Applications of Mathematics

Josef Dalík; Josef Daněček; Jiří Vala

Numerical solution of the Kiessl model

Applications of Mathematics, Vol. 45 (2000), No. 1, 3--17

Persistent URL: http://dml.cz/dmlcz/134426

NUMERICAL SOLUTION OF THE KIESSL MODEL

Josef Dalík, Josef Daněček, Jiří Vala, Brno

(Received April 28, 1997, revised January 19, 1998)

Abstract. The Kiessl model of moisture and heat transfer in generally nonhomogeneous porous materials is analyzed. A weak formulation of the problem of propagation of the state parameters of this model, which are so-called moisture potential and temperature, is derived. An application of the method of discretization in time leads to a system of boundary-value problems for coupled pairs of nonlinear second order ODE's. Some existence and regularity results for these problems are proved and an efficient numerical approach based on a certain special linearization scheme and the Petrov-Galerkin method is suggested.

Keywords: materials with pore structure, moisture and heat transport, nonlinear systems of partial differential equations, method of discretization in time

MSC 2000: 65M60

1. Introduction

The ceramic materials and most building materials have a part of their volumes filled with small pores connected by narrow channels. This pore structure is able to absorb, save, transport and discharge water both in the liquid and in the gas form. The amount of moisture in the pore structure, which is closely related to temperature, is essential for the durability of materials and stability of building constructions. Hence there is an extensive need for modelling of the process of moisture and heat transfer in porous materials. Because of this requirement, several mathematical models of this process have been developed. The first successful attempts appeared in Philip, de Vries [13] and Glaser [9] in the late fifties; more historical information can be found in [5] and [7]. In this article we pay attention to the model derived by Kiessl [11] in 1983, which has found a common acknowledgement for its ability to model the process in a broad class of cases. Nevertheless, we believe that it is possible to propose a more general, clear and physically better motivated model of moisture and heat transfer in porous materials; a preliminary version of such a model has been presented in [6].

The Kiessl model takes into account shapes and sizes of pores, simultaneous presence of moisture in various phases and interaction of the transport of heat with various kinds of the transport of moisture like capillary convection, diffusion, surface diffusion and effusion. It works with the following two state parameters: The moisture potential $\Phi = \Phi(x,t)$ [–] (x characterizes the location in the material body, t the actual time) and the temperature $\tau = \tau(x,t)$ [°C]. The basic equations of evolution are consequences of the physical laws of mass and energy preservation. The aim of our study is to derive a proper mathematical formulation of the problem, to obtain some existence and regularity results and to suggest a numerical approach for the construction of an accurate approximate solution.

2. Derivation of the Kiessl model

We restrict ourselves to a brief sketch of the physical background only. A complete derivation of this model can be found in [5].

For certain positive constants l and T, we consider $x \in \Omega = (0,l)$ and $t \in I = (0,T)$. We make use of the physical quantities

τ [°C]	temperature,
u [–]	relative part of water and (melted) ice in the porous material,
φ [–]	relative humidity of the air in pores,
M [kg/m^3]	amount of liquid and ice in 1 m^3 of the porous material,
H [J/m^3]	amount of (inner) energy in 1 m^3 of the porous material

and of the known constants

P [–]	material porosity,
ϱ_a [kg/m^3]	density and
c_a [J]	specific heat

of water for $a = W$, of ice for $a = E$ and of the porous material in the case $a = M$. Moreover, the analysis of phase changes of water deals with the following constants:

$L_{1,3}$	[J/kg]	specific heat of ice sublimation,
$L_{2,3}$	[J/kg]	specific heat of water evaporation,
$L_{1,2}$	[J/kg]	specific heat of ice melting.

Using the diffusion coefficients

$$D_u = \varrho_W \varepsilon_\kappa(\tau)\kappa(u),\ D_\varphi = \varrho_W \varepsilon_\varphi(\tau) k_{d\varphi}(u),\ D_\tau = \varrho_W \varepsilon_\tau(\tau) k_{d\tau}(u),\ \lambda = \lambda(\tau, u),$$

where the material characteristics $\varepsilon_\kappa(\tau)$, $\varepsilon_\varphi(\tau)$, $\varepsilon_\tau(\tau)$, $\kappa(u)$, $k_{d\varphi}(u)$, $k_{d\tau}(u)$ and $\lambda(\tau,u)$ have been established experimentally (graphs of some examples of these functions are reproduced in [5]), the following quantities can be introduced:

$D_u \partial u/\partial x$ [kg/(m^{2}s)] intensity of capillary flow of moisture,
$D_\varphi \partial\varphi/\partial x$ [kg/(m^{2}s)] intensity of flow of vapour at the gradient of relative humidity,
$D_\tau \partial\tau/\partial x$ [kg/(m^{2}s)] intensity of flow of vapour at the gradient of temperature,
$\lambda \partial\tau/\partial x$ [J/(m^{2}s)] intensity of flow of heat.

Now we are ready to formulate the laws of preservation of mass

$$\frac{\partial M}{\partial t} - \frac{\partial}{\partial x}\left(D_u\frac{\partial u}{\partial x} + D_\varphi\frac{\partial \varphi}{\partial x} + D_\tau\frac{\partial \tau}{\partial x}\right) = 0 \tag{1}$$

and of energy

$$\frac{\partial H}{\partial t} - \frac{\partial}{\partial x}\left(\lambda\frac{\partial \tau}{\partial x}\right) = L_{.,3}(\tau)\left[\varrho_W\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(D_u\frac{\partial u}{\partial x}\right)\right]. \tag{2}$$

