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DISCONTINUOUS WAVE EQUATIONS AND A TOPOLOGICAL
DEGREE FOR SOME CLASSES OF MULTI-VALUED MAPPINGS

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Abstract. The Leray-Schauder degree is extended to certain multi-valued mappings on separable Hilbert spaces with applications to the existence of weak periodic solutions of discontinuous semilinear wave equations with fixed ends.

Keywords: discontinuous wave equations, topological degree, multi-valued mappings

MSC 2000: 35L05, 47H17, 58C06

1. INTRODUCTION

In this paper we study the existence of weak 2π -periodic solutions to the discontinuous semilinear wave equation

$$\begin{aligned} \text{(Prob)} \quad & u_{tt} - u_{xx} + g(u) + f(x, t, u) = h(x, t), \\ & u(0, \cdot) = u(\pi, \cdot) = 0, \end{aligned}$$

where $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $\Omega = (0, \pi) \times (0, 2\pi)$ is Carathéodory continuous and nondecreasing in u , $g: \mathbb{R} \rightarrow \mathbb{R}$ is bounded nondecreasing and $h \in L^2(\Omega)$. Moreover, we suppose

$$|f(x, t, u)| \leq c_0|u| + h_0(x, t) \quad \forall u \in \mathbb{R}, \forall (x, t) \in \Omega$$

for a constant $c_0 > 0$ and $h_0 \in L^2(\Omega)$. (Prob) with $g = 0$, i.e. for the continuous case, was studied in [2], where a construction of a topological degree is introduced for a class of monotone single-valued mappings. The main purpose of this paper is to extend that method to monotone multi-valued mappings. Like in [2], we use a Galerkin projection method, which is however not at all the standard Galerkin approximation method of [2]. We construct continuous one-parametric generalized

Galerkin projections which we use for the derivation of a one-parametric family of multi-valued mappings possessing the Leray-Schauder degree [7]. The basic Lemma 6 below is the stabilization of this degree for large parameters. In this way, we can define a topological degree for our multi-valued mappings. The rest of the paper is the extension of the main results of [2] to the multi-valued case. We end the paper with a discontinuous equation of a vibrating string presenting a problem at resonance with an infinite-dimensional kernel. This result is motivated by [4]. Other topological degrees for single and multi-valued mappings have been introduced in [1], [3], [5], [6] and [9].

2. STANDARD CLASSES OF MAPPINGS

Let H be a real separable Hilbert space with an inner product (\cdot, \cdot) . In what follows we use the following notation:

- $\{a_n\}$ means $\{a_n\}_{n=1}^{\infty}$;
- $\lim a_n$ means $\lim_{n \rightarrow \infty} a_n$ (similarly \limsup and \liminf);
- $|\cdot|$ always denotes the norm in H ;
- $\|\cdot\|$ denotes the norm in the space of continuous linear mappings $\mathcal{L}(H)$.

A multi-valued mapping $F: H \rightarrow 2^H$ is

- *monotone* (denote $F \in (mMON)$), if

$$(f_u^* - f_v^*, u - v) \geq 0$$

for all $u, v \in H$ and all selections $f_u^* \in F(u)$, $f_v^* \in F(v)$;

- *quasimonotone* ($F \in (mQM)$), if for any sequence $\{u_n\}$ in H with $u_n \rightarrow u$ and for all selections $f_n^* \in F(u_n)$ we have

$$\liminf (f_n^*, u_n^* - u) \geq 0;$$

- *pseudomonotone* ($F \in (mPM)$), if for any sequence $\{u_n\}$ in H with $u_n \rightarrow u$, the existence of selections $f_n^* \in F(u_n)$ and a point $f^* \in H$ with $f_n^* \rightarrow f^*$ and $\limsup (f_n^*, u_n - u) \leq 0$ implies that $f^* \in F(u)$ and $(f_n^*, u_n) \rightarrow (f^*, u)$;
- *of class* (mS_+) ($F \in (mS_+)$), if for any sequence $\{u_n\}$ in H with $u_n \rightarrow u$, the existence of selections $f_n^* \in F(u_n)$ with $\limsup (f_n^*, u_n - u) \leq 0$ implies that $u_n \rightarrow u$;
- *compact* ($F \in (mCOMP)$), if for any bounded sequence $\{u_n\}$ in H and for any $f_n^* \in F(u_n)$ the sequence $\{f_n^*\}$ has a convergent subsequence;
- *of Leray-Schauder type* ($F \in (mLS)$), if $F = I + C$, where I is the identity on H , for some $C \in (mCOMP)$;

- *bounded*, if for any bounded set $B \subset H$ the set $\bigcup_{u \in B} F(u)$ is bounded;
- *convex-valued*, if $F(u)$ is a non-empty convex set in H for any $u \in H$;
- *upper semicontinuous* (F is usc), if $F^{-1}(A)$ is closed in H whenever $A \subset H$ is closed;
- *weak upper semicontinuous* (F is w-usc), if for any sequence $\{u_n\} \in H$, $u_n \rightarrow u \in H$, the existence of selections $f_n^* \in F(u_n)$ with $f_n^* \rightarrow f^* \in H$ implies $f^* \in F(u)$.

In what follows, we assume that all mappings used are bounded, w-usc and convex-valued. When a mapping is defined only on a subset of H , the above definitions can be modified in an obvious way.

Proposition 1. *For the classes defined above the following inclusions hold:*

$$(mLS) \subset (mS_+) \subset (mPM) \subset (mQM),$$

$$(mMON) \subset (mPM), \quad (mCOMP) \subset (mQM).$$

Proposition 2. *If $F \in (mQM)$ and $G \in (mS_+)$ then $F + G \in (mS_+)$.*

Proofs of Propositions 1 and 2 are similar to those for single-valued mappings [2].

