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TWO MAPPINGS RELATED TO SEMI-INNER PRODUCTS AND THEIR APPLICATIONS IN GEOMETRY OF NORMED LINEAR SPACES

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Abstract. In this paper we introduce two mappings associated with the lower and upper semi-inner product $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$ and with semi-inner products $[\cdot,\cdot]$ (in the sense of Lumer) which generate the norm of a real normed linear space, and study properties of monotonicity and boundedness of these mappings. We give a refinement of the Schwarz inequality, applications to the Birkhoff orthogonality, to smoothness of normed linear spaces as well as to the characterization of best approximants.

Keywords: lower and upper semi-inner product, semi-inner products, Schwarz inequality, smooth normed spaces, Birkhoff orthogonality, best approximants

MSC 2000: 46B20, 46B99, 46C99, 41A50

1. Introduction and preliminaries

In this paper we continue the study of mappings associated with the lower and upper semi-inner product in a normed space that started in [9], [10] and [11]. The main reason for studying these mappings is to obtain sharper estimates than those available using only the norm, and to attempt a clearer geometrical description of the normed space in the absence of a true inner product. We also consider semi-inner products in the sense of Lumer, which lie between the lower and upper semi-inner product, and define our mappings $\Psi^{[,]}_{x,y}$ in dependence on a particular semi-inner product. It turns out that the previously studied mappings $\Phi^i_{x,y}$ and $\Phi^s_{x,y}$ (see [10]) form the lower and upper envelope, respectively, for $\Psi^{[,]}_{x,y}$.

Our results will provide, in particular, refinements of the Schwarz inequality based on the lower and upper semi-inner product, and a characterization of best approximants.

Let $(X, \|\cdot\|)$ be a real normed linear space. We define the *lower* and *upper semi-inner product* by

$$(y,x)_i = \lim_{t\to 0-} \frac{\|x+ty\|^2 - \|x\|^2}{2t} \quad \text{and} \quad (y,x)_s = \lim_{t\to 0+} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

respectively. These limits are well defined for every pair $x, y \in X$ (see for example [5], [13]); the subscripts i and s stand for inferior and superior, respectively. We mention that $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$ are not semi-inner products in the sense of Lumer since they are not additive in the first variable (see (VII) below).

For the sake of completeness we list here some of the main properties of these products that will be used in the sequel (see [2], [5], [7], [8]), assuming that $p, q \in \{s, i\}$ and $p \neq q$:

(I) $(x,x)_p = \|x\|^2$ for all $x \in X$;

(II) $(\alpha x, \beta y)_p = \alpha\beta(x,y)_p$ if $\alpha\beta \geqslant 0$ and $x, y \in X$;

(III) $|(x,y)_p| \leqslant \|x\|\,\|y\|$ for all $x, y \in X$;

(IV) $(\alpha x + y, x)_p = \alpha(x,x)_p + (y,x)_p$ if x, y belong to X and α is a real number;

(V) $(-x, y)_p = -(x,y)_q$ for all $x, y \in X$;

(VI) $(x+y, z)_p \leqslant \|x\|\,\|z\| + (y,z)_p$ for all $x, y, z \in X$;

(VII) The mapping $(\cdot,\cdot)_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);

(VIII) The element $x \in X$ is Birkhoff orthogonal to the element $y \in X$ (that is,

$$\|x+ty\| \geqslant \|x\| \ \text{ for all } \ t \in \mathbb{R})$$

if and only if

$$(y,x)_i \leqslant 0 \leqslant (y,x)_s;$$

(IX) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \setminus \{0\}$ if and only if the mapping $y \mapsto (y, x_0)_p$ is linear, or if and only if $(y,x_0)_s = (y,x_0)_i$ for all $y \in X$;

(X) Let $\mathcal{J}$ be the normalized duality mapping on X, that is, let

$$\mathcal{J}(x) = \{f \in X^* \mid f(x) = \|x\|^2 \ \text{ and } \ \|f\| = \|x\|\};$$

note that, for every $x \in X$, $\mathcal{J}(x)$ is a nonempty convex subset of X^*, and

$$\mathcal{J}(\alpha x) = \alpha \mathcal{J}(x) \ \text{ for all } \ \alpha \in \mathbb{R} \ \text{ and all } \ x \in X.$$

Then for every pair $x, y \in X$ there exist $w_1, w_2 \in \mathcal{J}(x)$ so that

$$(y,x)_s = w_1(y), \qquad (y,x)_i = w_2(y);$$

(XI) We have the representations

$$(y,x)_s = \sup\{w(y) \mid w \in \mathcal{J}(x)\}, \quad (y,x)_i = \inf\{w(y) \mid w \in \mathcal{J}(x)\}.$$

Therefore

$$(y,x)_i \leqslant (y,x)_s \text{ for all } x,y \in X.$$

(XII) If the norm $\|\cdot\|$ is induced by an inner product $(\cdot,\cdot)$, then

$$(y,x)_i = (y,x) = (y,x)_s \text{ for all } x,y \in X.$$

The normalized duality maping is discussed in [2]; for other properties of $(\cdot,\cdot)_p$ see [1], [2], [5], [7], [8], [13], where further references are given.

The terminology throughout the paper is standard. We mention that for functions we use the terms 'increasing' (and 'strictly increasing'), 'decreasing' (and 'strictly decreasing'), thus avoiding 'nondecreasing' and 'nonincreasing'.

2. Properties of the mappings $\Phi_{x,y}^{[\,,\,]}$ and $\Psi_{x,y}^{[\,,\,]}$

First of all, let us recall the concept of semi-inner products on a real normed linear space introduced by Lumer [14].

