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LIMIT CYCLES IN THE EQUATION OF WHIRLING PENDULUM WITH AUTONOMOUS PERTURBATION

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Abstract. The two-parameter Hamiltonian system with the autonomous perturbation is considered. Via the Mel'nikov method, existence and uniqueness of a limit cycle of the system in a certain region of a two-dimensional space of parameters is proved.

Keywords: whirling pendulum, Hamiltonian system, autonomous perturbation, Melnikov function, limit cycle, homoclinic orbit, elliptic integral

MSC 2000: 58F21, 34C05

1. Introduction

We will consider the second-order differential equation

$$\ddot{x} = \sin x(\cos x - \gamma) + \varepsilon\dot{x}(\cos x + \alpha), \qquad x \in S^1. \tag{1}$$

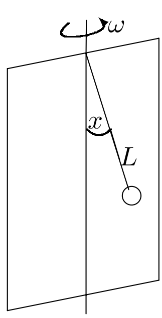
For $\varepsilon = 0$ this equation models the motion of a pendulum with length L in the plane rotating about its vertical axis with constant rotation rate ω. (The whirling pendulum with non-constant rotation rate is considered in [10].) The parameter γ is defined as $\gamma = g/L\omega^2$. The perturbation represents a small external forcing periodically dependent on the deviation x. Equations of such type were examined in the papers [6] and [8], which dealt with the perturbed harmonic oscillator $\ddot{x} = -\sin x$. The perturbation $g(x) = \dot{x}(\cos x + \alpha)$ is the same as in the above mentioned papers for a special choice of parameters. Particularly, in [8] the Josephson equation with the perturbation $g_1(x) = a - \dot{x}(1 + \gamma\cos x)$ is considered, where $\overline{a} = \varepsilon a$ represents a small output voltage in Josephson junction; we suppose $a = 0$. In [6], for the perturbation $g_2(x) = \cos nx + \alpha$, $n \in N$, it is shown that the system has n limit cycles; we suppose $n = 1$.

As a powerful tool, we use the Melnikov function, whose zeros correspond to limit cycles. However, looking for these zeros is not easy. It often leads to expressions with elliptic integrals, where the implicit relationship between the elliptic modulus k and the energy level h is complicated. That is why the direct calculations usually bring only incomplete results. A useful technique is deriving the so called Picard-Fuchs equations for integrals in the Melnikov function and then analyzing the equation which results from them. There are no known general techniques for the analysis of the resulting equation and one must solve it case by case. This approach was used in the study of limit cycles in the Josephson equation [8]. We have succeeded in applying it only partly for some values of the parameter γ. This is mainly caused by the complicated expression standing for the energy of the whirling pendulum. We have obtained more complete results by applying the Li and Zhang criterion (see [5]), which allows to examine the monotonicity of the ratio of two Abelian integrals without calculating them.

The paper is organized as follows. In Section 2 we shortly describe the whirling pendulum and its dynamics. In Section 3 we examine local bifurcations of a perturbed system. The Melnikov function and some of its properties are derived in Section 4; we give a full description of the phase portrait of (1) for some values of the parameter γ there. The other cases are considered in Section 5, where the Li and Zhang criterion is described and used to determine intervals of monotonicity of the Melnikov function.

2. Whirling pendulum

The whirling pendulum is shown in Fig. 1. It consists of a rigid frame that freely rotates about a vertical axis with constant rotation rate ω, to which a planar pendulum with length L is attached, the pivot being on the vertical axis.



x–angle deviation
ω–rotation rate
L–length of pendulum

Figure 1. Whirling pendulum

If the angle deviation is denoted by x, the motion of the system can be described by the equation

$$\ddot{x} = -\frac{g}{L}\sin x + \omega^2 \sin x \cos x, \qquad x \in S^1 \tag{2}$$

(see [4], p. 272).

Introducing a new variable $y = \dot{x}$ and then changing the variables $y \to \omega y$, $t \to t/\omega$ converts (2) to an equivalent planar system of first-order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \sin x(\cos x - \gamma), \end{aligned} \tag{3}$$

where $\gamma = g/L\omega^2$.

This system is hamiltonian with the energy

$$H(x, y) = \frac{1}{2}y^2 - \gamma \cos x + \frac{1}{2}\cos^2 x + \gamma - \frac{1}{2}. \tag{4}$$

Its levels $H^{-1}(h) = \Gamma_h$ correspond to trajectories of the system (3).

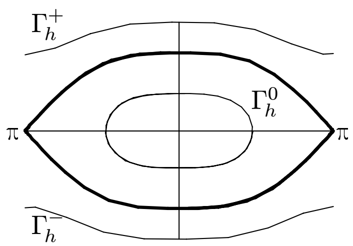


Figure 2. Phase portrait for $\gamma \geqslant 1$.

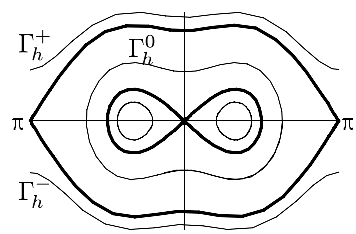


Figure 3. Phase portrait for $\gamma < 1$.

Depending on γ, we have two qualitatively different dynamics of (3):

Case (A) (see Fig. 2). For $\gamma \geqslant 1$ (i.e. for small rotation rate), dynamics is the same as that of a planar pendulum: it has two fixed points—a center $A_1 = (0, 0)$, a saddle $A_2 = (\pi, 0)$, and two types of periodic orbits:

(i) If $h \in (0, 2\gamma)$, then the level set $H^{-1}(h)$ is connected and it is equal to a periodic orbit Γ_h^0 of (3), i.e. the equation (3) possesses a family $\{\Gamma_h^0;\ h \in (0, 2\gamma)\}$ of periodic orbits. We call this family (in accordance with [3]), *the period annulus.* It corresponds to oscillations about the stable equilibrium A_1.

