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SOLUTION OF THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION

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Abstract. For open sets with a piecewise smooth boundary it is shown that a solution of the Dirichlet problem for the Laplace equation can be expressed in the form of the sum of the single layer potential and the double layer potential with the same density, where this density is given by a concrete series.

Keywords: Laplace equation, Dirichlet problem, single layer potential, double layer potential

MSC 2000: 31B10, 35J05, 35J25

1. Classical solutions

Suppose that $G \subset \mathbb{R}^m$ $(m > 2)$ is an open set with a non-void compact boundary ∂G such that $\partial G = \partial(\operatorname{cl} G)$. For a given $g \in \mathscr{C}(\partial G)$ (= the space of all continuous functions on the boundary of G equipped with the maximum modulus norm) we formulate the Dirichlet problem for the Laplace equation with boundary condition g as follows: Find $u \in \mathscr{C}(\operatorname{cl} G) \cap \mathscr{C}^2(G)$ such that

$$(1) \qquad \Delta u \equiv \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} = 0 \text{ in } G, \qquad u = g \text{ on } \partial G.$$

If G is unbounded we add the condition

$$(2) \qquad \lim_{x \in G, x \to \infty} u(x) = 0.$$

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Looking for a solution in the form of the double layer potential is a classical method. It was shown by J. Král (see [21]) and independently by Yu. D. Burago and V. G. Maz'ya (see [7]) that it is possible to define the double layer potential on G as a continuously extendable function on $\operatorname{cl} G$ for each density $f \in \mathscr{C}(\partial G)$ if and only if the cyclic variation of G is bounded, where

$$v^G(x) = \sup\left\{ \int_G \nabla\varphi \cdot \nabla h_x \, \mathrm{d}\mathscr{H}_m \, ; \; \varphi \in \mathscr{D}, |\varphi| \leqslant 1, \operatorname{spt}\varphi \subset \mathbb{R}^m \setminus \{x\} \right\} \tag{3}$$

is the cyclic variation of G at the point $x \in \mathbb{R}^m$. Here

$$h_x(y) = (m-2)^{-1} A^{-1} |x-y|^{2-m}, \tag{4}$$

A is the area of the unit sphere in $\mathbb{R}^m$, $\mathscr{D}$ is the space of all compactly supported infinitely differentiable functions in $\mathbb{R}^m$, $\mathscr{H}_k$ is the k-dimensional Hausdorff measure normalized such that $\mathscr{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$.

If $z \in \mathbb{R}^m$ and θ is a unit vector such that the symmetric difference of G and the half-space $\{x \in \mathbb{R}^m \, ; \; (x-z) \cdot \theta > 0\}$ has m-dimensional density zero at z then $n^G(z) = \theta$ is termed the interior normal of G at z in Federer's sense. (The symmetric difference of B and C is equal to $(B \setminus C) \cup (C \setminus B)$.) If there is no interior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in $\mathbb{R}^m$.

If the cyclic variation of G is bounded then for $x \in \operatorname{cl} G$ and $f \in \mathscr{C}(\partial G)$ the double layer potential $W^G f(x)$ with density f at the point x has the form

$$W^G f(x) = (1 - d_G(x)) f(x) + \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_{m-1}(y), \tag{5}$$

where

$$d_G(y) = \lim_{r \to 0_+} \frac{\mathscr{H}_m(\mathscr{U}(y;r) \cap G)}{\mathscr{H}_m(\mathscr{U}(y;r))}$$

exists for each $y \in \mathbb{R}^m$ (see [20], Lemma 2.9). Here $\mathscr{U}(y;r)$ denotes the ball with the centre y and the radius r. The operator $\widehat{W}^G$ which maps f onto the restriction of $W^G f$ onto the boundary of G is a bounded operator on $\mathscr{C}(\partial G)$ (see [20]).

If L is a bounded linear operator on the complex Banach space X we denote by $\|L\|_{\text{ess}}$ the essential norm of L, i.e. the distance of L from the space of all compact linear operators on X. The essential radius of L is defined by

$$r_{\text{ess}} L = \lim_{n \to \infty} (\|L^n\|_{\text{ess}})^{1/n}.$$

Denote by X' the dual space of X and by L' the adjoint operator of L.

In [26] the following theorem is proved:

Theorem. *If $r_{\mathrm{ess}}(\widehat{W}^G - \frac{1}{2}I) < 1/2$, where I denotes the identity operator, and the set $\mathbb{R}^m \setminus G$ is unbounded and connected and $g \in \mathscr{C}(\partial G)$ then the double layer potential $W^G f$ is a solution of the Dirichlet problem for the Laplace equation with the boundary condition g, where*

$$f = g + \sum_{j=0}^{\infty} (I - 2\widehat{W}^G)^j (2I - 2\widehat{W}^G) g.$$

The condition that the set $\mathbb{R}^m \setminus G$ is unbounded and connected is necessary for expressing the solution of the Dirichlet problem for the Laplace equation in the form of the double layer potential for each boundary condition. If we want to calculate the solution for an open set with holes we must modify the double layer potential. We will prove that we can express a solution of the Dirichlet problem in the form of the sum of the single layer potential and the double layer potential with the same density where the corresponding density is given by the concrete series. This method was used in [29].

Fix a nonnegative element λ of $\mathscr{C}'(\partial G)$ (= the Banach space of all finite signed Borel measures with support in ∂G with the total variation as a norm) and suppose that the single layer potential $\mathscr{U}\lambda$ is bounded and continuous on ∂G. Here

$$\mathscr{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y)\,\mathrm{d}\nu(y),$$

where $\nu \in \mathscr{C}'(\partial G)$,

$$h_x(y) = (m-2)^{-1} A^{-1} |x-y|^{2-m},$$

A is the area of the unit sphere in $\mathbb{R}^m$. It was shown in [32] that $\mathscr{U}\lambda$ is bounded and continuous on ∂G if and only if

$$\lim_{r \to 0_+} \sup_{y \in \partial G} \int_{\mathscr{U}(y;r)} h_y(x)\,\mathrm{d}\lambda(x) = 0,$$

where $\mathscr{U}(x;r) = \{y \in \mathbb{R}^m;\ |y-x| < r\}$. According to [20], Lemma 2.18 this is true if there are constants $\alpha > m-2$ and $k > 0$ such that $\lambda(\mathscr{U}(x;r)) \leqslant kr^\alpha$ for all $x \in \mathbb{R}^m$ and all $r > 0$.

We will look for a solution of (1) in the form of the sum of the double layer potential with density f and the single layer potential corresponding to the sign measure $f\lambda$. If $f \in \mathscr{C}(\partial G)$ then $\mathscr{U}(f\lambda)$ is a continuous function in $\mathbb{R}^m$ by [32],Proposition 6 and

$$\sup_{x \in \mathbb{R}^m} |\mathscr{U}(f\lambda)(x)| \leqslant \sup_{x \in \mathbb{R}^m} \mathscr{U}\lambda(x) \sup_{x \in \partial G} |f(x)|. \tag{6}$$

So this sum of the double layer potential and the single layer potential is a continuous function on $\operatorname{cl} G$ if and only if the cyclic variation of G is bounded. The cyclic variation of G is bounded if and only if $V^G < \infty$,where

$$V^G = \sup_{x\in\partial G} v^G(x)$$

(see [20], Theorem 2.16). There are more geometrical characterizations of $v^G(x)$ which ensure $V^G < \infty$ for G convex or for G with $\partial G \subset \cup_{i=1}^k L_i$, where L_i are $(m-1)$-dimensional Ljapunov surfaces (i.e. of class $\mathscr{C}^{1+\alpha}$). Denote by

$$\partial_e G = \left\{x \in \mathbb{R}^m\,;\ \bar{d}_G(x) > 0, \bar{d}_{\mathbb{R}^m\setminus G}(x) > 0\right\}$$

the essential boundary of G where

$$\bar{d}_M(x) = \limsup_{r\to 0_+} \frac{\mathscr{H}_m(M\cap \mathscr{U}(x;r))}{\mathscr{H}_m(\mathscr{U}(x;r))}$$

is the upper density of M at x. Then

$$v^G(x) = \frac{1}{A} \int\limits_{\partial\mathscr{U}(0;1)} n(\theta,x)\,\mathrm{d}\mathscr{H}_{m-1}(\theta),$$

where $n(\theta,x)$ is the number of all points of $\partial_e G\cap\{x+t\theta\,;\ t>0\}$ (see [8]). This expression is a modification of the similar expression in [20]. As a consequence we see that $V^G \leqslant 1/2$ if G is convex. Since $v^G(x)\leqslant V^G + 1/2$ by [20], Theorem 2.16 we see that if

$$\partial G \subset \bigcup_{i=1}^n \partial G_i$$

and $G_1,\dots,G_n$ are convex then $V^G \leqslant n$.

If $\mathscr{H}_{m-1}(\partial_e G)$, the perimeter of G, is finite then

$$v^G(x) = \int\limits_{\partial G} |n^G(y)\cdot\nabla h_x(y)|\,\mathrm{d}\mathscr{H}_{m-1}(y)$$

for each $x\in\mathbb{R}^m$ (see [20], Lemma 2.15).

Theorem 1. *Suppose that $V^G<\infty$, $r_{\mathrm{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$, where I is the identity operator. Denote $C = \mathbb{R}^m\setminus\operatorname{cl} G$ and suppose that $\lambda(H) > 0$ for each bounded component H of $\operatorname{cl} C$. Denote on $\mathscr{C}(\partial G)$ a bounded linear operator V*

$$Vg(x) = W^G g(x) + \mathscr{U}(g\lambda)(x). \tag{7}$$

Fix

$$\alpha > \frac{1}{2}\bigl(1 + \sup_{x\in\partial G} \mathscr{U}\lambda(x)\bigr). \tag{8}$$

Then there are constants $d_\alpha \in \langle 1,\infty)$, $q_\alpha \in (0,1)$ *such that for each* $g \in \mathscr{C}(\partial G)$ *and any positive integer number* n

$$\Bigl\|\Bigl(\frac{V-\alpha I}{\alpha}\Bigr)^n g\Bigr\|_{\mathscr{C}(\partial G)} \leqslant d_\alpha q_\alpha^n \|g\|_{\mathscr{C}(\partial G)}. \tag{9}$$

If $g \in \mathscr{C}(\partial G)$ *then there is a unique solution of the Dirichlet problem* (1) *and this solution has the form* $W^G f + \mathscr{U}(f\lambda)$, *where*

$$f = \sum_{n=0}^{\infty}\Bigl(-\frac{V-\alpha I}{\alpha}\Bigr)^n \frac{g}{\alpha}. \tag{10}$$

P r o o f. The uniqueness of the solution of the problem (1) follows by the maximum principle.