Definition 2.1. Let X be a real linear space. A mapping $[\cdot,\cdot] : X \times X \to \mathbb{R}$ is called a *semi-inner product* on X if it satisfies the following conditions:

(a) $[x,x] \geqslant 0$ for all $x \in X$, and $[x,x] = 0$ implies $x = 0$;
(b) $[\alpha x + \beta y, z] = \alpha[x,z] + \beta[y,z]$ for all $\alpha,\beta \in \mathbb{R}$ and all $x,y,z \in X$;
(c) $[x,\alpha y] = \alpha[x,y]$ for all $\alpha \in \mathbb{R}$ and all $x,y \in X$;
(d) $|[x,y]|^2 \leqslant [x,x][y,y]$ for all $x,y \in X$.

It is easy to see that if $[\cdot,\cdot]$ is a semi-inner product on X, then the mapping $\|\cdot\| : x \mapsto [x,x]^{1/2}$ for all $x \in X$ is a norm on X. Moreover, for any $y \in X$, the functional $f_y \colon x \mapsto [x,y]$ for all $x \in X$ is linear and continuous in the norm topology of $(X,\|\cdot\|)$, with $\|f_y\| = \|y\|$.

Conversely, if $(X,\|\cdot\|)$ is a normed linear space, then the norm can be always represented through a semi-inner product (Lumer [14]) in the form

$$[x,y] = \langle \widetilde{\mathcal{J}}(y), x \rangle \text{ for all } x,y \in X,$$

where $\widetilde{\mathcal{J}}$ is a section of the normalized duality mapping $\mathcal{J}$ defined in the introduction.

For the sake of completeness, we give a simple proof of the following known lemma.

Lemma 2.2. *Let $(X, \|\cdot\|)$ be a real normed linear space and $[\cdot,\cdot]$ a semi-inner product which generates the norm $\|\cdot\|$. Then*

$$(x,y)_i \leqslant [x,y] \leqslant (x,y)_s \text{ for all } x,y \in X. \tag{2.1}$$

Proof. We may assume that $y \neq 0$. The functional $f_y\colon X \to \mathbb{R}$ defined by $f_y(x) = [x,y]$ is linear, and

$$|f_y(x)| = |[x,y]| \leqslant \|x\|\,\|y\|\,, \text{ that is, } \|f_y\| \leqslant \|y\|\,.$$

Further,

$$\|f_y\| \geqslant \frac{|f_y(x)|}{\|y\|} = \frac{\|y\|^2}{\|y\|} = \|y\|\,,$$

which proves $\|f_y\| = \|y\|$. Also, $f_y(y) = \|f_y\|\,\|y\|$, and $f_y \in \mathcal{J}(y)$.

By property (XI) of the introduction,

$$(x,y)_s = \sup\{w(x) \mid w \in \mathcal{J}(y)\}, \quad x \in X,$$

and thus

$$(x,y)_s \geqslant f_y(x) = [x,y] \text{ for all } x \in X.$$

The inequality $(x,y)_i \leqslant [x,y]$ for all $x \in X$ follows from fact that

$$(x,y)_i = \inf\{w(x) \mid w \in \mathcal{J}(y)\}, \quad x \in X,$$

and the lemma is proved. □

For a given semi-inner product on X generating the norm of X we define the mappings

$$\Phi^{[,]}_{x,y}\colon \mathbb{R} \to \mathbb{R}, \qquad \Phi^{[,]}_{x,y}(t) := \frac{[y, x+ty]}{\|x+ty\|},$$

and

$$\Psi^{[,]}_{x,y}\colon \mathbb{R} \to \mathbb{R}, \qquad \Psi^{[,]}_{x,y}(t) := \frac{[x, x+ty]}{\|x+ty\|},$$

where x,y are assumed to be linearly independent in X. We will also make use of the mappings

$$\Phi^{p}_{x,y}(t) = \frac{(y, x+ty)_p}{\|x+ty\|}, \qquad \Psi^{p}_{x,y}(t) = \frac{(x, x+ty)_p}{\|x+ty\|},$$

where $p \in \{i,s\}$. (For more detailed properties of these functions see [10] and [17].)

The following proposition explores connections between all these mappings.

Proposition 2.3. *Let $(X, \|\cdot\|)$ be a real normed linear space, $[\cdot,\cdot]$ a semi-inner product generating the norm in X and x, y two linearly independent vectors in X. Then*

$$\|x+ty\| = \Psi_{x,y}^{[,]}(t) + t\Phi_{x,y}^{[,]}(t) \quad \text{for all } t \in \mathbb{R}, \tag{2.2}$$

and

$$\Phi_{x,y}^{[,]}(1/t) = \operatorname{sgn}(t)\Psi_{y,x}^{[,]}(t) \quad \text{for all } t \in \mathbb{R}\setminus\{0\}. \tag{2.3}$$

Proof. Let $t \in \mathbb{R}$. Then

$$\begin{aligned}\|x+ty\| &= \frac{\|x+ty\|^2}{\|x+ty\|} = \frac{[x+ty, x+ty]}{\|x+ty\|} \\ &= \frac{[x, x+ty] + t[y, x+ty]}{\|x+ty\|} = \Psi_{x,y}^{[,]}(t) + t\Phi_{x,y}^{[,]}(t),\end{aligned}$$

and equality (2.2) is proved.

Let $t \in \mathbb{R}\setminus\{0\}$. Then

$$\begin{aligned}\Phi_{x,y}^{[,]}(1/t) &= \frac{[y, x+y/t]}{\|x+y/t\|} = \frac{[y, y+tx]/t}{\|y+tx\|/|t|} = \operatorname{sgn}(t)\frac{[y, y+tx]}{\|y+tx\|} \\ &= \operatorname{sgn}(t)\Psi_{y,x}^{[,]}(t),\end{aligned}$$

where $\operatorname{sgn}(t) = |t|/t$ for $t \in \mathbb{R}\setminus\{0\}$. □

The following theorem summarizes the main properties of the mapping $\Phi_{x,y}^{[,]}$.