3. CLASSES OF ADMISSIBLE MAPPINGS

Let G be a bounded open subset in H , M a closed subspace of H and let Q and P stand for the orthogonal projections to M and M^\perp , respectively. The family

$$\mathcal{F}_G = \{F: \overline{G} \rightarrow 2^H \mid F = Qg + Pf \text{ for some } g \in (mLS) \text{ and } f \in (mS_+)\}$$

is called *the class of admissible mappings*. Other very important and more general classes are

$$\mathcal{F}_G(mQM) = \{F: \overline{G} \rightarrow 2^H \mid F = Qg + Pf \text{ for some } g \in (mLS) \text{ and } f \in (mQM)\}$$

and $\mathcal{F}_G(mPM)$ defined accordingly. Obviously $\mathcal{F}_G = \mathcal{F}_G(mS_+) \subset \mathcal{F}_G(mPM) \subset \mathcal{F}_G(mQM)$.

Let $L: H \supset D(L) \rightarrow H$ be a closed densely defined linear operator with $\text{Im } L = (\text{Ker } L)^\perp$. Let $L_0 = L/\text{Im } L$ and assume that the right inverse $L_0^{-1}: \text{Im } L \rightarrow \text{Im } L$ is compact. We choose $M = \text{Im } L$ and $M^\perp = \text{Ker } L$. Let $N: H \rightarrow 2^H$. Then similarly to [2], we consider the mapping

$$F = Q(I + L_0^{-1}QN) + PN.$$

Clearly, $F \in \mathcal{F}_G$ for $N \in (mS_+)$.

Lemma 3. *Let F and N be defined as above. Then for $h \in H$*

$$(S\text{Inc}) \quad h \in Lu + N(u) \quad \text{with } u \in D(L) \cap \overline{G}$$

if and only if

$$y \in F(u) \quad \text{with } u \in \overline{G} \text{ for } y = (L_0^{-1}Q + P)h.$$

The proof is the same as for single-valued mappings [2]. The inclusion $h \in Lu - N(u)$ can be treated analogously.

4. CONSTRUCTION OF THE DEGREE FOR $\mathcal{F}_G(mS_+)$

We construct a topological degree function for \mathcal{F}_G . First, we define a class of admissible homotopies. A mapping: $(t, u) \rightarrow f_t(u)$ from $[0, 1] \times \overline{G}$ to 2^H is a *(multi-) homotopy of the class (mS_+)* , if for any sequences $\{u_n\}$ in \overline{G} , $\{t_n\}$ in $[0, 1]$, $f_n^* \in f_{t_n}(u_n)$ with $u_n \rightarrow u$, $t_n \rightarrow t$ and $\limsup(f_n^*, u_n - u) \leq 0$ we have $u_n \rightarrow u$. We also assume that the mapping $(t, u) \rightarrow f_t(u)$ satisfies all the necessary conditions.

Similarly, a mapping: $(t, u) \rightarrow g_t(u) = (I + C_t)(u)$ from $[0, 1] \times \overline{G}$ to 2^H is called a *(multi-)homotopy of the Leray-Schauder type*, if the mapping $(t, u) \rightarrow C_t(u)$ is compact. We denote

$$\mathcal{H}_G = \{F_t \mid F_t = Qg_t + Pf_t\}$$

where g_t and f_t are homotopies of the Leray-Schauder type and of the class (mS_+) , respectively. The set \mathcal{H}_G is called *the class of admissible homotopies*. Obviously $F_t = (1 - t)F_1 + tF_2 \in \mathcal{H}_G$, $0 \leq t \leq 1$ for any $F_1, F_2 \in \mathcal{F}_G$.

We use Galerkin approximations [8] with respect to the subspace M^\perp . The space H ($M^\perp \subset H$) is separable so there exists a sequence $\{N_n\}$ of finite dimensional subspaces of M^\perp with $N_n \subset N_{n+1}$ for all n , and $\cup_{n=1}^\infty N_n$ is dense in M^\perp . We denote by P_n the orthogonal projection from H to N_n . We extend this to generalized Galerkin approximations defined by

$$P_\lambda = (\lambda - n)P_{n+1} + (n + 1 - \lambda)P_n \quad \text{for any } \lambda \in [n, n + 1].$$

We have the following obvious result.

Proposition 4. *The generalized Galerkin approximations satisfy*

- i) $(P_\lambda u, v) = (u, P_\lambda v)$ for every $\lambda \geq 1$, $u, v \in H$;
- ii) $\|P_\lambda\| \leq 1$ for all $\lambda \geq 1$;

- iii) $P_\lambda v \rightarrow Pv$ for every $v \in H$ as $\lambda \rightarrow \infty$;
- iv) $P_n P_\lambda = P_n$ for every $\lambda \geq n \in \mathbb{N}$;
- v) $(z, P_\lambda z) \geq 0$ for every $z \in H$ and $\lambda \geq 1$.

For each $F = Q(I + C) + Pf \in \mathcal{F}_G$, we define the approximations $\{F_\lambda \mid \lambda \geq 1\}$ by

$$F_\lambda = I + QC + \lambda P_\lambda f.$$

We note that $QC + \lambda P_\lambda f$ is compact, convex-valued and usc for each $\lambda \geq 1$.

Similarly, for each admissible homotopy $F_t = Q(I + C_t) + Pf_t$, $0 \leq t \leq 1$, we have

$$(F_t)_\lambda = I + QC_t + \lambda P_\lambda f_t,$$

which is obviously a homotopy of the Leray-Schauder type for any $\lambda \geq 1$. Finally, if $y \in H$ is given, we denote

$$y_\lambda = Qy + \lambda P_\lambda y \quad \text{for each } \lambda \geq 1.$$

It is clear that $(F - y)_\lambda = F_\lambda - y_\lambda$.

Proposition 5. *Let $\{u_k\}$, $u_k \in H$ and let $\{P_\lambda\}$ be the projections defined as above. Then*

- a) if $u_k \rightarrow u$ and $\lambda_k \rightarrow \infty$ then $P_{\lambda_k} u_k \rightarrow Pu$;
- b) if $u_k \rightarrow u$ and $\lambda_k \rightarrow \infty$ then $P_{\lambda_k} u_k \rightarrow Pu$.

Now we can formulate the basic lemma.