(ii) For $h > 2\gamma$, the set $H^{-1}(h)$ has two components—orbits Γ_h^+ for $y > 0$, corresponding to clockwise rotations of the pendulum, and orbits Γ_h^- for $y < 0$ corresponding to counterclockwise rotations. The boundary between the sets $\mathbf{P}^0 =$

$\{H^{-1}(h);\ h \in \langle 0, 2\gamma)\}$ and $\mathbf{P}^+ = \{H^{-1}(h);\ h > 2\gamma, y > 0\}$ is formed by a homoclinic orbit $\Gamma^+ = H^{-1}(2\gamma) \cap \{(x, y);\ y > 0\}$ (analogously for $y < 0$).

Case (B) (see Fig. 3). If $\gamma < 1$, i.e. if ω passes through the critical value $\omega_{krit} = (g/L)^{1/2}$, the situation inside the region bounded by homoclinic orbits Γ^+ and Γ^- changes: a stable center A_1 becomes an unstable saddle and two new equilibria (stable centers) appear at points $\pm A_3 = (\pm \arccos \gamma, 0)$.

In the sequel, we will take into consideration only the point A_3, since, due to symmetry, the results for $-A_3$ are analogous.

The center A_3 is surrounded by periodic orbits Γ_h^*, corresponding to oscillations for energy values $h \in \langle -0.5(1-\gamma)^2, 0)$. Each Γ_h^* represents one of two components of the set $H^{-1}(h)$ in the right part of the phase plane.

The saddle A_1 is connected to itself by two homoclinic orbits $+\Gamma^* = \{(x, y), x > 0\} \cap H^{-1}(0)$ and $-\Gamma^* = H^{-1}(0) \cap \{(x, y);\ x < 0\}$, which form a boundary between the sets $\mathbf{P}^0 = \{H^{-1}(h);\ h \in \langle 0, 2\gamma)\}$ and $\mathbf{P}^* = \{H^{-1}(h);\ h \in \langle -\frac{1}{2}(1-\gamma)^2, 0)\}$.

From here on we use the superscripts 0, $\pm$ and $*$ to denote, which Γ_h-family is being used; for instance, $\mathcal{A}^0(h)$ denotes a function $\mathcal{A}(h)$ restricted to $\mathbf{P}^0$.

3. Local bifurcations

The equation (1) can be written as a system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \sin x(\cos x - \gamma) + \varepsilon y(\cos x + \alpha), \end{aligned} \tag{5}$$

which is not hamiltonian. Its equilibria are the same as those of (3) (cases (A) and (B)), but their stability type depends on the values of α. Linearization of the system (5) at the point $(x, 0)$ is

$$L(x, 0) = \begin{pmatrix} 0 & 1 \\ \cos x(\cos x - \gamma) - \sin^2 x & \varepsilon(\cos x + \alpha) \end{pmatrix}.$$

In the case (A), the equilibrium A_1 is a sink for $\alpha < -1$ and a source for $\alpha > -1$. For $\alpha = -1$, $L(0, 0)$ has a pair of pure imaginary eigenvalues. By direct calculation one can observe that the conditions of the Poincaré-Andronov-Hopf theorem (see [11], p. 276) are satisfied. It means that for $\alpha > -1$ (sufficiently close to $\alpha = -1$) and for each $\varepsilon > 0$ sufficiently small, the system has a unique stable periodic orbit, which bifurcates from the center A_1 via the Hopf bifurcation.

In the case (B), the equilibrium A_3 is a sink for $\alpha < -\gamma$ and a source for $\alpha > -\gamma$. Similarly as in the case (A), it can be proved that the Hopf bifurcation occurs for $\alpha = -\gamma$ and a stable periodic orbit is born at the center A_3 for $\alpha > -\gamma$.

A local analysis of the system (5) shows that for values of the parameter α close to the bifurcation values ($\alpha = -1$ in the case (A) and $\alpha = -\gamma$ in the case (B)) there exists a limit cycle in (5). However, we have not yet answered the question if there exist limit cycles also for other values of α, and how many of them there are.

It is clear that for $\alpha < -1$ the sink A_1 in the case (A), resp. $\pm A_3$ in the case (B) is a global attractor. In this case the perturbation adds to $\dot{y}$ a negative term for $y > 0$ and a positive term for $y < 0$. Thus all periodic orbits are destroyed and the energy decreases along perturbed orbits. Similarly, for $\alpha > 1$ the source A_1 ($\pm A_3$) is a global repelor.

If $|\alpha| < 1$, the sign of the perturbation changes along orbits and depends on the values of parameters α, γ and the energy levels h. A phase portrait can be derived by computing the Melnikov function along each unperturbed solution.

Let $T(h)$ denote the period of an unperturbed orbit Γ_h on the energy level h and let the corresponding solution be $t \mapsto (x(t), y(t))$. Then the Melnikov function along Γ_h is

$$M(h) = \int_0^{T(h)} y^2(t)(\cos x(t) + \alpha)\,\mathrm{d}t. \tag{6}$$

If $\alpha < -1$, then the integrand is negative, and hence $M(h) < 0$ for any energy level h. According to [1] (see Theorems 4.6.2 and 4.5.3) or [2], there is neither a periodic orbit, nor a homoclinic orbit in (5). It is easy to obtain the same result for $\alpha > 1$.

Before starting to search zeros of $M(h)$ for $\alpha \in \langle -1, 1\rangle$, let us arrange (6) into a more suitable form. Denoting

$$\mathcal{B}(h) = \int_{\Gamma_h} y\,\mathrm{d}x, \quad \mathcal{C}(h) = \int_{\Gamma_h} y\cos x\,\mathrm{d}x, \tag{7}$$

we obtain

$$M(h) = \mathcal{C}(h) + \alpha\mathcal{B}(h). \tag{8}$$

Hence the Melnikov function equals zero for those values of the parameter α and the energy level h for which

$$\alpha = -\frac{\mathcal{C}(h)}{\mathcal{B}(h)}. \tag{9}$$

Thus, instead of solving the equation $M(h) = 0$, we can examine the function

$$\mathcal{A}(h) = -\frac{\mathcal{C}(h)}{\mathcal{B}(h)} \tag{10}$$

defined on $D(\mathcal{A}) = \langle h_{\min}, \infty)$ with

$$h_{\min} = \begin{cases} 0, & \text{for } \gamma \geqslant 1, \\ -0.5(1-\gamma)^2, & \text{for } \gamma < 1. \end{cases} \tag{11}$$

