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ON A DIFFERENTIAL-ALGEBRAIC PROBLEM

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Abstract. The method of quasilinearization is a procedure for obtaining approximate solutions of differential equations. In this paper, this technique is applied to a differential-algebraic problem. Under some natural assumptions, monotone sequences converge quadratically to a unique solution of our problem.

Keywords: differential-algebraic problem, monotone sequences, quadratic convergence

MSC 2000: 34A45, 34B99

SECTION 1

Consider the problem

$$(1) \quad \begin{cases} x'(t) = f(t, x(t), y(t)), & t \in J = [0, b], \\ x(0) = k_0, \\ y(t) = g(t, x(t), y(t)), & t \in J, \end{cases}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $k_0 \in \mathbb{R}$ are given. Problem (1) is called a differential-algebraic problem. A solution of (1) is a pair $(x, y) \in C^1(J, \mathbb{R}) \times C(J, \mathbb{R})$.

There are some methods which can be applied to construct approximate solutions of such problems. Convergence of approximate iterations for (1) is proved, for example, in papers [3], [6] (see also [1]). In [5], an approximate solution is constructed by some numerical methods. Another useful method is based on monotone iterations. The aim of this paper is to apply this procedure. Using the monotone iterative technique we can construct monotone sequences of approximate solutions that converge

uniformly to the unique solution of (1). As we shall see, the convergence is quadratic, which is useful in practice.

SECTION 2

Let $x_0, u_0 \in C^1(J, \mathbb{R})$, $y_0, v_0 \in C(J, \mathbb{R})$ be such that $x_0(t) \leq u_0(t)$, $y_0(t) \leq v_0(t)$ on J . Define the closed set

$$\Omega = \{(t, x, y) : t \in J, \quad x_0(t) \leq x \leq u_0(t), \quad y_0(t) \leq y \leq v_0(t)\}.$$

Theorem 1. *Let $f, g \in C(\Omega, \mathbb{R})$. Assume that f_x, f_y, g_x, g_y exist and $g_y(t, u, v) < 1$ on Ω . Then problem (1) has at most one solution.*

Proof. Assume that problem (1) has two distinct solutions (\bar{x}, \bar{y}) and $(\bar{\bar{x}}, \bar{\bar{y}})$. Put $p = \bar{x} - \bar{\bar{x}}$, $q = \bar{y} - \bar{\bar{y}}$. Note that $p(0) = 0$. Using the mean value theorem we obtain

$$\begin{aligned} p'(t) &= f(t, \bar{x}(t), \bar{y}(t)) - f(t, \bar{\bar{x}}(t), \bar{\bar{y}}(t)) \\ &= f(t, \bar{x}(t), \bar{y}(t)) - f(t, \bar{\bar{x}}(t), \bar{y}(t)) + f(t, \bar{\bar{x}}(t), \bar{y}(t)) - f(t, \bar{\bar{x}}(t), \bar{\bar{y}}(t)) \\ &= f_x(t, \xi, \bar{y}(t))p(t) + f_y(t, \bar{\bar{x}}(t), \sigma)q(t) \end{aligned}$$

and

$$\begin{aligned} q(t) &= g(t, \bar{x}(t), \bar{y}(t)) - g(t, \bar{\bar{x}}(t), \bar{\bar{y}}(t)) \\ &= g(t, \bar{x}(t), \bar{y}(t)) - g(t, \bar{\bar{x}}(t), \bar{y}(t)) + g(t, \bar{\bar{x}}(t), \bar{y}(t)) - g(t, \bar{\bar{x}}(t), \bar{\bar{y}}(t)) \\ &= g_x(t, \bar{\xi}, \bar{y}(t))p(t) + g_y(t, \bar{\bar{x}}(t), \bar{\sigma})q(t) \end{aligned}$$

with $\xi, \bar{\xi}$ between \bar{x} and $\bar{\bar{x}}$, and $\sigma, \bar{\sigma}$ between \bar{y} and $\bar{\bar{y}}$. Indeed,

$$q(t) = \frac{g_x(t, \bar{\xi}, \bar{y}(t))}{1 - g_y(t, \bar{\bar{x}}(t), \bar{\sigma})}p(t),$$

and hence we have

$$p'(t) = p(t)k(t) \quad \text{for} \quad k(t) = f_x(t, \xi, \bar{y}(t)) + \frac{f_y(t, \bar{\bar{x}}(t), \sigma)g_x(t, \bar{\xi}, \bar{y}(t))}{1 - g_y(t, \bar{\bar{x}}(t), \bar{\sigma})}.$$