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a real normed linear space, $[\cdot,\cdot]$ a semi-inner product which generates the norm of X, and x, y two linearly independent vectors in X. Then:*

(i) *The mapping $\Phi_{x,y}^{[,]}$ is bounded, and*

$$\left|\Phi_{x,y}^{[,]}(t)\right| \leqslant \|y\| \quad \text{for all } t \in \mathbb{R}; \tag{2.4}$$

(ii) *We have the inequalities*

$$\Phi_{x,y}^{i}(t) \leqslant \Phi_{x,y}^{[,]}(t) \leqslant \Phi_{x,y}^{s}(t) \quad \text{for all } t \in \mathbb{R}, \tag{2.5}$$

and the bounds

$$\Phi_{x,y}^{i}(t) = \inf\{\Phi_{x,y}^{[,]}(t) \mid [\cdot,\cdot] \in \mathcal{J}(X)\}, \tag{2.6}$$

$$\Phi_{x,y}^{s}(t) = \sup\{\Phi_{x,y}^{[,]}(t) \mid [\cdot,\cdot] \in \mathcal{J}(X)\}, \tag{2.7}$$

where $\mathcal{J}(X)$ *is the class of all semi-inner products generating the norm of* X*;*

(iii) *We have the limits*

$$\lim_{u\to 0-} \Phi_{x,y}^{[\,,\,]}(u) = \frac{(y,x)_i}{\|x\|}, \qquad \lim_{t\to 0+} \Phi_{x,y}^{[\,,\,]}(t) = \frac{(y,x)_s}{\|x\|} \tag{2.8}$$

and

$$\lim_{u\to -\infty} \Phi_{x,y}^{[\,,\,]}(u) = -\|y\|, \qquad \lim_{t\to +\infty} \Phi_{x,y}^{[\,,\,]}(t) = \|y\|; \tag{2.9}$$

(iv) *The mapping* $\Phi_{x,y}^{[\,,\,]}$ *is increasing on* $\mathbb{R}$.

Proof. (i) Follows from the Schwarz inequality (III).

(ii) Follows from Lemma 2.2.

(iii) According to Theorem 2.1 of [10],

$$\lim_{t\to 0+} \Phi_{x,y}^{p}(t) = \frac{(y,x)_s}{\|x\|}, \qquad p \in \{s,i\}. \tag{2.10}$$

Then an application of the inequalities (2.5) yields the second limit in (2.8). The first limit follows from $\Phi_{x,y}^{[\,,\,]}(-t) = -\Phi_{x,-y}^{[\,,\,]}(t)$.

The limits in (2.9) similarly follow from Theorem 2.1 of [10] by an application of (2.5).

(iv) Let $t_2 > t_1$. By the Schwarz inequality,

$$[x + t_2 y, x + t_1 y] \leqslant \|x + t_2 y\| \, \|x + t_1 y\|.$$

By the linearity of the semi-inner product in the first variable,

$$\begin{aligned}[x + t_2 y, x + t_1 y] &= [(t_2 - t_1)y + x + t_1 y, x + t_1 y] \\ &= \|x + t_1 y\|^2 + (t_2 - t_1)[y, x + t_1 y],\end{aligned}$$

and so, by the above inequality, we get

$$\|x + t_2 y\| \, \|x + t_1 y\| \geqslant \|x + t_1 y\|^2 + (t_2 - t_1)[y, x + t_1 y],$$

that is,

$$\|x + t_1 y\| \left(\|x + t_2 y\| - \|x + t_1 y\|\right) \geqslant (t_2 - t_1)[y, x + t_1 y],$$

which implies

$$\Phi_{x,y}^{[\,,\,]}(t_1) = \frac{[y, x + t_1 y]}{\|x + t_1 y\|} \leqslant \frac{\|x + t_2 y\| - \|x + t_1 y\|}{t_2 - t_1}.$$

On the other hand, for any $t>0$ we have

$$\begin{aligned} \|x\|\,\|x+ty\| &\geqslant (x,x+ty)_s = (x+ty-ty,x+ty)_s \\ &= \|x+ty\|^2 + (-ty,x+ty)_s = \|x+ty\|^2 - t(y,x+ty)_i; \end{aligned}$$

this implies

$$\frac{\|x+ty\|-\|x\|}{t} \leqslant \Phi^i_{x,y}(t) \quad (t>0). \tag{2.11}$$

In this inequality replace x by $x+t_1y$ and set $t := t_2 - t_1$; then

$$\begin{aligned} \frac{\|x+t_2y\|-\|x+t_1y\|}{t_2-t_1} &= \frac{\|x+t_1y+ty\|-\|x+t_1y\|}{t} \\ &\leqslant \Phi^i_{x+t_1y,y}(t) = \Phi^i_{x,y}(t_2). \end{aligned}$$

As $\Phi^i_{x,y}(t_2) \leqslant \Phi^{[,]}_{x,y}(t_2)$, we conclude that

$$\Phi^{[,]}_{x,y}(t_1) \leqslant \Phi^{[,]}_{x,y}(t_2),$$

which shows that $\Phi^{[,]}_{x,y}$ is increasing. □

The following theorem summarizes properties of $\Psi^{[,]}_{x,y}$.