It is clear that the Melnikov function is zero exactly for those values of the parameter α for which there exists an energy level h such that $\alpha = \mathcal{A}(h)$. In addition, from (8) we have

$$\frac{\mathrm{d}M}{\mathrm{d}h} = M'(h) = \mathcal{C}'(h) + \alpha \mathcal{B}'(h),$$

and hence, after recalling $\alpha = -\mathcal{C}(h)/\mathcal{B}(h)$ we find that $M'(h) \neq 0$ if and only if

$$\mathcal{B}\mathcal{C}' - \mathcal{C}\mathcal{B}' \neq 0,$$

and this is equivalent to the condition $\mathcal{A}' \neq 0$. This condition guarantees, for given values of parameters and for sufficiently small ε, the existence of exactly one limit cycle in the system (see [1], Theorem 4.6.2).

Inside the period annuli, the function $\mathcal{B}(h)$ (and similarly $\mathcal{C}(h)$) can be expressed as follows:

$$\begin{aligned} \mathcal{B}^*(h) &= 2\int_{x_h^1}^{x_h^2} y^+ \,\mathrm{d}x, \\ \mathcal{B}^0(h) &= 4\int_0^{x_h} y^+ \,\mathrm{d}x, \\ \mathcal{B}^\pm(h) &= 2\int_0^{\pi} y^\pm \,\mathrm{d}x, \end{aligned} \tag{12}$$

where

$$\begin{aligned} y^\pm &= \pm\sqrt{2h - 2\gamma + 1 + 2\gamma\cos x - \cos^2 x}, \\ x_h &= \arccos\bigl(\gamma - \sqrt{(\gamma-1)^2 + 2h}\bigr), \\ x_h^{1,2} &= \arccos\bigl(\gamma \pm \sqrt{(\gamma-1)^2 + 2h}\bigr). \end{aligned} \tag{13}$$

Using (14) and realizing that $\mathcal{C}(h)$ may be expressed in a similar way, we have $\mathcal{A}^+(h) = \mathcal{A}^-(h)$. Taking into account the expressions for $\mathcal{B}(h)$ and $\mathcal{C}(h)$, we have the following obvious result:

Lemma 1. *For any value of the parameter γ and for any $h \in D(\mathcal{A})$,*

$$|\mathcal{A}(h)| \leqslant 1.$$

Using [1], Theorem 4.5.3 and Theorem 4.6.2, we obtain

Proposition 1. *If $|\alpha| > 1$, then the system (5) has neither a periodic nor a homoclinic orbit.*

The next lemma describes how the function $\mathcal{A}(h)$ behaves at boundary points of its domain, i.e. at the points $-0.5(1-\gamma)^2$, 0, 2γ, ∞.

Lemma 2. *The function $\mathcal{A}(h)$ is continuous on its whole domain with*

$$\lim_{h\to\infty} \mathcal{A}(h) = 0$$

and

$$\lim_{h\to h_{\min}} \mathcal{A}(h) = \begin{cases} -1, & \text{for } \gamma \geqslant 1, \\ -\gamma, & \text{for } \gamma < 1. \end{cases} \tag{14}$$

P r o o f. Since $\mathcal{B}(h) = 0$ only for $h = h_{\min}$, it suffices to prove the continuity of $\mathcal{A}(h)$ only on the boundaries of the period annuli. Using the definition of $\mathcal{B}(h)$, we obtain for the energy levels corresponding to the homoclinic orbits (i.e. $h = 0$ and $h = 2\gamma$):

$$\lim_{h\to 0^-} \mathcal{B}^*(h) = 2\lim_{h\to 0^-}\int_{x_h^1}^{x_h^2} y\,\mathrm{d}x = 2\int_0^{\arccos(2\gamma-1)} y(0)\,\mathrm{d}x,$$

$$\lim_{h\to 0^+} \mathcal{B}^0(h) = 4\lim_{h\to 0^+}\int_0^{x_h} y\,\mathrm{d}x = 4\int_0^{\arccos(2\gamma-1)} y(0)\,\mathrm{d}x,$$

$$\lim_{h\to 2\gamma^-} \mathcal{B}^0(h) = 4\lim_{h\to 2\gamma^-}\int_0^{x_h} y\,\mathrm{d}x = 4\int_0^{\pi} y(2\gamma)\,\mathrm{d}x,$$

$$\lim_{h\to 2\gamma^+} \mathcal{B}^{\pm}(h) = 2\lim_{h\to 2\gamma^+}\int_0^{\pi} y\,\mathrm{d}x = 2\int_0^{\pi} y(2\gamma)\,\mathrm{d}x.$$

Similar expressions can be obtained for the function $\mathcal{C}(h)$. The continuity on the boundaries of the period annuli then immediately follows from (10).

The limit $\lim\limits_{h\to h_{\min}} \mathcal{A}(h)$ can be derived using the Stokes theorem. Let us denote by $D(h)$ the region in the phase space bounded by the trajectory Γ_h^0. Then

$$\begin{aligned}\lim_{h\to h_{\min}} \mathcal{A}(h) &= -\frac{\lim\limits_{h\to h_{\min}} \int_{\Gamma_h} y\cos x\,\mathrm{d}x}{\lim\limits_{h\to h_{\min}} \int_{\Gamma_h} y\,\mathrm{d}x} \\ &= -\frac{\lim\limits_{h\to h_{\min}} \int_{D_h} \cos x\,\mathrm{d}y\,\mathrm{d}x}{\lim\limits_{h\to h_{\min}} \int_{D_h} \mathrm{d}y\,\mathrm{d}x} = \begin{cases} -1, \text{for } \gamma \geqslant 1, \\ -\gamma, \text{for } \gamma < 1. \end{cases}\end{aligned}$$