Solving the linear differential equation we get

$$p(t) = p(0)e^{\int_0^t k(\tau) d\tau}, \quad t \in J,$$

so $p(t) = q(t) = 0$, $t \in J$. This proves Theorem 1. □

SECTION 3

Definition 1. A pair $(x_0, y_0) \in C^1(J, \mathbb{R}) \times C(J, \mathbb{R})$ is said to be a lower solution of problem (1) if

$$(2) \quad \begin{cases} x'_0(t) \leq f(t, x_0(t), y_0(t)), & t \in J, \\ x_0(0) \leq k_0, \\ y_0(t) \leq g(t, x_0(t), y_0(t)), & t \in J \end{cases}$$

and a pair (u_0, v_0) is an upper solution of (1) if the inequalities are reversed.

Theorem 2. Assume that $f, g \in C(\Omega, \mathbb{R})$, and

1. $(x_0, y_0) \in C^1(J, \mathbb{R}) \times C(J, \mathbb{R})$ and $(u_0, v_0) \in C^1(J, \mathbb{R}) \times C(J, \mathbb{R})$ are lower and upper solutions of problem (1), respectively, such that $x_0(t) \leq u_0(t)$, $y_0(t) \leq v_0(t)$, $t \in J$,
2. $f_{xx}, f_{xy}, f_{yx}, f_{yy}, g_{xx}, g_{xy}, g_{yx}, g_{yy}$ exist, are continuous and
3. $f_{xx}(t, u, v) \geq 0$, $f_{xy}(t, u, v) \geq 0$, $f_{yy}(t, u, v) \geq 0$, $f_y(t, u, v) \geq 0$, $g_{xx}(t, u, v) \geq 0$, $g_{xy}(t, u, v) \geq 0$, $g_{yy}(t, u, v) \geq 0$, $g_x(t, u, v) \geq 0$, $g_y(t, x, y) \leq B < 1$ on Ω .

Then there exist monotone sequences (x_n, y_n) , (u_n, v_n) which converge uniformly and monotonically on J to the unique solution of problem (1), and this convergence is quadratic.

Proof. For $n = 0, 1, \dots$ and $t \in J$, let us define the following sequences:

$$\begin{aligned} x'_{n+1}(t) &= f(t, x_n(t), y_n(t)) + f_x(t, x_n(t), y_n(t))[x_{n+1}(t) - x_n(t)] \\ &\quad + f_y(t, x_n(t), y_n(t))[y_{n+1}(t) - y_n(t)], \quad x_{n+1}(0) = k_0, \\ y_{n+1}(t) &= g(t, x_n(t), y_n(t)) + g_x(t, x_n(t), y_n(t))[x_{n+1}(t) - x_n(t)] \\ &\quad + g_y(t, x_n(t), y_n(t))[y_{n+1}(t) - y_n(t)], \end{aligned}$$

and

$$\begin{aligned} u'_{n+1}(t) &= f(t, u_n(t), v_n(t)) + f_x(t, x_n(t), y_n(t))[u_{n+1}(t) - u_n(t)] \\ &\quad + f_y(t, x_n(t), y_n(t))[v_{n+1}(t) - v_n(t)], \quad u_{n+1}(0) = k_0, \\ v_{n+1}(t) &= g(t, u_n(t), v_n(t)) + g_x(t, x_n(t), y_n(t))[u_{n+1}(t) - u_n(t)] \\ &\quad + g_y(t, x_n(t), y_n(t))[v_{n+1}(t) - v_n(t)]. \end{aligned}$$