Lemma 6. *Let $F_t = Q(I + C_t) + Pf_t$ be an admissible homotopy, y_t ($0 \leq t \leq 1$) a continuous curve in H and let A be a closed subset in \overline{G} . If $y_t \notin F_t(A)$ for all $t \in [0, 1]$, then there exists $n_0 \in \mathbb{N}$ such that*

$$(y_t)_\lambda \notin (F_t)_\lambda(A) \quad \text{for all } t \in [0, 1] \text{ and } \lambda \geq n_0.$$

Proof. Since also $F_t - y_t$ defines an admissible homotopy and $(F_t - y_t)_\lambda = (F_t)_\lambda - (y_t)_\lambda$, we may assume, without loss of generality, that $y_t \equiv 0$.

If the assertion were false, there would exist sequences $\{u_k\}$ in A and $\{\lambda_k\}$ in $[1, \infty)$, $\lambda_k \rightarrow \infty$, and $\{t_k\}$ in $[0, 1]$ such that $0 \in (F_{t_k})_{\lambda_k}(u_k)$. This is equivalent to the existence of selections $g_k^* \in f_{t_k}(u_k)$ and $c_k^* \in C_{t_k}(u_k)$ for which

$$u_k + Qc_k^* + \lambda_k P_{\lambda_k} g_k^* = 0.$$

Writing this equation in both subspaces M and M^\perp we get

$$(1) \quad Qu_k + Qc_k^* = 0,$$

$$(2) \quad Pu_k + \lambda_k P_{\lambda_k} g_k^* = 0.$$

The sequence u_k is bounded therefore we can (taking a subsequence, if necessary) assume that $u_k \rightarrow u$ for some $u \in H$. Similarly we can assume $t_k \rightarrow t, t \in [0, 1]$. We have $c_k^* \in C_{t_k}(u_k)$. Since C is a compact mapping we can assume $c_k^* \rightarrow z^*, z^* \in H$ and, since the sequence g_k^* is bounded, $g_k^* \rightarrow g^*, g^* \in H$.

Hence we have $Qc_k^* \rightarrow Qz^*$ and by (1), $Qu_k \rightarrow Qu$. By (2),

$$\frac{1}{\lambda_k} Pu_k + P_{\lambda_k} g_k^* = 0.$$

The set $\{u_k \mid k = 1, 2, \dots\}$ is bounded, thus $\frac{1}{\lambda_k} Pu_k \rightarrow 0$ for $k \rightarrow \infty$. This leads to $P_{\lambda_k} g_k^* \rightarrow 0$. On the other hand, for $g_k^* \rightarrow g^*$ we conclude $P_{\lambda_k} g_k^* \rightarrow Pg^*$, which yields $Pg^* = 0$. Hence we have $Pg_k^* \rightarrow Pg^* = 0$ followed by $\lim(g_k^*, Pu) = 0, \forall u \in H$.

We continue with calculating of $\limsup(g_k^*, u_k - u)$:

$$(3) \quad \limsup(g_k^*, u_k - u) = \limsup(g_k^*, Pu_k - Pu) = \limsup(g_k^*, Pu_k).$$

From (2) we obtain $Pu_k = -\lambda_k P_{\lambda_k} g_k^*$. Inserting it in to the last term of (3) we get

$$\limsup(g_k^*, u_k - u) = -\liminf \lambda_k (g_k^*, P_{\lambda_k} g_k^*).$$

By v) of Proposition 4, $\lambda_k (g_k^*, P_{\lambda_k} g_k^*) \geq 0$ and it immediately follows that

$$\limsup(g_k^*, u_k - u) \leq 0.$$

Hence, by the definition of the homotopy f_t of the class (mS_+) we have $u_k \rightarrow u, u \in A$. We have $t_n \rightarrow t, g_k^* \rightarrow g^*$ and $u_k \rightarrow u$. Since $(t, u) \rightarrow f_t(u)$ is w-usc, we get $g^* \in f_t(u)$. Similarly, we get $z^* \in C_t(u)$. From $Qu_k \rightarrow Qu, Qc_k^* \rightarrow Qz^*$ using (1), we obtain $Qu + Qz^* = 0$. Since $Pg^* = 0$ we have

$$0 = Qu + Qz^* + Pg^* \in Qu + QC_t(u) + Pf_t(u) = F_t(u),$$

a contradiction. The proof is complete. □

If we choose a constant homotopy $F_t = F = Qg + Pf$, $y_t = y$, and $A = \partial G$, we obtain the stabilization of a degree. If $y \notin F(\partial G)$, then there exists $\lambda_0 \geq 1$ such that $y_\lambda \notin F_\lambda(\partial G)$ for all $\lambda \geq \lambda_0$.

Lemma 7. *Let $F \in \mathcal{F}_G$ and $y \notin F(\partial G)$. Then there exists $\lambda_1 \in [1, \infty)$ such that*

$$d(F_\lambda, G, y_\lambda) = \text{constant} \quad \text{for all } \lambda \geq \lambda_1.$$

Due to Lemma 7 we can define a degree function for the class \mathcal{F}_G . We put

$$(4) \quad d(F, G, y) = \lim_{\lambda \rightarrow \infty} d(F_\lambda, G, y_\lambda)$$

for any given $F \in \mathcal{F}_G$ and $y \in H$ with $y \notin F(\partial G)$. In the next lemma we show that the degree function defined by (4) has all the usual properties. The proof is the same as in [2] with the use of the degree theory for multi-valued mappings [7].

Lemma 8. *Let G be an open bounded subset of H and $F \in \mathcal{F}_G(mS_+)$. Then*

- i) $d(F, G, y) \neq 0$ implies that there exists $y \in F(\overline{G})$.
- ii) $d(F, G, y) = d(F, G_1, y) + d(F, G_2, y)$ (thus $F \in \mathcal{F}_{G_1}$ and $F \in \mathcal{F}_{G_2}$), whenever G_1 and G_2 are disjoint open subsets of G such that $y \notin F(\overline{G} \setminus (G_1 \cup G_2))$.
- iii) $d(F_t, G, y_t)$ is independent of $t \in J = [0, 1]$ if $F_t \in \mathcal{H}_G$, y_t is a continuous curve in H and $y_t \notin F_t(\partial G)$ for all $t \in J$.
- iv) $d(I, G, w) = 1$ if and only if $w \in G$ (normalization).

We can also simply prove the *Borsuk Theorem* for the class $\mathcal{F}_G(mS_+)$.