Now we shall prove that $\lim\limits_{h\to\infty} \mathcal{A}(h) = 0$. In $\mathbf{P}^+$ we have

$$\sqrt{2h-4\gamma} \leqslant y^+(h) \leqslant \sqrt{2h+1}. \tag{15}$$

After integrating $\mathcal{C}(h)$ by parts we obtain

$$\mathcal{C}(h)=\int_{\Gamma_h} y\cos x\,\mathrm{d}x=-\int_{\Gamma_h}\frac{\mathrm{d}y}{\mathrm{d}x}\sin x\,\mathrm{d}x=\int_{\Gamma_h}\frac{\sin^2 x}{y}(\gamma-\cos x)\,\mathrm{d}x. \tag{16}$$

From (12) and (13) it follows that

$$2\pi\sqrt{2h-4\gamma}\leqslant\mathcal{B}^+(h)\leqslant 2\pi\sqrt{2h+1}$$

and

$$\mathcal{C}^+(h)\leqslant 2\pi\frac{\gamma+1}{\sqrt{2h-4\gamma}}.$$

From the last two inequalities and from (10) we obtain that

$$|\mathcal{A}^+(h)|\leqslant\frac{\gamma+1}{2h-4\gamma}\to 0.$$

□

Lemma 3. *If $\gamma\geqslant 1$, then the function $\mathcal{A}(h)$ is negative for $h\in\langle 0,\infty)$ and strictly increasing on the interval $(2\gamma,\infty)$.*

P r o o f. Obviously $\mathcal{B}^*(h)\geqslant 0$, $\mathcal{B}^+(h)\geqslant 0$, $\mathcal{B}^-(h)\leqslant 0$, and from (16) it follows that also $\mathcal{C}^*(h)\geqslant 0$, $\mathcal{C}^+(h)\geqslant 0$, $\mathcal{C}^-(h)\leqslant 0$ for all $h\in\langle 0,\infty)$. Both functions are zero only at $h=h_{\min}=0$, where $\lim\limits_{h\to 0}\mathcal{A}(h)=-1$ (see (14) in Lemma 2). So $\mathcal{A}(h)$ is negative for $h\in\langle 0,\infty)$.

The derivative of the function $\mathcal{A}(h)$ is

$$\mathcal{A}'=\frac{\mathcal{B}'\mathcal{C}-\mathcal{C}'\mathcal{B}}{\mathcal{B}^2}. \tag{17}$$

We obtain from (20) that if $h\in(2\gamma,\infty)$, then

$$\mathcal{C}'(h)=-2\int_0^{\pi}\frac{\sin^2 x}{y^3}(\gamma-\cos x)\,\mathrm{d}x<0$$

and

$$\mathcal{B}'(h)=2\int_0^{\pi}\frac{\mathrm{d}x}{y}>0.$$

These two inequalities together with (21) give $\mathcal{A}'(h)>0$, i.e. $\mathcal{A}^+(h)$ is a strictly increasing function for any $\gamma\geqslant 1$. □

Lemma 4. *For $\gamma = 1$ the function $\mathcal{A}(h)$ satisfies the differential equation*

$$2h\mathcal{A}' = 2\mathcal{A}(1-\mathcal{F}) + 2\mathcal{F} - 1, \tag{18}$$

where $\mathcal{F}(h)$ is strictly increasing on $\langle 0, 2)$ with the minimal value $\mathcal{F}(0) = \frac{3}{4}$.

Proof. From (4) it follows that $\cos^2 x = (2h - 2\gamma + 1) + 2\gamma \cos x - y^2$. Putting it into (16), we obtain for $\gamma = 1$

$$\mathcal{C}(h) = \int_{\Gamma_h} \frac{1-\cos^2 x}{y}(1-\cos x)\,\mathrm{d}x = \int_{\Gamma_h} \frac{2-2h-2\cos x + y^2}{y}(1-\cos x)\,\mathrm{d}x.$$

While $\mathcal{B}'(h) = \int_{\Gamma_h} \mathrm{d}x/y$ and $\mathcal{C}'(h) = \int_{\Gamma_h} \cos x\,\mathrm{d}x/y$, the last expression results in

$$\mathcal{C} = (2-2h)\mathcal{B}' + (2h-4)\mathcal{C}' + \mathcal{B} - \mathcal{C} + 2\int_{\Gamma_h} \frac{\cos^2 x\,\mathrm{d}x}{y} = 2h(\mathcal{B}' + \mathcal{C}') - \mathcal{B} - \mathcal{C}$$

and this yields

$$2\mathcal{C} + \mathcal{B} = 2h(\mathcal{B}' + \mathcal{C}'). \tag{19}$$

Using (17) and (19) we obtain

$$2h\mathcal{A}' = 2h\frac{\mathcal{B}'\mathcal{C} - \mathcal{B}\mathcal{C}'}{\mathcal{B}^2} = 2\mathcal{A}(1 - h\frac{\mathcal{B}'}{\mathcal{B}}) + 2h\frac{\mathcal{B}'}{\mathcal{B}} - 1.$$

If we indicate

$$\mathcal{F}(h) = h\frac{\mathcal{B}'}{\mathcal{B}}, \tag{20}$$

we obtain the equation (18). Now, it remains just to show that $\mathcal{F}(h)$ has the above mentioned properties. First, let us compute $\mathcal{B}'(h)$.

On the interval $(0, 2)$ we have

$$\mathcal{B}'(h) = 4\int_0^{x_h} \frac{\mathrm{d}x}{\sqrt{2h - 1 + 2\cos x - \cos^2 x}}.$$

After standard tedious arrangements (see for instance [9]) we obtain

$$\mathcal{B}'(h) = \frac{4}{\sqrt[4]{2h}}\int_0^1 \frac{\mathrm{d}s}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} = \frac{4}{\sqrt[4]{2h}}K(k), \tag{21}$$

where $K(k)$ is the complete elliptic integral of the first kind with the elliptic modulus

$$k = \sqrt{\frac{h + \sqrt{2h}}{2\sqrt{2h}}}.$$

With h increasing on $\langle 0, 2\rangle$, the elliptic modulus k increases on $\langle 0.5, 1\rangle$.