Theorem 2.5. *Let $(X,\|\cdot\|)$ be real normed linear space, $[\cdot,\cdot]$ a semi-inner product which generates the norm of X, and x,y linearly independent vectors in X. Then:*

(i) *The mapping $\Psi^{[,]}_{x,y}$ is bounded, and*

$$\left|\Psi^{[,]}_{x,y}(t)\right| \leqslant \|x\| \quad \text{for all } t\in\mathbb{R}; \tag{2.12}$$

(ii) *We have the inequalitites*

$$\Psi^i_{x,y}(t) \leqslant \Psi^{[,]}_{x,y}(t) \leqslant \Psi^s_{x,y}(t) \quad \text{for all } t\in\mathbb{R}, \tag{2.13}$$

and the bounds

$$\Psi^i_{x,y}(t) = \inf\{\Psi^{[,]}_{x,y}(t) \mid [\cdot,\cdot]\in\mathcal{J}(X)\}, \tag{2.14}$$

$$\Psi^s_{x,y}(t) = \sup\{\Psi^{[,]}_{x,y}(t) \mid [\cdot,\cdot]\in\mathcal{J}(X)\}; \tag{2.15}$$

(iii) *We have the limits*

$$\lim_{t\to+\infty}\Psi^{[,]}_{x,y}(t) = \frac{(x,y)_s}{\|y\|}, \qquad \lim_{u\to-\infty}\Psi^{[,]}_{x,y}(u) = -\frac{(x,y)_i}{\|y\|}; \tag{2.16}$$

(iv) *The mapping $\Psi^{[,]}_{x,y}$ is continuous at 0;*

(v) *$\Psi^{[,]}_{x,y}$ is increasing on $(-\infty,0]$ and decreasing on $(0,\infty)$.*

P r o o f. (i) Follows from the Schwarz inequality.

(ii) Can be deduced from Lemma 2.2; we omit the details.

(iii) We have

$$\begin{aligned}\lim_{t\to+\infty} \Psi^p_{x,y}(t) &= \lim_{t\to+\infty} \frac{(x, x+ty)_p}{\|x+ty\|} = \lim_{t\to+\infty} \frac{t(x, y+x/t)_p}{t\,\|y+x/t\|} \\ &= \lim_{\alpha\to 0+} \frac{(x, y+\alpha x)_p}{\|y+\alpha x\|} = \lim_{\alpha\to 0+} \Phi^p_{y,x}(\alpha)\,.\end{aligned}$$

By (2.10), $\lim_{\alpha\to 0+} \Phi^p_{y,x}(\alpha) = (x,y)_s/\|y\|$, and the result follows from inequality (2.13).

(iv) First we observe that

$$\Psi^{[\,,\,]}_{x,y}(t) = \|x+ty\| - t\Phi^{[\,,\,]}_{x,y}(t) \quad \text{for all } t\in\mathbb{R} \tag{2.17}$$

by Proposition 2.3. By (2.8) and the definition of $\Psi^{[\,,\,]}_{x,y}$,

$$\lim_{t\to 0} \Psi^{[\,,\,]}_{x,y}(t) = \|x\| = \Psi^{[\,,\,]}_{x,y}(0)\,.$$

(v) Let $t_1 < t_2 \leqslant 0$. Using (2.17), we deduce

$$\frac{\Psi^{[\,,\,]}_{x,y}(t_2) - \Psi^{[\,,\,]}_{x,y}(t_1)}{t_2-t_1} = \frac{\|x+t_2y\| - \|x+t_1y\|}{t_2-t_1} - \frac{t_2\Phi^{[\,,\,]}_{x,y}(t_2) - t_1\Phi^{[\,,\,]}_{x,y}(t_1)}{t_2-t_1}.$$

In the proof of Theorem 2.5 we established that

$$\Phi^{[\,,\,]}_{x,y}(t_1) \leqslant \frac{\|x+t_2y\| - \|x+t_1y\|}{t_2-t_1} \leqslant \Phi^{[\,,\,]}_{x,y}(t_2)$$

for all $t_1 < t_2$; by (2.2) we get

$$\begin{aligned}\frac{\Psi^{[\,,\,]}_{x,y}(t_2) - \Psi^{[\,,\,]}_{x,y}(t_1)}{t_2-t_1} &\geqslant \Phi^{[\,,\,]}_{x,y}(t_1) - \frac{t_2\Phi^{[\,,\,]}_{x,y}(t_2) - t_1\Phi^{[\,,\,]}_{x,y}(t_1)}{t_2-t_1} \\ &= \frac{t_2\Phi^{[\,,\,]}_{x,y}(t_1) - t_1\Phi^{[\,,\,]}_{x,y}(t_1) - t_2\Phi^{[\,,\,]}_{x,y}(t_2) + t_1\Phi^{[\,,\,]}_{x,y}(t_1)}{t_2-t_1} \\ &= \frac{t_2(\Phi^{[\,,\,]}_{x,y}(t_1) - \Phi^{[\,,\,]}_{x,y}(t_2))}{t_2-t_1} \geqslant 0\end{aligned}$$

as $t_2 \leqslant 0$ and $\Phi^{[\,,\,]}_{x,y}(t_1) \leqslant \Phi^{[\,,\,]}_{x,y}(t_2)$.

If $t_2 > t_1 > 0$, then by (2.2)

$$\begin{aligned}\frac{\Psi_{x,y}^{[,]}(t_2) - \Psi_{x,y}^{[,]}(t_1)}{t_2 - t_1} &\leqslant \Phi_{x,y}^{[,]}(t_2) - \frac{t_2\Phi_{x,y}^{[,]}(t_2) - t_1\Phi_{x,y}^{[,]}(t_1)}{t_2 - t_1} \\ &= \frac{t_2\Phi_{x,y}^{[,]}(t_2) - t_1\Phi_{x,y}^{[,]}(t_2) - t_2\Phi_{x,y}^{[,]}(t_2) + t_1\Phi_{x,y}^{[,]}(t_1)}{t_2 - t_1} \\ &= \frac{t_1(\Phi_{x,y}^{[,]}(t_1) - \Phi_{x,y}^{[,]}(t_2))}{t_2 - t_1} \leqslant 0\end{aligned}$$

as $t_1 > 0$ and $\Phi_{x,y}^{[,]}(t_1) \leqslant \Phi_{x,y}^{[,]}(t_2)$.

