

# Applications of Mathematics

---

Ivan Hlaváček

Finite element analysis of a static contact problem with Coulomb friction

*Applications of Mathematics*, Vol. 45 (2000), No. 5, 357--379

Persistent URL: <http://dml.cz/dmlcz/134445>

## Terms of use:

© Institute of Mathematics AS CR, 2000

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

FINITE ELEMENT ANALYSIS OF A STATIC CONTACT PROBLEM  
WITH COULOMB FRICTION\*

IVAN HLAVÁČEK, Praha

(Received March 4, 1999)

*Abstract.* A unilateral contact problem with a variable coefficient of friction is solved by a simplest variant of the finite element technique. The coefficient of friction may depend on the magnitude of the tangential displacement. The existence of an approximate solution and some a priori estimates are proved.

*Keywords:* unilateral contact, Coulomb friction, finite elements, existence proofs

*MSC 2000:* 65N30, 73T05

INTRODUCTION

The problem of a unilateral contact with Coulomb friction attracted attention of many research workers both in engineering and mathematics. Among the numerous literature we have chosen the paper by Licht, Pratt and Raous [7], who proposed an efficient approximate method of solution on the basis of a simplest variant of the finite element method. They justified the method by numerical experiments and presented some theoretical numerical analysis, namely the proof of existence of a solution and some conditions guaranteeing its uniqueness. They restricted themselves, however, to a constant coefficient  $\mathcal{F}$  of the Coulomb friction. See also the papers by Haslinger [5], [6] for similar results.

The aim of the present paper is to extend the above-mentioned results to the cases when the coefficient  $\mathcal{F}$  is not constant, but depends on (i) the place ( $\mathcal{F} =$

---

\* This research was supported by grant No. 201/97/0217 of the Grant Agency of the Czech Republic.

$\mathcal{F}(x)$ ) or (ii) on the place and on the magnitude of the tangential displacement, i.e.  $\mathcal{F} = \mathcal{F}(x, |u_T|)$ .

The first section contains the definition of a continuous unilateral problem of contact with a variable coefficient of friction. In the second section an approximate problem is formulated by means of a simple finite element technique. We prove the existence of an approximate solution and some a priori estimates for the case  $\mathcal{F} = \mathcal{F}(x)$ . The proof is based on a fixed point theorem, like in [7] for  $\mathcal{F} = \text{const}$ . The uniqueness is guaranteed if the ratio  $\|\mathcal{F}\|_\infty^2/h_0$  is sufficiently small. (Here  $\|\cdot\|_\infty$  is the standard norm in  $C(\Gamma_C)$  and  $h_0$  is the norm of the triangulation near the contact boundary  $\Gamma_C$ .)

The third section contains a proof of the existence theorem and some a priori estimates for the case  $\mathcal{F} = \mathcal{F}(x, |u_T|)$ . We employ the same method of proof as that used by Eck and Jarušek in [2], [3], i.e., a penalization and regularization, followed by a successive limiting process.

## 1. SETTING OF A CONTINUOUS CONTACT PROBLEM

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ , be a polyhedral domain with Lipschitz boundary  $\partial\Omega$ . Assume that

$$\partial\Omega = \Gamma_U \cup \Gamma_F \cup \Gamma_C$$

is a mutually disjoint partition,  $\Gamma_U$ ,  $\Gamma_F$ ,  $\Gamma_C$  are of positive surface measure. Moreover, let  $\Gamma_C$  be an open subset of a straight line or of a plane

$$\{x: x = (x_1, \dots, x_{d-1}, 0)\}.$$

Let the body occupying the domain  $\Omega$  be elastic, so that the stress-strain relations are

$$(1.1) \quad \sigma_{ij} = a_{ijkl} e_{kl},$$

where

$$e_{km} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_m} + \frac{\partial u_m}{\partial x_k} \right)$$

and  $u$  is the displacement vector,

$$a_{ijkl} = a_{jikl} = a_{klij} \in L_\infty(\Omega),$$

$$a_{ijkl} \tau_{ij} \tau_{kl} \geq \alpha_0 \tau_{ij} \tau_{ij} \text{ for all symmetric } \tau_{ij} \text{ and a.a. } x \in \Omega,$$

with some positive  $\alpha_0$ . Here we use the summation convention for repeated indices within the range  $\{1, \dots, d\}$ .

The equations of equilibrium are

$$(1.2) \quad \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \quad \text{in } \Omega, \quad 1 \leq i \leq d,$$

where  $f \in [L_2(\Omega)]^d$  are given body forces. We consider the boundary conditions

$$\begin{aligned} u &= 0 \quad \text{on } \Gamma_U, \\ \sigma_{ij} n_j &= (T_0)_i, \quad 1 \leq i \leq d \quad \text{on } \Gamma_F, \end{aligned}$$

where  $T_0 \in [L_2(\Gamma_F)]^d$  are given surface tractions and  $\mathbf{n}$  denotes the unit outward normal vector.

On the part  $\Gamma_C$  a unilateral contact with friction is considered:

$$(1.3) \quad u_N \leq 0, \quad \sigma_N \leq 0, \quad u_N \sigma_N = 0$$

$$(1.4) \quad \begin{aligned} |\sigma_T| &\leq \mathcal{F}(u_T) |\sigma_N|, \\ u_T = 0 &\Rightarrow |\sigma_T| < \mathcal{F}(0) |\sigma_N|, \\ u_T \neq 0 &\Rightarrow \sigma_T = -\mathcal{F}(u_T) |\sigma_N| u_T / |u_T|. \end{aligned}$$

Here

$$\begin{aligned} u_N &= u_i n_i, \quad u_{Ti} = u_i - u_N n_i, \\ \sigma_N &= \sigma_{ij} n_i n_j, \quad \sigma_{Ti} = \sigma_{ij} n_j - \sigma_N n_i, \quad 1 \leq i \leq d; \end{aligned}$$

$\mathcal{F}$  is the coefficient of the Coulomb friction, such that  $\mathcal{F}(u_T) \equiv \mathcal{F}(x, |u_T|)$  is a bounded nonnegative function on  $\Gamma_C \times [0, \infty)$  and  $\mathcal{F}(x, \cdot)$  is Lipschitz continuous for almost all  $x \in \Gamma_C$  with a constant  $C_L$  independent of  $x$ ;  $\mathcal{F}(\cdot, \xi)$  has a compact support in  $\Gamma_C$ .

We define the subspace

$$\mathbf{V} = \{v \in [H^1(\Omega)]^d : v = 0 \text{ on } \Gamma_U\},$$

the subset

$$\mathbf{K} = \{v \in \mathbf{V} : v_N \leq 0 \text{ on } \Gamma_C\},$$

the bilinear form

$$a(u, v) = \int_{\Omega} a_{ijkl} e_{ij}(u) e_{kl}(v) \, dx$$

and the linear functional

$$L(v) = \int_{\Omega} f_i v_i \, dx + \int_{\Gamma_F} T_{0i} v_i \, ds.$$

If  $\omega \in \mathbf{V}$ ,  $\sigma_{ij}(\omega) = a_{ijkm} e_{km}(\omega)$  and  $\partial\sigma_{ij}(\omega)/\partial x_j + f_i = 0$  in  $\Omega$ , the Green formula enables us to define a functional  $t(\omega) = t(\sigma(\omega)) \in \mathbf{H}^{-1/2}(\Gamma_C)$  as follows:

$$(1.5) \quad \langle t(\omega), v \rangle = a(\omega, \mathbf{P}v) - L(\mathbf{P}v) \quad \forall v \in [H_0^{1/2}(\Gamma_C)]^d,$$

where  $\mathbf{P}v \in \mathbf{V}$  is any extension of  $v$  such that  $\mathbf{P}v = 0$  on  $\Gamma_F$ , and  $H_0^{1/2}(\Gamma_C)$  is the subspace of traces of functions from  $H^1(\Omega)$  vanishing on  $\Gamma_U \cup \Gamma_F$ .