The integral $K(k)$ can be expressed via the infinite series

$$K(k) = \frac{\pi}{2}\left(1 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^4 + \ldots\right) = \sum_{n=0}^{\infty} C_n k^{2n},$$

which is increasing for $k \in \langle 0, 1\rangle$ with

$$\lim_{k \to 0^+} K(k) = K(0) = \frac{\pi}{2}, \quad \lim_{k \to 1^-} K(k) = +\infty.$$

The integral $K(k)$ can be estimated by

$$K(k) < \ln \frac{4}{\sqrt{1 - k^2}}\left(1 + \frac{1}{4}k'^2\right) \tag{22}$$

(see [7], Theorem 1.3), which guarantees the convergence of the sum (21) for any given $h \in (0, 2)$.

Substituting $K(k)$ into (21) and using the binomial expansion we obtain

$$\mathcal{B}'(h) = \frac{2\pi}{h^{\frac{1}{4}}\sqrt[4]{2}} \sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \binom{n}{k} \left(\sqrt{\frac{h}{2}}\right)^k. \tag{23}$$

Integration of the last equation gives

$$\mathcal{B}(h) = \frac{8\pi h^{\frac{3}{4}}}{\sqrt[4]{2}} \sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2k+3} \left(\sqrt{\frac{h}{2}}\right)^k. \tag{24}$$

The infinite series in (24) is convergent for any given h, since the series in (23) is its majorant.

If we substitute (23) and (24) into (20), we obtain

$$\mathcal{F}(h) = \frac{1}{4} \frac{\sum\limits_{n=0}^{\infty} C_n \sum\limits_{k=0}^{n} \binom{n}{k} \left(\sqrt{\frac{h}{2}}\right)^k}{\sum\limits_{m=0}^{\infty} C_m \sum\limits_{l=0}^{m} \binom{m}{l} \frac{1}{2l+3} \left(\sqrt{\frac{h}{2}}\right)^l}. \tag{25}$$

From the last equation it is easy to see that $\mathcal{F}(0) = \frac{3}{4}$. By differentiating (25) and after some arrangements we obtain

$$\mathcal{F}'(h) = \frac{1}{16} \frac{\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n C_m \sum_{k=0}^{n} \sum_{l=0}^{m} \binom{n}{k} \binom{m}{l} \left(\sqrt{\frac{h}{2}}\right)^{k+l-2} \frac{k-l}{2l+3}}{\left(\sum_{m=0}^{\infty} C_m \sum_{l=0}^{m} \binom{m}{l} \frac{1}{2l+3} \left(\sqrt{\frac{h}{2}}\right)^{l}\right)^2}. \tag{26}$$

Now, we modify this expression in the following way: each summand in the numerator determined by the quadruple (n, m, k, l) will be added to the summand determined by the quadruple (m, n, l, k). These summands are almost the same; the difference is only in the last fraction. Summing these fractions gives

$$\frac{k-l}{2l+3} + \frac{l-k}{2k+3} = \frac{2(k-l)^2}{(2l+3)(2k+3)}.$$

Then (26) can be arranged as an infinite series with nonegative terms, which means that $\mathcal{F}'(h) > 0$ and hence $\mathcal{F}'(h)$ is strictly increasing. □

Theorem 1. *The function $\mathcal{A}(h)$ is for $\gamma = 1$ strictly increasing on its domain with the range $\langle -1, 0)$.*

P r o o f. Lemma 1 and Lemma 3 imply that $\mathcal{A}(h) \in \langle -1, 0)$. Substituting it in (18), we obtain the estimate

$$2h\mathcal{A}' > 4\mathcal{F} - 3 > 0,$$

which is valid for any $h \in (0, 2)$. This means that $\mathcal{A}^0(h)$ is a strictly increasing function. Lemma 2 implies that the range of $\mathcal{A}$ is the whole interval $\langle -1, 0)$. □

Corollary 1. *(Phase portrait of* (5) *for $\gamma = 1$.) If $\gamma = 1$, then for any $\alpha \in (-1, 0)$ and for any sufficiently small ε the system* (5) *has exactly one ω-limit closed phase curve (periodic or homoclinic orbit), which is ε-close to Γ_h with the energy level $h = \mathcal{A}^{-1}(\alpha)$.*

P r o o f. To prove existence and uniqueness of a limit cycle or a homoclinic orbit, it is sufficient to show (see [1], Theorem 4.5.3 and Theorem 4.6.2) that the zeros of the Melnikov integral are simple, that is, if

$$M(h) = \mathcal{C}(h) + \alpha\mathcal{B}(h) = 0,$$

then

$$M'(h) = \mathcal{C}'(h) + \alpha\mathcal{B}'(h) \neq 0.$$

Since at the zero point we have $\alpha = -\mathcal{C}(h)/\mathcal{B}(h)$, the last inequality is true only if

$$\mathcal{B}\mathcal{C}' - \mathcal{C}\mathcal{B}' \neq 0.$$

However, this inequality is true due to the fact that $\mathcal{A}(h)$ is strictly increasing, so $\mathcal{A}'(h) \neq 0$. □

We have not succeeded in using the same procedure as above for a parameter $\gamma \neq 1$. Difficulties are caused by a complicated expression for the function $\mathcal{B}'$, which is, for instance, for $\gamma = 1 + \mu$, $\mu > 0$, given by

$$\mathcal{B}'(h) = \frac{4}{\sqrt[4]{2h + \mu^2}} K(k),$$

where $k = \sqrt{\frac{h - \mu + \sqrt{2h+\mu^2}}{2\sqrt{2h+\mu^2}}}$.

In the next section we describe a quite general method which allows us to analyze the intervals of monotonicity of the function $\mathcal{A}(h)$ also for $\gamma \neq 1$.

5. Criterion of Li and Zhang

First, we describe the criterion given by Li and Zhang in [5]. Then we apply it to the system (5).

Consider the system

$$\begin{aligned} \dot{x} &= \Psi'(y) \\ \dot{y} &= -\Phi'(x) + \varepsilon g(y)(\alpha f_1(x) + f_2(x)), \end{aligned} \tag{27}$$

where $\Phi \in C^2[a_1, b_1]$, $f_1, f_2 \in C^1[a_1, b_1]$, $\Psi, g \in C^2[a_2, b_2]$.