Since $r_{\rm ess}(\widehat{W}^G - \frac12 I) < \frac12$ the operator $\widehat{W}^G - \beta I$ is Fredholm for each $|\beta - 1/2| \geqslant 1/2$ by [17], Satz 51.8. Since $V - \widehat{W}^G$ is a compact operator by [32], Proposition 9 the operators $V - \beta I$ and $V' - \beta I$ are Fredholm for each $|\beta - 1/2| \geqslant 1/2$ by [40], Chapter V, Theorem 3.1, Chapter VII, Theorem 3.5. For each $\nu \in \mathscr{C}'(\partial G)$ (= the Banach space of all finite signed Borel measures with support in ∂G with the total variation as a norm) and each Borel set M we have

$$\begin{aligned} V'(\nu) &= \int_M (1 - d_G(x))\,\mathrm{d}\nu(x) + \int_{\partial G}\int_{(\partial G\cap M)} n^G(y)\cdot\nabla h_x(y)\,\mathrm{d}\mathscr{H}_{m-1}(y)\,\mathrm{d}\nu(x) + \int_M \mathscr{U}\nu\,\mathrm{d}\lambda \\ &= \int_M d_C(x)\,\mathrm{d}\nu(x) - \int_{\partial C}\int_{(\partial C\cap M)} n^C(y)\cdot\nabla h_x(y)\,\mathrm{d}\mathscr{H}_{m-1}(y)\,\mathrm{d}\nu(x) + \int_M \mathscr{U}\nu(x)\,\mathrm{d}\lambda(x) \end{aligned}$$

(see [35], Proposition 8), because the Lebesgue measure of ∂G is equal to 0 by [27], Lemma 4. Since the Lebesgue measure of ∂G is equal to 0 we have $V^G = V^C < \infty$. Denote by $\sigma(V)$ the spectrum of the operator V. According to [17], Satz 51.1, [27], Lemma 4, Lemma 5, Lemma 10, Lemma 11, Theorem 1, [42], Chapter VIII, §6, Theorem 2 we have $\sigma(V) \cap \{\beta \in \mathbb{C};\ |\beta - 1/2| \geqslant 1/2\} \subset \{\beta \in \mathbb{C};\ \beta \geqslant 1/2\}$. Since the spectral radius of $W^G - \frac12 I$ is smaller or equal to 1/2 (see [26], Proposition 1) and

$$\|W^G - V\| \leqslant \sup_{x\in\partial G} \mathscr{U}\lambda(x) (\equiv c_\lambda)$$

by the maximum principle, the spectral radius of $V - \frac12 I$ is smaller than or equal to $1/2 + c_\lambda$. Thus $\sigma(V) \cap \{\beta \in \mathbb{C};\ |\beta - 1/2| \geqslant 1/2\} \subset \{\beta \in \mathbb{C};\ 1/2 \leqslant \beta \leqslant 1 + c_\lambda\}$. If

$\alpha > \frac{1}{2}(1+c_\lambda)$ then the spectrum of V is disjoint with the set $\{\beta \in \mathbb{C};\ |\beta - \alpha| \geqslant \alpha\}$. Since the spectral radius of the operator $V - \alpha I$ is smaller than α there are constants $d_\alpha \in \langle 1, \infty)$, $q_\alpha \in (0,1)$ such that (9) holds for each $g \in \mathscr{C}(\partial G)$ and any positive integer number n. Easy calculation yields that f given by (10) is a solution of the equation $Vf = g$. Since $W^G f + \mathscr{U}(f\lambda)$ is a continuous function on the closure of G which is harmonic on G and $W^G f + \mathscr{U}(f\lambda) = Vf$ on the boundary of G, the function $W^G f + \mathscr{U}(f\lambda)$ is a solution of the problem (1). $\square$

Remark 1. It is well-known that the condition $r_{\text{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$ is fulfilled for sets with a smooth boundary (of class $C^{1+\alpha}$) (see [21]) and for convex sets (see [31]). R. S. Angell, R. E. Kleinman, J. Král and W. L. Wendland proved that rectangular domains (i.e. formed from rectangular parallelepipeds) in $\mathbb{R}^3$ have this property (see [2], [22]). A. Rathsfeld showed in [38], [39] that polyhedral cones in $\mathbb{R}^3$ have this property. (By a polyhedral cone in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface (i.e. every point of $\partial\Omega$ has a neighbourhood in $\partial\Omega$ which is homeomorphic to $\mathbb{R}^2$) and $\partial\Omega$ is formed by a finite number of plane angles. By a polyhedral open set with bounded boundary in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface and $\partial\Omega$ is formed by a finite number of polygons.) N. V. Grachev and V. G. Maz'ya obtained independently an analogous result for polyhedral open sets with bounded boundary in $\mathbb{R}^3$ (see [16]). (Let us note that there is a polyhedral set in $\mathbb{R}^3$ which has not a locally Lipschitz boundary (see [27], Example 2).) In [25] it was shown that the condition $r_{\text{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$ has a local character. As a consequence we obtain that this condition is fulfilled for $G \subset \mathbb{R}^3$ such that for each $x \in \partial G$ there are $r(x) > 0$, a domain D_x which is polyhedral or smooth or convex or a complement of a convex domain and a diffeomorphism $\psi_x\colon \mathscr{U}(x; r(x)) \to \mathbb{R}^3$ of class $C^{1+\alpha}$,where $\alpha > 0$, such that $\psi_x(G \cap \mathscr{U}(x; r(x))) = D_x \cap \psi_x(\mathscr{U}(x; r(x)))$. V. G. Maz'ya and N. V. Grachev proved this condition for several types of sets with "piecewise-smooth" boundary in the general Euclidean space (see [13]–[15]).

Corollary 1. *Suppose that $V^G < \infty$, $r_{\text{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$. Denote by $\mathscr{H}$ the restriction of $(m-1)$-dimensional Hausdorff measure onto ∂G. Then $\mathscr{H}(\partial G) < \infty$. Fix $c > 0$. Suppose that K is such a constant that $\mathscr{H}(\mathscr{U}(x; r)) \leqslant K r^{m-1}$ for each $x \in \mathbb{R}^m$, $r > 0$. (This condition is true for $K = Am(m+2)^m(V^G + 1/2)$.) Put $\lambda = c\mathscr{H}$. Then $\mathscr{U}\lambda$ is a continuous function in $\mathbb{R}^m$ and*

$$\sup_{x \in \partial G} \mathscr{U}\lambda(x) \leqslant cK 2^{m-1} A^{-1} (m-2)^{-1} \operatorname{diam} \partial G, \tag{11}$$

where

$$\operatorname{diam} \partial G = \sup_{x, y \in \partial G} |x - y|.$$

If

$$\alpha > \frac{1}{2}\bigl(1 + \sup_{x\in\partial G} \mathscr{U}\lambda(x)\bigr)$$

then there are constants $d_\alpha \in \langle 1,\infty), q_\alpha \in (0,1)$ *such that for each* $g \in \mathscr{C}(\partial G)$ *and any positive integer number* n

$$\left\| \left(\frac{V-\alpha I}{\alpha}\right)^n g \right\|_{\mathscr{C}(\partial G)} \leqslant d_\alpha q_\alpha^n \|g\|_{\mathscr{C}(\partial G)}.$$

If $g \in \mathscr{C}(\partial G)$ *then there is a unique solution of the Dirichlet problem* (1) *and this solution has the form* $W^G f + \mathscr{U}(f\lambda)$, *where*

$$f = \sum_{n=0}^{\infty} \left(-\frac{V-\alpha I}{\alpha}\right)^n \frac{g}{\alpha}.$$

P r o o f. $\mathscr{H}(\partial G) < \infty$ by [27], Corollary 1. The function $\mathscr{U}\lambda$ is continuous and (11) is true by [27], Remark 6. For $K = Am(m+2)^m(V^G + 1/2)$ we have $\mathscr{H}(\mathscr{U}(x;r)) \leqslant K r^{m-1}$ for each $x \in \mathbb{R}^m$, $r > 0$ by [20], Corollary 2.17. The rest is a consequence of Theorem 1. □

Corollary 2. *Suppose that* $V^G < \infty$, $r_{\text{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$. *Fix* $c > 0$. *Put* $\lambda = c\mathscr{H}$. *Let*

$$\alpha > \frac{1}{2}\left(1 + \sup_{x\in\partial G} \mathscr{U}\lambda(x)\right).$$

For $x, y \in G$ *we define*

$$f_x = \sum_{n=0}^{\infty} \left(-\frac{V-\alpha I}{\alpha}\right)^n \frac{h_x}{\alpha}, \tag{12}$$

$$\mathscr{G}(x,y) = h_x(y) - W^G f_x(y) - \mathscr{U}(f_x\lambda)(y). \tag{13}$$

Then $\mathscr{G}$ *is the Green function for* G.

E x a m p l e 1. Put $G = \{x \in R^3;\ 1/2 < |x_j| < 1, j = 1,2,3\}$. Let $f \in \mathscr{C}(\partial G)$, $g \in \mathscr{C}(\operatorname{cl} G)$. We want to solve the following problem: Find $u \in \mathscr{C}(\operatorname{cl} G) \cap \mathscr{C}^2(G)$ such that

$$\Delta u = g \text{ in } G,$$
$$u = f \text{ on } \partial G.$$

Since ∂G is a subset of 12 planes we have $\mathscr{H}(\mathscr{U}(x;r)) \leqslant 12\pi r^2$ for each $r > 0$, $x \in \partial G$. Fix $c > 0$. Put $\lambda = c\mathscr{H}$. Then $\mathscr{U}\lambda$ is a continuous function in R^3 and

$$\sup_{x\in\partial G} \mathscr{U}\lambda(x) \leqslant 50c$$

by Corollary 1. Put, for example, $c = 0.1$. Fix $\alpha > 3$. Put

$$h = \sum_{n=0}^{\infty} \Big(-\frac{V - \alpha I}{\alpha} \Big)^n \frac{f}{\alpha}.$$

Then $W^G h + \mathscr{U}(h\lambda)$ is a classical solution of the problem $\Delta v = 0$ in G, $v = f$ in ∂G. If we define for $x, y \in G$ the function f_x by (12) and $\mathscr{G}(x, y)$ by (13) then $\mathscr{G}$ is a Green function for G. The solution u of our problem has the form

$$u(x) = -\int_G \mathscr{G}(x, y) g(y) \, \mathrm{d}\mathscr{H}_m(y) + W^G h(x) + \mathscr{U}(h\lambda)(x).$$

2. Generalized solution

Let g be an arbitrary extended real-valued function defined on ∂G. We denote by $\overline{U}_g^G$ the set of all hyperharmonic functions u on G which are lower bounded on G, non-negative outside the trace on G of a compact set of $\mathbb{R}^m$ and such that for any $y \in \partial G$

$$\liminf_{x \to y} u(x) \geqslant g(y).$$

The function u on G is hyperharmonic if it is lower-semicontinuous and for each $x \in G$ and $r > 0$ such that $\operatorname{cl} \mathscr{U}(x; r) \subset G$ it satisfies $u(x) > -\infty$ and

$$u(x) \geqslant \frac{1}{\mathscr{H}_m(\mathscr{U}(x; r))} \int_{\mathscr{U}(x;r)} u(y) \, \mathrm{d}\mathscr{H}_m(y).$$

We put $\underline{U}_g^G = -\overline{U}_{(-g)}^G$ and denote by $\overline{H}_g^G$ (resp. $\underline{H}_g^G$) the greatest (resp. least upper) bound of $\overline{U}_g^G$ (resp. $\underline{U}_g^G$). (Compare [6], [18].)

A function g on ∂G is said to be resolutive (relative to G), if $\overline{H}_g^G = \underline{H}_g^G$ and $|\overline{H}_g^G(x)| < \infty$ for any $x \in G$. We set $H_g^G = \overline{H}_g^G$, the generalized solution of the Dirichlet problem for the Laplace equation with the boundary condition g, provided g is resolutive. If $g \in \mathscr{C}(\partial G)$ and u is a classical solution of the Dirichlet problem for the Laplace equation with the boundary condition g then g is resolutive and $H_g^G = u$. It is worth noting that any bounded Baire function on ∂G is resolutive ([6], Theorem 6 and the text on p. 94).