Proposition 9. *Let G be an open bounded symmetric subset of H containing the origin and let F be a multi-mapping $\overline{G} \rightarrow 2^H$ satisfying $F(-u) = -F(u)$ for all $u \in \partial G$. Then if $F \in \mathcal{F}_G(mS_+)$, the inclusion $0 \in F(u)$ admits a solution u in \overline{G} and $d(F, G, 0)$ is odd whenever defined.*

We have already proved all the desired properties of the degree function for the class \mathcal{F}_G . We formulate our result in the next theorem.

Theorem 10. *Let H be a real separable Hilbert space, G a bounded open subset of H , \mathcal{F}_G the class of admissible mappings and \mathcal{H}_G the class of admissible homotopies defined above. Then there exists a classical topological degree function d on \mathcal{F}_G satisfying the properties i), ii), iii) and iv) from Lemma 8 with respect to \mathcal{H}_G and the normalizing mapping I .*

5. A DEGREE THEORY FOR $\mathcal{F}_G(mQM)$ AND $\mathcal{F}_G(mPM)$

The most common method of studying the existence of solutions of differential equations (inclusions) applying a degree theory is to use a homotopy between a mapping and the normalizing mapping—the identity. However, it is not always possible to use such a homotopy. Sometimes we can substitute the identity with an other mapping with a nonzero degree. A mapping $R \in \mathcal{F}_G(mS_+)$ with $d(R, G, y) \neq 0$ for all $y \in R(G)$ is called a *reference mapping*.

The next basic theorem immediately follows from the properties of the degree function d (see [2, Theorem 2]).

Theorem 11. *Let G be an open bounded set, $G \subset H$, $R \in \mathcal{F}_G$ a reference mapping and $F \in \mathcal{F}_G$. If for a given $y \in H$ there exists $w \in R(G)$ such that*

$$(1-t)w + ty \notin (1-t)R(u) + tF(u) \quad \text{for all } u \in \partial G \text{ and } t \in [0, 1],$$

then $d(F, G, y) \neq 0$ and hence the inclusion $y \in F(u)$ admits a solution u in G .

We already have constructed a degree function for the class $\mathcal{F}_G = \mathcal{F}_G(mS_+)$. By using this we can also build a similar degree function for the larger class $\mathcal{F}_G(mQM)$. By Proposition 2 we can use an (mS_+) approximation $f_\varepsilon = f + \varepsilon I$ for any given $f \in (mQM)$ and $\varepsilon > 0$. In fact, we generalize Theorem 11 in the following form (see [2, Theorem 3]).

Theorem 12. *Let G be an open bounded subset of H , $R \in \mathcal{F}_G(mS_+)$ a reference mapping and $F \in \mathcal{F}_G(mQM)$. If for a given $y \in H$ there exists $w \in R(G)$ such that the condition*

$$(5) \quad (1-t)w + ty \notin (1-t)R(u) + tF(u) \quad \text{for all } u \in \partial G \text{ and } t \in [0, 1]$$

holds, then the inclusion $y \in F(u)$ is almost solvable, i.e. $y \in \overline{F(\overline{G})}$.

P r o o f. Without loss of generality, we can assume that $y = 0$ and $w = 0 \in R(G)$. If $0 \in \overline{F(\partial G)} \subset \overline{F(\overline{G})}$, the assertion is true. Thus let $0 \notin \overline{F(\partial G)}$. Since $F \in \mathcal{F}_G(mQM)$, F has a representation $F = Q(I + C) + Pf$ for some $C \in (mCOMP)$ and $f \in (mQM)$. Similar argument leads to $R = Q(I + C_0) + Pf_0$ for some $C_0 \in (mCOMP)$ and $f_0 \in (mS_+)$. Now we introduce an $\mathcal{F}_G(mS_+)$ approximation of F , for each $\varepsilon > 0$ we denote

$$F_\varepsilon = F + \varepsilon PR = Q(I + C) + P(f + \varepsilon f_0).$$

By Proposition 2, $f + \varepsilon f_0 \in (mS_+)$, therefore $F_\varepsilon \in \mathcal{F}_G(mS_+)$ for all $\varepsilon > 0$. For applying Theorem 11 to the mapping F_ε we show there exists ε_0 such that

$$(6) \quad 0 \notin (1-t)R(u) + tF_\varepsilon(u) \quad \text{for all } u \in \partial G \text{ and } t \in [0, 1] \text{ and } 0 < \varepsilon < \varepsilon_0.$$

If (6) were not true, we could find sequences $\{\varepsilon_n\}, \{t_n\} \subset [0, 1]$ and $\{u_n\} \subset \partial G$ for which

$$(7) \quad 0 \in (1-t_n)R(u_n) + t_n F_{\varepsilon_n}(u_n) \quad \text{for all } n \in \mathbb{N}.$$

Taking subsequences, if necessary, we can assume that $\varepsilon_n \rightarrow 0^+$, $t_n \rightarrow t$ and $u_n \rightarrow u$. Clearly $t \neq 1$, since otherwise $0 \in \overline{F(\partial G)}$ which contradicts the assumption. Thus $t \in [0, 1)$. Writing (6) in the components and taking selections $c_n^* \in C(u_n)$, $d_n^* \in C_0(u_n)$, $f_n^* \in f(u_n)$ and $g_{n,1}^*, g_{n,2}^* \in f_0(u_n)$ we have

$$(8) \quad Qu_n + (1-t_n)Qd_n^* + t_n Qc_n^* = 0,$$

$$(9) \quad (1-t_n)Pg_{n,1}^* + t_n \varepsilon_n Pg_{n,2}^* + t_n Pf_n^* = 0$$

with $r_n^* = Q(u_n + d_n^*) + Pg_{n,1}^* \in R(u_n)$ and $h_n^* = Q(u_n + c_n^*) + P(f_n^* + \varepsilon_n g_{n,2}^*) \in F_{\varepsilon_n}(u_n)$. Without loss of generality, we can assume that $c_n^* \rightarrow c^*$ and $d_n^* \rightarrow d^*$ for some $c^*, d^* \in H$, which yields $r_n^* \rightarrow r^*$ and $h_n^* \rightarrow h^*$. Therefore from (8) we have $Qu_n \rightarrow Qu$ ($u_n \rightarrow u$). By (9)

$$Pg_{n,1}^* = -\frac{t_n}{1-t_n}Pf_n^* - \frac{t_n \varepsilon_n}{1-t_n}Pg_{n,2}^*.$$