Here

$$L_{.,3}(\tau) = (1-\chi(\tau))L_{1,3} + \chi(\tau)L_{2,3}$$

and $\chi(\tau)$ [–] is the relative part of water mass in total water and ice mass. This relation can be represented by some smoothening of the Heaviside function. We approximate the partial pressure $c_S(\tau)$ [kg/m^3] of saturated vapour in the air by a certain exponential function (see e.g. [5]) with a high accuracy. By means of the notations

$$\hat\chi(\tau) = (1-\chi(\tau))(\varrho_W/\varrho_E - 1)$$

and

$$C(\tau) = (1-\chi(\tau))c_E\tau + \chi(\tau)(c_W\tau + L_{1,2}),$$

we can express the amounts M and H in the forms

$$M = \varrho_W u + c_S(\tau)\varphi\,(P - u - \hat\chi(\tau)u) \tag{3}$$

and

$$H = \varrho_M c_M\tau + \varrho_W u C(\tau). \tag{4}$$

The moisture potential Φ has been defined in [11] by the formula

$$\Phi = \begin{cases} \varphi & \text{for} \quad \varphi \leqslant 0.9, \\ 1.7 + 0.1\log r & \text{for} \quad 0.9 < \varphi, \end{cases}$$

where the quantity r [m] characterizes the mean value of pore diameters. The moisture potential is connected with u via a so-called sorption isotherm $u = f(\Phi)$ which has to be found experimentally for each sort of material.

In the following considerations, we transform the laws of preservation (1) and (2) to a pair of evolution equations for the unknown state variables τ and Φ.

If we insert the composite functions

$$\begin{aligned} g(\Phi) &= \varphi(f(\Phi)), & K_{d\varphi}(\Phi) &= k_{d\varphi}(f(\Phi)), \\ S(\Phi) &= \kappa(f(\Phi)), & K_{d\tau}(\Phi) &= k_{d\tau}(f(\Phi)), \end{aligned}$$
$$\Lambda(\tau, \Phi) = \lambda(\tau, f(\Phi))$$

into (3) and (4), then we obtain the expressions

$$M(\tau, \Phi) = \varrho_W G_2(\tau, \Phi)$$

with

$$\begin{aligned} G_2(\tau, \Phi) &= f(\Phi) + G(\tau, \Phi), \\ G(\tau, \Phi) &= \varrho_W^{-1} c_S(\tau) g(\Phi)[P - (1 + \hat{\chi}(\tau)) f(\Phi)] \end{aligned}$$

and

$$H(\tau, \Phi) = \varrho_M c_M \tau + \varrho_W C(\tau) f(\Phi).$$

Inserting the above formula for M into the law (1) and dividing by ϱ_W, we obtain the equation

$$\begin{aligned} (5) \quad & \frac{\partial}{\partial t} G_2(\tau, \Phi) \\ & = \frac{\partial}{\partial x} \left(\varepsilon_\kappa(\tau) S(\Phi) f'(\Phi) \frac{\partial \Phi}{\partial x} + \varepsilon_\varphi(\tau) K_{d\varphi}(\Phi) g'(\Phi) \frac{\partial \Phi}{\partial x} + \varepsilon_\tau(\tau) K_{d\tau}(\Phi) \frac{\partial \tau}{\partial x} \right). \end{aligned}$$

If we substitute the above expression of H into (2) and divide by

$$R(\tau) = \varrho_W L_{.,3}(\tau),$$

we obtain the equation

$$\begin{aligned} & R^{-1}(\tau) \frac{\partial}{\partial t} G_2(\tau, \Phi) - \frac{\partial}{\partial t} f(\Phi) \\ & \qquad = R^{-1}(\tau) \frac{\partial}{\partial x} \left(\Lambda(\tau, \Phi) \frac{\partial \tau}{\partial x} \right) - \frac{\partial}{\partial x} \left(\varepsilon_\kappa(\tau) S(\Phi) f'(\Phi) \frac{\partial \Phi}{\partial x} \right), \end{aligned}$$

whose physical unit [s^{-1}] is the same as that of the equation (5). Hence we can sum up the last two equations with the result

$$
\begin{aligned}
(6)\ & R^{-1}(\tau)\frac{\partial}{\partial t}H(\tau,\Phi)+\frac{\partial}{\partial t}G(\tau,\Phi)\\
&=R^{-1}(\tau)\frac{\partial}{\partial x}\left(\Lambda(\tau,\Phi)\frac{\partial\tau}{\partial x}\right)+\frac{\partial}{\partial x}\left(\varepsilon_\varphi(\tau)K_{d\varphi}(\Phi)g'(\Phi)\frac{\partial\Phi}{\partial x}+\varepsilon_\tau(\tau)K_{d\tau}(\Phi)\frac{\partial\tau}{\partial x}\right).
\end{aligned}
$$

An application of the notations

$$
\begin{aligned}
&P_1^2(\tau,\Phi)=\varepsilon_\varphi(\tau)K_{d\varphi}(\Phi)g'(\Phi), && P_1^1(\tau,\Phi)=P_2^1(\tau,\Phi)+R^{-1}(\tau)\Lambda(\tau,\Phi),\\
&P_2^1(\tau,\Phi)=\varepsilon_\tau(\tau)K_{d\tau}(\Phi), && P_2^2(\tau,\Phi)=P_1^2(\tau,\Phi)+\varepsilon_\kappa(\tau)S(\Phi)f'(\Phi)\\
&\qquad\text{and}\qquad Q(\tau,\Phi)=R^{-2}(\tau)\ R'(\tau)\Lambda(\tau,\Phi)
\end{aligned}
$$

simplifies the equation (5) to the following *moisture equation*

$$
\frac{\partial}{\partial t}G_2(\tau,\Phi)-\frac{\partial}{\partial x}\left(P_2^1(\tau,\Phi)\frac{\partial\tau}{\partial x}+P_2^2(\tau,\Phi)\frac{\partial\Phi}{\partial x}\right)=0. \tag{7}
$$