For fixed $x \in G$ there is a unique Borel measure μ_x^G on ∂G such that

$$H_g^G(x) = \int g \, \mathrm{d}\mu_x^G \tag{14}$$

for each $g \in \mathscr{C}(\partial G)$. The relation (14) holds for each resolutive g (see [18], Satz 1,2).

Let us note that we have tacitly used the fact that $\mathbb{R}^m$ is a strong harmonic space in the sense of the theory of harmonic spaces (see [3], p. 61).

In the rest of the paper we will suppose that $V^G < \infty$. Denote by $\widehat{\mathscr{H}}$ the restriction of the $(m-1)$-dimensional Hausdorff measure onto $\partial_e G$. Fix $c \geqslant 0$ and suppose that if $c = 0$ then $\mathbb{R}^m \setminus G$ is unbounded and connected. Put $\lambda = c\widehat{\mathscr{H}}$. Then the single layer potential $\mathscr{U}\lambda$ is bounded and continuous on ∂G (see [20], Lemma 2.17, Lemma 2.18) and $\lambda(H) > 0$ for each bounded component H of $\operatorname{cl} G$. Note that if $r_{\text{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$ then $\mathscr{H} = \widehat{\mathscr{H}}$ (see [27], Corollary 1).

Theorem 2. *Suppose $r_{\text{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$. Fix*

$$\alpha > \frac{1}{2}\Bigl(1 + \sup_{x \in \partial G} \mathscr{U}\lambda(x)\Bigr).$$

Then for each $g \in \mathscr{C}(\partial G)$

$$H_g^G = W^G f + \mathscr{U}(f\lambda),$$

where

$$f = \sum_{n=0}^{\infty} \Bigl(-\frac{V - \alpha I}{\alpha}\Bigr)^n \frac{g}{\alpha}. \tag{15}$$

Define on $L^1(\mathscr{H})$ a bounded operator T by

$$Tg(x) = \frac{1}{2}g(x) + \int_{\partial G} g(y) n^G(x) \cdot \nabla h_y(x) \,\mathrm{d}\mathscr{H}(y) + cg(x) \int_{\partial G} h_x \,\mathrm{d}\mathscr{H}.$$

Then there are constants $d_\alpha \in \langle 1, \infty), q_\alpha \in (0,1)$ such that for each $g \in L^1(\mathscr{H})$ and any positive integer number n

$$\Bigl\| \Bigl(\frac{T - \alpha I}{\alpha}\Bigr)^n g \Bigr\|_{L^1(\mathscr{H})} \leqslant d_\alpha q_\alpha^n \|g\|_{L^1(\mathscr{H})}. \tag{16}$$

For fixed $x \in G$ put

$$\begin{aligned} g_x(y) &= n^G(y) \cdot \nabla h_x(y) + ch_x(y), \\ k_x &= \sum_{n=0}^{\infty} \Bigl(-\frac{T - \alpha I}{\alpha}\Bigr)^n \frac{g_x}{\alpha}. \end{aligned} \tag{17}$$

Then $\mu_x^G = k_x \mathscr{H}$. If $c = 0$ then there are constants $d \in \langle 1, \infty), q \in (0,1)$ such that for each $g \in L^1(\mathscr{H})$ and any positive integer number n

$$\bigl\|(I - 2T)^n 2(I - T)g\bigr\|_{L^1(\mathscr{H})} \leqslant d_\alpha q_\alpha^n \|g\|_{L^1(\mathscr{H})} \tag{18}$$

and

$$k_x = g_x + \sum_{j=0}^{\infty}(I-2T)^j(2I-2T)g_x. \tag{19}$$

P r o o f. Since V is invertible, V' is invertible as well and $(V')^{-1}$ is the adjoint operator of V^{-1}. Let $x \in G$, $g \in \mathscr{C}(\partial G)$, $\varepsilon > 0$, f being given by (15). Then $W^G f + \mathscr{U}(f\lambda) + \varepsilon \in \overline{U}_g^G$ and $W^G f + \mathscr{U}(f\lambda) - \varepsilon \in \underline{U}_g^G$. Thus $H_g^G = W^G f + \mathscr{U}(f\lambda)$ and

$$\begin{aligned}\int g \,\mathrm{d}\mu_x &= H_g^G(x) = W^G(V^{-1}g)(x) + \mathscr{U}\left[c(V^{-1}g)\mathscr{H}\right](x) \\ &= \int_{\partial G}\left(V^{-1}g\right)(y) g_x(y)\,\mathrm{d}\mathscr{H}(y) = \int_{\partial G} g \,\mathrm{d}\left[(V')^{-1}(g_x\mathscr{H})\right].\end{aligned}$$

According to [27], Corollary 1, T is a bounded invertible operator on $L^1(\mathscr{H})$, $(V')^{-1}(g_x\mathscr{H}) = (T^{-1}g_x)\mathscr{H}$, (16) holds and

$$T^{-1}g_x = \sum_{n=0}^{\infty}\left(-\frac{T-\alpha I}{\alpha}\right)^n \frac{g_x}{\alpha} = k_x.$$

If $c = 0$ then (18), (19) hold by [27], Corollary 1. $\square$

N o t a t i o n. For a bounded $\widehat{\mathscr{H}}$-measurable function f on ∂G define the double layer potential $W^G f$ on $\operatorname{cl} G$ by formula (5). The function $W^G f + \mathscr{U}(f\lambda)$ is harmonic on G. Denote $Vf(x) = W^G f(x) + \mathscr{U}(f\lambda)(x)$ for $x \in \partial G$. Then Vf is a bounded $\widehat{\mathscr{H}}$-measurable function on ∂G. Denote by $\tilde{f}$ the class in $L^\infty(\widehat{\mathscr{H}})$ corresponding to f (i.e. $\tilde{f} = \{g;\ \widehat{\mathscr{H}}(\{x \in \partial G;\ f(x) \neq g(x)\}) = 0\}$). Since for $g \in \tilde{f}$ we have $W^G f + \mathscr{U}(f\lambda) = W^G g + \mathscr{U}(g\lambda)$ on G we denote $W^G \tilde{f} + \mathscr{U}(\tilde{f}\lambda) \equiv W^G f + \mathscr{U}(f\lambda)$ on G. Since for $g \in \tilde{f}$ we have $\widetilde{Vf} = \widetilde{Vg}$ we define on $L^\infty(\widehat{\mathscr{H}})$ a bounded operator $\tilde{V}$ by $\tilde{V}\tilde{f} = \widetilde{Vf}$. Let $\mathscr{B}(\partial G)$ we denote the Banach space of all bounded Baire functions on ∂G equipped with the supremum norm. If $f \in \mathscr{B}(\partial G)$ then $Vf \in \mathscr{B}(\partial G)$ and $\|Vf\| \leqslant \|f\|(1 + V^G + \|\mathscr{U}\lambda\|)$ on ∂G.

Lemma 1. *Let $x \in \partial G$, $f \in \mathscr{B}(\partial G)$, f being continuous in x. Then*

$$\lim_{y \in G, y \to x}[W^G f(y) + \mathscr{U}(f\lambda)(y)] = Vf(x).$$

P r o o f. $\mathscr{U}(f\lambda)$ is a continuous function in $\mathbb{R}^m$ by [32], Proposition 6. The rest is a consequence of [10], Proposition 1.1 and [20], Theorem 2.16. $\square$

Definition. For $x \in \partial G$, $\beta > 0$ denote by $\Gamma_\beta(x) = \{y \in G;\ (1+\beta)\operatorname{dist}(y,\partial G) > |x-y|\}$ the nontangential region of opening β at the point x. Here $\operatorname{dist}(y,\partial G) = \sup\{|y-z|;\ z \in \partial G\}$ is the distance of y from ∂G. If f is a function on G, $x \in \partial G$ and there is $\alpha > 0$ such that $x \in \operatorname{cl}(\Gamma_\alpha(x))$ we say that c is *the nontangential limit of f in x* if

$$\lim_{y \in \Gamma_\beta(x), y \to x} f(y) = c$$

for each $\beta > 0$ such that $x \in \operatorname{cl}(\Gamma_\beta(x))$.

Let us note that if $S \subset G$, $x \in S$ and S is a set such that for each series x_n of points of $\partial G \setminus \{x\}$ and each series y_n of points of S such that $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = x$, $\lim\limits_{n\to\infty} (x_n - x)/|x_n - x| = \theta_1$, $\lim\limits_{n\to\infty} (y_n - x)/|y_n - x| = \theta_2$ we have $\theta_1 \neq \theta_2$, then there are $\beta > 0$, $\delta > 0$ such that $S \cap \mathscr{U}(x;\delta) \subset \Gamma_\beta(x)$ (see [10], Proposition 0.1).

The following lemma is a moderate modification of Lemma 2.1 in [10]. Let us note that $\widehat{\mathscr{H}}\bigl(\mathscr{U}(x;\delta)\bigr) > 0$ for each $x \in \partial G$ and $\delta > 0$ (see Isoperimetric Lemma in [20], p. 50).

Lemma 2. *Let $x \in \partial G$, let f be an $\widehat{\mathscr{H}}$-measurable bounded function on ∂G,*

$$\lim_{\delta\to 0_+} \frac{1}{\widehat{\mathscr{H}}\bigl(\mathscr{U}(x;\delta)\bigr)} \int\limits_{\partial G \cap \mathscr{U}(x;\delta)} |f(y) - f(x)| \,\mathrm{d}\widehat{\mathscr{H}}(y) = 0. \tag{20}$$

If there is $\alpha > 0$ such that $x \in \operatorname{cl}(\Gamma_\alpha(x))$ then $Vf(x)$ is the nontangential limit of $W^G f + \mathscr{U}(f\lambda)$ at x.