By using this and $Qu_n \rightarrow Qu$ we can calculate

$$\begin{aligned} \limsup(g_{n,1}^*, u_n - u) &= \limsup(Pg_{n,1}^*, u_n - u) \\ &= \limsup\left\{-\frac{t_n}{1-t_n}(Pf_n^*, u_n - u) - \frac{t_n \varepsilon_n}{1-t_n}(Pg_{n,2}^*, u_n - u)\right\} \\ &= -\frac{t}{1-t} \liminf(f_n^*, u_n - u). \end{aligned}$$

Since $f \in (mQM)$, we conclude that

$$\limsup(g_{n,1}^*, u_n - u) = -\frac{t}{1-t} \liminf(f_n^*, u_n - u) \leq 0.$$

By using the (mS_+) property of f_0 we obtain $u_n \rightarrow u$. Clearly $u \in \partial G$. From the usc property of the mapping R we get $r_n^* \rightarrow r^* \in R(u)$ and similarly $h_n^* \rightarrow h^* \in F(u)$. Consequently,

$$0 = (1-t_n)r_n^* + t_n h_n^* \rightarrow (1-t)r^* + th^*$$

giving $0 = (1 - t)r^* + th^*$ and hence

$$0 \in (1 - t)R(u) + tF(u) \quad \text{for some } u \in \partial G.$$

That gives a contradiction to (5) ($y = w = 0$). Hence we have proved (6). Thus applying Theorem 11 we find $u_\varepsilon \in G$ such that

$$(10) \quad 0 \in F_\varepsilon(u_\varepsilon) \quad \text{for every } \varepsilon \in (0, \varepsilon_0).$$

Choosing an arbitrary sequence $\{\varepsilon_i\}$ with $\varepsilon_i \rightarrow 0^+$ along with solutions u_{ε_i} from (10) we get

$$F(u_{\varepsilon_i}) = F_{\varepsilon_i}(u_{\varepsilon_i}) - \varepsilon_i PR(u_{\varepsilon_i})$$

where the set $\cup_i PR(u_{\varepsilon_i})$ is bounded. Thus $0 \in \overline{F(\overline{G})}$. \square

Because the condition of quasimonotonicity is in applications the easiest to verify (e.g. it is implied by the monotonicity), we need a result for $\mathcal{F}_G(mPM)$. By using similar argument as in Theorem 12, we obtain the following result (see [2, Theorem 4]).

Theorem 13. *Let the assumptions of Theorem 12 be satisfied and let, in addition, G be convex and $F \in \mathcal{F}_G(mPM)$. If for a given $y \in H$ there exists $w \in R(G)$ such that the condition (5) holds, then the inclusion $y \in F(u)$ does admit a solution u in G .*

Now we can generalize the *Borsuk Theorem* to the class $\mathcal{F}_G(mQM)$ (see [2, Theorem 5]).

Theorem 14. *Let G be an open bounded symmetric subset of H containing the origin and let F be a multi-mapping $\overline{G} \rightarrow 2^H$ satisfying $F(-u) = -F(u)$ for all $u \in \partial G$. Then if $F \in \mathcal{F}_G(mQM)$, the inclusion $0 \in F(u)$ is almost solvable in \overline{G} , i.e., $0 \in \overline{F(\overline{G})}$.*

For the class $\mathcal{F}_G(mQM)$ we have the following surjectivity theorem (see [2, Theorem 6]).

Theorem 15. *Let $F: H \rightarrow 2^H$ be a mapping of the class $\mathcal{F}_H(mQM)$ satisfying the following condition:*

for any $K \in \mathbb{R}^+$ there exists $R_0 \in \mathbb{R}$ such that

$$(A) \quad |f^*| > K \quad \text{for all selections } f^* \in F(u), |u| \geq R_0.$$

Assume moreover that there exists a positive real constant r_0 such that one of the following condition holds:

$$(B) \quad \frac{(f^*, u)}{|u|} + |f^*| > 0 \quad \text{for all } |u| \geq r_0 \text{ and for all } f^* \in F(u);$$

$$(C) \quad F(-u) = -F(u) \quad \text{for all } |u| \geq r_0.$$

Then $\overline{F(H)} = H$.

Proof. Let $y \in H$. By (A), there exists $r_1 > r_0$ such that $0 \notin \overline{F(\partial B_r)}$ for all $r \geq r_1$, where B_r is the ball $B_r = \{u \in H \mid |u| < r\}$.

Assume first that (B) holds. It gives

$$0 \notin (1-t)u + tF(u) \quad \text{for all } t \in [0, 1] \text{ and } u \in \partial B_r.$$

By using the same argument as in the proof of Theorem 12 (with $R = I$) we can conclude (see (6)) that there exists a positive constant $\varepsilon_0 = \varepsilon_0(r)$ such that

$$0 \notin (1-t)u + t(F(u) + \varepsilon Pu) \quad \text{for all } |u| = r, r > r_1, t \in [0, 1] \text{ and } 0 < \varepsilon < \varepsilon_0.$$

Hence

$$d(F + \varepsilon P, B_r, 0) = +1 \quad \text{for all } r > r_1 \text{ and } 0 < \varepsilon < \varepsilon_0.$$

We take $r_2 > r_1$ such that $|f^*| \geq (3|y|+1)$ for all $f^* \in F(u)$, $|u| \geq r_2$. Let $0 < \varepsilon_1 < \varepsilon_0$ be such that $\varepsilon_1 r_2 < |y| + 1$. Consequently,

$$|f^* + \varepsilon Pu| \geq 3|y| - |y| = 2|y|$$

for all $f^* \in F(u)$, $u \in \partial B_{r_2}$ and $0 < \varepsilon < \varepsilon_1$. Since $|ty| \leq |y|$ we have

$$ty \notin (F + \varepsilon P)(\partial B_{r_2})$$

and

$$d(F + \varepsilon P, B_{r_2}, y) = d(F + \varepsilon P, B_{r_2}, 0) = +1 \quad \text{for all } 0 < \varepsilon < \varepsilon_1.$$

Hence $y \in (F + \varepsilon P)(B_{r_2})$ for all $0 < \varepsilon < \varepsilon_1$. It follows that $y \in \overline{F(B_{r_2})} \subset \overline{F(H)}$.