First we shall show that

$$(3) \quad x_0(t) \leq x_1(t) \leq u_1(t) \leq u_0(t),$$

$$(4) \quad y_0(t) \leq y_1(t) \leq v_1(t) \leq v_0(t)$$

on J . Let $p(t) = x_0(t) - x_1(t)$, $q(t) = y_0(t) - y_1(t)$ on J . Note that

$$\begin{aligned}
 p'(t) &= x'_0(t) - x'_1(t) \\
 &\leq f(t, x_0(t), y_0(t)) - f(t, x_0(t), y_0(t)) \\
 &\quad - f_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t)] - f_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t)] \\
 &= f_x(t, x_0(t), y_0(t))p(t) + f_y(t, x_0(t), y_0(t))q(t), \\
 q(t) &= y_0(t) - y_1(t) \\
 &\leq g(t, x_0(t), y_0(t)) - g(t, x_0(t), y_0(t)) \\
 &\quad - g_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t)] - g_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t)] \\
 &= g_x(t, x_0(t), y_0(t))p(t) + g_y(t, x_0(t), y_0(t))q(t)
 \end{aligned}$$

since (x_0, y_0) is a lower solution of problem (1). Hence

$$q(t) \leq \frac{g_x(t, x_0(t), y_0(t))}{1 - g_y(t, x_0(t), y_0(t))}p(t), \quad t \in J,$$

so

$$p'(t) \leq K(t)p(t), \quad t \in J, \quad p(0) \leq 0,$$

where

$$K(t) = f_x(t, x_0(t), y_0(t)) + \frac{f_y(t, x_0(t), y_0(t))g_x(t, x_0(t), y_0(t))}{1 - g_y(t, x_0(t), y_0(t))}.$$

This yields the inequality

$$p(t) \leq p(0)e^{\int_0^t K(\tau) d\tau} \leq 0, \quad t \in J,$$

so $x_0(t) \leq x_1(t)$ and $y_0(t) \leq y_1(t)$ on J .

If we now put $p(t) = u_1(t) - u_0(t)$, $q(t) = v_1(t) - v_0(t)$, $t \in J$, then

$$\begin{aligned}
 p'(t) &= u'_1(t) - u'_0(t) \\
 &\leq f(t, u_0(t), v_0(t)) + f_x(t, x_0(t), y_0(t))[u_1(t) - u_0(t)] \\
 &\quad + f_y(t, x_0(t), y_0(t))[v_1(t) - v_0(t)] - f(t, u_0(t), v_0(t)) \\
 &= f_x(t, x_0(t), y_0(t))p(t) + f_y(t, x_0(t), y_0(t))q(t), \quad p(0) \leq 0,
 \end{aligned}$$

and

$$\begin{aligned}
 q(t) &= v_1(t) - v_0(t) \\
 &\leq g(t, u_0(t), v_0(t)) + g_x(t, x_0(t), y_0(t))[u_1(t) - u_0(t)] \\
 &\quad + g_y(t, x_0(t), y_0(t))[v_1(t) - v_0(t)] - g(t, u_0(t), v_0(t)) \\
 &= g_x(t, x_0(t), y_0(t))p(t) + g_y(t, x_0(t), y_0(t))q(t).
 \end{aligned}$$

Similarly as in the previous case, we immediately have $p(t) \leq 0$, $q(t) \leq 0$ on J , so $u_1(t) \leq u_0(t)$, $v_1(t) \leq v_0(t)$, $t \in J$.

To show that $x_1(t) \leq u_1(t)$, $y_1(t) \leq v_1(t)$ on J we need some relations on f and g . Observe that Taylor's formula yields

$$\begin{aligned} f(t, u, \alpha) &= f(t, u, \alpha) - f(t, v, \alpha) + f(t, v, \alpha) - f(t, v, \beta) + f(t, v, \beta) \\ &= f(t, v, \beta) + f_x(t, v, \alpha)(u - v) + \frac{1}{2}f_{xx}(t, \xi, \alpha)(u - v)^2 \\ &\quad + f_y(t, v, \beta)(\alpha - \beta) + \frac{1}{2}f_{yy}(t, v, \delta)(\alpha - \beta)^2 \end{aligned}$$

where ξ is between u and v , while δ is between α and β . Assume that $u \geq v$, $\alpha \geq \beta$. Since $f_{xx} \geq 0$, $f_{yy} \geq 0$, $f_{xy} \geq 0$, we have

$$(5) \quad f(t, u, \alpha) \geq f(t, v, \beta) + f_x(t, v, \beta)(u - v) + f_y(t, v, \beta)(\alpha - \beta).$$