P r o o f. Fix $\alpha > 0$ such that $x \in \operatorname{cl}(\Gamma_\alpha(x))$. Fix $\varepsilon > 0$. Since $\mathscr{U}(f\lambda)$ is a continuous function in $\mathbb{R}^m$ by [32], Proposition 6, there is $r_1 > 0$ such that $|\mathscr{U}(f\lambda)(x) - \mathscr{U}(f\lambda)(y)| < \varepsilon/4$ for $y \in \Gamma_\alpha(x) \cap \mathscr{U}(x;r_1)$. According to [20], Corollary 2.17 there is a positive constant C such that

$$\widehat{\mathscr{H}}\bigl(\mathscr{U}(x;r)\bigr) \leqslant C r^{m-1}$$

for each $r > 0$. Since

$$v^G(x) = \frac{1}{A} \int\limits_{\partial G} \left| n^G(y) \cdot \frac{x-y}{|x-y|^m} \right| \mathrm{d}\widehat{\mathscr{H}}(y) < \infty$$

there is $R \in (0, r_1)$ such that

$$\frac{1}{A} \int\limits_{\partial G \cap \mathscr{U}(x;R)} \left| n^G(y) \cdot \frac{x-y}{|x-y|^m} \right| |f(y) - f(x)| \,\mathrm{d}\widehat{\mathscr{H}}(y) < \frac{\varepsilon}{4(2+\alpha)^m + 4}$$

and

$$\int\limits_{\partial G\cap\mathscr{U}(x;r)} |f(y)-f(x)|\,\mathrm{d}\widehat{\mathscr{H}}(y) \leqslant \frac{\varepsilon}{C2^{m+3}(2+\alpha)^m}\widehat{\mathscr{H}}\bigl(\mathscr{U}(x;r)\bigr)$$

for each $r\in(0,2R)$. Put

$$f_1(y)=\begin{cases} f(x) & \text{on } \partial G\cap\mathscr{U}(x;R),\\ f(y) & \text{on } \partial G\setminus\mathscr{U}(x;R),\end{cases}$$

$$f_2(y)=\begin{cases} f(y)-f(x) & \text{on } \partial G\cap\mathscr{U}(x;R),\\ 0 & \text{on } \partial G\setminus\mathscr{U}(x;R).\end{cases}$$

Since f_1 is continuous in x there is $r_2\in(0,R)$ such that $|W^Gf_1(x)-W^Gf_1(y)|<\varepsilon/4$ for each $y\in\Gamma_\alpha(x)\cap\mathscr{U}(x;r_2)$ (see [10], Proposition 1.1 and [20], Theorem 2.16). If $y\in\Gamma_\alpha(x)\cap\mathscr{U}(x;r_2)$ then

$$\begin{aligned}&|W^Gf_2(y)-W^Gf_2(x)|\\ &=\frac{1}{A}\Biggl|\int\limits_{\partial G\cap\mathscr{U}(x;R)}(f(z)-f(x))n^G(z)\cdot\left[\frac{z-x}{|z-y|^m}+\frac{x-y}{|z-y|^m}-\frac{z-x}{|z-x|^m}\right]\mathrm{d}\widehat{\mathscr{H}}(z)\Biggr|.\end{aligned}$$

If $z\in\partial G$ then $|y-x|\leqslant(1+\alpha)|z-y|$ and thus $|x-z|\leqslant|x-y|+|y-z|\leqslant(2+\alpha)|y-z|$. Thus

$$\begin{aligned}&|W^Gf_2(y)-W^Gf_2(x)|\\ &\quad\leqslant\frac{\bigl[1+(2+\alpha)^m\bigr]}{A}\int\limits_{\partial G\cap\mathscr{U}(x;R)}|f(z)-f(x)|\left|n^G(z)\cdot\frac{z-x}{|z-x|^m}\right|\mathrm{d}\widehat{\mathscr{H}}(z)\\ &\quad+\frac{(2+\alpha)^m}{A}\int\limits_{\partial G\cap\mathscr{U}(x;R)\setminus\mathscr{U}(x;|x-y|)}|f(z)-f(x)|\frac{|y-x|}{|z-x|^m}\,\mathrm{d}\widehat{\mathscr{H}}(z)\\ &\quad+\frac{(1+\alpha)^m}{A}\int\limits_{\partial G\cap\mathscr{U}(x;|x-y|)}|f(z)-f(x)||y-x|^{1-m}\,\mathrm{d}\widehat{\mathscr{H}}(z).\end{aligned}$$

Fix an integer k such that $2^{k-1}|x-y|<R\leqslant 2^k|x-y|$. Then

$$\begin{aligned}&|W^Gf_2(y)-W^Gf_2(x)|\\ &\leqslant\frac{\varepsilon}{4}+\frac{(2+\alpha)^m}{A}\sum_{j=1}^{k}\frac{|x-y|^{1-m}}{2^{(j-1)m}}\int\limits_{\mathscr{U}(x;2^j|x-y|)}|f(z)-f(x)|\,\mathrm{d}\widehat{\mathscr{H}}(z)\\ &\quad+\frac{C(1+\alpha)^m}{A\widehat{\mathscr{H}}(\mathscr{U}(x;|x-y|))}\int\limits_{\mathscr{U}(x;|x-y|)}|f(z)-f(x)|\,\mathrm{d}\widehat{\mathscr{H}}(z)\leqslant\frac{\varepsilon}{4}+\frac{\varepsilon}{8}\sum_{j=1}^{k}2^{-j}+\frac{\varepsilon}{8}\leqslant\frac{\varepsilon}{2}.\end{aligned}$$

Thus

$$|W^G f(y) + \mathscr{U}(f\lambda)(y) - Vf(x)| < \varepsilon$$

for $y \in \Gamma_\alpha(x) \cap \mathscr{U}(x; r_2)$. $\square$

Lemma 3. *Let $x \in \partial G$, $d_G(x) > 0$. Suppose that there are positive constants R, C such that if $y \in \partial G \cap \mathscr{U}(x; R)$, $r \in (0, R)$ then $\widehat{\mathscr{H}}(\mathscr{U}(y; r)) \geqslant Cr^{m-1}$. Then there is $\alpha > 0$ (depending only on C, m, $d_G(x)$, V^G) such that $x \in \operatorname{cl} \Gamma_\alpha$. Moreover,*

$$\lim_{\gamma \to \infty} \bar{d}_{G \setminus \Gamma_\gamma}(x) = 0.$$

If f is an $\widehat{\mathscr{H}}$-measurable bounded function on ∂G,

$$\lim_{\delta \to 0_+} \frac{1}{\widehat{\mathscr{H}}\bigl(\mathscr{U}(x; \delta)\bigr)} \int\limits_{\partial G \cap \mathscr{U}(x;\delta)} |f(y) - f(x)| \, \mathrm{d}\widehat{\mathscr{H}}(y) = 0$$

then $Vf(x)$ is the nontangential limit of $W^G f + \mathscr{U}(f\lambda)$ at x and

$$\lim_{\delta \to 0_+} \frac{1}{\mathscr{H}_m\bigl(\mathscr{U}(x; \delta) \cap G\bigr)} \int\limits_{G \cap \mathscr{U}(x;\delta)} |W^G f(y) + \mathscr{U}(f\lambda)(y) - Vf(x)| \, \mathrm{d}\mathscr{H}_m(y) = 0.$$

P r o o f. Fix $\beta \in (0, 1)$. We show that there is a positive constant K such that

$$\int\limits_{\mathscr{U}(x;r) \cap G} \operatorname{dist}(y, \partial G)^{\beta - 1} \, \mathrm{d}\mathscr{H}_m(y) \leqslant K \mathscr{H}_m(\mathscr{U}(x; r)) r^{\beta - 1} \tag{21}$$

for $r \in (0, R/3)$. Fix $r \in (0, R/3)$. Put

$$E_n = \{y \in G \cap \mathscr{U}(x; r);\ \operatorname{dist}(y, \partial G) < 2^{1-n} r\}$$

for a positive integer n. Fix n. Compactness of $\partial G \cap \operatorname{cl} \mathscr{U}(x; 2r)$ yields that there are points $z_1, \ldots, z_k$ in $\partial G \cap \operatorname{cl} \mathscr{U}(x; 2r)$ such that $|z_i - z_j| > 2^{1-n} r$ for $i \neq j$ and for each $y \in \partial G \cap \operatorname{cl} \mathscr{U}(x; 2r)$ there is $j \in \{1, \ldots, k\}$ such that $|z_j - y| \leqslant 2^{1-n} r$. Since $\mathscr{U}(z_1; 2^{-n} r), \ldots, \mathscr{U}(z_k; 2^{-n} r)$ are disjoint and

$$E_n \subset \bigcup_{j=1}^{k} \mathscr{U}(z_j; 2^{2-n} r)$$

we have

$$\mathscr{H}_m(E_n) \leqslant k2^{m(2-n)}\mathscr{H}_m(\mathscr{U}(x;r)) \leqslant \mathscr{H}_m(\mathscr{U}(x;r))C^{-1}r^{1-m}2^{2m-n}\sum_{j=1}^{k}\widehat{\mathscr{H}}(\mathscr{U}(z_j;2^{-n}r))$$
$$\leqslant \mathscr{H}_m(\mathscr{U}(x;r))C^{-1}r^{1-m}2^{2m-n}\widehat{\mathscr{H}}(\partial G\cap\mathscr{U}(x;3r))$$
$$\leqslant Am(m+2)^m\left(\frac{1}{2}+V^G\right)3^{m-1}\mathscr{H}_m(\mathscr{U}(x;r))C^{-1}2^{2m-n}$$

by [20], Corollary 2.17 and

$$\int\limits_{\mathscr{U}(x;r)\cap G}\operatorname{dist}(y,\partial G)^{\beta-1}\,\mathrm{d}\mathscr{H}_m(y) \leqslant \sum_{n=1}^{\infty}2^{-n(\beta-1)}r^{\beta-1}\mathscr{H}(E_n\setminus E_{n+1})$$
$$\leqslant \sum_{n=1}^{\infty}2^{-n\beta+2m}Am(m+2)^m\left(\frac{1}{2}+V^G\right)3^{m-1}C^{-1}\mathscr{H}_m(\mathscr{U}(x;r))r^{\beta-1}.$$

Relation (21) holds with

$$K = Am(m+2)^m\left(\frac{1}{2}+V^G\right)3^{3m}C^{-1}\sum_{n=1}^{\infty}(2^{\beta})^{-n}.$$

If $r\in(0,R/3)$ then

$$K\mathscr{H}_m(\mathscr{U}(x;r))r^{\beta-1} \geqslant \int\limits_{G\cap\mathscr{U}(x;r)\setminus\Gamma_\alpha(x)}[\operatorname{dist}(y,\partial G)]^{\beta-1}\,\mathrm{d}\mathscr{H}_m(y)$$
$$\geqslant (1+\alpha)^{1-\beta}\int\limits_{G\cap\mathscr{U}(x;r)\setminus\Gamma_\alpha(x)}|x-y|^{\beta-1}\,\mathrm{d}\mathscr{H}_m(y)$$
$$\geqslant (1+\alpha)^{1-\beta}r^{\beta-1}\mathscr{H}_m(G\cap\mathscr{U}(x;r)\setminus\Gamma_\alpha(x)).$$

If $\alpha > \left(\frac{K}{d_G(x)}\right)^{\frac{1}{1-\beta}}$ then

$$\frac{\mathscr{H}_m(G\cap\mathscr{U}(x;r)\setminus\Gamma_\alpha(x))}{\mathscr{H}_m(\mathscr{U}(x;r))} \leqslant \frac{K}{(1+\alpha)^{1-\beta}} < d_G(x)$$

and $x\in\operatorname{cl}\Gamma_\alpha$. Moreover,

$$\lim_{\gamma\to\infty}\bar{d}_{G\setminus\Gamma_\gamma}(x) = 0.$$

The rest is a consequence of this fact, the boundedness of $W^Gf+\mathscr{U}(f\lambda)$ and Lemma 2. $\square$

Lemma 4. *Let f be an $\widehat{\mathscr{H}}$-measurable bounded function on ∂G. Then there is $M \subset \partial G$ with $\widehat{\mathscr{H}}(M) = 0$ such that if $x \in \partial G \setminus M$ and there is $\alpha > 0$ such that $x \in \mathrm{cl}(\Gamma_\alpha(x))$ then $Vf(x)$ is the nontangential limit of $W^G f + \mathscr{U}(f\lambda)$ at x.*

P r o o f. We use Lemma 2 and the fact that the relation (20) holds at $\widehat{\mathscr{H}}$-a.a. points of ∂G. This fact is a moderate modification of results of A. S. Besicovitch and A. P. Morse and is in fact proved in the proof of Theorem 1.3.8 in [43]. $\square$