If $\varepsilon = 0$, then the system (27) is hamiltonian with the energy

$$H(x, y) = \Phi(x) + \Psi(y). \tag{28}$$

Let us suppose that the levels of the energy $h = H(x, y)$ on the interval $(a_1, b_1) \times (a_2, b_2)$ are changing along an interval (h_1, h_2). Each compact component Γ_h of the level curve

$$H^{-1}(h) = \{(x, y);\ H(x, y) = h\}, \quad h \in (h_1, h_2)$$

corresponds to a closed orbit of the system (27). Denote by $l(h)$ and $u(h)$ the border points of the variable x on the orbit Γ_h, while the border points of the variable y will be denoted by $L(h)$ and $U(h)$ (see Fig. 4). Assume that there exists a point $(x_0, y_0) \in (a_1, b_1) \times (a_2, b_2)$ such that the following hypothesis is satisfied:

(H) $\Phi'(x)(x - x_0) > 0$ (or < 0) and $\Psi'(y)(y - y_0) > 0$ (or < 0), where $(x, y) \in (a_1, b_1) \times (a_2, b_2) \setminus \{(x_0, y_0)\}$.

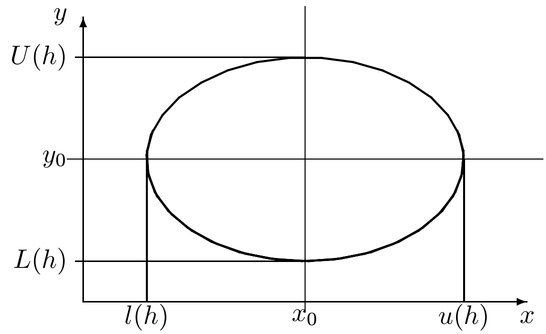


Figure 4. Phase portrait of (31) with $\varepsilon = 0$

This hypothesis requires the orbits of (27) to be

- *symmetric* in the following sense: there exist one-to-one mappings

$$\langle l(h), x_0\rangle \longrightarrow \langle x_0, u(h)\rangle\colon\; x \mapsto \tilde{x},$$

$$\langle L(h), y_0\rangle \longrightarrow \langle y_0, U(h)\rangle\colon\; y \mapsto \tilde{y},$$

  for which

$$\begin{gathered}\Phi(x) = \Phi(\tilde{x}), \qquad \Psi(y) = \Psi(\tilde{y}) \\ \frac{\mathrm{d}\tilde{x}}{\mathrm{d}x} = \frac{\Phi'(x)}{\Phi'(\tilde{x})} < 0, \quad \frac{\mathrm{d}\tilde{y}}{\mathrm{d}y} = \frac{\Psi'(y)}{\Psi'(\tilde{y})} < 0.\end{gathered} \tag{29}$$

  Since, for given h, (28) gives $y(x) = \Psi^{-1}(h - \Phi(x))$, we have $y(x) = y(\tilde{x})$, $\tilde{y}(x) = \tilde{y}(\tilde{x})$ for $x \in (l(h), x_0)$. It implies that Γ_h consists of two branches $y(x)$ and $\tilde{y}(x)$, which are symmetric to each other with respect to $y = y_0$, and each of them is symmetric with respect to $x = x_0$.
- *monotonic* on intervals $(l(h), x_0)$, $(x_0, u(h))$, $(L(h), y_0)$, $(y_0, U(h))$—$\Phi'(x)$ and $\Psi'(x)$ do not change sign on these intervals.

For $\varepsilon \neq 0$ the existence and the number of periodic orbits of the system (27) can be found by computing zeros of the Melnikov function along each orbit Γ_h, i.e. by solving the equation

$$M(h) = \int_{\Gamma_h} g(y)\,(\alpha f_1(x) + f_2(x))\;\mathrm{d}x = 0.$$

This equation is equivalent to the equation

$$\alpha = -\frac{I_2(h)}{I_1(h)}, \qquad \text{where } I_k(h) = \int_{\Gamma_h} f_k(x) g(y)\,\mathrm{d}x,\;\; k = 1, 2.$$

Similarly as in §4, we are interested in the behaviour of the function

$$\mathcal{A}(h) = -\frac{I_2(h)}{I_1(h)}, \qquad h \in (h_1, h_2).$$

Theorem 2. (See [5], Theorem 1.) *Let* $f_1(x)f_1(\tilde{x}) > 0$, $g'(y)g'(\tilde{y}) > 0$ *for each* $(x, y) \in (l(h), x_0) \times (L(h), y_0)$ *and let* (H) *be satisfied. Denote*

$$\xi(x) = \frac{f_2(x)\Phi'(\tilde{x}) - f_2(\tilde{x})\Phi'(x)}{f_1(x)\Phi'(\tilde{x}) - f_1(\tilde{x})\Phi'(x)}, \tag{30}$$

$$\eta(y) = \frac{(g(\tilde{y}) - g(y))\Psi'(\tilde{y})\Psi'(y)}{g'(\tilde{y})\Psi'(y) - g'(y)\Psi'(\tilde{y})}. \tag{31}$$

Then $\xi'(x)\eta'(y) > 0$ (< 0) *implies* $\mathcal{A}'(h) < 0$ (> 0).

Now, we apply Theorem 2 to the system (5). Since the assumptions require symmetry and monotonicity related to both variables, we can use it only in the regions $\mathbf{P}^0$ in the case (A) and $\mathbf{P}^*$ in the case (B).