In the same way, we can prove that

$$(6) \quad g(t, u, \alpha) \geq g(t, v, \beta) + g_x(t, v, \beta)(u - v) + g_y(t, v, \beta)(\alpha - \beta)$$

provided $u \geq v$ and $\alpha \geq \beta$.

Now, put $p(t) = x_1(t) - u_1(t)$, $q(t) = y_1(t) - v_1(t)$. Note that $p(0) = 0$. Basing on (5) and (6), we have

$$\begin{aligned} p'(t) &= f(t, x_0(t), y_0(t)) + f_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t)] \\ &\quad + f_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t)] - f(t, u_0(t), v_0(t)) \\ &\quad - f_x(t, x_0(t), y_0(t))[u_1(t) - u_0(t)] - f_y(t, x_0(t), y_0(t))[v_1(t) - v_0(t)] \\ &\leq -f_x(t, x_0(t), y_0(t))[u_0(t) - x_0(t)] - f_y(t, x_0(t), y_0(t))[v_0(t) - y_0(t)] \\ &\quad + f_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t) - u_1(t) + u_0(t)] \\ &\quad + f_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t) - v_1(t) + v_0(t)] \\ &= f_x(t, x_0(t), y_0(t))p(t) + f_y(t, x_0(t), y_0(t))q(t), \\ q(t) &= g(t, x_0(t), y_0(t)) + g_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t)] \\ &\quad + g_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t)] - g(t, u_0(t), v_0(t)) \\ &\quad - g_x(t, x_0(t), y_0(t))[u_1(t) - u_0(t)] - g_y(t, x_0(t), y_0(t))[v_1(t) - v_0(t)] \\ &\leq -g_x(t, x_0(t), y_0(t))[u_0(t) - x_0(t)] - g_y(t, x_0(t), y_0(t))[v_0(t) - y_0(t)] \\ &\quad + g_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t) - u_1(t) + u_0(t)] \\ &\quad + g_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t) - v_1(t) + v_0(t)] \\ &= g_x(t, x_0(t), y_0(t))p(t) + g_y(t, x_0(t), y_0(t))q(t). \end{aligned}$$

From the above, we have $p(t) \leq 0$, $q(t) \leq 0$ on J , so $x_1(t) \leq u_1(t)$, $y_1(t) \leq v_1(t)$, $t \in J$. This proves that (3) and (4) are satisfied.

In the next step we have to show that (x_1, y_1) and (u_1, v_1) are lower and upper solutions of problem (1), respectively. To show this we will use (5) and (6), obtaining

$$\begin{aligned}
 x_1'(t) &= f(t, x_0(t), y_0(t)) + f_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t)] \\
 &\quad + f_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t)] \\
 &\leq f(t, x_1(t), y_1(t)) - f_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t)] \\
 &\quad - f_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t)] + f_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t)] \\
 &\quad + f_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t)] \\
 &= f(t, x_1(t), y_1(t)), \\
 y_1(t) &= g(t, x_0(t), y_0(t)) + g_x(t, x_0(t), y_0(t))[x_1(t) - x_0(t)] \\
 &\quad + g_y(t, x_0(t), y_0(t))[y_1(t) - y_0(t)] \\
 &\leq g(t, x_1(t), y_1(t)),
 \end{aligned}$$