If  $\sigma_{ij}(\omega) \in H^1(\Omega)$ , the standard formula for surface stress vector holds:

$$t_i(\omega) = \sigma_{ij}(\omega)n_j \in L_2(\Gamma_C), \quad 1 \leq i \leq d,$$

and  $\langle \cdot, \cdot \rangle$  reduces to the inner product in  $[L_2(\Gamma_C)]^d$ .

Finally, we define the normal component of the surface stress vector

$$(1.6) \quad \langle t_N(\omega), w \rangle = \langle t(\omega), \mathbf{n}w \rangle \quad \forall w \in H_0^{1/2}(\Gamma_C).$$

The *weak solution* of the contact problem is a function  $u \in \mathbf{K}$  such that

$$(1.7) \quad a(u, v - u) - \langle t_N(u), \mathcal{F}(u_T)(|v_T| - |u_T|) \rangle \geq L(v - u) \quad \forall v \in \mathbf{K}.$$

For the existence and regularity of a weak solution we refer to Eck and Jarušek [2], [3], who considered even more general domains  $\Omega$  and functions  $\mathcal{F}(x, |u_T|)$ .

## 2. APPROXIMATE CONTACT PROBLEM

We shall approximate the problem (1.7) by a simplest finite element technique, i.e., by means of linear simplicial elements.

Assume that  $\{\mathcal{T}_h\}, h \rightarrow 0+$ , is a quasi-uniform (strongly regular) family of triangulations of the domain  $\Omega$  (see [1], (17.13) for the definition). We introduce the following finite element spaces on simplexes  $T \in \mathcal{T}_h$ :

$$\begin{aligned} X_h &= \{w \in C(\bar{\Omega}) : w|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h\}, \\ \mathbf{V}_h &= \{w \in [X_h]^d : w = 0 \text{ on } \Gamma_U\}, \\ \mathbf{K}_h &= \{v \in \mathbf{V}_h : v_N \leq 0 \text{ on } \Gamma_C\}, \\ \tilde{X}_h &= \{w|_{\Gamma_C} : w \in X_h, w = 0 \text{ on } \partial\Gamma_C\} = X_h|_{\Gamma_C} \cap H_0^{1/2}(\Gamma_C). \end{aligned}$$

The following discrete analog of the definitions (1.5), (1.6) will be used:

$$(2.1) \quad \langle \langle t^h(u), \tilde{v} \rangle \rangle = a(u, \mathbf{R}\tilde{v}) - L(\mathbf{R}\tilde{v}), \quad \tilde{v} \in [\tilde{X}_h]^d, \quad u \in \mathbf{V}_h,$$

$$(2.2) \quad \langle \langle t_N^h(u), \tilde{w} \rangle \rangle = \langle \langle t^h(u), \tilde{w}\mathbf{n} \rangle \rangle, \quad \tilde{w} \in \tilde{X}_h, \quad u \in \mathbf{V}_h,$$

where  $\mathbf{R}: [\tilde{X}_h]^d \rightarrow \mathbf{V}_h$  is a linear mapping such that  $\mathbf{R}\tilde{v}(a_i) = \tilde{v}(a_i)$  at the nodes  $a_i \in \Gamma_C$  and  $\mathbf{R}\tilde{v} = 0$  at the other nodes of the triangulation  $\tilde{\mathcal{T}}_h$ .

Let  $\Pi^h$  denote the Lagrange interpolation operator of  $X_h$  restricted to the part  $\Gamma_C$  of the boundary,  $\Pi^h: C^0(\overline{\Gamma_C}) \rightarrow \tilde{X}_h$ , where  $C^0$  denotes the space of continuous functions vanishing on  $\partial\Gamma_C$ .

The *approximate solution* is a function  $u^h \in \mathbf{K}_h$  such that

$$(2.3) \quad a(u^h, v - u^h) - \langle t_N^h(u^h), \Pi^h(\mathcal{F}(u_T^h)(|v_T| - |u_T^h|)) \rangle \geq L(v - u^h) \quad \forall v \in \mathbf{K}_h.$$

The main result of the section is represented by the following

**Theorem 2.1.** *There exists at least one approximate solution  $u^h$  of (2.3). Positive constants  $C_0$  and  $M$  exist, independent of  $\mathcal{F}$  and such that*

$$\begin{aligned} \|u^h\|_{1,\Omega} &\leq \|L\|_{-1}/C_0, \\ \|t_N^h(u^h)\|_* &\leq M\|L\|_{-1}h_0^{-1/2}, \end{aligned}$$

where

$$C_0 = \inf_{v \in V \setminus \{0\}} \frac{a(v, v)}{\|v\|_{1,\Omega}^2},$$

$\|L\|_{-1}$  is the norm of  $L$  in the dual space  $([H^1(\Omega)]^d)'$ ;  $\|\cdot\|_*$  is the norm in  $(\tilde{X}_h)'$ ;

$$\begin{aligned} \|g\|_* &= \sup_{\tilde{v} \in \tilde{X}_h} \frac{\langle g, \tilde{v} \rangle}{\|\tilde{v}\|_{0,\Gamma_C}}, \\ h_0 &= \max_{T \subset \text{supp } \mathbf{R}\tilde{v}} (\text{diam } T). \end{aligned}$$

Let  $\mathcal{R}: \tilde{X}_h \rightarrow X_h$  be the extension determined by the nodal values of  $\tilde{z} \in \tilde{X}_h$  on  $\Gamma_C$  and by zero values at the other nodes of  $\tilde{\mathcal{T}}_h$ .

**Lemma 2.1.** *There exists a positive constant  $\hat{C}$ , independent of  $h_0$  and such that*

$$(2.4) \quad \|\mathcal{R}\tilde{z}\|_{0,\Omega} \leq \hat{C}h_0^{1/2}\|\tilde{z}\|_{0,\Gamma_C} \quad \forall \tilde{z} \in \tilde{X}_h.$$

**Proof.** (i) Let  $d = 2$ . Consider a triangle  $T_1(a_1a_2a_3)$ ,  $a_1 = (0, 0)$ ,  $a_2 = (a_{12}, 0)$ ,  $a_3 = (a_{13}, a_{23})$  and the barycentric coordinates

$$\lambda_1 = 1 - \lambda_2 - \lambda_3, \quad \lambda_2 = (x_1 - a_{13}x_2/a_{23})/a_{12}, \quad \lambda_3 = x_2/a_{23}.$$

We find that

$$(2.5) \quad \int_{T_1} \lambda_i^2 dx = \frac{1}{6} \text{meas } T_1 = \frac{1}{4} a_{23} \int_0^{a_{12}} \tilde{\lambda}_i^2 dx_1, \quad i = 1, 2,$$

where  $\tilde{\lambda}_i = \lambda_i|_{x_2=0}$ . Furthermore, we have

$$(2.6) \quad \int_{T_1} \lambda_1 \lambda_2 dx = \frac{1}{4} a_{23} \int_0^{a_{12}} \tilde{\lambda}_1 \tilde{\lambda}_2 dx_1 = \frac{1}{12} \text{meas } T_1.$$