We have

$$f_1(x) = 1, \ f_2(x) = \cos x, \ g(y) = y,$$
$$\Phi(x) = \frac{1}{2}\cos^2 x - \gamma\cos x, \ \Psi(y) = \frac{1}{2}y^2 + \gamma - \frac{1}{2},$$
$$\Phi'(x) = \sin x(\gamma - \cos x), \ \Psi'(y) = y.$$

Case (A). In the region $\mathbf{P}^0$ we have $\langle a_1, b_1\rangle = \langle -\pi, \pi\rangle$, $\langle a_2, b_2\rangle = \langle -2\sqrt{\gamma}, 2\sqrt{\gamma}\rangle$ and

$$(x_0, y_0) = (0, 0), \ \tilde{x} = -x, \ \tilde{y} = -y.$$

Then

- $\Phi'(x)(x - x_0) = x\sin x(\gamma - \cos x) > 0$,
- $\Psi'(y)(y - y_0) = y^2 > 0$,
- $\Phi(x) = \Phi(\tilde{x})$, $\Psi(y) = \Psi(\tilde{y})$,
- $\Phi'(x)/\Phi'(\tilde{x}) = \sin x(\gamma - \cos x)/\sin\tilde{x}(\gamma - \cos\tilde{x}) = -1 < 0$,
- $\Psi'(y)/\Psi'(\tilde{y}) = y/\tilde{y} = -1 < 0$,
- $g'(y)g'(\tilde{y}) = 1 > 0$, $f_1(x)f_1(\tilde{x}) = 1 > 0$.

Let us compute the functions $\xi(x)$ and $\eta(y)$ for $(x, y) \in (-\pi, 0) \times (-2\sqrt{\gamma}, 0)$:

$$\xi(x) = \frac{\cos x\sin\tilde{x}(\gamma - \cos\tilde{x}) - \cos\tilde{x}\sin x(\gamma - \cos x)}{\sin\tilde{x}(\gamma - \cos\tilde{x}) - \sin x(\gamma - \cos x)} = \cos x,$$
$$\eta(y) = \frac{(\tilde{y} - y)\tilde{y}y}{y - \tilde{y}} = y^2.$$

Then $\xi'(x)\,\eta'(y) = (-\sin x)2y < 0$ and Theorem 2 implies $\mathcal{A}'(h) > 0$.

Case (B). In the region $\mathbf{P}^*$ we have $\langle a_1, b_1\rangle = \langle 0, \arccos(2\gamma - 1)\rangle$, $\langle a_2, b_2\rangle = \langle \gamma - 1, 1 - \gamma\rangle$ and

$$(x_0, y_0) = (x_\gamma, 0),\ \tilde{x} = \arccos(2\gamma - \cos x),\ \tilde{y} = -y.$$

Then

- $\Phi'(x)(x - x_0) = (x - x_\gamma)\sin x(\gamma - \cos x) > 0$,
- $\Psi'(y)(y - y_0) = y^2 > 0$,
- $\Phi(\tilde{x}) = \frac{1}{2}(2\gamma - \cos x)^2 - \gamma(2\gamma - \cos x) = \frac{1}{2}\cos^2 x - \gamma\cos x = \Phi(x)$,
- $\Psi(y) = \Psi(\tilde{y})$,
- $\Phi'(x)/\Phi'(\tilde{x}) = \sin x(\gamma - \cos x)/\sin\tilde{x}(\gamma - (2\gamma - \cos x)) = -\sin x/\sin\tilde{x} < 0$, since x and $\tilde{x}$ are in the interval $(0, \pi)$.
- $\Psi'(y)/\Psi'(\tilde{y}) = y/\tilde{y} = -1 < 0$,
- $g'(y)g'(\tilde{y}) = 1 > 0$,
- $f_1(x)f_1(\tilde{x}) = 1 > 0$.

Let us compute the functions $\xi(x)$ and $\eta(y)$ for $(x, y) \in (0, x_\gamma) \times (\gamma - 1, 0)$:

$$\xi(x) = \frac{\cos x\sin\tilde{x}(\gamma - (2\gamma - \cos x)) - \cos\tilde{x}\sin x(\gamma - \cos x)}{\sin\tilde{x}(\gamma - (2\gamma - \cos x)) - \sin x(\gamma - \cos x)} = \frac{\sin(x + \tilde{x})}{\sin x + \sin\tilde{x}},$$

$$\eta(y) = \frac{(\tilde{y} - y)\tilde{y}y}{y - \tilde{y}} = y^2.$$

The derivative of the function $\xi(x)$ is

$$\xi'(x) = \frac{\cos(x + \tilde{x})(1 + \frac{\mathrm{d}\tilde{x}}{\mathrm{d}x})(\sin x + \sin\tilde{x}) - (\cos x + \cos\tilde{x}\frac{\mathrm{d}\tilde{x}}{\mathrm{d}x})\sin(x + \tilde{x})}{(\sin x + \sin\tilde{x})^2}.$$

To find the sign of the derivative, we arrange the nominator of it. Using (29), we obtain

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}x} = \frac{\Phi'(x)}{\Phi'(\tilde{x})} = -\frac{\sin x}{\sin\tilde{x}}.$$

Hence

$$\xi'(x) = \frac{4\gamma(1 - \cos x\cos\tilde{x} + \sin x\sin\tilde{x})(\gamma - \cos x)}{\sin\tilde{x}(\sin x + \sin\tilde{x})^2}.$$

Since $x \in (0, \arccos\gamma)$, we have $\gamma - \cos x < 0$, and $\sin x\sin\tilde{x} > 0$. So $\xi'(x) < 0$. Then $\xi'(x)\eta'(y) > 0$ and Theorem 2 implies $\mathcal{A}'(h) < 0$.