and

$$\begin{aligned}
 u_1'(t) &= f(t, u_0(t), v_0(t)) + f_x(t, x_0(t), y_0(t))[u_1(t) - u_0(t)] \\
 &\quad + f_y(t, x_0(t), y_0(t))[v_1(t) - v_0(t)] \\
 &\geq f(t, u_1(t), v_1(t)) + f_x(t, u_1(t), v_1(t))[u_0(t) - u_1(t)] \\
 &\quad + f_y(t, u_1(t), v_1(t))[v_0(t) - v_1(t)] + f_x(t, x_0(t), y_0(t))[u_1(t) - u_0(t)] \\
 &\quad + f_y(t, x_0(t), y_0(t))[v_1(t) - v_0(t)] \\
 &\geq f(t, u_1(t), v_1(t)), \\
 v_1(t) &= g(t, u_0(t), v_0(t)) + g_x(t, x_0(t), y_0(t))[u_1(t) - u_0(t)] \\
 &\quad + g_y(t, x_0(t), y_0(t))[v_1(t) - v_0(t)] \\
 &\geq g(t, u_1(t), v_1(t)),
 \end{aligned}$$

since f_x , f_y , g_x and g_y are nondecreasing with respect to the last two variables. This shows that (9) and (10) are satisfied.

Let us assume that

$$\begin{aligned}
 x_0(t) &\leq x_1(t) \leq \dots \leq x_k(t) \leq u_k(t) \leq \dots \leq u_1(t) \leq u_0(t), \\
 y_0(t) &\leq y_1(t) \leq \dots \leq y_k(t) \leq v_k(t) \leq \dots \leq v_1(t) \leq v_0(t),
 \end{aligned}$$

$t \in J$, and let (x_k, y_k) , (u_k, v_k) be lower and upper solutions of problem (1), respectively, for some $k > 1$.

We shall prove that

$$(7) \quad \begin{cases} x_k(t) \leq x_{k+1}(t) \leq u_{k+1}(t) \leq u_k(t), & t \in J, \\ y_k(t) \leq y_{k+1}(t) \leq v_{k+1}(t) \leq v_k(t), & t \in J. \end{cases}$$

As before, we set $p(t) = x_k(t) - x_{k+1}(t)$, $q(t) = y_k(t) - y_{k+1}(t)$, $t \in J$. We see that $p(0) = 0$ and

$$\begin{aligned} p'(t) &\leq f(t, x_k(t), y_k(t)) - f(t, x_k(t), y_k(t)) - f_x(t, x_k(t), y_k(t))[x_{k+1}(t) - x_k(t)] \\ &\quad - f_y(t, x_k(t), y_k(t))[y_{k+1}(t) - y_k(t)] \\ &= f_x(t, x_k(t), y_k(t))p(t) + f_y(t, x_0(t), y_0(t))q(t), \\ q(t) &\leq g(t, x_k(t), y_k(t)) - g(t, x_k(t), y_k(t)) - g_x(t, x_k(t), y_k(t))[x_{k+1}(t) - x_k(t)] \\ &\quad - g_y(t, x_k(t), y_k(t))[y_{k+1}(t) - y_k(t)] \\ &= g_x(t, x_k(t), y_k(t))p(t) + g_y(t, x_k(t), y_k(t))q(t). \end{aligned}$$

Hence we have $p(t) \leq 0$ and $q(t) \leq 0$ on J , so $x_k(t) \leq x_{k+1}(t)$, $y_k(t) \leq y_{k+1}(t)$, $t \in J$. Similarly, we can show that $u_{k+1}(t) \leq u_k(t)$ and $v_{k+1}(t) \leq v_k(t)$, $t \in J$. Now let $p(t) = x_{k+1}(t) - u_{k+1}(t)$, $q(t) = y_{k+1}(t) - v_{k+1}(t)$, $t \in J$. Then in view of (5) and (6) we get

$$\begin{aligned} p'(t) &= f(t, x_k(t), y_k(t)) + f_x(t, x_k(t), y_k(t))[x_{k+1}(t) - x_k(t)] \\ &\quad + f_y(t, x_k(t), y_k(t))[y_{k+1}(t) - y_k(t)] - f(t, u_k(t), v_k(t)) \\ &\quad - f_x(t, x_k(t), y_k(t))[u_{k+1}(t) - u_k(t)] - f_y(t, x_k(t), y_k(t))[v_{k+1}(t) - v_k(t)] \\ &\leq f_x(t, x_k(t), y_k(t))p(t) + f_y(t, x_k(t), y_k(t))q(t), \end{aligned}$$