R e m a r k 2. Denote

$$v_r^G(x) = \int_{\partial G \cap \mathscr{U}(x;r)} |n^G(y) \cdot \nabla h_x(y)| \,\mathrm{d}\mathscr{H}_{m-1}(y)$$

for $x \in \mathbb{R}^m$, $r > 0$. Suppose that

$$\lim_{r \to 0+} \sup_{x \in \partial G} v_r^G(x) < \frac{1}{2}.$$

(This condition is stronger then the condition $r_{\mathrm{ess}}(\widehat{W}^G - \frac{1}{2}I) < 1/2$.) Denote $\widehat{\partial} G = \{y \in \partial G;\ n^G(y) \neq 0\}$. Then $\mathscr{H}_{m-1}(\partial G \setminus \widehat{\partial} G) = 0$ (see [20], Isoperimetric Lemma on page 50). According to [21], Lemma 3.10 and [28], Theorem 5.6 (compare [21], Theorem 3.13) there exists a positive constant c such that the following holds: For each $y \in \widehat{\partial} G$ there is a positive number δ, a neighbourhood U of y in $\{x \in \mathbb{R}^m;\ (x-y) \cdot n^G(y) = 0\}$ and a Lipschitz function f on U with the Lipschitz constant c such that $\partial G \cap \mathrm{cl}\,\mathscr{U}(y;\delta) = \{u + f(u) n^G(y);\ u \in U\}$. The suppositions of Lemma 3 are satisfied for each $x \in \widehat{\partial} G$ with C which depends only on the set G. If g is an $\widehat{\mathscr{H}}$-measurable bounded function on ∂G then $Vg(x)$ is the nontangential limit of $W^G g + \mathscr{U}(g\lambda)$ at x for $\widehat{\mathscr{H}}$ a.a. points $x \in \partial G$.

E x a m p l e 2. The following example shows that the condition

$$\lim_{r \to 0+} \frac{1}{\widehat{\mathscr{H}}(\mathscr{U}(x;r))} \int_{\mathscr{U}(x;r) \cap \partial G} f(y) \,\mathrm{d}\widehat{\mathscr{H}}(y) = f(x)$$

is not sufficient for the existence of the nontangential limit of $W^G f(y) + \mathscr{U}(f\lambda)(y)$ at x. Put $G = \{[z_1, z_2, z_3];\ \max |z_j| \leqslant 1\}$, $x = [0,0,1]$, $f(z_1, z_2, z_3) = z_1/|z_1|$ for

$z_1 \neq 0$, $z_3 = 1$, $f(z_1, z_2, z_3) = 0$ elsewhere. Fix $a \in (0,1)$. Then

$$\begin{aligned}
&\lim_{t\to 0+} \bigl[W^G f\bigl([ta, 0, 1-t(1-a)]\bigr) + \mathscr{U}(f\lambda)([ta, 0, 1-t(1-a)])\bigr] \\
&\quad = \mathscr{U}(f\lambda)(x) + \lim_{t\to 0_+} \frac{1}{A} \int_{-1}^{1} \biggl\{ \int_{-1}^{0} \frac{t(1-a)}{\bigl[(z_1 - ta)^2 + z_2^2 + t^2(1-a)^2\bigr]^{3/2}} \,\mathrm{d}z_1 \\
&\qquad - \int_0^1 \frac{t(1-a)}{\bigl[(z_1 - ta)^2 + z_2^2 + t^2(1-a)^2\bigr]^{3/2}} \,\mathrm{d}z_1 \biggr\} \,\mathrm{d}z_2 \\
&\quad = \mathscr{U}(f\lambda)(x) + \lim_{t\to 0_+} \frac{1}{A} \int_{\frac{-1}{t(1-a)}}^{\frac{1}{t(1-a)}} \biggl\{ \int_{\frac{-1-ta}{t(1-a)}}^{\frac{-a}{1-a}} \frac{1}{\bigl[z_1^2 + z_2^2 + 1\bigr]^{3/2}} \,\mathrm{d}z_1 \\
&\qquad - \int_{\frac{-a}{1-a}}^{\frac{1-ta}{t(1-a)}} \frac{1}{\bigl[z_1^2 + z_2^2 + 1\bigr]^{3/2}} \,\mathrm{d}z_1 \biggr\} \,\mathrm{d}z_2 \\
&\quad = \mathscr{U}(f\lambda)(x) + \lim_{t\to 0_+} \frac{1}{A} \int_{\frac{-1}{t(1-a)}}^{\frac{1}{t(1-a)}} \biggl\{ \int_{\frac{1-ta}{t(1-a)}}^{\frac{1+ta}{t(1-a)}} \frac{1}{\bigl[z_1^2 + z_2^2 + 1\bigr]^{3/2}} \,\mathrm{d}z_1 \\
&\qquad - \int_{\frac{-a}{1-a}}^{\frac{a}{1-a}} \frac{1}{\bigl[z_1^2 + z_2^2 + 1\bigr]^{3/2}} \,\mathrm{d}z_1 \biggr\} \,\mathrm{d}z_2 \\
&\quad = \mathscr{U}(f\lambda)(x) - \frac{2}{A} \int_{-\infty}^{\infty} \int_0^{\frac{a}{1-a}} \frac{1}{\bigl[z_1^2 + z_2^2 + 1\bigr]^{3/2}} \,\mathrm{d}z_1 \,\mathrm{d}z_2.
\end{aligned}$$

Since

$$\lim_{t\to 0+} \bigl[W^G f\bigl([0,0,1-t]\bigr) + \mathscr{U}(f\lambda)([0,0,1-t])\bigr] = \mathscr{U}(f\lambda)(x),$$

the nontangential limit of $W^G f + \mathscr{U}(f\lambda)$ in x does not exist.

E x a m p l e 3. The following example shows that the condition

$$\lim_{r\to 0+} \frac{1}{\widehat{\mathscr{H}}(\mathscr{U}(x;r))} \int_{\mathscr{U}(x;r)\cap\partial G} |f(y) - f(x)| \,\mathrm{d}\widehat{\mathscr{H}}(y) = 0$$

is not sufficient for the existence of the limit of $W^G f(y) + \mathscr{U}(f\lambda)(y)$ over G at x. Put $G = \{[z_1, z_2, z_3];\ \max |z_j| \leqslant 1\}$, $\lambda \equiv 0$, $x = [0,0,1]$, $f(z_1, z_2, z_3) = 1$ for $z_3 = 1$, $2^{-n} - 2^{-2n} < z_1^2 + z_2^2 < 2^{-n}$ and an integer n, $f(z_1, z_2, z_3) = 0$ elsewhere. Lemma 2 yields that the nontangential limit of $W^G f$ at x is equal to 0 and the nontangential limit of $W^G f$ at $[2^{-n} - 2^{-3n}, 0, 1]$ is equal to $1/2$. Thus there are $y_n \in G$ such that y_n tend to x and $\lim W^G f(y_n) = 1/2$. Therefore the limit of $W^G f$ over G in x does not exist.

Lemma 5. *Denote by $V_{\mathscr{C}}$ the operator V considered as an operator on $\mathscr{C}(\partial G)$ and by $V_{\mathscr{B}}$ the operator V mentioned as an operator on $\mathscr{B}(\partial G)$. Then*

$$r_{\text{ess}}\left(\widehat{W}^G - \frac{1}{2}I\right) = r_{\text{ess}}\left(V_{\mathscr{C}} - \frac{1}{2}I\right) = r_{\text{ess}}\left(V_{\mathscr{B}} - \frac{1}{2}I\right) = r_{\text{ess}}\left(\tilde{V} - \frac{1}{2}I\right).$$

P r o o f. Since $V_{\mathscr{C}} - \widehat{W}^G$ is a compact operator (see [32], Proposition 9), $r_{\text{ess}}(\widehat{W}^G - \frac{1}{2}I) = r_{\text{ess}}(V_{\mathscr{C}} - \frac{1}{2}I)$. Since $r_{\text{ess}}(V_{\mathscr{B}} - \frac{1}{2}I) = r_{\text{ess}}(V'_{\mathscr{C}} - \frac{1}{2}I)$ by [25], Lemma 1.5, we have $r_{\text{ess}}(V_{\mathscr{B}} - \frac{1}{2}I) = r_{\text{ess}}(V_{\mathscr{C}} - \frac{1}{2}I)$ by [40], Chapter IX, Theorem 2.1, Theorem 1.3 and Chapter VII, Theorem 3.5. Define an operator j from $\mathscr{C}(\partial G)$ to $L^\infty(\widehat{\mathscr{H}})$ by $j(g) = \tilde{g}$. Since $\widehat{\mathscr{H}}(\mathscr{U}(x;\delta) \cap \partial G) > 0$ for each $x \in \partial G, \delta > 0$ by Isoperimetric Lemma (see [20], p. 50) we have $\|j(g)\| = \|g\|$ for each $g \in \mathscr{C}(\partial G)$. Thus

$$r_{\text{ess}}\left(V_{\mathscr{C}} - \frac{1}{2}I\right) = r_{\text{ess}}\left(\left[\tilde{V} - \frac{1}{2}I\right] / j\left(\mathscr{C}(\partial G)\right)\right) \leqslant r_{\text{ess}}\left(\tilde{V} - \frac{1}{2}I\right)$$

by [19], Lemma 15. If $\alpha \in \mathbb{C}, |\alpha - 1/2| > r_{\text{ess}}(V_{\mathscr{B}} - \frac{1}{2}I)$ then $V_{\mathscr{B}} - \alpha I$ is a Fredholm operator by [17], Satz 51.8 and thus $(V_{\mathscr{B}} - \alpha I)(\mathscr{B}(\partial G))$ has a finite codimension in $\mathscr{B}(\partial G)$. Since $L^\infty(\widehat{\mathscr{H}}) = \{\tilde{f};\ f \in \mathscr{B}(\partial G)\}$, the subspace $(\tilde{V} - \alpha I)(L^\infty(\widehat{\mathscr{H}}))$ has a finite codimension in $L^\infty(\widehat{\mathscr{H}})$. Since $\tilde{V} - \alpha I$ is semi-Fredholm it is a Fredholm operator because the index is constant on each component of the semi-Fredholmness (see [12], Theorem 2.2, Theorem 8.1). Thus $r_{\text{ess}}(\tilde{V} - \frac{1}{2}I) \leqslant r_{\text{ess}}(V_{\mathscr{B}} - \frac{1}{2}I)$ by [17], Satz 51.8. □

Lemma 6. *Suppose that $r_{\text{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$. Fix α such that*

$$\alpha > \frac{1}{2}\bigl(1 + \sup_{x \in \partial G} \mathscr{U}\lambda(x)\bigr).$$

Then there are constants $d_\alpha \in \langle 1, \infty)$, $q_\alpha \in (0,1)$ such that for each $g \in \mathscr{B}(\partial G)$ and each positive integer number n

$$\left\|\left(\frac{V - \alpha I}{\alpha}\right)^n g\right\|_{\mathscr{B}(\partial G)} \leqslant d_\alpha q_\alpha^n \|g\|_{\mathscr{B}(\partial G)}. \tag{22}$$

If $g \in \mathscr{B}(\partial G)$ then there is a unique $f \in \mathscr{B}(\partial G)$ such that $Vf = g$ and this f is given by the series

$$f = \sum_{n=0}^{\infty}\left(-\frac{V - \alpha I}{\alpha}\right)^n \frac{g}{\alpha}. \tag{23}$$