To make the passage to the variational formulation easy, we modify the right-hand side of (6) by means of the obvious identity

$$
\frac{1}{L_{.,3}(\tau)}\frac{\partial}{\partial x}\left(\Lambda(\tau,\Phi)\frac{\partial\tau}{\partial x}\right)=\frac{\partial}{\partial x}\left[\frac{1}{L_{.,3}(\tau)}\left(\Lambda(\tau,\Phi)\frac{\partial\tau}{\partial x}\right)\right]+\frac{L'_{.,3}(\tau)}{L^2_{.,3}(\tau)}\Lambda(\tau,\Phi)\left(\frac{\partial\tau}{\partial x}\right)^2.
$$

This modification leads to the *temperature equation*

$$
\begin{aligned}
R^{-1}(\tau)\frac{\partial}{\partial t}H(\tau,\Phi)+\frac{\partial}{\partial t}G(\tau,\Phi)-\frac{\partial}{\partial x}\left(P_1^1(\tau,\Phi)\frac{\partial\tau}{\partial x}+P_1^2(\tau,\Phi)\frac{\partial\Phi}{\partial x}\right)&\\
-Q(\tau,\Phi)\left(\frac{\partial\tau}{\partial x}\right)^2&=0.
\end{aligned} \tag{8}
$$

For given functions $\tau_0(x)$, $\Phi_0(x)$, $g_1(x,t,\tau,\Phi)$ and $g_2(x,t,\tau,\Phi)$, the state parameters Φ and τ have to satisfy the *initial conditions*

$$
\tau(x,0)=\tau_0(x),\quad \Phi(x,0)=\Phi_0(x) \tag{9}
$$

for all $x\in\Omega$ and the *boundary conditions*

$$
P_j^1(\tau,\Phi)\frac{\mathrm{d}\tau}{\mathrm{d}n}+P_j^2(\tau,\Phi)\frac{\mathrm{d}\Phi}{\mathrm{d}n}=g_j(x,t,\tau,\Phi) \tag{10}
$$

for all $j\in\{1,2\}$, $t\in I$, $x=0$ and $x=l$. Here $\dfrac{\mathrm{d}.}{\mathrm{d}n}$ means $\partial./\partial x$ for $x=l$ and $-\partial./\partial x$ for $x=0$.

3. Formulation of the problem and discretization in time

Let us accept the following assumptions on physical characteristics:

1. The constants ϱ_M, c_M, ϱ_W, c_W, ϱ_E, c_E, $L_{1,2}$ and P are positive and satisfy $\varrho_E < \varrho_W$, $c_E < c_W$.
2. The functions f, g, $L_{.,3}$ (and R, consequently) belong to $C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $0 \leqslant f \leqslant P$, $f = 0$ on $(-\infty, 0)$ and f is nondecreasing for positive arguments, $0 \leqslant g \leqslant 1$, $g = 0$ on $(-\infty, -b]$ and g is increasing on $(-b, b)$ for some $b \geqslant 1$, $g(0)$ is positive and sufficiently small, $g(b) = 1$.
3. The functions Λ, ε_τ, ε_φ, ε_κ, $K_{\mathrm{d}\varphi}$, $K_{\mathrm{d}\tau}$, S belong to $C^{0,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $c_s \geqslant 0$, $\tau c_s(\tau) \to 0$ for $\tau \to -\infty$ and there exist real numbers c', c'' satisfying $0 < c' \leqslant L_{.,3}$, Λ, R, $R' \leqslant c''$.

In what follows, the Einstein summation is used.

The problem is to find a pair of functions $(\tau, \varPhi)$ which satisfies the system of evolution equations (7), (8) for all $x \in \Omega$ and $t \in I$ together with the initial and boundary conditions (9) and (10). We assume that $\tau_0, \varPhi_0 \in C^{1,\gamma}(\overline{\Omega}), \gamma \in (0,1]$ and

$$|\tau_0(x)| \leqslant a, \quad |\varPhi_0(x)| \leqslant b' \qquad \forall x \in \overline{\Omega}$$

for some $a > 0$ and for $0 < b' < b$ (b appears in the above assumptions).

Let $0 = t_0 < t_1 < \ldots < t_r = T$ be an equidistant mesh with step $0 < k \leqslant 1$. We take a fixed integer i such that $0 \leqslant i < r$ and abbreviate $\zeta(., t_i)$ by $\zeta_i(.)$ for any function ζ.

For this fixed i, we assume that that the functions $\tau_i(x)$ and $\varPhi_i(x)$ are known. In the following, the problem of existence (Sect. 4) and numerical approximation (Sect. 5) of the functions $\tau(x, t_{i+1})$, $\varPhi(x, t_{i+1})$ will be investigated. For simplicity, we use the notation

$$\tau(x) = \tau(x, t_{i+1}), \qquad \varPhi(x) = \varPhi(x, t_{i+1}).$$

Suitable numerical integration of the equations (8), (7) from $t = t_i$ to $t = t_{i+1}$ leads to the following nonlinear differential equations

$$\begin{aligned}(11)\quad &\frac{H(\tau,\varPhi) - H(\tau_i,\varPhi_i)}{R(\tau_i)} + G(\tau,\varPhi) - G(\tau_i,\varPhi_i) \\ &\qquad - k\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(P^1_{1i}(x)\tau'(x) + P^2_{1i}(x)\varPhi'(x)\right) + Q_i(x)\tau_i'(x)\,\tau'(x)\right) = 0,\end{aligned}$$

$$(12)\quad G_2(\tau,\varPhi) - G_2(\tau_i,\varPhi_i) - k\frac{\mathrm{d}}{\mathrm{d}x}\left(P^1_{2i}(x)\tau'(x) + P^2_{2i}(x)\varPhi'(x)\right) = 0.$$