The theorem is thus proved. □

In the case that $(X, \|\cdot\|)$ is a real inner product space, the mappings $\Phi_{x,y}^{[,]}$ and $\Psi_{x,y}^{[,]}$ assume the following form:

$$\Phi_{x,y}(t) = \frac{(y,x) + t\|y\|^2}{\|x + ty\|}, \qquad \Psi_{x,y}(t) = \frac{\|x\|^2 + t(x,y)}{\|x + ty\|}.$$

It is then possible to calculate the second derivatives of these mappings, and determine their convexity and concavity. The details of this investigation appeared in [10] and [17].

Proposition 2.6 ([10], [17])**.** *Let $(X; (\cdot,\cdot))$ be a real inner product space, and x, y two linearly independent vectors in X. Then:*

(i) $\Phi_{x,y}^p$ *is strictly convex on the interval* $(-\infty, -(x,y)/\|y\|^2)$*, and strictly concave on the interval* $(-(x,y)/\|y\|^2, +\infty)$.

(ii) $\Psi_{x,y}^p$ *is strictly convex on the set* $(-\infty, t_1) \cup (t_2, +\infty)$ *and strictly concave on the interval* (t_1, t_2)*, where*

$$t_1 = \frac{-(x,y) - \sqrt{\Delta_{x,y}}}{4\|y\|^2}, \qquad t_2 = \frac{-(x,y) + \sqrt{\Delta_{x,y}}}{4\|y\|^2}$$

and $\Delta_{x,y} = 8\|x\|^2\|y\|^2 + (x,y)^2 > 0$.

3. A refinement of the Schwarz inequality

Using the results of the preceding sections, it is possible to give a refinement of the Schwarz inequality involving a semi-inner product generating the norm of the given normed linear space. First we observe that, for any $t > 0$,

$$\begin{aligned}\|x+2ty\|\,\|x+ty\| &\geqslant (x+2ty, x+ty)_s = (x+ty+ty, x+ty)_s \\ &= \|x+ty\|^2 + (ty, x+ty)_s,\end{aligned}$$

and

$$\Phi^s_{x,y}(t) \leqslant \frac{\|x+2ty\| - \|x+ty\|}{t} \quad (t > 0). \tag{3.1}$$

The following theorem is then a direct consequence of Theorem 2.4, (2.11) and (3.1).

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space and $[\cdot,\cdot]$ a semi-inner product generating the norm of X. Given two linearly independent vectors $x, y \in X$, then for all $t > 0 > u$ we have the following inequalities:*

$$\begin{aligned}\|x\|\,\|y\| &\geqslant \frac{\|x+2ty\| - \|x+ty\|}{t}\,\|x\| \geqslant \frac{(y, x+ty)_s}{\|x+ty\|}\,\|x\| \\ &\geqslant \frac{[y, x+ty]}{\|x+ty\|}\,\|x\| \geqslant \frac{(y, x+ty)_i}{\|x+ty\|}\,\|x\| \geqslant \frac{\|x+ty\| - \|x\|}{t}\,\|x\| \\ &\geqslant (y,x)_s \geqslant [y,x] \geqslant (y,x)_i \geqslant \frac{\|x+uy\| - \|x\|}{u}\,\|x\| \\ &\geqslant \frac{(y, x+uy)_s}{\|x+uy\|}\,\|x\| \geqslant \frac{[y, x+uy]}{\|x+uy\|}\,\|x\| \geqslant \frac{(y, x+uy)_i}{\|x+uy\|}\,\|x\| \\ &\geqslant \frac{\|x+2uy\| - \|x+uy\|}{u} \\ &\geqslant -\|x\|\,\|y\|.\end{aligned} \tag{3.2}$$

Example 3.2. Let Ω be a compact metric space and $C(\Omega)$ the space of all real valued functions on Ω equipped with the norm $\|x\| = \sup\limits_{s\in\Omega} |x(s)|$. For any $x \in C(\Omega)$ write

$$\Omega_x = \{s \in \Omega\colon\ |x(s)| = \|x\|\}.$$

For each $x \in C(\Omega)$ select a finite Borel measure μ_x on Ω satisfying

$$|\mu_x|(\Omega) = \|x\|, \quad \operatorname{supp} \mu_x \subset \Omega_x, \quad \mu_x \operatorname{sgn}(x) \geqslant 0.$$

In view of [2], Example 12.2,

$$[y, x] = \int_\Omega y \,\mathrm{d}\mu_x$$

is a semi-inner product in $C(\Omega)$ generating the norm of $C(\Omega)$. Also,

$$(y,x)_i = \inf\{x(s)y(s)\colon\ s\in\Omega_x\}, \qquad (y,x)_s = \sup\{x(s)y(s)\colon\ s\in\Omega_x\}.$$

If x, y are linearly independent elements of $C(\Omega)$ and $t>0$, we have the inequalities

$$\begin{aligned}
\sup_{s\in\Omega}|x(s)|\sup_{s\in\Omega}|y(s)| &\geqslant \frac{\sup_{s\in\Omega}|x(s)+2ty(s)| - \sup_{s\in\Omega}|x(s)+ty(s)|}{t}\sup_{s\in\Omega}|x(s)| \\
&\geqslant \frac{\sup_{s\in\Omega_{x+ty}} y(s)(x(s)+ty(s))}{\sup_{s\in\Omega}|x(s)+ty(s)|}\sup_{s\in\Omega}|x(s)| \\
&\geqslant \frac{\int_\Omega y\,\mathrm{d}\mu_{x+ty}}{\sup_{s\in\Omega}|x(s)+ty(s)|}\sup_{s\in\Omega}|x(s)| \\
&\geqslant \frac{\inf_{s\in\Omega_{x+ty}} y(s)(x(s)+ty(s))}{\sup_{s\in\Omega}|x(s)+ty(s)|}\sup_{s\in\Omega}|x(s)| \\
&\geqslant \frac{\sup_{s\in\Omega}|x(s)+ty(s)| - \sup_{s\in\Omega}|x(s)|}{t}\sup_{s\in\Omega}|x(s)| \\
&\geqslant \sup_{s\in\Omega_x} x(s)y(s) \geqslant \int_\Omega y\,\mathrm{d}\mu_x \geqslant \inf_{s\in\Omega_x} x(s)y(s).
\end{aligned}$$

Analogous inequalities hold for $u<0$.