Consequently, we obtain for  $\mathcal{R}\tilde{z} = z_1 \lambda_1 + z_2 \lambda_2$ ,  $\tilde{z} = z_1 \tilde{\lambda}_1 + z_2 \tilde{\lambda}_2$

$$(2.7) \quad \int_{T_1} (\mathcal{R}\tilde{z})^2 dx = \frac{1}{4} a_{23} \int_0^{a_{12}} \tilde{z}^2 dx_1.$$

For the adjacent triangle  $T_2(a_1 a_3 a_4)$  (with  $a_{24} > 0$ ) we derive

$$\int_{T_2} (\mathcal{R}\tilde{z})^2 dx = \int_{T_2} z_1^2 \mu_1^2(x) dx = \frac{1}{6} z_1^2 \text{meas } T_2,$$

where  $\mu_1$  is a barycentric coordinate and (2.5) has been used. Since the family of triangulations is strongly regular,

$$\text{meas } T_2 \leq C \text{meas } T_1$$

holds with the constant  $C$  independent of  $h$  and therefore

$$(2.8) \quad \int_{T_2} (\mathcal{R}\tilde{z})^2 dx \leq \frac{1}{6} z_1^2 C \text{meas } T_1 \leq \tilde{C} z_1^2 a_{23} \int_0^{a_{12}} \tilde{\lambda}_1^2 dx_1.$$

Due to the regularity of the family of triangulations, there exist at most  $M$  triangles with the vertex  $a_1$ ,  $M$  being independent of  $h$ . Since  $a_{23} \leq h_0$ , adding the estimates of the type (2.7) and (2.8) we arrive at

$$\sum_j \int_{T_j} (\mathcal{R}\tilde{z})^2 dx \leq h_0 \left( \frac{1}{4} + M\tilde{C} \right) \int_{\Gamma_C} \tilde{z}^2 dx_1,$$

so that (2.4) follows.

(ii)  $d = 3$ . Consider a tetrahedron  $T_1(a_1, a_2, a_3, a_4)$ , where  $a_1 = (0, 0, 0)$ ,  $a_2 = (a_{12}, 0, 0)$ ,  $a_3 = (a_{13}, a_{23}, 0)$ ,  $a_4 = (a_{14}, a_{24}, a_{34})$ ,  $a_{34} > 0$ ,  $a_{12} > 0$ . Using the barycentric coordinates  $\lambda_i$ , we derive

$$(2.9) \quad \int_{T_1} \lambda_i^2 dx = \frac{1}{5} a_{34} \int_{\tilde{T}_1} \tilde{\lambda}_i^2 dx_1 dx_2, \quad 1 \leq i \leq 3,$$

$$(2.10) \quad \int_{T_1} \lambda_i \lambda_j dx = \frac{1}{5} a_{34} \int_{\tilde{T}_1} \tilde{\lambda}_i \tilde{\lambda}_j dx_1 dx_2, \quad i \neq j, \quad 1 \leq i, j \leq 3,$$

where  $\tilde{T}_1 = \tilde{T}_1(a_1, a_2, a_3)$ . Then for  $\mathcal{R}\tilde{z} = \sum_{i=1}^3 z_i \lambda_i$ ,  $\tilde{z} = \sum_{i=1}^3 z_i \tilde{\lambda}_i$ ,  $\tilde{\lambda}_i = \lambda_i|_{x_3=0}$  we obtain

$$(2.11) \quad \int_{T_1} (\mathcal{R}\tilde{z})^2 dx = \frac{1}{5} a_{34} \int_{\tilde{T}_1} \tilde{z}^2 dx_1 dx_2.$$

Next, let us consider the tetrahedron  $T_2(a_2, a_3, a_4, b)$ , where  $b = (b_1, b_2, b_3)$ ,  $b_3 > 0$ . We may write

$$(2.12) \quad \int_{T_2} (\mathcal{R}\tilde{z})^2 dx = z_2^2 \int_{T_2} \mu_2^2 dx + z_3^2 \int_{T_2} \mu_3^2 dx + 2z_2 z_3 \int_{T_2} \mu_2 \mu_3 dx.$$

Using (2.9), we obtain

$$(2.13) \quad \int_{T_2} \mu_2^2 dx \leq \frac{1}{5} h_0 \int_{\Delta} \tilde{\mu}_2^2 dS, \quad \Delta = \Delta(a_2, a_3, a_4).$$

The results of part (i) and the definition of a strongly regular family of triangulations imply that

$$\int_{\Delta} \tilde{\mu}_2^2 dS = \frac{1}{6} \text{meas } \Delta \leq \frac{1}{12} h_0^2 = \tilde{C} \text{meas } \tilde{T}_1 = C \int_{\tilde{T}_1} \tilde{\lambda}_2^2 dx_1 dx_2.$$

Substituting this estimate into (2.13), we arrive at

$$(2.14) \quad \int_{T_2} \mu_2^2 dx \leq \frac{1}{5} Ch_0 \int_{\tilde{T}_1} \tilde{\lambda}_2^2 dx_1 dx_2.$$

In the same way we derive that

$$(2.15) \quad \int_{T_2} \mu_2 \mu_3 dx \leq \frac{1}{5} h_0 \int_{\Delta} \tilde{\mu}_2 \tilde{\mu}_3 dS = \frac{1}{60} h_0 \text{meas } \Delta \leq \frac{1}{5} Ch_0 \int_{\tilde{T}_1} \tilde{\lambda}_2 \tilde{\lambda}_3 dx_1 dx_2.$$

There exist at most  $M$  tetrahedrons with the vertex  $a_i$ ,  $i = 1, 2, 3$ , where  $M$  is independent of  $h$ . Combining the estimates (2.11), (2.12), (2.14) and (2.15), we are led to the estimate (2.4).  $\square$

### The case $\mathcal{F} = \mathcal{F}(x)$ .

First we introduce an auxiliary problem of unilateral contact with a given slip stress.



Let  $G$  be the set of *positive* linear functionals  $g$  on  $\tilde{X}_h$ . For any  $g \in G$  let us define the problem  $\mathbf{P}_g^h$  to find  $u_g \in \mathbf{K}_h$  such that

$$(2.16) \quad a(u_g, v - u_g) + \langle g, \Pi^h(\mathcal{F}(|v_T| - |u_{gT}|)) \rangle \geq L(v - u_g) \quad \forall v \in \mathbf{K}_h.$$

**Proposition 2.1.** *The problem  $(\mathbf{P}_g^h)$  has a unique solution for any  $g \in G$ .*

*P r o o f.* Let us denote

$$J_1(u) = \langle g, \Pi^h(\mathcal{F}|u_T|) \rangle, \quad J_2(u) = \frac{1}{2}a(u, u) - L(u).$$

Since  $J_1$  is convex,  $J_2$  strictly convex and differentiable on  $\mathbf{V}_h$ , the inequality in  $(\mathbf{P}_g^h)$  is equivalent to the minimization of the sum  $J = J_1 + J_2$  over the set  $\mathbf{K}_h$ .

We can show that the functional  $J_1$  is Lipschitz continuous on  $\mathbf{V}_h$ , i.e.,

$$(2.17) \quad |J_1(u) - J_1(v)| \leq C_g \|\mathcal{F}\|_\infty \|u - v\|_{1,\Omega} \quad \forall u, v \in \mathbf{V}_h,$$

where  $\|\cdot\|_\infty$  denotes the standard norm in  $C(\overline{\Gamma_C})$ .