If the set $\mathbb{R}^m \setminus G$ is unbounded and connected and $c = 0$ then there are constants $d \in \langle 1, \infty)$, $q \in (0,1)$ such that for each $g \in \mathscr{B}(\partial G)$ and each positive integer number n,

$$\bigl\|\bigl(I - 2V\bigr)^n (2I - 2V)g\bigr\|_{\mathscr{B}(\partial G)} \leqslant dq^n \|g\|_{\mathscr{B}(\partial G)}. \tag{24}$$

If $g \in \mathscr{B}(\partial G)$ then there is a unique $f \in \mathscr{B}(\partial G)$ such that $Vf = g$ and this f is given by the series

$$f = g + \sum_{j=0}^{\infty} (I - 2V)^j (2I - 2V) g. \tag{25}$$

P r o o f. Denote by $V_{\mathscr{C}}$ the operator V considered as an operator on $\mathscr{C}(\partial G)$ and by $V_{\mathscr{B}}$ the operator V considered as an operator on $\mathscr{B}(\partial G)$. Lemma 5 and [27], Theorem 2 yield that the spectral radius of the operator $V'_{\mathscr{C}} - \alpha I$ is smaller than $|\alpha|$ and thus the spectral radius of the operator $V''_{\mathscr{C}} - \alpha I$ is smaller than $|\alpha|$ by [42], Chapter VII, § 1, Theorem 2'. Since $V_{\mathscr{B}}$ is the restriction of the operator $V''_{\mathscr{C}}$ onto $\mathscr{B}$ the operator $V_{\mathscr{B}} - \beta I$ is injective for each $\beta \in \mathbb{C}$ such that $|\beta - \alpha| \geqslant |\alpha|$. Since $V_{\mathscr{B}} - \beta I$ is a Fredholm operator with index 0 by Lemma 5, [17], Satz 51.8, Satz 51.1, the spectral radius of the operator $V_{\mathscr{B}} - \alpha I$ is smaller than $|\alpha|$. Therefore there are constants $d_\alpha \in \langle 1, \infty), q_\alpha \in (0,1)$ such that the relation (22) holds for each $g \in \mathscr{B}(\partial G)$ and an integer number n. Since V is invertible in $\mathscr{B}$ there is a unique $f \in \mathscr{B}(\partial G)$ such that $Vf = g$ for each $g \in \mathscr{B}(\partial G)$. Easy calculation yields that f given by the series (23) fulfils $Vf = g$.

Suppose now that $\mathbb{R}^m \setminus G$ is unbounded and connected. We can suppose that $\lambda \equiv 0$. Then the operator $V'_{\mathscr{C}} - \beta I$ is invertible for each $\beta \in \mathbb{C}$ such that $|\beta - 1/2| \geqslant 1/2$, $\beta \neq 0$, $\beta \neq 1$ by Lemma 5, [42], Chapter VII, § 1, Theorem 2', [17], Satz 51.8, Satz 51.1, [26], Proposition 1, and the operator $V''_{\mathscr{C}} - \beta I$ is invertible by [42], Chapter VIII, § 6, Theorem 1. Since $V_{\mathscr{B}}$ is the restriction of the operator $V''_{\mathscr{C}}$ onto $\mathscr{B}$ the operator $V_{\mathscr{B}} - \beta I$ is injective. Since $V_{\mathscr{B}} - \beta I$ is a Fredholm operator with index 0 by Lemma 5, [17], Satz 51.8, Satz 51.1, the operator $V_{\mathscr{B}} - \beta I$ is invertible. The operator $V_{\mathscr{C}}$ is invertible by [17], Satz 51.8, Satz 51.1, [26], Theorem 2 and the operator $V_{\mathscr{C}''}$ is invertible by [42], Chapter VIII, § 6, Theorem 1. Since $V_{\mathscr{B}}$ is the restriction of the operator $V''_{\mathscr{C}}$ onto $\mathscr{B}$ the operator $V_{\mathscr{B}}$ is injective. Since $V_{\mathscr{B}}$ is a Fredholm operator with index 0 by Lemma 5, [17], Satz 51.8, Satz 51.1, the operator $V_{\mathscr{B}}$ is invertible. Since $\operatorname{Ker}(V'_{\mathscr{C}} - I)^2 = \operatorname{Ker}(V'_{\mathscr{C}} - I)$ by [26], Proposition 2, we have $\operatorname{Ker}(V_{\mathscr{C}} - I)^2 = \operatorname{Ker}(V_{\mathscr{C}} - I)$ by [17], Satz 51.8, Satz 51.1. Since $\operatorname{Ker}(V''_{\mathscr{C}} - I) = \operatorname{Ker}(V_{\mathscr{C}} - I)$, $\operatorname{Ker}(V''_{\mathscr{C}} - I)^2 = \operatorname{Ker}(V_{\mathscr{C}} - I)^2$ by [40], Chapter VII, Theorem 3.1, Theorem 3.2 and $V_{\mathscr{B}}$ is the restriction of the operator $V''_{\mathscr{C}}$ onto $\mathscr{B}$ we have $\operatorname{Ker}(V_{\mathscr{B}} - I)^2 = \operatorname{Ker}(V''_{\mathscr{C}} - I)^2 = \operatorname{Ker}(V''_{\mathscr{C}} - I) = \operatorname{Ker}(V_{\mathscr{B}} - I)$. The space $\mathscr{B}(\partial G)$ is the direct sum of $\operatorname{Ker}(V_{\mathscr{B}} - I)$ and $(V_{\mathscr{B}} - I)(\mathscr{B}(\partial G))$ by [17], Satz 50.2. Denote by $\check{V}$ the restriction of $V_{\mathscr{B}}$ onto $(V_{\mathscr{B}} - I)(\mathscr{B}(\partial G))$. Since $\check{V} - \beta I$ is invertible for each $\beta \in \mathbb{C}$, $|\beta - 1/2| \geqslant 1/2$, the spectral radius of $\check{V} - \frac{1}{2} I$ is smaller than $\frac{1}{2}$ and there are constants $d \in \langle 1, \infty)$, $q \in (0,1)$ such that the relation (24) holds. Easy calculation yields that f given by the series (25) fulfils $Vf = g$. $\square$

Theorem 3. *Suppose that $r_{\mathrm{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$. Fix α such that*

$$\alpha > \frac{1}{2}\Bigl(1 + \sup_{x\in\partial G} \mathscr{U}\lambda(x)\Bigr).$$

Let g be a bounded $\mathscr{H}$-measurable function. Then

$$H_g^G = W^G\tilde{f} + \mathscr{U}(\tilde{f}\lambda),$$

where

$$\tilde{f} = \sum_{n=0}^{\infty}\Bigl(-\frac{\tilde{V}-\alpha I}{\alpha}\Bigr)^n\frac{\tilde{g}}{\alpha}. \tag{26}$$

There is $M \subset \partial G$ with $\mathscr{H}(M) = 0$ (depending on g) such that if $x \in \partial G \setminus M$ and there is $\beta > 0$ such that $x \in \mathrm{cl}(\Gamma_\beta(x))$ then $g(x)$ is the nontangential limit of H_g^G at x. Further,

$$\sup_{y\in\partial G} H_g^G(y) \leqslant \|g\|_{L^\infty(\mathscr{H})}.$$

If $g \in \mathscr{B}(\partial G)$ then

$$H_g^G = W^G f + \mathscr{U}(f\lambda),$$

where

$$f = \sum_{n=0}^{\infty}\Bigl(-\frac{V-\alpha I}{\alpha}\Bigr)^n\frac{g}{\alpha}. \tag{27}$$

If the set $\mathbb{R}^m \setminus G$ is unbounded and connected and $c = 0$ then

$$f = g + \sum_{j=0}^{\infty}(2I - V)^j(2I - 2V)g. \tag{28}$$

P r o o f. Let $g \in \mathscr{B}(\partial G)$. Then f given by the series (27) is a solution of the equation $Vf = g$ by Lemma 6. If the set $\mathbb{R}^m \setminus G$ is unbounded and connected then the relation (28) holds by Lemma 6. Let us denote by $\mathscr{S}$ the set of $g \in \mathscr{B}(\partial G)$ for which $H_g^G = W^G f + \mathscr{U}(f\lambda)$. Theorem 2 yields that $\mathscr{C}(\partial G) \subset \mathscr{S}$.

Let $f, g \in \mathscr{B}(\partial G)$, $Vf = g$, let $\{g_n\}$ be a uniformly bounded sequence of elements of $\mathscr{S}$, $\lim g_n = g$ pointwise on ∂G. Let $\{f_n\}$ be a sequence of elements of $\mathscr{B}(\partial G)$ such that $Vf_n = g_n$, $H_{g_n}^G = W^G f_n + \mathscr{U}(f_n\lambda)$. Fix $z \in G$ and observe that

$$\lim_{n\to\infty} H_{g_n}^G(z) = \lim_{n\to\infty}\int g_n \,\mathrm{d}\mu_z^G = H_g^G(z)$$

by the Lebesgue dominated convergence theorem. Put

$$\nu_z = n^G(y) \cdot \nabla h_z(y) \mathscr{H} + h_z \lambda.$$

Denote by $V_{\mathscr{C}}$ the operator V considered as an operator on $\mathscr{C}(\partial G)$ and by $V_{\mathscr{B}}$ the operator V considered as an operator on $\mathscr{B}(\partial G)$. There is $\mu \in \mathscr{C}'(\partial G)$ such that $V'_{\mathscr{C}}\mu = \nu_z$ by [27], Theorem 1. Since $V'_{\mathscr{C}}$ is the restriction of $V'_{\mathscr{B}}$ onto $\mathscr{C}'(\partial G)$ (see [35], Proposition 8) the Lebesgue dominated convergence theorem yields that

$$\begin{aligned} H_g^G(z) &= \lim_{n\to\infty} [W^G f_n(z) + \mathscr{U}(f_n\lambda)(z)] = \lim_{n\to\infty} \int f_n \,\mathrm{d}\nu_z = \lim_{n\to\infty} \int f_n \,\mathrm{d}V'_{\mathscr{B}}\mu \\ &= \lim_{n\to\infty} \int g_n \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu = \int V f \,\mathrm{d}\mu = \int f \,\mathrm{d}V'_{\mathscr{B}}\mu = \int f \,\mathrm{d}\nu_z \\ &= W^G f(z) + \mathscr{U}(f\lambda)(z). \end{aligned}$$

Therefore $\mathscr{S} = \mathscr{B}(\partial G)$.