A specialization of Theorem 3.1 to a real inner product space yields the following result.

Corollary 3.3. *Let $(X;(\cdot,\cdot))$ be a real inner product space. For any two linearly independent vectors $x,y\in X$ and any $t>0>u$ we have the following refinement of the Schwarz inequality:*

$$\begin{aligned}
\|x\|\,\|y\| &\geqslant \frac{[3t\,\|y\|^2+2(x,y)]\,\|x\|}{\|x+2ty\|+\|x+ty\|} \geqslant \frac{[t\,\|y\|^2+(x,y)]\,\|x\|}{\|x+ty\|} \\
&\geqslant \frac{[t\,\|y\|^2+2(x,y)]\,\|x\|}{\|x+ty\|+\|x\|} \\
&\geqslant (x,y) \\
&\geqslant \frac{[u\,\|y\|^2+2(x,y)]\,\|x\|}{\|x+uy\|+\|x\|} \geqslant \frac{[u\,\|y\|^2+2(x,y)]\,\|x\|}{\|x+uy\|} \\
&\geqslant \frac{[3u\,\|y\|^2+2(x,y)]\,\|x\|}{\|x+2uy\|+\|x+uy\|} \\
&\geqslant -\|x\|\,\|y\|.
\end{aligned}$$

4. A property of a semi-inner product

Suppose that $(X, \|\cdot\|)$ is a normed space, and $[\cdot,\cdot]$ a semi-inner product generating the norm of X. In this section we consider the following problem regarding a condition which involves the second (nonlinear) argument of the semi-inner product:

$$\text{(Q)} \qquad \text{Does } [y, x+ty] = 0 \text{ for all } t \in \mathbb{R} \text{ imply } y = 0?$$

If $[\cdot,\cdot]$ is a true inner product (linear in the second argument), then the preceding question has a positive answer as

$$0 = [y, x+ty] = [y, x] + t\,\|y\|^2 \quad \text{for all } t \in \mathbb{R},$$

which obviously implies $y = 0$. We show that the mappings introduced and studied in the present paper, in particular the refinement of the Schwarz inequality obtained in the preceding section, can be used to provide an affirmative answer to (Q) in the general case of a semi-inner product.

We start by proving the following theorem.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a normed linear space and $[\cdot,\cdot]$ a semi-inner product generating the norm of X. Given two linearly independent vectors $x, y \in X$, the following statements are equivalent:*

(i) $\|x+ty\| = \|x\|$ *for all* $t \in \mathbb{R}$*;*
(ii) $\|x+2ty\| = \|x+ty\|$ *for all* $t \in \mathbb{R}$*;*
(iii) $(y, x+ty)_i = 0$ *for all* $t \in \mathbb{R}$*;*
(iv) $[y, x+ty] = 0$ *for all* $t \in \mathbb{R}$*;*
(v) $(y, x+ty)_s = 0$ *for all* $t \in \mathbb{R}$*.*

P r o o f. In the proof we use Theorem 3.1.

(i) $\iff$ (ii). If $\|x+ty\| = \|x\|$ for all $t \in \mathbb{R}$, then also $\|x+2ty\| = \|x\|$ for all $t \in \mathbb{R}$, and (ii) holds. Conversely, if (ii) holds, then (3.2) gives

$$0 \geqslant \frac{\|x+ty\| - \|x\|}{t} \geqslant \frac{\|x+uy\| - \|x\|}{u} \geqslant 0,$$

which implies $\|x+ty\| = \|x\|$ for all $t \in \mathbb{R} \setminus \{0\}$; the latter equality is true also for $t = 0$, and (i) holds.

From (3.2) we deduce that (ii) implies (iii), (iv) and (v), while any of the conditions (iii), (iv), (v) implies (i). This completes the proof. $\square$

The theorem enables us to give an answer to the problem (Q).

Theorem 4.2. *Let* $(X, \|\cdot\|)$ *be a normed linear space, and* $[\cdot,\cdot]$ *a semi-inner product that generates the norm of* X*. Then the following is true:*

$$[y, x+ty] = 0 \ \textit{ for all } \ t \in \mathbb{R} \ \textit{ implies } \ y = 0. \tag{4.1}$$

More generally, any of the conditions (i)–(v) *of Theorem* 4.1 *implies* $y = 0$.

P r o o f. (a) If x, y are linearly dependent, then $y = \alpha x$ for some $\alpha \neq 0$. If $[y, x+ty] = 0$ for all $t \in \mathbb{R}$, then

$$0 = [y, x+ty] = [y, (\alpha + t)y] = (\alpha + t)\,\|y\|^2 \ \text{ for all } \ t \in \mathbb{R},$$

which implies $y = 0$.

(b) Suppose that x, y are linearly independent and that $[y, x+ty] = 0$ for all $t \in \mathbb{R}$. By the preceding theorem $\|x+ty\| = \|x\|$ for all $t \in \mathbb{R}$. But

$$\|x+ty\| = \|x - (-t)y\| \geqslant |\,\|x\| - |t|\,\|y\|\,|,$$

and $\|x\| \geqslant |\,\|x\| - |t|\,\|y\|\,|$ for all $t \in \mathbb{R}$, which implies

$$|t|\,\|y\| \leqslant 2\,\|x\| \ \text{ for all } t \in \mathbb{R};$$

hence $y = 0$. □

5. New characterizations of the Birkhoff orthogonality

Let us recall the concept of orthogonality in the sense of Birkhoff which can be defined in normed linear spaces.

Definition 5.1. Let $(X, \|\cdot\|)$ be a real normed linear space and x, y two elements of X. The vector x is called *Birkhoff orthogonal to* y if

$$\|x + \alpha y\| \geqslant \|x\| \ \text{ for all } \ \alpha \in \mathbb{R}. \tag{5.1}$$

We use notation $x \perp y$ (B).