and

$$\begin{aligned} q(t) &= g(t, x_k(t), y_k(t)) + g_x(t, x_k(t), y_k(t))[x_{k+1}(t) - x_k(t)] \\ &\quad + g_y(t, x_k(t), y_k(t))[y_{k+1}(t) - y_k(t)] - g(t, u_k(t), v_k(t)) \\ &\quad - g_x(t, x_k(t), y_k(t))[u_{k+1}(t) - u_k(t)] - g_y(t, x_k(t), y_k(t))[v_{k+1}(t) - v_k(t)] \\ &\leq g_x(t, x_k(t), y_k(t))p(t) + g_y(t, x_k(t), y_k(t))q(t). \end{aligned}$$

As a result we have

$$x_{k+1}(t) \leq u_{k+1}(t) \quad \text{and} \quad y_{k+1}(t) \leq v_{k+1}(t), \quad t \in J$$

so (7) holds.

Basing on (5) and (6) we can show that (x_{k+1}, y_{k+1}) , (u_{k+1}, v_{k+1}) are lower and upper solutions of problem (1), respectively.

Hence, by induction, we have

$$\begin{aligned}x_0(t) &\leq x_1(t) \leq \dots \leq x_n(t) \leq u_n(t) \leq \dots \leq u_1(t) \leq u_0(t), \\y_0(t) &\leq y_1(t) \leq \dots \leq y_n(t) \leq v_n(t) \leq \dots \leq v_1(t) \leq v_0(t),\end{aligned}$$

$t \in J$, for all $n > 1$, and $(x_n(t), y_n(t))$, $(u_n(t), v_n(t))$ are lower and upper solutions of problem (1), respectively.

Employing Dini's theorem we can show that the sequences (x_n, y_n) , (u_n, v_n) converge uniformly and monotonically to the corresponding solutions of problem (1). Since problem (1) has at most one solution (x, y) , so (x_n, y_n) , (u_n, v_n) converge to the unique solution of (1).

We shall prove that the convergence of (x_n, y_n) , (u_n, v_n) to (x, y) is quadratic. First, we put $p_{n+1}(t) = x(t) - x_{n+1}(t) \geq 0$, $q_{n+1}(t) = y(t) - y_{n+1}(t) \geq 0$, $t \in J$. Note that $p_{n+1}(0) = 0$. Then, by the mean value theorem and the monotonicity of f_x , f_y , g_x , g_y we have

$$\begin{aligned}p'_{n+1}(t) &= f(t, x(t), y(t)) - f(t, x_n(t), y_n(t)) - f_x(t, x_n(t), y_n(t))[x_{n+1}(t) - x_n(t)] \\&\quad - f_y(t, x_n(t), y_n(t))[y_{n+1}(t) - y_n(t)] \\&= f(t, x(t), y(t)) - f(t, x_n(t), y(t)) + f(t, x_n(t), y(t)) - f(t, x_n(t), y_n(t)) \\&\quad - f_x(t, x_n(t), y_n(t))[x_{n+1}(t) - x(t) + x(t) - x_n(t)] \\&\quad - f_y(t, x_n(t), y_n(t))[y_{n+1}(t) - y(t) + y(t) - y_n(t)] \\&= f_x(t, \xi_1, y(t))p_n(t) + f_y(t, x_n(t), \delta_1)q_n(t) \\&\quad + f_x(t, x_n(t), y_n(t))[p_{n+1}(t) - p_n(t)] + f_y(t, x_n(t), y_n(t))[q_{n+1}(t) - q_n(t)] \\&\leq [f_x(t, x(t), y(t)) - f_x(t, x_n(t), y(t))] \\&\quad + f_x(t, x_n(t), y(t)) - f_x(t, x_n(t), y_n(t))]p_n(t) \\&\quad + f_x(t, x_n(t), y_n(t))p_{n+1}(t) + f_y(t, x_n(t), y_n(t))q_{n+1}(t) \\&= [f_{xx}(t, \xi_2, y(t))p_n(t) + f_{xy}(t, x_n(t), \delta_2)q_n(t)]p_n(t) \\&\quad + f_x(t, x_n(t), y_n(t))p_{n+1}(t) + f_y(t, x_n(t), y_n(t))q_{n+1}(t),\end{aligned}$$