Let us put $R_i(x) = R(\tau_i)$, $G_1(x,\tau,\Phi) = H(\tau,\Phi)/R_i(x) + G(\tau,\Phi)$, $A_i^1(x) = Q_i(x)\tau_i'(x)$. We can see that for $i = 0,1,\dots,r-1$, the equations (11), (12) and the boundary conditions (10) form the following boundary-value problem (13–15) for a coupled pair of nonlinear ODE's:

$$(13)\quad -k\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{1i}^1(x)\tau'(x) + P_{1i}^2(x)\Phi'(x)\right) - kA_i^1(x)\tau'(x) + G_1(x,\tau,\Phi) = G_{1i}(x),$$

$$(14)\quad -k\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{2i}^1(x)\tau'(x) + P_{2i}^2(x)\Phi'(x)\right) + G_2(\tau,\Phi) = G_{2i}(x),$$

$$(15)\quad P_{1i}^1(x)\frac{\mathrm{d}\tau}{\mathrm{d}n} + P_{1i}^2(x)\frac{\mathrm{d}\Phi}{\mathrm{d}n} = g_{1i}(x),\quad P_{2i}^1(x)\frac{\mathrm{d}\tau}{\mathrm{d}n} + P_{2i}^2(x)\frac{\mathrm{d}\Phi}{\mathrm{d}n} = g_{2i}(x)$$

for $t \in I$, $x = 0$ and $x = l$.

4. Weak solution of the problem (13–15)

We say that the pair (τ,Φ) of functions from $H^1(\Omega)$ is a *weak solution* of the problem (13–15), whenever

$$(16)\quad \begin{aligned} &k\int_\Omega \left[\left(P_{1i}^1\tau' + P_{1i}^2\Phi'\right)\varphi_1' + \left(P_{2i}^1\tau' + P_{2i}^2\Phi'\right)\varphi_2'\right]\mathrm{d}x - k\int_\Omega A_i^1\tau'\varphi_1\,\mathrm{d}x \\ &\quad + \int_\Omega (G_1(x,\tau,\Phi)\varphi_1 + G_2(\tau,\Phi)\varphi_2)\,\mathrm{d}x - k\int_{\partial\Omega}(g_{1i}\varphi_1 + g_{2i}\varphi_2)\,\mathrm{d}H^{n-1} \\ &= \int_\Omega (G_{1i}\varphi_1 + G_{2i}\varphi_2)\,\mathrm{d}x \end{aligned}$$

for all $\varphi_1,\varphi_2 \in H^1(\Omega)$. For the sake of simplicity we put $u = (u_1,u_2) = (\tau,\Phi)$, $\varphi = (\varphi_1,\varphi_2)$, $V = H^1(\Omega,\mathbb{R}^2)$ and $V^* = \left(H^1(\Omega,\mathbb{R}^2)\right)^*$. The space V^* is dual to V and we denote by $\langle .,. \rangle$ the related duality. Now we can define an operator $T\colon V \to V^*$ and a functional $F \in V^*$ by

$$(17)\quad \begin{aligned} \langle Tu,\varphi\rangle = k&\int_\Omega \left[\left(P_{1i}^1\tau' + P_{1i}^2\Phi'\right)\varphi_1' + \left(P_{2i}^1\tau' + P_{2i}^2\Phi'\right)\varphi_2'\right]\mathrm{d}x \\ &- k\int_\Omega A_i^1\tau'\varphi_1\,\mathrm{d}x + \int_\Omega (G_1(x,\tau,\Phi)\varphi_1 + G_2(\tau,\Phi)\varphi_2)\,\mathrm{d}x \end{aligned}$$

and

$$(18)\quad \langle F,\varphi\rangle = \int_\Omega (G_{1i}\varphi_1 + G_{2i}\varphi_2)\,\mathrm{d}x + k\int_{\partial\Omega}(g_{1i}\varphi_1 + g_{2i}\varphi_2)\,\mathrm{d}H^{n-1}$$

for all $\varphi \in V$. Here the symbol H^{n-1} denotes the Hausdorff measure of dimension $n-1$. A vector-valued function u is a weak solution of our problem (13–15) whenever it is a solution of the operator equation

$$Tu = F.$$

Let us verify the following properties for $i = 0, 1, \ldots, r$ (it is necessary to remember that $|\tau_i(x)| \leqslant a$ and $|\Phi_i(x)| \leqslant b'$):

Smoothness conditions

(19)
$$\tau_i(x), \Phi_i(x) \in C^{1,\gamma}\left(\overline{\Omega}\right) \Longrightarrow P^l_{ji}(x), A^1_i(x), G_{ji}(x), g_{ji}(x) \in C^{0,\gamma}\left(\overline{\Omega}\right) \quad \forall\, j, l = 1, 2,$$

for $\gamma \in (0, 1]$.

P r o o f. See [1], p. 186. □

Growth conditions

(20)
$$\begin{aligned}&\left|P^1_{ji}(x)p_1\right|, \left|P^2_{ji}(x)p_2\right|, \left|A^1_i(x)p_1\right|, \left|G_1(x,u)\right|, \left|N(u)\right| \\ &\qquad \leqslant c_0\left(1 + |u| + |p|\right) \quad \forall x \in \Omega,\ \forall u, p \in \mathbb{R}^2,\ j = 1, 2,\ c_0 > 0.\end{aligned}$$

P r o o f. These conditions are easy consequences of the properties of functions formulated at the beginning of Section 3. □

Ellipticity condition

(21)
$$P^l_{ki}(x)p_k p_l \geqslant c_1 |p|^2 \quad \forall x \in \Omega,\ \forall p \in \mathbb{R}^2,\ c_1 > 0.$$