Indeed, let  $d = 2$ . For any  $v \in H^1(\Gamma_C)$  we have

$$\|\Pi^h v - v\|_{0,\Gamma_C} \leq C_\pi h_0 |v|_{1,\Gamma_C}$$

so that

$$(2.18) \quad \|\Pi^h v\|_{0,\Gamma_C} \leq C_\pi h_0 |v|_{1,\Gamma_C} + \|v\|_{0,\Gamma_C}.$$

We may write

$$(2.19) \quad \begin{aligned} |J_1(u) - J_1(v)| &\leq \|g\|_* \|\Pi^h(\mathcal{F}(|u_T| - |v_T|))\|_{0,\Gamma_C} \\ &\leq \|g\|_* \|\mathcal{F}\|_\infty \|\Pi^h(|u_T| - |v_T|)\|_{0,\Gamma_C} \\ &\leq \|g\|_* \|\mathcal{F}\|_\infty \|\Pi^h(|w_T|)\|_{0,\Gamma_C} \end{aligned}$$

since

$$|\Pi^h(\mathcal{F}(|u_T| - |v_T|))| \leq \|\mathcal{F}\|_\infty \Pi^h(|u_T| - |v_T|) \leq \|\mathcal{F}\|_\infty \Pi^h(|w_T|),$$

where  $w := u - v$ . For  $w_j \in X_h|_{\Gamma_C}$  the ‘‘inverse inequality’’

$$(2.20) \quad \|w_j\|_{1,\Gamma_C} \leq Ch_0^{-1} \|w_j\|_{0,\Gamma_C}$$

holds [1]. Using (2.18), (2.20) and the Trace Theorem, we obtain

$$(2.21) \quad \begin{aligned} \|\Pi^h(|w_T|)\|_{0,\Gamma_C} &\leq C_\pi h_0 \|w_1\|_{1,\Gamma_C} + \|w_1\|_{0,\Gamma_C} \\ &\leq Ch_0 \|w_1\|_{1,\Gamma_C} + \|w_1\|_{0,\Gamma_C} \\ &\leq \tilde{C} \|w_1\|_{0,\Gamma_C} \leq \tilde{C} C \|w\|_{1,\Omega}. \end{aligned}$$

Inserting (2.21) into (2.19), we arrive at (2.17).

Next, let  $d = 3$ . Let us consider

$$v := |w_j|, \quad w_j \in X_h|_{\Gamma_C}, \quad (j = 1, 2),$$

and realize that for any triangle  $K \in \Gamma_C$  we may write (cf. [1], Theorem 3.16)

$$(i) \quad \|\Pi_K v - v\|_{0,2,K}^2 \leq C(\text{meas } K)^{1-2/(2+\varepsilon)} h_K^2 |v|_{1,2+\varepsilon,K}^2, \quad \varepsilon > 0.$$

Since we have

$$(ii) \quad \begin{aligned} \left| \frac{\partial w_j}{\partial x_i} \right| &= \left| \frac{\partial |w_j|}{\partial x_i} \right| \quad \text{a.e. in } K \quad (i, j = 1, 2), \\ |v|_{1,2+\varepsilon,K}^2 &= |w_j|_{1,2+\varepsilon,K}^2 \end{aligned}$$

holds. By means of the ‘‘inverse assumption’’ (cf. [1], (3.2.33)), we may write

$$(iii) \quad |w_j|_{1,2+\varepsilon,K}^2 \leq C(h_0^2)^{2/(2+\varepsilon)-1} |w_j|_{1,2,K}^2.$$

Inserting (ii) and (iii) into (i), we obtain

$$\|\Pi_K v - v\|_{0,2,K}^2 \leq Ch_K^2 |w_j|_{1,2,K}^2.$$

Summing over all  $K \in \Gamma_C$ , we arrive at the estimate

$$\|\Pi^h |w_j| - |w_j|\|_{0,\Gamma_C} \leq Ch_0 |w_j|_{1,\Gamma_C}, \quad j = 1, 2.$$

As a consequence, we have

$$\|\Pi^h |w_j|\|_{0,\Gamma_C} \leq \|w_j\|_{0,\Gamma_C} + Ch_0 |w_j|_{1,\Gamma_C} \leq \tilde{C} \|w_j\|_{0,\Gamma_C}.$$

Since

$$\Pi^h(|w_T|) \leq \sum_{j=1}^2 \Pi^h(|w_j|),$$

we obtain

$$(2.21a) \quad \|\Pi^h(|w_T|)_{0,\Gamma_C} \leq \sum_{j=1}^2 \|\Pi^h(|w_j|)\|_{0,\Gamma_C} \leq \tilde{C} \sum_{j=1}^2 \|w_j\|_{0,\Gamma_C} \leq \tilde{C}C \|w\|_{1,\Omega}.$$

Combining (2.21a) with (2.19), (2.17) follows.

As a consequence, the functional  $J$  is continuous and coercive on  $V_h$  by virtue of Korn's inequality and the non-negativeness of  $J_1(u)$ . Since the set  $\mathbf{K}_h$  is convex and closed, a minimizer exists. The uniqueness follows from the fact that  $J_2$  is strictly convex and  $J_1$  is convex.  $\square$

Next let us define a mapping  $\mathbf{T}: G \rightarrow (X_h)'$  by the formula

$$(2.22) \quad \mathbf{T}(g) = -t_N^h(u_g).$$

**Lemma 2.2.**

$$\mathbf{T}(G) \subset G.$$

*Proof.* Let  $\tilde{w} \in \tilde{X}_h$ ,  $\tilde{w} \geq 0$ . We may write

$$(2.23) \quad \langle \mathbf{T}(g), \tilde{w} \rangle = \langle -t_N^h(u_g), \tilde{w} \rangle = a(u_g, \mathbf{R}(-\tilde{w}\mathbf{n})) - L(\mathbf{R}(-\tilde{w}\mathbf{n})).$$

If  $v = u_g + \mathbf{R}(-\tilde{w}\mathbf{n})$ , then  $v \in \mathbf{K}_h$ , since  $(\mathbf{R}(-\tilde{w}\mathbf{n}))_N \leq 0$  on  $\Gamma_C$ . From the inequality  $(\mathbf{P}_g^h)$  we deduce

$$a(u_g, \mathbf{R}(-\tilde{w}\mathbf{n})) - L(\mathbf{R}(-\tilde{w}\mathbf{n})) \geq -\langle g, \Pi^h(\mathcal{F}(|u_{gT} + \mathbf{R}_T(-\tilde{w}\mathbf{n})| - |u_{gT}|)) \rangle = 0,$$

since  $\mathbf{R}_T(-\tilde{w}\mathbf{n}) = 0$ . Inserting this into (2.23), we obtain

$$\langle \mathbf{T}(g), \tilde{w} \rangle \geq 0.$$

$\square$

**Lemma 2.3.** *The mapping  $\mathbf{T}$  is Lipschitz continuous, i.e.,*

$$\|\mathbf{T}(g_2) - \mathbf{T}(g_1)\|_* \leq Ch_0^{-1/2} \|\mathcal{F}\|_\infty \|g_2 - g_1\|_*,$$

where  $C$  is independent of  $h_0$ ,  $\mathcal{F}$ ,  $g_1$ ,  $g_2$ .