Now we assume that (C) holds. We can use similar argument that in the first case. By applying Theorem 14, from (C) we obtain

$$d(F + \varepsilon P, B_{r_2}, y) = d(F + \varepsilon P, B_{r_2}, 0) = \text{odd}$$

for all $0 < \varepsilon < \varepsilon_1$. Hence $y \in (F + \varepsilon P)(B_{r_2})$ for all $0 < \varepsilon < \varepsilon_1$ and the conclusion $y \in \overline{F(H)}$ follows. Thus the proof is completed. \square

It is not easy to check whether a multi-mapping F is of the class $\mathcal{F}_G(mQM)$ or $\mathcal{F}_G(mS_+)$. The simplest condition to check is the monotonicity and therefore the most useful theorems concern the class $\mathcal{F}_G(mPM)$. We prove the following existence result (see [2, Theorem 9]).

Theorem 16. *Let $N \in (mPM)$, let $L: D(L) \subset H \rightarrow H$ be the linear mapping defined as above and let $c > 0$, $-c \notin \sigma(L)$, where $\sigma(L)$ is the spectrum of L . If there exist real positive constants τ , $\tau < \text{dist}(-c, \sigma(L))$ and K such that*

$$|f^* - cu| \leq (\tau|u| + K) \quad \text{for all } u \in H, f^* \in N(u),$$

then for all $h \in H$ the inclusion (SInc) admits a solution in H , i.e. there exists $u \in D(L) \cap G$ such that $h \in Lu + N(u)$.

Proof. We consider the homotopy

$$F_t(u) = Lu + cu + t(N(u) - cu) \quad \text{for all } t \in [0, 1]$$

and $y_t = th$. Clearly $F_0(u) = Lu + cu$ and $F_1(u) = Lu + N(u)$. Since $-c \notin \sigma(L)$ the operator $(L + cI)^{-1}$ exists and is bounded. Thus we can put $r = \|(L + cI)\|^{-1}(K + |h| + 1)/(1 - \tau\|(L + cI)^{-1}\|)$ and $G = B_r$. If for some $u \in \partial G \cap D(L)$ we have

$$th \in Lu + cu + t(N(u) - cu)$$

for some $t \in [0, 1]$ then

$$u = (L + cI)^{-1}(-t(f^* - cu) + th), \quad f^* \in N(u).$$

Hence

$$r = |u| \leq \|(L + cI)^{-1}\|(\tau r + K + |h|) < r,$$

a contradiction. Thus

$$th \notin Lu + cu + t(N(u) - cu) \quad \text{for all } t \in [0, 1] \text{ and } u \in \partial G \cap D(L).$$

For $t = 0$ we have $Lu + cu = 0$ not possessing a non-zero solution. The mapping $L + cI$ is linear, odd and $cI \in (mS_+)$, therefore using Lemma 3, Proposition 9 and then applying Theorem 13, we conclude $h \in F_1(G)$, i.e. $h \in Lu + N(u)$ for some $u \in D(L) \cap G$. \square

6. M -REGULAR MULTI-FUNCTIONS

A function $p: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is called *superpositionally measurable* [6] if $p(x, t, u(x, t))$ is measurable for any measurable function $u: \Omega \rightarrow \mathbb{R}$. A multi-function $S: \Omega \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is called *measurable-bounded* if there exist two superpositionally measurable functions $q_-(x, t, u)$ and $q_+(x, t, u)$ such that

$$q_-(x, t, u) \leq q_+(x, t, u) \quad \text{and} \quad S(x, t, u) = [q_-(x, t, u), q_+(x, t, u)]$$

for any $(x, t, u) \in \Omega \times \mathbb{R}$ where the function $q_-(x, t, u)$ is lsc in u , the function $q_+(x, t, u)$ is usc in u and there exist positive constants d_1, d_2 and $c_1, c_2 \in L^2(\Omega)$ such that

$$|q_-(x, t, u)| \leq c_1(x, t) + d_1|u| \quad \text{and} \quad |q_+(x, t, u)| \leq c_2(x, t) + d_2|u|$$

for any $(x, t, u) \in \Omega \times \mathbb{R}$. We denote by (mMB) the set of all measurable-bounded multi-functions. By using a multi-function $S \in (mMB)$, for any $u \in L^2(\Omega)$ we put

$$\begin{aligned} N(u) &= \{v \in L^2(\Omega) \mid v(x, t) \in S(x, t, u(x, t))\} \\ &= \{v \in L^2(\Omega) \mid q_-(x, t, u(x, t)) \leq v(x, t) \leq q_+(x, t, u(x, t))\} \end{aligned}$$

and call it *an M -regular multi-function*. We denote the set of all such multi-functions by (mMr) . $N(u)$ is nonempty because $q_{\pm}(x, t, u(x, t)) \in N(u)$. The degree theory we have constructed works just for multi-mappings that are w-usc.

Lemma 17. *Let $N \in (mMr)$ and $u_n \rightarrow u$ in $L^2(\Omega)$. If a sequence $\{w_n^*\}$ satisfies $w_n^* \in N(u_n)$ and $w_n^* \rightarrow w^*$ in $L^2(\Omega)$ then $w^* \in N(u)$, i.e. $N: L^2(\Omega) \rightarrow 2^{L^2(\Omega)}$ is w-usc.*

Proof. First, we have

$$(11) \quad q_-(x, t, u_n(x, t)) \leq w_n^*(x, t)$$

for every $n \in \mathbb{N}$. Since $w_n^* \rightarrow w^*$, using the Mazur theorem we can choose a sequence $\{v_n\}$, $v_n \in \text{conv}\{w_n^*, w_{n+1}^*, \dots\}$ such that $v_n \rightarrow w^*$ almost everywhere in $L^2(\Omega)$. Thus we have

$$v_n = \sum_{k=n}^{m_n} \lambda_{n,k} w_k^*; \quad 0 \leq \lambda_{n,k} \leq 1; \quad \sum_{k=n}^{m_n} \lambda_{n,k} = 1 \quad \text{for } n < m_n \in \mathbb{N}, n \leq k \leq m_n.$$