P r o o f. The ellipticity condition is equivalent to the following two conditions: $P^1_{1i}(x) > 0$ and $\det\left(P^l_{ki}(x)\right) > 0$, for all $x \in \overline{\Omega}$. It follows from the above assumptions that

$$P^1_{1i}(x) = \varepsilon_{\tau i}(x) K_{\mathrm{d}\tau i}(x) + R_i^{-1}(x)\Lambda_i(x) > 0,$$

since $\varepsilon_{\tau i}(x) K_{\mathrm{d}\tau i}(x) \geqslant 0$ and $R_i^{-1}(x)\Lambda_i(x) > 0$ on $\overline{\Omega}$. Similarly, we can see that

$$\begin{aligned}\det\left(P^l_{ki}(x)\right) &= \varepsilon_{\varphi i}(x) K_{\mathrm{d}\varphi i}(x) R_i^{-1}(x)\Lambda_i(x) g'_i(x) \\ &\quad + \left[\varepsilon_{\tau i}(x) K_{\mathrm{d}\tau i}(x) + R_i^{-1}(x)\Lambda_i(x)\right] \varepsilon_{\kappa i}(x) S_i(x) f'_i(x) > 0\end{aligned}$$

because the first term is positiv and the second term is nonnegativ on $\overline{\Omega}$. □

Coercivity condition

(22)
$$\begin{aligned}&kP^l_{ki}(x)p_k p_l - kA^1_i\,(x)\,u_1 p_1 \\ &\quad + G_1(x,u)u_1 + N(u)u_2 \geqslant kc_2 |p|^2 - c_3 \quad \forall x \in \Omega,\ \forall u, p \in \mathbb{R}^2,\ c_2, c_3 > 0.\end{aligned}$$

P r o o f. It follows from the ellipticity condition (21) that $kP^l_{ki}(x)p_k p_l \geqslant kc_1 |p|^2$. Putting $m_i = \sup\limits_{x \in \overline{\Omega}} \left|A^1_i(x)\right|$, we obtain

$$-kA^1_i(x)u_1 p_1 \geqslant -\frac{k\left|A^1_i(x)\right|}{2}\left(u_1^2 + p_1^2\right) \geqslant -\frac{km_i}{2}\left(u_1^2 + p_1^2\right).$$

Further,

$$
\begin{aligned}
&kP^l_{ki}(x)p_kp_l - kA^1_i(x)\,u_1p_1 + G_1(x,u)u_1 + N(u)u_2 \\
&\quad\geqslant kc_1\,|p|^2 - \frac{km_i}{2}p_1^2 + \left(\frac{\varrho_M c_M}{R_i(x)} - \frac{km_i}{2}\right)u_1^2 \\
&\qquad + \frac{1}{\varrho_W}\left(P - \frac{1}{\varrho_E}f(u_2)\left(\varrho_W + (\varrho_W - \varrho_E)\chi(u_1)\right)\right)g(u_2)c_s(u_1)u_1 \\
&\qquad + \left[f(u_2) + \frac{1}{\varrho_W}\left(P - \frac{1}{\varrho_E}f(u_2)\left(\varrho_W + (\varrho_W - \varrho_E)\chi(u_1)\right)\right)g(u_2)c_s(u_1)\right]u_2 \\
&\quad\geqslant k\left(c_1 - \frac{m_i}{2}\right)|p|^2 + \left(\frac{\varrho_M c_M}{R_i(x)} - \frac{km_i}{2}\right)u_1^2 - c_3 \\
&\quad\geqslant kc_2\,|p|^2 - c_3,
\end{aligned}
$$

where $c_2 = c_1 - m_i/2 > 0$ and $\varrho_M c_M/R_i(x) - km_i/2 \geqslant 0$. □

Pseudomonotony condition

$$P^l_{ki}(x)p_kp_l > 0 \quad \forall x \in \Omega,\ \forall p \in \mathbb{R}^2,\ p \neq 0. \tag{23}$$

P r o o f. This is a consequence of the ellipticity condition (21). □

Theorem 1. *There exists a solution $u \in H^1(\Omega, \mathbb{R}^2)$ of the equation $Tu = F$.*

P r o o f. According to [12], p. 60, this statement is a consequence of the conditions (19), (20), (22) and (23). □

The system (13–15) is a special case of the problem

$$-D_\alpha\left(A^{\alpha\beta}_{ij}(x)D_\beta u^j\right) = -D_\alpha a^\alpha_i(x,u) + a_i(x,u,Du), \quad i = 1,\dots N \tag{S}$$

in $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$ and with the nonlinear Neumann boundary condition

$$A^{\alpha\beta}_{ij}(x)\nu_\alpha(x)D_\beta u^j = g_i(x,u) + a^\beta_i(x,u)\nu_\beta(x), \quad x \in \partial\Omega,\ i = 1,\dots N, \tag{BC}$$

where $\alpha,\ \beta = 1,\dots,n,\ i,j = 1,\dots,N,\ n \geqslant 1,\ N \geqslant 1$ and $a^\alpha_i(x,u),\ a_i(x,u,p)$, $g_i(x,u)$ are *the Caratheodory functions,* i.e. they are measurable with respect to x and continuous with respect to u,p. Further, $A^{\alpha\beta}_{ij}(x) \in C^{0,\gamma}\left(\overline{\Omega}\right), \gamma \in (0,1]$ and the following Legendre-Hadamard condition

$$A^{\alpha\beta}_{ij}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \geqslant \nu_0\,|\xi|^2\,|\eta|^2 \qquad \forall x \in \Omega, \forall \xi \in \mathbb{R}^n,\ \forall \eta \in \mathbb{R}^N;\ \nu_0 > 0 \tag{L-H}$$

is satisfied in our case.

R e m a r k. The condition (L-H) follows from the standard condition of ellipticity. The opposite implication is not true.