*Proof.* Denote  $u^1 := u_{g_1}$ ,  $u^2 := u_{g_2}$  and choose an arbitrary  $\tilde{w} \in \tilde{X}_h$ . It is readily seen that

$$(2.24) \quad |\langle t_N^h(u^1) - t_N^h(u^2), \tilde{w} \rangle| = |a(u^1 - u^2, \mathbf{R}(\tilde{w}\mathbf{n}))| \leq C_1 |u^1 - u^2|_{1,\Omega} |\mathcal{R}\tilde{w}|_{1,\Omega},$$

since  $n_j = 0$  and  $\mathbf{R}_j(\tilde{w}n) = 0$  for  $1 \leq j \leq d-1$ ,  $n_d = -1$ ,  $\mathbf{R}_d(\tilde{w}n) = -\mathcal{R}\tilde{w}$ . Lemma 2.1 and the inverse inequality for elements of  $X_h$  yield

$$(2.25) \quad |\mathcal{R}\tilde{w}|_{1,\Omega} \leq C_2 h_0^{-1} \|\mathcal{R}\tilde{w}\|_{0,\Omega} \leq C_2 \hat{C} h_0^{-1/2} \|\tilde{w}\|_{0,\Gamma_C}.$$

Thus we have the following estimate from (2.24) and (2.25):

$$(2.26) \quad \|\mathbf{T}(g_1) - \mathbf{T}(g_2)\|_* \leq C_3 h_0^{-1/2} |u^1 - u^2|_{1,\Omega}.$$

On the other hand, the definition (2.16) and Korn's inequality imply

$$(2.27) \quad \begin{aligned} C_0 \|u^1 - u^2\|_{1,\Omega}^2 &\leq a(u^1 - u^2, u^1 - u^2) \\ &\leq \langle g_1 - g_2, \Pi^h(\mathcal{F}(|u_T^2| - |u_T^1|)) \rangle \\ &\leq \|g_1 - g_2\|_* \|\Pi^h((|u_T^2| - |u_T^1|)\mathcal{F})\|_{0,\Gamma_C}. \end{aligned}$$

Using (2.20) and (2.21) or (2.21a), we obtain

$$\|\Pi^h(\mathcal{F}(|u_T^2| - |u_T^1|))\|_{0,\Gamma_C} \leq \|\mathcal{F}\|_\infty \|\Pi^h(|w_T|)\|_{0,\Gamma_C} \leq C \|\mathcal{F}\|_\infty \|u^2 - u^1\|_{1,\Omega}$$

so that (2.27) yields

$$(2.28) \quad C_0 \|u^2 - u^1\|_{1,\Omega} \leq C \|\mathcal{F}\|_\infty \|g_1 - g_2\|_*.$$

Combining (2.26) and (2.28), we arrive at

$$\|\mathbf{T}(g_1) - \mathbf{T}(g_2)\|_* \leq C_0^{-1} C \|\mathcal{F}\|_\infty h_0^{-1/2} \|g_1 - g_2\|_*.$$

□

**Lemma 2.4.** *There exists a constant  $M > 0$ , independent of  $h_0$  and  $\mathcal{F}$ , such that*

$$\|\mathbf{T}(g)\|_* \leq M \|L\|_{-1} h_0^{-1/2} \quad \forall g \in G.$$

*Proof.* Setting  $v := 0$  in the definition (2.16) and using Korn's inequality, we obtain

$$C_0 \|u_g\|_{1,\Omega}^2 \leq a(u_g, u_g) \leq L(u_g) - \langle g, \Pi^h(\mathcal{F}|u_{gT}|) \rangle \leq L(u_g) \leq \|L\|_{-1} \|u_g\|_{1,\Omega}$$

so that

$$(2.29) \quad \|u_g\|_{1,\Omega} \leq C_0^{-1} \|L\|_{-1}$$

holds for all  $g \in G$ . We may write

$$\begin{aligned}
 (2.30) \quad |\langle \mathbf{T}(g), \tilde{w} \rangle| &= |a(u_g, \mathbf{R}(\mathbf{n}\tilde{w})) - L(\mathbf{R}(\mathbf{n}\tilde{w}))| \\
 &\leq C_1 \|u_g\|_{1,\Omega} \|\mathcal{R}\tilde{w}\|_{1,\Omega} + \|L\|_{-1} \|\mathcal{R}\tilde{w}\|_{1,\Omega} \\
 &\leq (C_0^{-1}C_1 + 1) \|L\|_{-1} \|\mathcal{R}\tilde{w}\|_{1,\Omega}.
 \end{aligned}$$

On the other hand,

$$\|\mathcal{R}\tilde{w}\|_{1,\Omega} \leq C_2 h_0^{-1} \|\mathcal{R}\tilde{w}\|_{0,\Omega} \leq C_2 \hat{C} h_0^{-1/2} \|\tilde{w}\|_{0,\Gamma_C}$$

follows from the inverse inequality on the domain  $\text{supp}(\mathcal{R}\tilde{w})$  and from Lemma 2.1. Inserting this into (2.30), we arrive at

$$\|\mathbf{T}(g)\|_* \leq (1 + C_1/C_0) C_3 h_0^{-1/2} \|L\|^{-1}.$$

□

**Proof of Theorem 2.1 in case  $\mathcal{F} = \mathcal{F}(x)$ .** Let us denote

$$B(h_0) = \{g \in G: \|g\|_* \leq M \|L\|_{-1} h_0^{-1/2}\},$$

where the constant  $M$  is that of Lemma 2.4. Since the set  $B(h_0)$  is bounded and closed in the dual space  $(\tilde{X}_h)'$ ,  $B(h_0)$  is compact and convex. By virtue of Lemma 2.3 the mapping  $\mathbf{T}$  is continuous and  $\mathbf{T}(B(h_0)) \subset B(h_0)$  holds by virtue of Lemma 2.4. As a consequence, the Brouwer Theorem yields the existence of a fixed point of  $\mathbf{T}$ .

It is easy to see that a solution of the problem (2.3) exists if and only if there exists a fixed point of  $T$ .

The a priori estimates of Theorem 2.1 follow from (2.29) and Lemma 2.4. □

**Theorem 2.2.** *There exists a positive constant  $C$ , independent of  $h_0$ ,  $\mathcal{F}$ , and  $L$  such that the problem (2.3) has at most one solution provided*

$$h_0 > C \|\mathcal{F}\|_{\infty}^2.$$

**Proof.** If  $u$  and  $\bar{u}$  are two solutions of (2.3), then

$$\begin{aligned}
 a(u, \bar{u} - u) - \langle t_N^h(u), \Pi^h(\mathcal{F}(|\bar{u}_T| - |u_T|)) \rangle &\geq L(\bar{u} - u), \\
 a(\bar{u}, u - \bar{u}) - \langle t_N^h(\bar{u}), \Pi^h(\mathcal{F}(|u_T| - |\bar{u}_T|)) \rangle &\geq L(u - \bar{u}).
 \end{aligned}$$

By addition, we derive that

$$a(u - \bar{u}, \bar{u} - u) + \langle t_N^h(\bar{u}) - t_N^h(u), \Pi^h(\mathcal{F}(|\bar{u}_T| - |u_T|)) \rangle \geq 0.$$

By definitions (1.5), (1.6) we may therefore write

$$a(u - \bar{u}, u - \bar{u}) \leq a(\bar{u} - u, \mathbf{R}(\mathbf{n}\Pi^h(\mathcal{F}(|\bar{u}_T| - |u_T|))))).$$