Let now g be a bounded $\mathscr{H}$-measurable function. Then there is $h \in \mathscr{B}(\partial G)$ such that $h = g$ at $\mathscr{H}$-a.a. points of ∂G. Since μ_x^G is absolutely continuous with respect to $\mathscr{H}$ for each $x \in G$ by Theorem 2, we have $H_g^G = H_h^G$ and relation (26) holds. Lemma 4 yields that there is $M \subset \partial G$ with $\mathscr{H}(M) = 0$ such that if $x \in \partial G \setminus M$ and there is $\beta > 0$ such that $x \in \mathrm{cl}(\Gamma_\beta(x))$ then $g(x)$ is the nontangential limit of $H_g^G = W^G f + \mathscr{U}(f\lambda)$ at x. Fix $y \in G$, put φ identically equal to $\|g\|_{L^\infty(\mathscr{H})}$. Using the maximum principle for H_φ^G and the fact that μ_y^G is absolutely continuous with respect to $\mathscr{H}$, we conclude

$$|H_g^G(y)| = \left| \int g \,\mathrm{d}\mu_y^G \right| \leqslant \int \varphi \,\mathrm{d}\mu_y^G = H_\varphi^G(y) \leqslant \|g\|_{L^\infty(\mathscr{H})}.$$

□

Lemma 7. *Let $x \in \partial G$, $R > 0$, let $\psi\colon \mathscr{U}(x; 2R) \to \mathbb{R}^m$ be a diffeomorphism of class $C^{1+\alpha}$, where $\alpha > 0$ and $D\psi(x) = I$, where $D\psi(x)$ is the total differential of ψ at the point x. Denote $H = \psi(G \cap \mathscr{U}(x; R))$, $u = \psi(x)$. Suppose that if $f \in \mathscr{B}(\partial G)$ and*

$$\lim_{\delta\to 0_+} \frac{1}{\widehat{\mathscr{H}}\big(\mathscr{U}(x;\delta)\big)} \int_{\partial G \cap \mathscr{U}(x;\delta)} |f(y) - f(x)| \,\mathrm{d}\widehat{\mathscr{H}}(y) = 0 \tag{29}$$

then

$$\lim_{\delta\to 0_+} \frac{1}{\widehat{\mathscr{H}}\big(\mathscr{U}(x;\delta)\big)} \int_{\partial G \cap \mathscr{U}(x;\delta)} |\widehat{W}^G f(y) - \widehat{W}^G f(x)| \,\mathrm{d}\widehat{\mathscr{H}}(y) = 0. \tag{30}$$

Denote by $\tilde{\mathscr{H}}$ the restriction of the $(m-1)$-dimensional Hausdorff measure onto $\partial_e H$. If $f \in \mathscr{B}(\partial H)$ and

$$\lim_{\delta\to 0_+} \frac{1}{\tilde{\mathscr{H}}\big(\mathscr{U}(u;\delta)\big)} \int\limits_{\partial H\cap \mathscr{U}(u;\delta)} |f(y)-f(u)|\,\mathrm{d}\tilde{\mathscr{H}}(y) = 0 \tag{31}$$

then

$$\lim_{\delta\to 0_+} \frac{1}{\tilde{\mathscr{H}}\big(\mathscr{U}(u;\delta)\big)} \int\limits_{\partial H\cap \mathscr{U}(u;\delta)} |\widehat{W}^H f(y)-\widehat{W}^H f(u)|\,\mathrm{d}\tilde{\mathscr{H}}(y) = 0. \tag{32}$$

P r o o f. Let $f \in \mathscr{B}(H)$ and let the relation (31) hold. Fix $\varepsilon > 0$. Then there is $r > 0$ such that for each $y \in \partial H \cap \mathscr{U}(u;r)$ we have

$$\begin{aligned}\Bigg| &\int\limits_{\mathscr{U}(u;r)\cap\partial H} [f(z)-f(u)]\nabla h_{\psi(y)}(z)\cdot n^H(z)\,\mathrm{d}\tilde{\mathscr{H}}(z) \\ &- \int\limits_{\psi^{-1}(\mathscr{U}(u;r))\cap\partial G} [f(\psi(z))-f(u)]\nabla h_y(z)\cdot n^G(z)\,\mathrm{d}\widehat{\mathscr{H}}(z)\Bigg| < \frac{\varepsilon}{4}\end{aligned}$$

(see [25], Lemma 2.1). Put

$$\begin{aligned} f_1(y) &= \begin{cases} f(u) & \text{on } \partial H\cap\mathscr{U}(u;r),\\ f(y) & \text{on } \partial H\setminus\mathscr{U}(u;r),\end{cases}\\ f_2(y) &= \begin{cases} f(y)-f(u) & \text{on } \partial H\cap\mathscr{U}(u;r),\\ 0 & \text{on } \partial H\setminus\mathscr{U}(u;r).\end{cases}\end{aligned}$$

Since $\psi(\partial_e G\cap\mathscr{U}(x;R)) = \partial_e H\cap\psi(\mathscr{U}(x;R))$ and $D\psi(x) = I$, for each $\delta > 0$ there is $r_1 > 0$ such that if E is a Borel subset of $\mathscr{U}(x;r_1)$ then $(1-\delta)\widehat{\mathscr{H}}(E) \leqslant \tilde{\mathscr{H}}(\psi(E)) \leqslant (1+\delta)\widehat{\mathscr{H}}(E)$. Easy calculation yields that (31) implies

$$\lim_{\delta\to 0_+} \frac{1}{\widehat{\mathscr{H}}\big(\mathscr{U}(x;\delta)\big)} \int\limits_{\partial G\cap \mathscr{U}(x;\delta)} |f_2(\psi((y))-f_2(\psi((x))|\,\mathrm{d}\widehat{\mathscr{H}}(y) = 0.$$

Since $|W^H f_2(y) - \widehat{W}^G(f_2 \circ \psi)(\psi^{-1}(y))| < \varepsilon/4$ for $y \in \partial H \cap \mathscr{U}(u;r)$ we obtain using the fact that $D\psi(x) = I$

$$\begin{aligned}
&\lim_{\delta\to 0_+} \frac{1}{\tilde{\mathscr{H}}\big(\mathscr{U}(u;\delta)\big)} \int\limits_{\partial H\cap \mathscr{U}(u;\delta)} |W^H f_2(y) - W^H f_2(u)| \, \mathrm{d}\tilde{\mathscr{H}}(y) \\
&\quad \leqslant \lim_{\delta\to 0_+} \int\limits_{\mathscr{U}(u;\delta)} \frac{|W^H f_2(y) - W^H f_2(u) - \widehat{W}^G f_2 \circ \psi(\psi^{-1}(y)) + \widehat{W}^G f_2 \circ \psi(x)|}{\tilde{\mathscr{H}}\big(\mathscr{U}(u;\delta)\big)} \, \mathrm{d}\tilde{\mathscr{H}}(y) \\
&\qquad + \lim_{\delta\to 0_+} \frac{1}{\tilde{\mathscr{H}}\big(\mathscr{U}(u;\delta)\big)} \int\limits_{\partial H\cap \mathscr{U}(u;\delta)} |\widehat{W}^G(f_2 \circ \psi)(\psi^{-1}(y)) - \widehat{W}^G(f_2 \circ \psi)(x)| \, \mathrm{d}\tilde{\mathscr{H}}(y) \\
&\quad \leqslant \frac{\varepsilon}{2} + 2 \lim_{\delta\to 0_+} \frac{1}{\widehat{\mathscr{H}}\big(\mathscr{U}(x;\delta)\big)} \int\limits_{\mathscr{U}(x;\delta)} |\widehat{W}^G(f_2 \circ \psi)(y) - \widehat{W}^G(f_2 \circ \psi)(x)| \, \mathrm{d}\widehat{\mathscr{H}}(y) = \frac{\varepsilon}{2}.
\end{aligned}$$

Since f_1 is continuous at u there is $r_1 > 0$ such that $|W^H f_1(y) - W^H f_1(u)| < \varepsilon/4$ for $y \in \partial H \cap \mathscr{U}(u;r_1)$. The relation (32) holds. $\square$

Lemma 8. *Suppose that* $r_{\mathrm{ess}}(\widehat{W}^G - \frac{1}{2}I) < \frac{1}{2}$, $g, f \in \mathscr{B}(\partial G)$, $Vf = g$, $x \in \partial G$. *Then* g *is continuous at* x *if and only if* f *is continuous at* x. *If* g *is continuous at* x *then*

$$\lim_{y\in G, y\to x} H_g^G(y) = g(y).$$

Suppose that $\psi(\partial G \cap \mathscr{U}(x;R))$ *is a subset of the union of finite number of hyperplanes, where* $\psi\colon \mathscr{U}(x;R) \to \mathbb{R}^m$ *is a diffeomorphism of class* $C^{1+\beta}$ $(R, \beta > 0)$. *Then*

$$\lim_{\delta\to 0_+} \frac{1}{\mathscr{H}\big(\mathscr{U}(x;\delta)\big)} \int\limits_{\partial G\cap \mathscr{U}(x;\delta)} |f(y) - f(x)| \, \mathrm{d}\mathscr{H}(y) = 0 \tag{33}$$

if and only if

$$\lim_{\delta\to 0_+} \frac{1}{\mathscr{H}\big(\mathscr{U}(x;\delta)\big)} \int\limits_{\partial G\cap \mathscr{U}(x;\delta)} |g(y) - g(x)| \, \mathrm{d}\mathscr{H}(y) = 0. \tag{34}$$

If relation (34) holds then $g(x)$ *is the nontangential limit of* H_g^G *at* x *and*

$$\lim_{\delta\to 0_+} \frac{1}{\mathscr{H}_m\big(\mathscr{U}(x;\delta)\cap G\big)} \int\limits_{G\cap \mathscr{U}(x;\delta)} |H_g^G(y) - g(x)| \, \mathrm{d}\mathscr{H}_m(y) = 0. \tag{35}$$

P r o o f. Suppose that f is continuous at x. Fix $\varepsilon > 0$. Then there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/[6(\|V\| + 1)]$ for $y \in \partial G$, $|y - x| < \delta$. Put

$$f_1(y) = f(x) \qquad \text{for } y \in \partial G,$$
$$f_2(y) = \begin{cases} f(y) - f(x) & \text{if } y \in \partial G, |y - x| > \delta, \\ 0 & \text{if } y \in \partial G, |y - x| \leqslant \delta, \end{cases}$$
$$f_3(y) = \begin{cases} 0 & \text{if } y \in \partial G, |y - x| > \delta, \\ f(y) - f(x) & \text{if } y \in \partial G, |y - x| \leqslant \delta. \end{cases}$$

Since Vf_1, Vf_2 are continuous at x, there is $r \in (0, \delta)$ such that $|Vf_1(y) - Vf_1(x)| < \varepsilon/3$, $|Vf_2(y) - Vf_2(x)| < \varepsilon/3$ for $y \in \partial G$, $|x - y| < r$. For such y we have $|Vf(x) - Vf(y)| \leqslant |Vf_1(x) - Vf_1(y)| + |Vf_2(x) - Vf_2(y)| + |Vf_3(x)| + |Vf_3(y)| < \varepsilon$. Therefore $g = Vf$ is continuous at x.

Suppose now that g is continuous at x. Lemma 6 yields that

$$f = \sum_{n=0}^{\infty} \Big(-\frac{V - \alpha I}{\alpha} \Big)^n \frac{g}{\alpha} \tag{36}$$

and

$$\Big\| \Big(\frac{V - \alpha I}{\alpha} \Big)^n g \Big\|_{\mathscr{B}(\partial G)} \leqslant d_\alpha q_\alpha^n \|g\|_{\mathscr{B}(\partial G)}, \tag{37}$$

where $q_\alpha \in (0, 1)$. Fix $\varepsilon > 0$. Then there is n_0 such that

$$\sum_{n=n_0}^{\infty} \Big\| \Big(-\frac{V - \alpha I}{\alpha} \Big)^n \frac{g}{\alpha} \Big\|_{\mathscr{B}(\partial G)} < \varepsilon. \tag{38}$$

We have proved that

$$\sum_{n=0}^{n_0} \Big(-\frac{V - \alpha I}{\alpha} \Big)^n \frac{g}{\alpha}$$

is continuous at x. Therefore f is continuous at x, too. Lemma 1 and Theorem 3 yield that

$$\lim_{y \in G, y \to x} H_g^G(y) = g(y).$$

(This assertation is known (see [18], Satz 3).)