We know that in each real normed linear space $(X, \|\cdot\|)$ there exists at least one semi-inner product $[\cdot,\cdot]$ which generates the norm $\|\cdot\|$, that is, $\|x\| = [x,x]^{1/2}$ for all $x \in X$, and that such semi-inner product is unique if and only if X is smooth.

The following concept of orthogonality is well known (see [14], [12]).

Definition 5.2. Let $[\cdot,\cdot]$ be a semi-inner product which generates the norm of X, and let $x, y \in X$. The vector x is said to be *orthogonal to y in the sense of Lumer* (relative to the semi-inner product $[\cdot,\cdot]$) if $[y, x] = 0$. We denote this by $x \perp y$ (L).

The following connection between Birkhoff's and Lumer's orthogonality holds.

Proposition 5.3. *Let $(X, \|\cdot\|)$ be a real normed linear space, and x, y two vectors in X. Then $x \perp y$ (B) if and only if $x \perp y$ (L) relative to some semi-inner product $[\cdot,\cdot]$ which generates the norm $\|\cdot\|$.*

Proof. Assume that $[\cdot,\cdot]$ is a semi-inner product which generates the norm of X, and that $x \perp y$ (L), that is, $[y, x] = 0$. Then

$$\|x\|^2 = [x, x] = [x + \lambda y, x] \leqslant \|x\| \, \|x + \lambda y\|$$

for all $\lambda \in \mathbb{R}$, that is, $\|x\| \leqslant \|x + \lambda y\|$ for $\lambda \in \mathbb{R}$, which is equivalent to $x \perp y$ (B).

Conversely suppose that $x \perp y$ (B). Then (5.1) holds, and by the Hahn-Banach theorem there is $f_x \in X^*$ such that

$$f_x(y) = 0, \quad f_x(x) = \|x\|^2, \quad \|f_x\| = \|x\|.$$

Hence we can choose a section $\widetilde{\mathcal{J}}$ of the normalized duality mapping $\mathcal{J}$ (see (X)) so that $\widetilde{\mathcal{J}}(x) = f_x$. The semi-inner product

$$[u, v] = \langle \widetilde{\mathcal{J}}(v), u \rangle, \qquad u, v \in X,$$

generates the norm of X, and

$$[y, x] = \langle \widetilde{\mathcal{J}}(x), y \rangle = f_x(y) = 0.$$

Consequently $x \perp y$ (L) relative to $[\cdot,\cdot]$. □

The following counterexample shows that $x \perp y$ (B) need not imply $[y, x] = 0$ for *every* semi-inner product generating the norm of X.

Example 5.4. Let us consider the normed linear space $(\mathbb{R}^3, \|\cdot\|_1)$. It is easy to check that

$$[\overline{y}, \overline{x}] = \|\overline{x}\|_1 \sum_{x_k \neq 0} \frac{x_k y_k}{|x_k|}$$

is a semi-inner product which generates the norm $\|\cdot\|_1$. Consider the vectors $\overline{x} = (1, 0, 0)$ and $\overline{y} = (1, 1, 0)$. We have

$$\|\overline{x}\|_1 = 1, \quad \overline{x} + \lambda \overline{y} = (1 + \lambda, \lambda, 0), \quad \|\overline{x} + \lambda \overline{y}\|_1 = |1 + \lambda| + |\lambda|.$$

Now it is clear that

$$\|\overline{x}+\lambda\overline{y}\|_1 = |1+\lambda|+|\lambda| \geqslant 1 = \|\overline{x}\|_1 \text{ for all } \lambda\in\mathbb{R},$$

that is, $x \perp y$ (B). On the other hand,

$$[\overline{y},\overline{x}] = 1 \neq 0,$$

which shows that $\overline{x}$ is not Lumer orthogonal to $\overline{y}$.

Remark 5.5. For a given $y \in X$ define

$$B(y) = \{x\in X \mid x\perp y \ \text{(B)}\} \quad \text{and} \quad L^{[\,,\,]}(y) = \{x\in X \mid [y,x]=0\},$$

where $[\cdot,\cdot]$ belongs to $\mathcal{J}(X)$, the class of all semi-inner products generating the norm of X. With this notation, Proposition 5.3 can be expressed as

$$B(y) = \bigcup_{[\,,\,]\in\mathcal{J}(X)} L^{[\,,\,]}(y).$$

The following proposition also holds.

Proposition 5.6. *Let $(X,\|\cdot\|)$ be a real normed linear space and x,y two elements of X. The following statements are equivalent:*

(i) $x\perp y$ (B)*;*

(ii) *For every semi-inner product $[\cdot,\cdot]$ which generates the norm of X we have the inequalities*

$$[y,x+uy] \leqslant 0 \leqslant [y,x+ty] \text{ for all } u<0<t. \tag{5.2}$$

Proof. (i) $\Longrightarrow$ (ii) If $x\perp y$ (B), then we have $(y,x)_i \leqslant 0 \leqslant (y,x)_s$. By (2.5) we have that

$$\Phi^i_{x,y}(t) \leqslant \Phi^{[\,,\,]}_{x,y}(t) \leqslant \Phi^s_{x,y}(t), \quad t\in\mathbb{R},$$

that is,

$$\frac{(y,x+ty)_i}{\|x+ty\|} \leqslant \frac{[y,x+ty]}{\|x+ty\|} \leqslant \frac{(y,x+ty)_s}{\|x+ty\|}, \quad t\in\mathbb{R}.$$

If $u<0$, then

$$\frac{[y,x+uy]}{\|x+uy\|} \leqslant \frac{(y,x+uy)_s}{\|x+uy\|} \leqslant (y,x)_i \leqslant 0;$$

if $t>0$, then

$$\frac{[y,x+ty]}{\|x+ty\|} \geqslant \frac{(y,x+ty)_i}{\|x+ty\|} \geqslant (y,x)_s \geqslant 0,$$

and the inequality (5.2) is obtained.