Theorem 2. *Let Ω be a bounded subset of $\mathbb{R}$ and $u \in H^1\left(\Omega, \mathbb{R}^N\right)$ be a weak solution of the nonlinear system* (S) *with the boundary condition* (BC) *under the hypotheses* (L-H). *Let*

$$(*) \qquad |a_i^\alpha(x,u)| \leqslant f_i^\alpha(x) + |u|^q$$

and

$$(**) \qquad |a_i(x,u,p)| \leqslant f_i(x) + c\left(|u|^q + |p|^3\right)$$

for almost every $x \in \Omega$, all $u \in \mathbb{R}^N$, $p \in \mathbb{R}^N$ and suppose that $g_i(x,u) \in C^{0,1}\left(\overline{\Omega} \times \mathbb{R}^N\right)$, $f_i^\alpha(x) \in \mathcal{L}^{2,1+2\gamma}(\Omega)$, $f_i(x) \in L^{1,\gamma}(\Omega)$ and q is arbitrary. Then $u \in C^{1,\gamma}\left(\overline{\Omega}, \mathbb{R}^N\right)$ and

$$\|u\|_{C^{1,\gamma}\left(\overline{\Omega},\mathbb{R}^N\right)} \leqslant C\left(\nu_0, N, q, \operatorname{diam}\Omega, \|f_i^\alpha\|_{\mathcal{L}^{2,1+2\gamma}(\Omega)}, \|f_i\|_{L^{1,\gamma}(\Omega)}\right).$$

P r o o f. See [2], p. 111, and [8]. □

R e m a r k. For the definition and properties of the Morrey-Campanato spaces $\mathcal{L}^{2,1+2\gamma}(\Omega)$ and $L^{1,\gamma}(\Omega)$ see [12], pp. 35–38.

Now we can formulate our main result:

Theorem 3. *For every $\tau_i, \Phi_i \in C^{1,\gamma}\left(\overline{\Omega}\right)$ satisfying $|\tau_i(x)| \leqslant a$, $|\Phi_i(x)| \leqslant b'$, there exists a weak solution u of the system* (13–15) *such that $u = (\tau, \Phi) \in C^{1,\gamma}\left(\overline{\Omega}\right) \times C^{1,\gamma}\left(\overline{\Omega}\right)$.*

P r o o f. The existence follows from Theorem 1. Conditions $(*)$, $(**)$ are simple consequences of (20) and, therefore, regularity follows from Theorem 2. □

5. Numerical solution of the problem (13–15)

The nonlinear problem (13–15) has been solved numerically by the following iteration scheme: Let us denote

$$\lambda^j_{i0} = \frac{\partial G_j}{\partial \tau}(\tau_i, \Phi_i), \quad \varrho^j_{i0} = \frac{\partial G_j}{\partial \Phi}(\tau_i, \Phi_i)$$

for $j = 1, 2$, choose a fixed integer $L > 1$ and put

$$\overline{\lambda}^j_{is} = \begin{cases} \dfrac{G_j(\tau^s_h, \Phi_i) - G_j(\tau_i, \Phi_i)}{\tau^s_h - \tau_i} & \text{for } \tau^s_h \neq \tau_i, \\ \lambda^j_{i0} & \text{for } \tau^s_h = \tau_i, \end{cases}$$

$$\lambda^j_{is} = \begin{cases} \overline{\lambda}^j_{is} & \text{for } s \leqslant L, \\ \lambda^j_{i,s-1} & \text{for } L < s,\ |\lambda^j_{i,s-1}| < |\overline{\lambda}^j_{is}|, \\ \overline{\lambda}^j_{is} & \text{for } L < s,\ |\overline{\lambda}^j_{is}| \leqslant |\lambda^j_{i,s-1}|, \end{cases}$$

$$\overline{\varrho}^j_{is} = \begin{cases} \dfrac{G_j(\tau_i, \Phi^s_h) - G_j(\tau_i, \Phi_i)}{\Phi^s_h - \Phi_i} & \text{for } \Phi^s_h \neq \Phi_i, \\ \varrho^j_{i0} & \text{for } \Phi^s_h = \Phi_i, \end{cases}$$

$$\varrho^j_{is} = \begin{cases} \overline{\varrho}^j_{is} & \text{for } s \leqslant L, \\ \varrho^j_{i,s-1} & \text{for } L < s,\ |\varrho^j_{i,s-1}| < |\overline{\varrho}^j_{is}|, \\ \overline{\varrho}^j_{is} & \text{for } L < s,\ |\overline{\varrho}^j_{is}| \leqslant |\varrho^j_{i,s-1}|, \end{cases}$$

for $s = 1, 2, \ldots$ consecutively. Here the functions τ^s_h, Φ^s_h $(s = 1, 2, \ldots)$ are approximations of the exact solutions of the following linear differential problem (24–26):

$$(24)\quad \lambda^1_{i,s-1}(\tau^s - \tau_i) + \varrho^1_{i,s-1}(\Phi^s - \Phi_i) - k\Big[\frac{\mathrm{d}}{\mathrm{d}x}(P^1_{1i}(\tau^s)'(x) + P^2_{1i}(\Phi^s)'(x)) + A^1_i(x)(\tau^s)'(x)\Big] = 0,$$

$$(25)\quad \lambda^2_{i,s-1}(\tau^s - \tau_i) + \varrho^2_{i,s-1}(\Phi^s - \Phi_i) - k\frac{\mathrm{d}}{\mathrm{d}x}(P^1_{2i}(\tau^s)'(x) + P^2_{2i}(\Phi^s)'(x)) = 0,$$

$$(26)\qquad P^1_{ji}\frac{\mathrm{d}\tau^s}{\mathrm{d}n} + P^2_{ji}\frac{\mathrm{d}\Phi^s}{\mathrm{d}n} = g_{ji}, \quad j = 1, 2.$$

We put $\tau = \tau^s_h$ and $\Phi = \Phi^s_h$ whenever $|\lambda^j_{is} - \lambda^j_{i,s-1}| < k^d$ and $|\varrho^j_{is} - \varrho^j_{i,s-1}| < k^d$ for $j = 1, 2$ and a fixed $d \in (0, 1\rangle$. A scheme of this kind has been used by J. Kačur in [10].