From (11) we have

$$\sum_{k=n}^{m_n} \lambda_{n,k} q_-(x, t, u_k(x, t)) \leq \sum_{k=n}^{m_n} \lambda_{n,k} w_k^*(x, t) = v_n(x, t).$$

By virtue of the convergence in the measure we can assume that $v_n(x, t) \rightarrow w^*(x, t)$ almost everywhere in (x, t) [10]. Let $(x_0, t_0) \in \Omega$ be such an element. The mapping $s \rightarrow q_-(x_0, t_0, s)$ is lsc and so for every $\varepsilon > 0$ there exists a positive integer n_0 such that for every $k \geq n_0$ we have

$$q_-(x_0, t_0, u(x_0, t_0)) - \varepsilon \leq q_-(x_0, t_0, u_k(x_0, t_0)).$$

Summing this inequality for $k = n, n+1, \dots, m_n$ with weights $\lambda_{n,k}$ we get

$$\begin{aligned} \sum_{k=n}^{m_n} \lambda_{n,k} (q_-(x_0, t_0, u(x_0, t_0)) - \varepsilon) &\leq \sum_{k=n}^{m_n} \lambda_{n,k} q_-(x_0, t_0, u_k(x_0, t_0)), \\ q_-(x_0, t_0, u(x_0, t_0)) - \varepsilon &\leq v_n(x_0, t_0) \end{aligned}$$

for all $n \geq n_0$. Hence by the convergence $v_n(x_0, t_0) \rightarrow w^*(x_0, t_0)$ we have

$$q_-(x_0, t_0, u(x_0, t_0)) - \varepsilon \leq w^*(x_0, t_0) \text{ for every } \varepsilon > 0.$$

Finally, we get

$$q_-(x_0, t_0, u(x_0, t_0)) \leq w^*(x_0, t_0)$$

as we need. Similar argument leads to

$$w^*(x_0, t_0) \leq q_+(x_0, t_0, u(x_0, t_0)).$$

Thus $w^* \in N(u)$. □

7. SEMILINEAR WAVE EQUATIONS

We show how the previous results can be applied to the semilinear wave equation (Prob). We state the precise setting of (Prob) by putting

$$\begin{aligned} q_-(x, t, u) &= g_-(u) + f(x, t, u), & q_+(x, t, u) &= g_+(u) + f(x, t, u), \\ g_+(u) &= \limsup_{s \rightarrow u} g(s), & g_-(u) &= \liminf_{s \rightarrow u} g(s). \end{aligned}$$

We note that g_{\pm} are Borel measurable. By Lemma 17, the Nemytskij operator $N: H \rightarrow 2^H$, $H = L^2(\Omega)$ defined by

$$N(u) = \{v \in L^2(\Omega) \mid q_-(x, t, u(x, t)) \leq v(x, t) \leq q_+(x, t, u(x, t))\}$$

is bounded and w-usc. $N \in (mMON)$ and hence by Proposition 1, $N \in (mPM)$. Let C^2 be the set of twice continuously differentiable functions $v: [0, \pi] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying $v(0, \cdot) = v(\pi, \cdot) = 0$ and 2π -periodic in $t \in \mathbb{R}$.

A weak 2π -periodic solution of (Prob) for $h \in H$ is any $u \in H$ satisfying

$$(wProb) \quad (u, v_{tt} - v_{xx}) + (u^*, v) = (h, v) \quad \text{for some } u^* \in N(u) \text{ and for all } v \in C^2.$$

Let $\varphi_{m,n}(x, t) = \pi^{-1}e^{imt} \sin nx$ for all $m \in \mathbb{Z}, n \in \mathbb{Z}^+$. Each $u \in L^2(\Omega)$ has a representation

$$u = \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^+} (n^2 - m^2) u_{m,n} \varphi_{m,n},$$

where $u_{m,n} = (u, \varphi_{m,n})$ and $\bar{u}_{m,n} = u_{m,n}$, since u is a real function. The abstract realization of the wave operator $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ in $L^2(\Omega)$ is the linear operator $L: D(L) \rightarrow L^2(\Omega)$ defined by

$$Lu = \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^+} (n^2 - m^2) u_{m,n} \varphi_{m,n},$$

where

$$D(L) = \left\{ u \in L^2(\Omega) \mid \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}^+} |n^2 - m^2|^2 |u_{m,n}|^2 < \infty \right\}.$$

It can be shown that $u \in L^2(\Omega)$ is a weak solution of (wProb) if and only if $h \in Lu + N(u)$ with $u \in D(L)$. Moreover, L is densely defined, self-adjoint, closed, $\text{Im } L = (\text{Ker } L)^\perp$ and L has a pure point spectrum of eigenvalues $\sigma(L) = \{n^2 - m^2 \mid m \in \mathbb{Z}, n \in \mathbb{Z}^+\}$ with the corresponding eigenfunctions $\{\varphi_{m,n}\}$. Clearly $\sigma(L)$ is unbounded both from above and from below, any eigenvalue $\lambda \neq 0$ has a finite multiplicity, but $\text{Ker } L$ is infinite dimensional. The right inverse of $L_0, L_0^{-1}: \text{Im } L \rightarrow \text{Im } L$ is compact. Hence (Prob) fulfils all conditions for the inclusion (SInc) of Lemma 3. Theorem 16 immediately implies the following result (see [2, pp. 961–962]).

Theorem 18. *The semilinear wave equation (Prob) under the conditions mentioned above admits a weak 2π -periodic solution for any $h \in L^2(\Omega)$ provided that in addition*

$$a \leq f(x, t, u)/u \leq b \quad \forall (x, t) \in \Omega \text{ and } \forall u \in \mathbb{R}, |u| > R$$

for positive constants a, b, R such that $m^2 - n^2 \notin [a, b]$ for all $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$.