Combining these conclusions with those of §4, we obtain the following theorem:

Theorem 3.

i) *If $\gamma \geqslant 1$, then the function $\mathcal{A}(h)$ is strictly increasing on its domain with the range $\langle -1, 0)$.*

ii) *If $\gamma < 1$, then the function $\mathcal{A}(h)$ is strictly decreasing on $\langle -\frac{1}{2}(1-\gamma)^2, 0\rangle$ with the range $\langle \mathcal{A}(0), -\gamma\rangle$.*

Corollary 2. *(Phase portrait of the system* (5)*)*

i) *If $\gamma \geqslant 1$, then for any $\alpha \in (-1, \mathcal{A}(2\gamma))$ and for any ε sufficiently small, the system* (5) *has exactly one periodic orbit $\mathcal{K}(\alpha)$ which is ε-close to Γ_h with energy level $h = \mathcal{A}^{-1}(\alpha)$. The family of orbits $\{\mathcal{K}(\alpha);\ \alpha \in (-1, \mathcal{A}(2\gamma))\}$ is born via Hopf bifurcation at $\alpha_b = -1$, and vanishes at $\alpha = \mathcal{A}(2\gamma)$ where the homoclinic orbit (which is ε-close to the homoclinic orbit of the unperturbed system) is formed.*

For any $\alpha \in (\mathcal{A}(2\gamma), 0)$, there are exactly two periodic orbits of the system (5)*—the first in $\mathbf{P}^+$, the other in $\mathbf{P}^-$.*

For other parameter values of α, there is neither a periodic nor a homoclinic orbit.

ii) *If $\gamma < 1$, then, in the region $\mathbf{P}^*$, the system* (5) *has exactly one periodic orbit for any $\alpha \in (\mathcal{A}(0), -\gamma)$ and for any ε sufficiently small, and no periodic orbit for other parameter values of α.*

In addition, there exist parameter values α_1 and α_2 close to the values $\mathcal{A}(0)$ and $\mathcal{A}(2\gamma)$, for which the system (5) *has homoclinic orbits, which are ε-close to the homoclinic orbits of the unperturbed system.*

P r o o f. The arguments are the same as those in the proof of Corollary 1 and Proposition 1. The existence of homoclinic orbits can be derived from continuity of the function $\mathcal{A}(h)$ at the point $h = 2\gamma$ in the case (A), and at the points $h = 0$, $h = 2\gamma$ in the case (B). □

There still remains an unsolved part of the problem: we have not yet said how the function $\mathcal{A}(h)$ behaves for $\gamma < 1$ on the interval $(0, \infty)$. In numerical experiments it has the same course as the function $\mathcal{A}(h)$ for $\gamma \geqslant 1$ on the interval $(2\gamma, \infty)$. However, we have not succeeded in proving it analytically. In the end, we want to show the problem which we have to face in order to prove the monotonicity of the function $\mathcal{A}(h)$ in the region $\mathbf{P}^0$, where the trajectories are not monotonic.

The sign of the derivative $\mathcal{A}(h)$ depends only on the sign of the expression

$$G(h) = \mathcal{B}(h)\mathcal{C}'(h) - \mathcal{C}(h)\mathcal{B}'(h). \tag{32}$$

In the region $\mathbf{P}^0$ we have

$$\mathcal{B}(h) = 2\int_{-x_h}^{x_h} y^+\,\mathrm{d}x \qquad \mathcal{C}(h) = 2\int_{-x_h}^{x_h} y^+ \cos x\,\mathrm{d}x.$$

Using (12) we obtain

$$\mathcal{B}'(h) = 2\int_{-x_h}^{x_h} \frac{\mathrm{d}x}{y}, \qquad \mathcal{C}'(h) = 2\int_{-x_h}^{x_h} \frac{\cos x}{y}\,\mathrm{d}x,$$

where $y = y^+$. Substituting it into (32), we obtain

$$\begin{aligned}\frac{1}{4}G(h) &= \int_{-x_h}^{x_h} y\,\mathrm{d}x \int_{-x_h}^{x_h} \frac{\cos x}{y}\,\mathrm{d}x - \int_{-x_h}^{x_h} y\cos x\,\mathrm{d}x \int_{-x_h}^{x_h} \frac{\mathrm{d}x}{y}\\ &= \frac{1}{2}\int_{-x_h}^{x_h}\int_{-x_h}^{x_h} (\cos x_1 - \cos x_2)\left(\frac{y(x_2)}{y(x_1)} - \frac{y(x_1)}{y(x_2)}\right)\mathrm{d}x_1\,\mathrm{d}x_2\\ &= \frac{1}{2}\int_{-x_h}^{x_h}\int_{-x_h}^{x_h} \frac{(\cos x_1 - \cos x_2)^2}{y(x_2)y(x_1)}(\cos x_1 + \cos x_2 - 2\gamma)\,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

If we recall (13) and realize that $\gamma < 1$, $h \in \langle 0, 2\gamma\rangle$, we find that the expression $(\cos x_1 + \cos x_2 - 2\gamma)$ changes its sign, and we are not able to say anything about the sign of $G(h)$. We must leave this case of symmetric and nonmonotonic trajectories for further investigation, as well as the case of free rotations, i.e. the trajectories in the regions $\mathbf{P}^+$ and $\mathbf{P}^-$.

References

[1] *J. Guckenheimer and P. J. Holmes*: Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Springer Verlag, New York, Heidelberg, Berlin, 1983.

[2] *C. Chicone*: Bifurcations of nonlinear oscillations and frequency entrainment near resonance. SIAM J. Math. Anal. *23(6)* (1992), 1577–1608.

[3] *C. Chicone*: On bifurcation of limit cycles from centers. Lecture Notes in Math., 1455. 1991, pp. 20–43.

[4] *H. Kauderer*: Nichtlineare mechanik. Springer Verlag, Berlin, Gottingen, Heidelberg, 1958.

[5] *Ch. Li and Z.-F. Zhang*: A criterion for determining the monotonicity of the ratio of two abelian integrals. Journal of Differential Equations *124* (1996), 407–424.

[6] *A. D. Morozov*: On limit cycles and chaos in equations of pendulum type. Prikladnaja matematika i mechanika *53(5)* (1989), 721–730. (In Russian.)

[7] *S.-L.Qiu and M. K. Vamanamurthy*: Sharp estimates for complete elliptic integrals. Siam J. Math. Anal. *27(3)* (1996), 823–834.

[8] *J.A. Sanders and R. Cushman*: Limit cycles in the Josephson equation. SIAM J. Math. Anal. *17(3)* (1986), 495–511.

[9] *E. T. Whittaker and G. N. Watson*: A Course of Modern Analysis. Cambridge at the University Press, 1927.

[10] *S. Wiggins*: Global Bifurcations and Chaos: Analytical Methods. Springer Verlag, New York, Heidelberg, Berlin, 1988.
[11] *S. Wiggins*: Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer Verlag, New York, Heidelberg, Berlin, 1990.

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