Denoting  $w := \bar{u} - u$ , we obtain

$$(2.31) \quad C_0 \|w\|_{1,\Omega}^2 \leq C_1 \|w\|_{1,\Omega} |\mathcal{U}_d|_{1,\Omega},$$

where

$$\mathcal{U}_d = \mathcal{R}(\Pi^h(\mathcal{F}(|\bar{u}_T| - |u_T|))).$$

Since  $\mathcal{U}_d \in X_h$ , the inverse inequality and Lemma 2.1 imply

$$(2.32) \quad |\mathcal{U}_d|_{1,\Omega} \leq C_2 h_0^{-1} \|\mathcal{U}_d\|_{0,\Omega} \leq C_2 \hat{C} h_0^{-1/2} \|\Pi^h(\mathcal{F}(|\bar{u}_T| - |u_T|))\|_{0,\Gamma_C}.$$

Arguing as in the derivation of the estimates (2.20), (2.21), we obtain

$$(2.33) \quad \|\Pi^h(\mathcal{F}(|\bar{u}_T| - |u_T|))\|_{0,\Gamma_C} \leq C_3 \|\mathcal{F}\|_{\infty} \|w\|_{1,\Omega}.$$

Combining (2.31), (2.32) and (2.33), we arrive at

$$(2.34) \quad \|w\|_{1,\Omega} \leq C_0^{-1} C_1 C_2 \hat{C} C_3 h_0^{-1/2} \|\mathcal{F}\|_{\infty} \|w\|_{1,\Omega}.$$

Let us denote  $C_4 := C_0^{-1} C_1 C_2 \hat{C} C_3$  and assume that

$$(2.35) \quad C_4 h_0^{-1/2} \|\mathcal{F}\|_{\infty} < 1.$$

Then  $w = 0$  follows from (2.34). □

**Remark 2.1.** It is easy to see that the mapping  $\mathbf{T}$  defined by (2.22) is contractive if (2.35) holds. □

### 3. THE CASE $\mathcal{F} = \mathcal{F}(x, |u_T|)$

Following the line of thoughts used by Eck and Jarušek in [2] and [3] for the continuous problem (1.7), we shall prove Theorem 2.1. Thus we will apply a penalization with respect to  $t_N^h(u)$  and a regularization of the absolute values in the definition (2.3). After that, we will pass to the limit with the parameters of regularization and penalization.

**Remark 3.1.** The approach of the previous section, based on the fixed point, fails in the present case since we are not able to prove the continuity of the mapping  $\mathbf{T}$  outside a small ball in  $(\tilde{X}_h)'$ , where the uniqueness for  $(\mathbf{P}_g^h)$  is guaranteed. □

Let us introduce the functionals

$$\begin{aligned}\Phi_\delta(u, v) &= \int_{\Gamma_C} \Pi^h(\delta^{-1}[u_N]_+ v_N) \, ds, \\ j_\delta(u, v) &= \int_{\Gamma_C} \Pi^h(\delta^{-1}[u_N]_+ \mathcal{F}(u_T) |v_T|) \, ds,\end{aligned}$$

where  $\delta$  is a positive parameter, and the problem  $(P_\delta)$ : find  $u \in \mathbf{V}_h$  such that

$$(3.1) \quad a(u, v - u) + \Phi_\delta(u, v - u) + j_\delta(u, v) - j_\delta(u, u) \geq L(v - u) \quad \forall v \in \mathbf{V}_h.$$

Let  $\varepsilon > 0$  and let

$$\varphi_\varepsilon(t) = \begin{cases} |t| & \text{for } |t| \geq \varepsilon, \\ -\frac{|t|^4}{8\varepsilon^3} + \frac{3|t|^2}{4\varepsilon} + \frac{3}{8}\varepsilon & \text{for } |t| \leq \varepsilon \end{cases}$$

be a regularization of the absolute value  $|t|$ .

We define also

$$j_{\delta,\varepsilon}(u, v) = \int_{\Gamma_C} \Pi^h(\delta^{-1}[u_N]_+ \mathcal{F}(u_T) \varphi_\varepsilon(v_T)) \, ds$$

and

$$\begin{aligned}\psi_{\delta,\varepsilon} &= \lim_{\lambda \rightarrow 0^+} (j_{\delta,\varepsilon}(u, u + \lambda v) - j_{\delta,\varepsilon}(u, u)) \\ &= \int_{\Gamma_C} \Pi^h(\delta^{-1}[u_N]_+ \mathcal{F}(u_T) \operatorname{grad} \varphi_\varepsilon(u_T) \cdot v_T) \, ds.\end{aligned}$$

The regularized problem (3.1), where  $j_\delta$  is replaced by  $j_{\delta,\varepsilon}$ , is equivalent to the following variational equation  $(P_{\delta,\varepsilon})$ : find  $u \in \mathbf{V}_h$ , such that

$$(3.2) \quad a(u, v) + \Phi_\delta(u, v) + \psi_{\delta,\varepsilon}(u, v) = L(v) \quad \forall v \in \mathbf{V}_h.$$

In what follows, we prove the existence of a solution of (3.2). Then passing to the limit successively with  $\varepsilon \rightarrow 0^+$  and  $\delta \rightarrow 0^+$ , we obtain the existence of a solution of the problem (2.3).

Let us introduce the operators

$$A: \mathbf{V}_h \rightarrow \mathbf{V}'_h, \quad Q: \mathbf{V}_h \rightarrow \mathbf{V}'_h, \quad F: \mathbf{V}_h \rightarrow \mathbf{V}'_h$$

by the formulae

$$\langle Au, v \rangle = a(u, v), \quad \langle Qu, v \rangle = \Phi_\delta(u, v), \quad \langle Fu, v \rangle = \psi_{\delta,\varepsilon}(u, v)$$

and the operator  $T: \mathbf{V}_h \rightarrow \mathbf{V}_h'$ ,  $T = A + Q + F$ .

We can show that the operator  $T$  is continuous and coercive. To this end we need an auxiliary

**Lemma 3.1.** *For any  $u, v, w \in [X_h]^d$ , we have*

$$\begin{aligned} [u_N]_+ &\leq |u_N| = |u_d|, \\ |v_T| = |v_1| \quad \text{for } d = 2 \quad &\text{and} \quad |v_T| \leq |v_1| + |v_2| \quad \text{for } d = 3, \\ |\Pi^h(|u_N| + v_N)| &\leq \Pi^h([u_N]_+ |v_N|) \leq \Pi^h(|u_d| |v_d|) \leq \|u_d\|_\infty \|v_d\|_\infty, \\ \Pi^h(|u_j| |w_T|) &\leq \|u_j\|_\infty \left( \sum_{j=1}^{d-1} \|w_j\|_\infty \right). \end{aligned}$$

*Proof* is obvious. □

**Lemma 3.2.** *The following assertions hold:*

- (i)  $A$  is continuous, linear and elliptic,
- (ii)  $Q$  is continuous and  $\langle Qv, v \rangle \geq 0$  for all  $v \in \mathbf{V}_h$ ,
- (iii)  $F$  is continuous and  $\langle Fv, v \rangle \geq 0$  for all  $v \in \mathbf{V}_h$ .

*Proof.* (i) is obvious.