Suppose that $\psi(\partial G \cap \mathscr{U}(x; R))$ is a subset of the union of a finite number of hyperplanes, where $\psi\colon \mathscr{U}(x; R) \to \mathbb{R}^m$ is a diffeomorphism of class $C^{1+\beta}$ $(R, \beta > 0)$. Suppose that relation (33) holds. With respect to Lemma 7 and the fact that $\mathscr{U}(f\lambda)$ is continuous, we can suppose that

$$\partial G \cap \mathscr{U}(x; R) \subset \bigcup_{j=1}^{n} L_j,$$

where L_j are different hyperplanes intersecting x. Put

$$f_0(y) = \begin{cases} f(x) & \text{for } y \in \partial G \cap \mathscr{U}(x;R), \\ f(y) & \text{for } y \in \partial G \setminus \mathscr{U}(x;R), \end{cases}$$

$$f_j(y) = \begin{cases} f(y) - f(x) & \text{for } y \in L_j \cap \partial G \cap \mathscr{U}(x;R), \\ 0 & \text{elsewhere.} \end{cases}$$

Since f_0 is a continuous fuction the function Vf_0 is continuous, too. If $y \in L_j$ then $Vf_j(y) - Vf_j(x) = (1 - d_G(y))(f(y) - f(x)) + \mathscr{U}(f_j\lambda)(y) - \mathscr{U}(f_j\lambda)(x)$ and thus relation (33) yields

$$\lim_{\delta \to 0_+} \frac{1}{\mathscr{H}\big(\mathscr{U}(x;\delta)\big)} \int\limits_{\partial G \cap \mathscr{U}(x;\delta) \cap L_j} |Vf_j(y) - Vf_j(x)| \,\mathrm{d}\mathscr{H}(y) = 0,$$

because $\mathscr{U}(f_j\lambda)$ is a continuous function. Let now $i \neq j$. Let M be a component of $\mathscr{U}(x;R) \setminus L_j$, let $\tilde{\mathscr{H}}$ be the restriction of $\mathscr{H}_{m-1}$ onto ∂M. Put

$$h_j(y) = \begin{cases} f_j(y) - f_j(x) & \text{if } y \in \partial G \cap \partial M, n^G(y) = n^M(y), \\ f_j(x) - f_j(y) & \text{if } y \in \partial G \cap \partial M, n^G(y) = -n^M(y), \\ 0 & \text{elsewhere.} \end{cases}$$

If $y \in M \cap L_i$ then $Vf_j(y) - Vf_j(x) = W^M h_j(y) - W^M h_j(x) + \mathscr{U}(f_j\lambda)(y) - \mathscr{U}(f_j\lambda)(x)$. Since $\mathscr{U}(f_j\lambda)$, $\mathscr{U}(h_j\tilde{\mathscr{H}})$ are continuous functions, Lemma 2 yields that

$$\lim_{y \in L_i \cap M, y \to x} Vf_j(y) = Vf_j(x).$$

Therefore (34) holds.

Suppose now that (34) holds. Then f is given by the series (36) by Lemma 6 and relation (37) holds, where $q_\alpha \in (0,1)$. Fix $\varepsilon > 0$. Then there is n_0 such that relation (38) holds. We have proved that

$$\lim_{\delta \to 0_+} \frac{1}{\mathscr{H}\big(\mathscr{U}(x;\delta)\big)} \int\limits_{\partial G \cap \mathscr{U}(x;\delta)} \left| \sum_{n=0}^{n_0} \Big(-\frac{V - \alpha I}{\alpha}\Big)^n \frac{g}{\alpha}(y) - \Big(-\frac{V - \alpha I}{\alpha}\Big)^n \frac{g}{\alpha}(x) \right| \mathrm{d}\mathscr{H}(y) = 0.$$

For sufficiently small δ we have

$$\frac{1}{\mathscr{H}\big(\mathscr{U}(x;\delta)\big)} \int\limits_{\partial G \cap \mathscr{U}(x;\delta)} |f(y) - f(x)| \,\mathrm{d}\mathscr{H}(y) < \varepsilon$$

and relation (33) holds. Therefore $g(x)$ is the nontangential limit of H_g^G at x by Theorem 3 and Lemma 2. Relation (35) holds by Lemma 3. $\square$

References

[1] *F. Ahner*: The exterior Dirichlet problem for the Helmholtz equation. Journal of mathematical analysis and application *52* (1975), 415–429.

[2] *R. S. Angell, R. E. Kleinman, J. Král*: Layer potentials on boundaries with corners and edges. Čas. pěst. mat. *113* (1988), 387–402.

[3] *H. Bauer*: Harmonische Räume und ihre Potentialtheorie. Springer Verlag, Berlin, 1966.

[4] *A. S. Besicovitch*: A general form of the covering principle and relative differentiation of additive functions I. Proc. Cambridge Phil. Soc. *41* (1945), 103–110.

[5] *A. S. Besicovitch*: A general form of the covering principle and relative differentiation of additive functions II. Proc. Cambridge Phil. Soc. *42* (1946), 1–10.

[6] *N. Boboc, C. Constantinescu, A. Cornea*: On the Dirichlet problem in the axiomatic theory of harmonic functions. Nagoya Math. J. *23* (1963), 73–96.

[7] *Yu. D. Burago, V. G. Maz' ya*: Potential theory and function theory for irregular regions. Leningrad, 1969. (In Russian Seminars in mathematics V. A. Steklov Mathematical Institute.)

[8] *M. Chlebík*: Tricomi potentials. Thesis, Mathematical Institute of the Czechoslovak Academy of Sciences, Praha, 1988. (In Slovak.)

[9] *D. Colton, R. Kress*: Iterative methods for solving the exterior Dirichlet problem for the Helmholtz equation with applications to the inverse scattering problem for low frequency acoustic waves. Journal of mathematical analysis and applications *77* (1980), 60–72.

[10] *M. Dont*: Non-tangential limits of the double layer potentials. Čas. pěst. mat. *97* (1972), 231–258.

[11] *H. Federer*: Geometric Measure Theory. Springer-Verlag, 1969.

[12] *I. C. Gohberg, M. S. Krein*: The basic propositions on defect numbers, root numbers and indices of linear operators. Amer. Math. Soc. Transl. *13* (1960), 185–264.

[13] *N. V. Grachev, V. G. Maz'ya*: On the Fredholm radius for operators of the double layer potential type on piecewise smooth boundaries. Vest. Leningrad. Univ. *19(4)* (1986), 60–64.

[14] *N. V. Grachev, V. G. Maz'ya*: Estimates for kernels of the inverse operators of the integral equations of elasticity on surfaces with conic points. Report LiTH-MAT-R-91-06, Linköping Univ., Sweden.

[15] *N. V. Grachev, V. G. Maz'ya*: Invertibility of boundary integral operators of elasticity on surfaces with conic points. Report LiTH-MAT-R-91-07, Linköping Univ., Sweden.

[16] *N. V. Grachev, V. G. Maz'ya*: Solvability of a boundary integral equation on a polyhedron. Report LiTH-MAT-R-91-50, Linköping Univ., Sweden.

[17] *H. Heuser*: Funktionalanalysis. Teubner, Stuttgart, 1975.

[18] *J. Köhn, M. Sieveking*: Zum Cauchyschen und Dirichletschen Problem. Math. Ann. *177* (1968), 133–142.

[19] *V. Kordula, V. Müller, V. Rakočević*: On the semi-Browder spectrum. Studia Math. *123* (1997), 1–13.

[20] *J. Král*: Integral Operators in Potential Theory. Lecture Notes in Mathematics 823. Springer-Verlag, Berlin, 1980.

[21] *J. Král*: The Fredholm method in potential theory. Trans. Amer. Math. Soc. *125* (1966), 511–547.

[22] *J. Král, W. L. Wendland*: Some examples concerning applicability of the Fredholm-Radon method in potential theory. Aplikace matematiky *31* (1986), 239–308.

[23] *R. Kress, G. F. Roach*: On the convergence of successive approximations for an integral equation in a Green's function approach to the Dirichlet problem. Journal of mathematical analysis and applications *55* (1976), 102–111.

[24] *V. G. Maz'ya*: Boundary Integral Equations. Sovremennyje problemy matematiki, fundamental'nyje napravlenija, 27. Viniti, Moskva (Russian), 1988.
[25] *D. Medková*: The third boundary value problem in potential theory for domains with a piecewise smooth boundary. Czechoslov. Math. J. *47* (1997), 651–679.
[26] *D. Medková*: Solution of the Neumann problem for the Laplace equation. Czechoslov. Math. J. *48* (1998), 768–784.
[27] *D. Medková*: Solution of the Robin problem for the Laplace equation. Appl. of Math. *43* (1998), 133–155.
[28] *M. Miranda*: Distribuzioni aventi derivate misure, Insiemi di perimetro localmente finito. Ann. Scuola norm. Sup. Pisa Cl. Sci. *18* (1964), 27–56.
[29] *D. Mitrea*: The method of layer potentials for non-smooth domains with arbitrary topology. Integr. equ. oper. theory *29* (1997), 320–338.
[30] *A. P. Morse*: Perfect blancets. Trans. Am. Math. Soc. *61* (1947), 418–442.
[31] *I. Netuka*: Fredholm radius of a potential theoretic operator for convex sets. Čas. pěst. mat. *100* (1975), 374–383.
[32] *I. Netuka*: An operator connected with the third boundary value problem in potential theory. Czechoslov. Math. J. *22(97)* (1972), 462–489.
[33] *I. Netuka*: The third boundary value problem in potential theory. Czechoslov. Math. J. *2(97)* (1972), 554–580.
[34] *I. Netuka*: Smooth surfaces with infinite cyclic variation (Czech). Čas. pěst. mat. *96* (1971), 86–101.
[35] *I. Netuka*: Generalized Robin problem in potential theory. Czechoslov. Math. J. *22(97)* (1972), 312–324.
[36] *I. Netuka*: Double layer potential representation of the solution of the Dirichlet problem. Comment. Math. Un. Carolinae *14* (1973), 183–186.
[37] *I. Netuka*: Double layer potentials and Dirichlet problem. Czechoslov. Math. J. *24(99)* (1974), 59–73.
[38] *A. Rathsfeld*: The invertibility of the double layer potential in the space of continuous functions defined on a polyhedron. The panel method. Applicable Analysis *45* (1992), 1–4, 135–177.
[39] *A. Rathsfeld*: The invertibility of the double layer potential in the space of continuous functions defined on a polyhedron. The panel method. Erratum. Applicable Analysis *56* (1995), 109–115.
[40] *M. Schechter*: Principles of Funtional Analysis. Academic Press, London, 1971.
[41] *H. Watanabe*: Double layer potentials for a bounded domain with fractal boundary. in Potential Theory. ICPT 94, ed. J. Král, J. Lukeš, I. Netuka, J. Veselý. Walter de Gruyter, Berlin, New York, 1996, pp. 463–471.
[42] *K. Yosida*: Functional Analysis. Springer Verlag, 1965.
[43] *W. P. Ziemer*: Weakly Differentiable Functions. Springer Verlag, 1989.

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