Our numerical solution of the problem (24–26) is based on the following weak formulation: Find the functions τ^s, Φ^s in $H^1(\Omega)$ such that

$$(27)\qquad a_1(\tau^s, v) + a_2(\Phi^s, v) = f(v),$$

$$(28)\qquad b_1(\tau^s, v) + b_2(\Phi^s, v) = g(v)$$

hold for all $v \in H^1(\Omega)$. Here

$$
\begin{aligned}
a_1(\tau, v) &= \int_0^l \Big[kP^1_{1i}\tau' \, v' - kA^1_i \tau' v + \lambda^1_{i,s-1}\tau v \Big] \, \mathrm{d}x, \\
a_2(\Phi, v) &= \int_0^l \Big[kP^2_{1i}\Phi' \, v' + \varrho^1_{i,s-1}\Phi v \Big] \, \mathrm{d}x, \\
f(v) &= \int_0^l (\lambda^1_{i,s-1}\tau_i + \varrho^1_{i,s-1}\Phi_i) v \, \mathrm{d}x + k\Big[g_{1i}(l)v(l) + g_{1i}(0)v(0) \Big], \\
b_1(\tau, v) &= \int_0^l \Big[kP^1_{2i}\tau' \, v' + \lambda^2_{i,s-1}\tau v \Big] \, \mathrm{d}x, \\
b_2(\Phi, v) &= \int_0^l \Big[kP^2_{2i}\Phi' \, v' + \varrho^2_{i,s-1}\Phi v \Big] \, \mathrm{d}x, \\
g(v) &= \int_0^l (\lambda^2_{i,s-1}\tau_i + \varrho^2_{i,s-1}\Phi_i) v \, \mathrm{d}x + k[g_{2i}(l)v(l) + g_{2i}(0)v(0)].
\end{aligned}
$$

Since the linear problem (24–26) consists of two equations in which absolute values of the coefficients $\lambda^j_{i,s-1}$, $\varrho^j_{i,s-1}$ $(j = 1, 2)$ are essentially larger than the absolute values of the other coefficients, standard Galerkin discretization of this problem leads to a system of equations with a non-monotone matrix. That is why an oscillating approximate solution appears typically. For this reason, problem (27), (28) has been solved approximately by the following iteration scheme:

Let us take a mesh $0 = x_0 < x_1 < \ldots < x_n = l$ on the interval $\langle 0, l\rangle$ such that each point, in which two different materials meet, is a node. Let $0 = y_0 < y_1 < \ldots < y_{2n} = l$ be such a mesh that $y_{2j} = x_j$ for $j = 0, \ldots, n$ and $y_{2j-1} = (x_{j-1} + x_j)/2$ for $j = 1, \ldots, n$. Let $w_0, \ldots, w_n$ and $z_0, \ldots, z_{2n}$ be the standard base functions with small supports in the space of linear splines related to the nodes $x_0, \ldots, x_n$ and $y_0, \ldots, y_{2n}$ respectively. The problem is to compute approximations τ^s_h, Φ^s_h of τ^s, Φ^s in the forms

$$\tau^s_h = \tau_0 w_0 + \ldots + \tau_n w_n, \quad \Phi^s_h = \Phi_0 w_0 + \ldots + \Phi_n w_n.$$

We compute the vector $(\tau_0, \Phi_0, \ldots, \tau_n, \Phi_n)$ related to time $(i+1)h$ (i can be an arbitrary integer) using the corresponding values from time ih by the following iteration scheme:

Step 1. Take small positive numbers ε_τ, ε_Φ and find

$$\tau^{(0)} = \tau^{(0)}_0 w_0 + \ldots + \tau^{(0)}_n w_n$$

as a solution of the system of equations

$$a_1(\tau^{(0)}, \mathcal{T}_a(w_j)) = f(\mathcal{T}_a(w_j)) - a_2(\Phi_i, \mathcal{T}_a(w_j)) \tag{29}$$

for $j = 0, 1, \ldots, n$. The test functions

$$\mathcal{T}_a(w_j) = \alpha_j z_{2j-1} + \beta_j z_{2j} + \gamma_j z_{2j+1} \quad \text{for} \quad j = 0, 1, \ldots, n \quad (\alpha_0 = 0 = \gamma_n)$$

are constructed in a way described in [3], Sect. 4, with the aim to make the matrix of (29) monotone. Further, find

$$\varPhi^{(0)} = \varPhi_0^{(0)} w_0 + \ldots + \varPhi_n^{(0)} w_n$$

as a solution of the system of equations

$$b_2(\varPhi^{(0)}, \mathcal{T}_b(w_j)) = g(\mathcal{T}_b(w_j)) - b_1(\tau^{(0)}, \mathcal{T}_b(w_j)) \tag{30}$$

for $j = 0, 1, \ldots, n$. Here the test functions

$$\mathcal{T}_b(w_j) = \overline{\alpha}_j z_{2j-1} + \overline{\beta}_j z_{2j} + \overline{\gamma}_j z_{2j+1} \quad \text{for} \quad j = 0, 1, \ldots, n \quad (\overline{\alpha}_0 = 0 = \overline{\gamma}_n)$$

are constructed as in [3], Sect. 3, with the aim to make the matrix of (30) monotone.

Step 2. For $k = 1, 2, \ldots$, compute

$$\tau^{(k)} = \tau_0^{(k)} w_0 + \ldots + \tau_n^{(k)} w_n \quad \text{and} \quad \varPhi^{(k)} = \varPhi_0^{(k)} w_0 + \ldots + \varPhi_n^{(k)} w_n$$

as a solution of the system of equations

$$a_1(\tau^{(k)}, \mathcal{T}_a(w_j)) = f(\mathcal{T}_a(w_j)) - a_2(\varPhi^{(k-1)}, \mathcal{T}_a(w_j)) \quad (j = 0, \ldots, n)$$

and

$$b_2(\varPhi^{(k)}, \mathcal{T}_b(w_j)) = g(\mathcal{T}_b(w_j)) - b_1(\tau^{(k)}, \mathcal{T}_b(w_j)) \quad (j = 0, \ldots, n)$$

consecutively with the test functions $\mathcal{T}_a(w_j)$ and $\mathcal{T}_b(w_j)$ constructed as in Step 1.