and

$$\begin{aligned}q_{n+1}(t) &= g(t, x(t), y(t)) - g(t, x_n(t), y(t)) + g(t, x_n(t), y(t)) - g(t, x_n(t), y_n(t)) \\&\quad - g_x(t, x_n(t), y_n(t))[x_{n+1}(t) - x(t) + x(t) - x_n(t)] \\&\quad - g_y(t, x_n(t), y_n(t))[y_{n+1}(t) - y(t) + y(t) - y_n(t)] \\&= g_x(t, \xi_3, y(t))p_n(t) + g_y(t, x_n(t), \delta_3)q_n(t) \\&\quad + g_x(t, x_n(t), y_n(t))[p_{n+1}(t) - p_n(t)] + g_y(t, x_n(t), y_n(t))[q_{n+1}(t) - q_n(t)]\end{aligned}$$

$$\begin{aligned}
&\leq [g_x(t, x(t), y(t)) - g_x(t, x_n(t), y(t)) + g_x(t, x_n(t), y(t)) \\
&\quad - g_x(t, x_n(t), y_n(t))]p_n(t) + g_x(t, x_n(t), y_n(t))p_{n+1}(t) \\
&\quad + g_y(t, x_n(t), y_n(t))q_{n+1}(t) + [g_y(t, x_n(t), y(t)) - g_y(t, x_n(t), y_n(t))]q_n(t) \\
&= [g_{xx}(t, \xi_4, y(t))p_n(t) + g_{xy}(t, x_n(t), \delta_4)q_n(t)]p_n(t) + g_{yy}(t, x_n(t), \delta_5)q_n^2(t) \\
&\quad + g_y(t, x_n(t), y_n(t))q_{n+1}(t) + g_x(t, x_n(t), y(t))p_{n+1}(t)
\end{aligned}$$

where $x_n(t) < \xi_1, \xi_2, \xi_3, \xi_4 < x(t)$, $y_n(t) < \delta_1, \delta_2, \delta_3, \delta_4, \delta_5 < y(t)$. Hence

$$\begin{aligned}
q_{n+1}(t) &\leq B_1 p_n^2(t) + B_2 p_n(t)q_n(t) + B_3 q_n^2(t) + Bq_{n+1}(t) + B_4 p_{n+1}(t) \\
&\leq \left(B_1 + \frac{1}{2}B_2\right)p_n^2(t) + \left(B_3 + \frac{1}{2}B_2\right)q_n^2(t) + Bq_{n+1}(t) + B_4 p_{n+1}(t),
\end{aligned}$$

so

$$(8) \quad q_{n+1}(t) \leq \frac{B_4}{1-B} p_{n+1}(t) + b_1 p_n^2(t) + b_2 q_n^2(t), \quad t \in J,$$

since $B < 1$, where

$$\begin{aligned}
|g_{xx}(t, u, v)| &\leq B_1, \quad |g_{xy}(t, u, v)| \leq B_2, \quad |g_{yy}(t, u, v)| \leq B_3, \\
|g_x(t, u, v)| &\leq B_4 \text{ on } \Omega, \\
b_1 &= \frac{B_1 + \frac{1}{2}B_2}{1-B}, \quad b_2 = \frac{B_3 + \frac{1}{2}B_2}{1-B}.
\end{aligned}$$