(ii) Since  $|[a]_+ - [b]_+| \leq |a - b|$  holds for all  $a, b \in \mathbb{R}$ , we have

$$\begin{aligned} |\langle Qu - Qw, v \rangle| &\leq \delta^{-1} \int_{\Gamma_C} |\Pi^h(([u_N]_+ - [w_N]_+)v_n)| \, ds \\ &\leq \delta^{-1} \int_{\Gamma_C} \Pi^h(|u_N - w_N| |v_N|) \, ds \leq C\delta^{-1} \|u_d - w_d\|_\infty \|v_d\|_\infty. \end{aligned}$$

Hence  $Q$  is Lipschitz continuous. Since

$$[a]_+ a = ([a]_+)^2 \geq 0,$$

we have

$$\langle Qv, v \rangle = \delta^{-1} \int_{\Gamma_C} \Pi^h(|v_N| + v_N) \, ds \geq 0.$$

(iii) We may write

$$\begin{aligned} (3.3) \quad |\langle Fu - Fw, v \rangle| &= \delta^{-1} \left| \int_{\Gamma_C} \left\{ \Pi^h([u_N]_+ \mathcal{F}(u_T) \nabla \varphi_\varepsilon(u_T) \cdot v_T \right. \right. \\ &\quad \left. \left. - \Pi^h([w_N]_+ \mathcal{F}(w_T) \nabla \varphi_\varepsilon(w_T) \cdot v_T) \right\} \, ds \right| \\ &\leq \delta^{-1} \int_{\Gamma_C} |\Pi^h(J_1 + J_2 + J_3)| \, ds \\ &\leq \delta^{-1} \int_{\Gamma_C} (|\Pi^h J_1| + |\Pi^h J_2| + |\Pi^h J_3|) \, ds, \end{aligned}$$



where

$$\begin{aligned} J_1 &= ([u_N]_+ - [w_N]_+) \mathcal{F}(u_T) \nabla \varphi_\varepsilon(u_T) \cdot v_T, \\ J_2 &= [w_N]_+ \mathcal{F}(u_T) (\nabla \varphi_\varepsilon(u_T) - \nabla \varphi_\varepsilon(w_T)) \cdot v_T, \\ J_3 &= [w_N]_+ (\mathcal{F}(u_T) - \mathcal{F}(w_T)) \nabla \varphi_\varepsilon(w_T) \cdot v_T. \end{aligned}$$

We have

$$\int_{\Gamma_C} |\Pi^h J_1| \, ds \leq C \|\mathcal{F}\|_\infty \|u_d - w_d\|_\infty \|v_T\|_\infty,$$

since  $|\nabla \varphi_\varepsilon| \leq 1$  everywhere;

$$\int_{\Gamma_C} |\Pi^h J_2| \, ds \leq C \|\mathcal{F}\|_\infty \|w_d\|_\infty \sum_{j=1}^{d-1} \|u_j - w_j\|_\infty \|v_T\|_\infty,$$

since

$$\begin{aligned} |\nabla \varphi_\varepsilon(u_T) - \nabla \varphi_\varepsilon(w_T)| &\leq \frac{3}{2\varepsilon} |u_T - w_T|; \\ \int_{\Gamma_C} |\Pi^h J_3| \, ds &\leq CC_L \|w_d\|_\infty \sum_{j=1}^{d-1} \|u_j - w_j\|_\infty \|v_T\|_\infty \end{aligned}$$

since

$$|\mathcal{F}(s) - \mathcal{F}(t)| \leq C_L |s - t| \quad \forall s, t \in [0, \infty) \text{ and a.a. } x \in \Gamma_C.$$

Inserting these estimates into (3.3), we obtain

$$\begin{aligned} (3.4) \quad |\langle Fu - Fw, v \rangle| &\leq C\delta^{-1} \\ &\times \left\{ \|\mathcal{F}\|_\infty \|u_d - w_d\|_\infty + (\|\mathcal{F}\|_\infty + C_L) \|w_d\|_\infty \sum_{j=1}^{d-1} \|u_j - w_j\|_\infty \right\} \|v_T\|_\infty \end{aligned}$$

where  $C \equiv C(\varepsilon)$ , so that  $F$  is continuous. Finally, we have

$$\langle Fv, v \rangle = \delta^{-1} \int_{\Gamma_C} \Pi^h([v_N]_+ \mathcal{F}(v_T) \nabla \varphi_\varepsilon(v_T) \cdot v_T) \, ds \geq 0,$$

since

$$\nabla \varphi_\varepsilon(v_T) \cdot v_T \geq 0.$$

In fact, the latter inequality follows from the convexity of  $\varphi_\varepsilon$  and the fact that  $\varphi_\varepsilon$  attains its minimum at the origin.  $\square$

**Proposition 3.1.** *The problem  $(P_{\delta,\varepsilon})$  (3.2) has at least one solution for any positive  $\delta$  and  $\varepsilon$ .*

*P r o o f* follows from a general theorem—see [4], Theorem 2.5, since the operator  $T = A + Q + F$  is continuous and coercive by Lemma 3.2.  $\square$

**Proposition 3.2.** *The problem (3.1)  $(P_\delta)$  has at least one solution for any positive  $\delta$ .*

*P r o o f.* Let us denote the solution of the problem (3.2) with parameters  $\delta, \varepsilon$  by  $u_\varepsilon$  and let us substitute  $v := u_\varepsilon$  in (3.2). We have

$$C_0 \|u_\varepsilon\|_{1,\Omega}^2 \leq \langle T u_\varepsilon, u_\varepsilon \rangle = L(u_\varepsilon) \leq \|L\|_{-1} \|u_\varepsilon\|_{1,\Omega}$$

so that

$$\|u_\varepsilon\|_{1,\Omega} \leq \|L\|_{-1}/C_0 \quad \forall \varepsilon > 0.$$

There exists an element  $\omega \in \mathbf{V}_h$  and a sequence  $\{\varepsilon_k\}, k \rightarrow \infty$ , such that  $\varepsilon_k \rightarrow 0$  and  $u_k \rightarrow \omega$  hold for  $u_k := u_{\varepsilon_k}$ .

The equation (3.2) is equivalent to the variational inequality

$$a(u_k, v - u_k) + \Phi_\delta(u_k, v - u_k) + j_{\delta,\varepsilon_k}(u_k, v) - j_{\delta,\varepsilon_k}(u_k, u_k) \geq L(v - u_k) \quad \forall v \in \mathbf{V}_h.$$

Let us pass to the limit with  $k \rightarrow \infty$  and use Lemma 3.2. Thus we obtain

$$(3.5) \quad a(u_k, v - u_k) \rightarrow a(\omega, v - \omega), \quad L(v - u_k) \rightarrow L(v - \omega),$$

$$\Phi_\delta(u_k, v - u_k) = \langle Q u_k, v - u_k \rangle \rightarrow \langle Q \omega, v - \omega \rangle = \Phi_\delta(\omega, v - \omega).$$

Next, we may write

$$\begin{aligned} & |j_{\delta,\varepsilon_k}(u_k, v) - j_\delta(\omega, v)| \\ & \leq |j_{\delta,\varepsilon_k}(u_k, v) - j_\delta(u_k, v)| + |j_\delta(u_k, v) - j_\delta(\omega, v)| \\ & = J_1 + J_2, \\ J_1 & = \delta^{-1} \left| \int_{\Gamma_C} \Pi^h([u_{kN}]_+ \mathcal{F}(u_{kT})(\varphi_{\varepsilon_k}(v_T) - |v_T|)) \, ds \right| \\ & \leq \delta^{-1} \|\mathcal{F}\|_\infty \int_{\Gamma_C} \Pi^h(|u_{kN}| |\varphi_{\varepsilon_k}(v_T) - |v_T||) \, ds \\ & \leq C \delta^{-1} \|\mathcal{F}\|_\infty \varepsilon_k \|u_{kd}\|_\infty \rightarrow 0, \end{aligned}$$

since

$$|\varphi_{\varepsilon_k}(v_T) - |v_T|| \leq \varepsilon_k;$$

$$\begin{aligned}
J_2 &\leq \delta^{-1} \int_{\Gamma_C} |\Pi^h((u_{kN}]_+ - |\omega_N|)\mathcal{F}(u_{kT})|v_T|)| \, ds \\
&\quad + \delta^{-1} \int_{\Gamma_C} |\Pi^h([\omega_N]_+(\mathcal{F}(u_{kT}) - \mathcal{F}(\omega_T))|v_T|)| \, ds \\
&\leq C\delta^{-1} \left\{ \|\mathcal{F}\|_\infty \|u_{kd} - \omega_d\|_\infty + C_L \|\omega_d\|_\infty \sum_{j=1}^{d-1} \|u_{kj} - \omega_j\|_\infty \right\} \|v_T\|_\infty \rightarrow 0.
\end{aligned}$$