Step 3. Repeat Step 2 as long as

$$\max_{0 \leqslant j \leqslant n} \left| \tau_j^{(k)} - \tau_j^{(k-1)} \right| \leqslant \varepsilon_\tau \quad \text{and} \quad \max_{0 \leqslant j \leqslant n} \left| \varPhi_j^{(k)} - \varPhi_j^{(k-1)} \right| \leqslant \varepsilon_\varPhi.$$

If this condition is satisfied, then put

$$\tau_j = \tau_j^{(k)} \quad \text{and} \quad \varPhi_j = \varPhi_j^{(k)} \quad \text{for} \quad j = 0, 1, \ldots, n.$$

In [3], the stability and accuracy of the Petrov-Galerkin method used above has been studied.

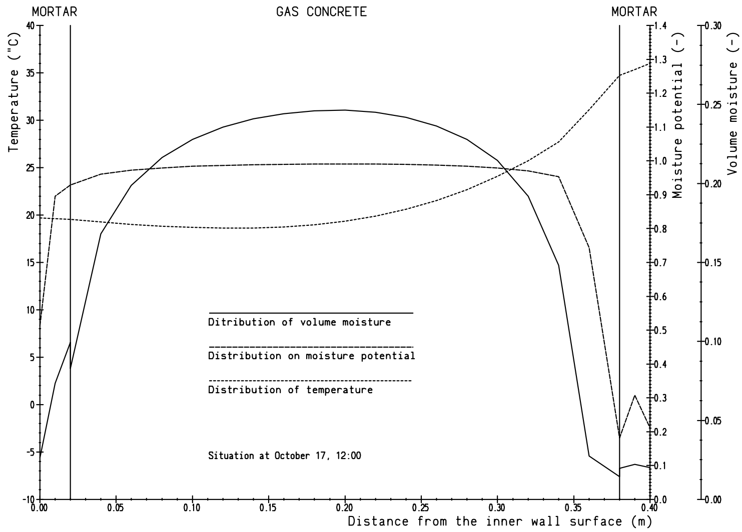

Figure 1. Distribution of moisture and temperature, October 17, 12:00.

E x a m p l e. By means of the method described above, the transport of heat and moisture across a vertical wall from lightconcrete (0.36 [m]) with plasters from mortar (each 0.02 [m]) has been modeled in the time-period of 180 days—from July 21 to January 17. The intervals [0, 0.02] and [0.38, 0.40] are divided with step 0.01 and the interval [0.02, 0.38] is divided with step 0.02. The interior surface corresponds to $x = 0$ and the exterior one corresponds to $x = 0.4$. Constant temperature 20°C of the air in the interior and certain typical temperatures of the air in the exterior have been assumed. The influence of sunshine on the exterior surface, which is oriented to the south, has been taken into account. For illustration, the distribution of moisture and temperature at the noon of October 17 is presented.

References

[1] *J. Appel, P. A. Zabrejko*: Nonlinear Superposition Operators. Cambridge tracts in mathematics 95, Cambridge University Press, Cambridge, 1990.

[2] *S. Campanato*: Sistemi Ellittici in Forma Divergenza. Regolarita all'interno. Quaderni, Pisa, 1980.

[3] *J. Dalík*: A Petrov-Galerkin approximation of convection-diffusion and reaction-diffusion problems. Appl. Math. *36* (1991), 329–352.

[4] *J. Dalík, J. Daněček, S. Šťastník*: The Kiessl model, existence of the classical solution. Sborník konf. Kočovce (1999), to appear.

[5] *J. Dalík, J. Daněček, S. Šťastník*: A model of simultaneous distribution of moisture and temperature in porous materials. Ceramics (Silikáty) *41 (2)* (1997), 41–46.

[6] *J. Dalík, J. Svoboda, S. Šťastník*: Návrh matematického modelu šíření vlhkosti a tepla v pórovitém prostředí. Preprint.

[7] *D. Gawin, P. Baggio, B. A. Schrefler*: Modelling heat and moisture transfer in deformable porous building materials. Arch. Civil Eng. *XLII, 3* (1996), 325–349.

[8] *M. Giaquinta, G. Modica*: Local existence for quasilinear parabolic systems under nonlinear boundary conditions. Ann. Mat. Pura Appl. *149* (1987), 41–59.

[9] *H. Glaser*: Graphisches Verfahren zur Untersuchung von Diffusionsvorgängen. Kältetechnik *H.10* (1959), 345–349.

[10] *J. Kačur*: Solution to strongly nonlinear parabolic problems by a linear approximation scheme. Preprint M2-96, Comenius University Bratislava, Faculty of Mathematics and Physics.

[11] *K. Kiessl*: Kapillarer und dampfförmiger Feuchtetransport in mehrschichtlichen Bauteilen. Dissertation, Universität in Essen.

[12] *J. Nečas*: Introduction to the theory of nonlinear elliptic equations. Teubner Texte zur Mathematik 52, Teubner Verlag, Leipzig, 1986.

[13] *J. R. Philip, D. A. de Vries*: Moisture movements in porous materials under temperature gradients. Am. Geophys. Union *38* (1957), 222–232.

Authors' addresses: *Josef Dalík*, Ústav matematiky a deskriptivní geometrie Fakulty stavební VUT, Žižkova 17, 602 00 Brno, e-mail: `mddal@fce.vutbr.cz`; *Josef Daněček*, Ústav matematiky a deskriptivní geometrie Fakulty stavební VUT, Žižkova 17, 602 00 Brno, e-mail: `mddan@fce.vutbr.cz`; *Jiří Vala*, Ústav matematiky a deskriptivní geometrie Fakulty stavební VUT, Žižkova 17, 602 00 Brno, e-mail: `mdval@fce.vutbr.cz`.