We leave to the reader other extensions of results on pp. 960–962 from [2] to (Prob). In the conclusion, we consider the nonlinear discontinuous vibrating string equation

$$\begin{aligned} u_{tt} - u_{xx} + g(u) &= f(x, t) \\ u(0, \cdot) &= u(\pi, \cdot) = 0, \end{aligned}$$

where $f \in L^\infty(\Omega)$, $g: \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and bounded on \mathbb{R} . This equation is considered in the form

$$(12) \quad \begin{aligned} u_{xx} - u_{tt} + f(x, t) &\in [g_-(u), g_+(u)] \\ u(0, \cdot) &= u(\pi, \cdot) = 0. \end{aligned}$$

We put $g(+\infty) = \sup_{\mathbb{R}} g$, $g(-\infty) = \inf_{\mathbb{R}} g$. We extend Theorem 1 of [4] to (12).

Theorem 19. *Let f have a decomposition*

$$f(x, t) = f_1(x, t) + f_2(x, t), \quad f_1 \in (\ker L)^\perp, \quad f_2 \in \ker L$$

together with

$$g(-\infty) + \delta \leq f_2(x, t) \leq g(+\infty) - \delta \quad \forall (x, t) \in \Omega$$

for some $\delta > 0$. Then (12) has a weak 2π -periodic solution.

P r o o f. For $1 > \varepsilon > 0$ fixed, we consider the problem

$$(13) \quad \begin{aligned} u_{xx} - u_{tt} + f(x, t) &\in [g_-(u), g_+(u)] + \varepsilon u \\ u(0, \cdot) &= u(\pi, \cdot) = 0. \end{aligned}$$

Since $1 > \varepsilon > 0$ and g is nondecreasing and bounded on \mathbb{R} , Theorem 16 implies the existence of a weak 2π -periodic solution u of (13). Now we show that u is bounded in $L^\infty(\Omega)$ uniformly for $\varepsilon > 0$ small. From (13) we have

$$(14) \quad -Lu + f = z + \varepsilon u, \quad z \in [g_-(u), g_+(u)].$$

We decompose $u = u_1 + u_2$, $z = z_1 + z_2$, $u_1, z_1 \in (\ker L)^\perp$, $u_2, z_2 \in \ker L$. Then (14) implies

$$(15) \quad Lu_1 + \varepsilon u_1 + z_1 = f_1$$

$$(16) \quad f_2 = z_2 + \varepsilon u_2.$$

Since $f_1, z \in L^\infty(\Omega)$ for $\varepsilon > 0$ small, we have $|u_1|_{L^\infty} < \tilde{C}$ (see [4, p. 417]). In what follows \tilde{C} denotes positive constants independent of ε . By (16) also

$$(17) \quad \int_0^{2\pi} \int_0^\pi f_2 u_2 \, dx \, dt = \int_0^{2\pi} \int_0^\pi z_2 u_2 \, dx \, dt + \varepsilon \int_0^{2\pi} \int_0^\pi u_2^2 \, dx \, dt \geq \int_0^{2\pi} \int_0^\pi z_2 u_2 \, dx \, dt.$$

We take $M > 0$ such that

$$g(v) \geq g(+\infty) - \frac{\delta}{2} \quad \forall v \geq M, \quad g(v) \leq g(-\infty) + \frac{\delta}{2} \quad \forall v \leq -M.$$

Let us define

$$\Sigma_+ = \{(x, t) \mid u_2(x, t) \geq M + \tilde{C}\}, \quad \Sigma_- = \{(x, t) \mid u_2(x, t) \leq -M - \tilde{C}\}.$$

Then we have

$$z(x, t) \geq g(+\infty) - \frac{\delta}{2} \quad \forall (x, t) \in \Sigma_+, \quad z(x, t) \leq g(-\infty) + \frac{\delta}{2} \quad \forall (x, t) \in \Sigma_-.$$

Now (17) gives

$$\begin{aligned} & (g(+\infty) - \delta) \int_{\Sigma_+} u_2 \, dx \, dt + (g(-\infty) + \delta) \int_{\Sigma_-} u_2 \, dx \, dt + \int_{(\Sigma_+ \cup \Sigma_-)'} f_2 u_2 \, dx \, dt \\ & \geq \int_0^{2\pi} \int_0^\pi f_2 u_2 \, dx \, dt \\ & \geq \int_{(\Sigma_+ \cup \Sigma_-)'} z_2 u_2 \, dx \, dt + \left(g(+\infty) - \frac{\delta}{2}\right) \int_{\Sigma_+} u_2 \, dx \, dt + \left(g(-\infty) + \frac{\delta}{2}\right) \int_{\Sigma_-} u_2 \, dx \, dt, \end{aligned}$$

hence we arrive at

$$\int_{\Sigma_+ \cup \Sigma_-} |u_2| \, dx \, dt \leq \tilde{C}.$$

Consequently, we obtain that $|u_2|_{L^1} < \tilde{C}$. Since $\dim \ker L = \infty$, this estimate is not enough. Now following the same argument like on p. 419 of [4], we obtain that $|u_2|_{L^\infty} < \tilde{C}$. Hence for any $\varepsilon > 0$ small, any solution of (14) satisfies $|u|_{L^\infty} < \tilde{C}$. Then clearly $|u|_{L^2} < \tilde{C}$. By passing to the limit $\varepsilon \rightarrow 0_+$ in (14) like for Theorem 13, we obtain a weak 2π -periodic solution of (12). \square

We remark according to Theorem 19: If $-g(-\infty) = g(+\infty) > 0$ then the condition

$$\left| \int_0^{2\pi} \int_0^{\pi} f_2(x, t) z(x, t) \, dx \, dt \right| \leq g(+\infty) \int_0^{2\pi} \int_0^{\pi} |z(x, t)| \, dx \, dt, \quad \forall z \in \ker L$$

is necessary, and the condition

$$\left| \int_0^{2\pi} \int_0^{\pi} f_2(x, t) z(x, t) \, dx \, dt \right| \leq (g(+\infty) - \delta) \int_0^{2\pi} \int_0^{\pi} |z(x, t)| \, dx \, dt, \quad \forall z \in \ker L$$

for some $0 < \delta < g(+\infty)$ is sufficient for the weak 2π -periodic solvability of (12). Finally, we note that similar results hold for (Prob) with nonincreasing f and g in u .

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