 0$ oscillate in a neighborhood of the origin. More precisely, we construct sequences $0 = t_0 < t_1 < t_2 \ldots$ and $q_0 > q_1 > q_2 > \ldots > 0$ with

$$\xi(t_i) = \begin{cases} -q_i & \text{for even } i, \\ 0 & \text{for odd } i, \end{cases} \quad \text{and} \quad \eta(t_i) = \begin{cases} 0 & \text{for even } i, \\ q_i & \text{for odd } i, \end{cases}$$

$$q_i \geq \frac{q_0}{1 + iq_0},$$

$$\int_{t_{i-1}}^{t_i} |u'(t)| dt \leq q_{i-1} \sqrt{2} \leq q_0 \sqrt{2},$$

$$\int_{t_{i-1}}^{t_i} |x'(t) - y'(t)| dt \geq \frac{\sqrt{2}}{3 \sqrt{3}} q_{i-1}^{3/2}.$$

We will show later that this construction ensures that the solutions satisfy (3.2).
With
\[
K = \frac{2\sqrt{2}}{3\sqrt{3}} \left( 1 - \frac{1}{\sqrt{1 + R/\sqrt{8}}} \right)
\]
we choose \(q_0 > 0\) sufficiently small such that
\[
q_0 < \min\left(\{1, \frac{K^2}{4M^2}, \frac{R}{\sqrt{8}}\}\right) \quad \text{and} \quad \sqrt{q_0^2 + \beta(q_0)^2} < 2q_0.
\]

We put \(t_0 = 0, x_0 = (-q_0, \beta(q_0))^T, y_0 = (0, 0)^T\) and proceed by induction. Suppose sequences \(t_i\) and \(q_i\) and a forcing function \(u \in W^{1,1}([0, t_n], \mathbb{R}^N)\) have been established such that the conditions (3.3), (3.4), (3.5) and (3.6) are satisfied up to \(t_n\). Without loss of generality we assume that \(n\) is even. The other case is treated similarly with the roles of \(x\) and \(y\) interchanged. We put \(t_{n+1} = t_n + q_n\) and continue the forcing function \(u\) on the interval \([t_n, t_{n+1}]\) by
\[
u'(t) = \begin{pmatrix}
1 \\
-\gamma(q_n - t_n + t)
\end{pmatrix}.
\]  

Put \(\xi(t) = -q_n + t - t_n\). Obviously \(x = (\xi(t), \beta(|\xi(t)|))^T\) satisfies \(x' = u'\), so that \(x = S(x_0, u)\). In particular \(\xi(t_{n+1}) = 0\). We obtain \(y(t)\) by
\[
y'(t) = \alpha(t) \begin{pmatrix}
1 \\
\gamma(\eta(t))
\end{pmatrix}
\]
with
\[
\alpha(t) = \frac{1 - \gamma(|\xi(t)|)\gamma(\eta(t))}{1 + \gamma^2(\eta(t))}.
\]

Consider the outward unit normal vector \(n(y(t))\) to \(\partial Z\) given by
\[
n(y(t)) = \frac{1}{\sqrt{1 + \gamma^2(\eta(t))}} \begin{pmatrix}
\gamma(\eta(t)) \\
-1
\end{pmatrix}
\]
and let
\[
\lambda(t) = \frac{\gamma(|\xi(t)|) + \gamma(\eta(t))}{\sqrt{1 + \gamma^2(\eta(t))}} \geq 0.
\]
A straightforward computation shows that \(y'(t) + \lambda(t)n(y(t)) = u'(t)\) so that \(y = S(y_0, u)\).

Since \(0 \leq \alpha(t) \leq 1\) we infer that \(\eta(t) \leq q_n\) for \(t \in [t_n, t_{n+1}]\). A more careful estimate shows now that
\[
\alpha(t) \geq \frac{1 - \gamma^2(q_n)}{1 + \gamma^2(q_n)} = \frac{1 - \frac{q_n}{q_n + 2}}{1 + \frac{q_n}{q_n + 2}} = \frac{1}{q_n + 1}.
\]
We put $q_{n+1} = \eta(t_n)$ and obtain

$$q_{n+1} \geq (t_{n+1} - t_n) \min_{t \in [t_n, t_{n+1}]} (\alpha(t)) \geq \frac{q_n}{q_{n+1}} \geq \frac{q_0}{1 + (n + 1)q_0}.$$  

Using the inequalities $q_0 \leq 1$ and $\gamma(\tau) \leq 1$ we obtain

$$\int_{t_n}^{t_{n+1}} |u'(t)| \, dt = \int_{t_n}^{t_{n+1}} \sqrt{1 + \gamma^2(|\xi(t)|)} \, dt \leq q_n \sqrt{2} \leq q_0 \sqrt{2},$$

and

$$\int_{t_n}^{t_{n+1}} |x'(t) - y'(t)| \, dt = \int_{t_n}^{t_{n+1}} \lambda(t) \, dt \geq \int_{t_n}^{t_{n+1}} \frac{\gamma(|\xi(t)|)}{\sqrt{2}} \, dt$$

$$= \frac{1}{\sqrt{2}} \int_{t_n}^{t_{n+1}} \sqrt{\frac{t_{n+1} - t}{t_{n+1} - t + 2}} = \frac{1}{\sqrt{2}} \int_{q_0}^{q_n} \sqrt{\frac{s}{s + 2}} \, ds$$

$$\geq \frac{1}{\sqrt{6}} \int_0^{q_n} \sqrt{s} \, ds = \frac{\sqrt{2}}{3\sqrt{3}} q_n^{3/2}.$$

At this point the inductive construction is complete.

We choose now an integer $n$ such that $nq_0 \sqrt{2} \leq R < (n+1)q_0 \sqrt{2}$. Since $q_0 \leq R/\sqrt{8}$ this implies $R/\sqrt{8} \leq nq_0 \leq R/\sqrt{2}$. From (3.5) we infer immediately

$$\int_0^{t_n} |u'(t)| \, dt \leq R.$$

From (3.4) and (3.6) we infer now

$$\int_0^{t_n} |x'(t) - y'(t)| \, dt \geq \frac{\sqrt{2}}{3\sqrt{3}} \sum_{i=0}^{n-1} \left( \frac{q_0}{1 + i q_0} \right)^{3/2}$$

$$\geq \frac{\sqrt{2}}{3\sqrt{3}} \int_0^n \left( \frac{q_0}{1 + s q_0} \right)^{3/2} \, ds = q_0^{1/2} \frac{2\sqrt{2}}{3\sqrt{3}} \left( 1 - (1 + nq_0)^{-1/2} \right)$$

$$= q_0^{1/2} \frac{2\sqrt{2}}{3\sqrt{3}} \left( 1 - \frac{1}{\sqrt{1 + R/\sqrt{8}}} \right) = K q_0^{1/2}$$

$$\geq 2Mq_0 \geq M|x_0 - y_0|.$$

\[\square\]

**Remark 3.2.** Again the domain in Example 3.2 can be modified to be bounded.
References


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