(ii) $\Longrightarrow$ (i) Suppose that (5.2) holds. Since

$$\lim_{u\to 0-} \frac{[y,x+uy]}{\|x+uy\|} = \frac{(y,x)_i}{\|y\|}, \quad \lim_{t\to 0+} \frac{[y,x+ty]}{\|x+ty\|} = \frac{(y,x)_s}{\|y\|}$$

by Theorem 2.4 (ii), we get $(y,x)_i \leqslant 0 \leqslant (y,x)_s$, and the proposition is proved. □

Remark 5.7. Condition (ii) of the above theorem can be replaced by the following weaker condition.

(ii′) *There exists a semi-inner product which generates the norm of X and $\varepsilon>0$ such that*

(5.2′) $$[y,x+uy] \leqslant 0 \leqslant [y,x+ty] \quad \textit{for } -\varepsilon<u<0<t<\varepsilon.$$

This follows from the monotonicity of the mapping $\Phi^{[,]}_{x,y}$ (see Theorem 2.4).

The mapping $\Phi^{[,]}_{x,y}$ provides a new characterization of smoothness.

Theorem 5.8. *Let $(X,\|\cdot\|)$ be a real normed linear space and let $x_0 \in X\setminus\{0\}$. Then the following conditions on x_0 are equivalent:*

(i) *X is smooth at the point x_0;*

(ii) *For some semi-inner product $[\cdot,\cdot]$ generating the norm of X, the mapping $\Phi^{[,]}_{x_0,y}$ is continuous at 0 for all $y\in X$.*

Proof. The equivalence follows from the equations

$$\lim_{u\to 0-} \Phi^{[,]}_{x,y}(u) = \frac{(y,x)_i}{\|x\|}, \quad \lim_{t\to 0+} \Phi^{[,]}_{x,y}(t) = \frac{(y,x)_s}{\|x\|}$$

established in Theorem 2.4. □

Birkhoff orthogonality has applications in the theory of best approximation in normed linear spaces. The preceding results can be used to give new characterizations of best approximants.

Definition 5.9. Let X be a normed linear space, G a set in X, and $x\in X$. An element $g_0\in G$ is called an *element of best approximation* of x (by the elements of the set G) if

(5.3) $$\|x-g_0\| = \inf_{g\in G} \|x-g\|.$$

We denote by $\mathcal{P}_G(x)$ the set of all such elements g_0, that is,

$$\mathcal{P}_G(x) = \{g_0 \in G \mid \|x - g_0\| = \inf_{g \in G} \|x - g\|\}. \tag{5.4}$$

For classical results on best approximation see the books [15], [16] by I. Singer. For some new results concerning the characterization of best approximants, proximinal, semičebyševian or čebyševian subspaces in terms of the upper and lower (as well as ordinary) semi-inner products we refer to the recent papers [3], [4], [6] of S. S. Dragomir.

We state here the following characterization of best approximants in terms of Birkhoff's orthogonality due to Singer [15], p. 92.

Lemma 5.10. *Let $(X, \|\cdot\|)$ be a normed linear space, G a linear subspace of X, $x \in X \setminus \overline{G}$ and $g_0 \in G$. Then*

$$g_0 \in \mathcal{P}_G(x) \text{ if and only if } x - g_0 \perp G \ \text{(B)}.$$

Combining this lemma with preceding characterizations of Birkhoff orthogonality, we obtain the following characterization of best approximants.

Theorem 5.11. *Let X, G, x and g_0 be as in Lemma 5.10. The following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_G(x)$;

(ii) *For some (in fact any) semi-inner product $[\cdot,\cdot]$ generating the norm of X, g_0 and x satisfy the inequality*

$$[x - g_0, x - g_0 + w] \leqslant \|x - g_0 + w\|^2 \text{ for all } w \in G. \tag{5.5}$$

P r o o f. By Lemma 5.10, (i) is equivalent to $x - g_0 \perp g$ (B) $= 0$ for all $g \in G$, which in turn is equivalent to

$$[g, x - g_0 + ug] \leqslant 0 \leqslant [g, x - g_0 + tg] \quad \text{if } u < 0 < t \tag{5.6}$$

by Proposition 5.6. But

$$[g, x - g_0 + tg] \geqslant 0, \quad t > 0, \tag{5.7}$$

is equivalent to $[tg, x - g_0 + tg] \geqslant 0$ for all $t > 0$. As

$$\begin{aligned} [tg, x - g_0 + tg] &= [x - g_0 + tg - x + g_0, x - g_0 + tg] \\ &= \|x - g_0 + tg\|^2 - [x - g_0, x - g_0 + tg], \end{aligned}$$

(5.7) is equivalent to

$$[x - g_0, x - g_0 + tg] \leqslant \|x - g_0 + tg\|^2 \tag{5.8}$$

for all $g \in G$, $t > 0$. Similarly, the relation

$$[g, x - g_0 + ug] \leqslant 0, \quad u < 0,$$

is equivalent to $[ug, x - g_0 + ug] \geqslant 0$ for all $u < 0$ in view of the linearity of $[\cdot,\cdot]$ in the first argument, and consequently to

$$[x - g_0, x - g_0 + ug] \leqslant \|x - g_0 + ug\|^2 \tag{5.9}$$

for all $g \in G$, $u < 0$.

Combining (5.8) and (5.9) and observing that (5.8) holds (with equality) also for $t = 0$, we conclude that

$$[x - g_0, x - g_0 + tg] \leqslant \|x - g_0 + tg\|^2$$

for all $g \in G$ and all $t \in \mathbb{R}$.

As $g \in G$ if and only if $tg \in G$ for $t \neq 0$, we deduce the desired equivalence, and the theorem is proved. □

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