Moreover, we have

$$(9) \quad \begin{aligned} p'_{n+1}(t) &\leq A_1 p_n^2(t) + A_2 p_n(t)q_n(t) + A_3 p_{n+1}(t) + A_4 q_{n+1}(t) \\ &\leq \left(A_1 + \frac{1}{2}A_2\right)p_n^2(t) + \frac{1}{2}A_2 q_n^2(t) + A_3 p_{n+1}(t) + A_4 q_{n+1}(t) \end{aligned}$$

where $|f_{xx}(t, u, v)| \leq A_1$, $|f_{xy}(t, u, v)| \leq A_2$, $|f_x(t, u, v)| \leq A_3$, $|f_y(t, u, v)| \leq A_4$ on Ω . Combining (8) and (9) we finally get

$$p'_{n+1}(t) \leq a_1 p_{n+1}(t) + a_2 p_n^2(t) + a_3 q_n^2(t), \quad t \in J,$$

where

$$a_1 = A_3 + \frac{B_4 A_4}{1-B}, \quad a_2 = A_1 + \frac{1}{2}A_2 + A_4 b_1, \quad a_3 = \frac{1}{2}A_2 + A_4 b_2.$$

By Gronwall's inequality, we see that

$$\begin{aligned}
p_{n+1}(t) &\leq \int_0^t [a_2 p_n^2(s) + a_3 q_n^2(s)] e^{a_1(t-s)} ds \\
&\leq [a_2 \max_s p_n^2(s) + a_3 \max_s q_n^2(s)] c, \quad c = be^{a_1 b}, \quad t \in J.
\end{aligned}$$

Thus

$$\max_{t \in J} |x_{n+1}(t) - x(t)| \leq c_2 \max_{t \in J} |x_n(t) - x(t)|^2 + c_3 \max_{t \in J} |y_n(t) - y(t)|^2,$$

where $c_i = ca_i$ for $i = 2, 3$. Hence and by (8), we directly obtain

$$\max_{t \in J} |y_{n+1}(t) - y(t)| \leq \bar{a}_2 \max_{t \in J} |x_n(t) - x(t)|^2 + \bar{a}_3 \max_{t \in J} |y_n(t) - y(t)|^2,$$

where

$$\bar{a}_i = \frac{B_4 c_i}{1 - B} + b_{i-1}, \quad i = 2, 3.$$

In the same way we can show that the convergence of (u_n, v_n) to (x, y) is quadratic, so

$$\begin{aligned} \max_{t \in J} |u_{n+1}(t) - x(t)| &\leq d_1 \max_{t \in J} |u_n(t) - x(t)|^2 + d_2 \max_{t \in J} |v_n(t) - y(t)|^2 \\ &\quad + d_3 \max_{t \in J} |x_n(t) - x(t)|^2 + d_4 \max_{t \in J} |y_n(t) - y(t)|^2, \\ \max_{t \in J} |v_{n+1}(t) - y(t)| &\leq e_1 \max_{t \in J} |u_n(t) - x(t)|^2 + e_2 \max_{t \in J} |v_n(t) - y(t)|^2 \\ &\quad + e_3 \max_{t \in J} |x_n(t) - x(t)|^2 + e_4 \max_{t \in J} |y_n(t) - y(t)|^2, \end{aligned}$$

where $|f_{yy}(t, u, v)| \leq A_5$ on Ω ,

$$\begin{aligned} d_1 &= c \left[A_2 + \frac{3}{2} A_1 + A_4 \frac{B_2 + \frac{3}{2} B_1}{1 - B} \right], & d_2 &= c \left[A_2 + \frac{3}{2} A_5 + A_4 \frac{B_2 + \frac{3}{2} B_3}{1 - B} \right], \\ d_3 &= \frac{1}{2} c \left[A_2 + A_1 + A_4 \frac{B_2 + B_1}{1 - B} \right], & d_4 &= \frac{1}{2} c \left[A_2 + A_5 + A_4 \frac{B_2 + B_3}{1 - B} \right], \end{aligned}$$

and

$$\begin{aligned} e_1 &= \frac{B_2 + \frac{3}{2} B_1 + d_1 B_4}{1 - B}, & e_2 &= \frac{B_2 + \frac{3}{2} B_3 + d_2 B_4}{1 - B}, \\ e_3 &= \frac{1}{2} \frac{B_2 + B_1 + d_3 B_4}{1 - B}, & e_4 &= \frac{1}{2} \frac{B_2 + B_3 + d_4 B_4}{1 - B}. \end{aligned}$$

□

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