As a consequence, we get

$$(3.6) \quad j_{\delta, \varepsilon_k}(u_k, v) \rightarrow j_\delta(\omega, v).$$

In a similar way, we can write

$$\begin{aligned}
|j_{\delta, \varepsilon_k}(u_k, u_k) - j_\delta(\omega, \omega)| &\leq |j_{\delta, \varepsilon_k}(u_k, u_k) - j_{\delta, \varepsilon_k}(u_k, \omega)| + |j_{\delta, \varepsilon_k}(u_k, \omega) - j_\delta(\omega, \omega)| \\
&= J_3 + J_4.
\end{aligned}$$

From (3.6),  $J_4 \rightarrow 0$  follows immediately. Finally, we have

$$\begin{aligned}
(3.7) \quad J_3 &= \delta^{-1} \left| \int_{\Gamma_C} \Pi^h([u_{kN}]_+ \mathcal{F}(u_{kT})(\varphi_{\varepsilon_k}(u_{kT}) - \varphi_{\varepsilon_k}(\omega_T))) \, ds \right| \\
&\leq C\delta^{-1} \|\mathcal{F}\|_\infty \|u_{kd}\|_\infty \sum_{j=1}^{d-1} \|u_{kj} - \omega_j\|_\infty \rightarrow 0
\end{aligned}$$

using Lemma 3.1 and the estimate

$$|\varphi_{\varepsilon_k}(u_{kT}) - \varphi_{\varepsilon_k}(\omega_T)| \leq ||u_{kT}| - |\omega_T|| \leq |u_{kT} - \omega_T|.$$

Combining (3.5)–(3.7), we arrive at the inequality

$$a(\omega, v - \omega) + \Phi_\delta(\omega, v - \omega) + j_\delta(\omega, v) - j_\delta(\omega, \omega) \geq L(v - \omega).$$

As a consequence,  $\omega$  is a solution of the problem  $(P_\delta)$  (3.1). □

Next let us consider a solution  $u := u_\delta$  of the problem (3.1) with a parameter  $\delta$  and substitute  $v := 0$  into (3.1). Then

$$\begin{aligned}
a(u, u) + \Phi_\delta(u, u) &\leq j_\delta(u, 0) - j_\delta(u, u) + L(u), \\
\Phi_\delta(u, u) &= \delta^{-1} \int_{\Gamma_C} \Pi^h([u_N]_+^2) \, ds, \\
j_\delta(u, 0) - j_\delta(u, u) &= -j_\delta(u, u) = -\delta^{-1} \int_{\Gamma_C} \Pi^h([u_N]_+ \mathcal{F}(u_T)|u_T|) \, ds \leq 0.
\end{aligned}$$

We arrive at the estimate

$$(3.8) \quad C_0 \|u\|_{1,\Omega}^2 + \delta^{-1} \int_{\Gamma_C} \Pi^h([u_N]_+^2) \, ds \leq \|L\|_{-1} \|u\|_{1,\Omega}$$

and at

**Lemma 3.3.** *There exists a positive constant  $C$  independent of  $\delta$  and such that*

$$\|u_\delta\|_{1,\Omega} + \delta^{-1} \int_{\Gamma_C} \Pi^h([u_{\delta N}]_+^2) \, ds \leq C$$

holds for all solutions  $u_\delta$  of the problem (3.1).

*Proof.* The estimate (3.8) yields that

$$(3.9) \quad \|u_\delta\|_{1,\Omega} \leq \|L\|_{-1}/C_0,$$

and inserting this into the right-hand side of (3.8) we get

$$\delta^{-1} \int_{\Gamma_C} \Pi^h([u_{\delta N}]_+^2) \, ds \leq \|L\|_{-1}^2/C_0.$$

□

As a consequence of Lemma 3.3, there exist  $u \in \mathbf{V}_h$  and a sequence  $\{\delta_k\}$ ,  $k \rightarrow \infty$ , such that  $\delta_k \rightarrow 0$  and

$$(3.10) \quad u_k := u_{\delta_k} \rightarrow u.$$

Let us denote

$$G_k = \delta_k^{-1} [u_{kN}]_+$$

and define functionals  $\mathcal{G}_k \in (\tilde{X}_h)'$  as follows:

$$\langle \mathcal{G}_k, \psi \rangle = \int_{\Gamma_C} \Pi^h(G_k \psi) \, ds, \quad \psi \in \tilde{X}_h.$$

Each  $\mathcal{G}_k$  is linear and bounded, since

$$|\langle \mathcal{G}_k, \psi \rangle| \leq C \|G_k\|_\infty \|\psi\|_\infty.$$

Let

$$\|\mathcal{G}_k\|' = \sup \langle \mathcal{G}_k, \psi \rangle / \|\psi\|_\infty \text{ for } \psi \in \tilde{X}_h \setminus \{0\}.$$

**Lemma 3.4.** *There exists a positive constant  $C$  such that*

$$\|\mathcal{G}_k\|' \leq C \quad \forall k \geq 1.$$

*Proof.* Let us insert

$$v = u_k \pm \mathbf{R}(\psi \mathbf{n}), \quad \psi \in \tilde{X}_h$$

into (3.1), where  $\mathbf{R}$  is the mapping from (2.1). We obtain

$$(3.11) \quad a(u_k, \mathbf{R}(\psi \mathbf{n})) + \Phi_{\delta_k}(u_k, \mathbf{R}(\psi \mathbf{n})) = L(\mathbf{R}(\psi \mathbf{n})),$$

since  $(\mathbf{R}(\psi \mathbf{n}))_T = 0$  and therefore

$$j_{\delta_k}(u_k, u_k \pm \mathbf{R}(\psi \mathbf{n})) = j_{\delta_k}(u_k, u_k).$$

The equation (3.11) implies that

$$(3.12) \quad \begin{aligned} |\Phi_{\delta_k}(u_k, \mathbf{R}(\psi \mathbf{n}))| &= |L(\mathbf{R}(\psi \mathbf{n})) - a(u_k, \mathbf{R}(\psi \mathbf{n}))| \\ &\leq (\|L\|_{-1} + C_1 \|u_k\|_{1,\Omega}) \|\mathbf{R}(\psi \mathbf{n})\|_{1,\Omega} \\ &\leq C_4 \|\mathcal{R}\psi\|_{1,\Omega} \leq C_5 \|\psi\|_{0,\Gamma_C}, \end{aligned}$$

where Lemma 3.3, the definition of  $\mathbf{R}$ , the inverse inequality and Lemma 2.1 have been used. Since  $(\mathbf{R}(\psi \mathbf{n}))_N = \psi$ , (3.12) and the definition of  $\Phi_\delta$  imply that

$$|\langle \mathcal{G}_k, \psi \rangle| = |\Phi_{\delta_k}(u_k, \mathbf{R}(\psi \mathbf{n}))| \leq C_6 \|\psi\|_\infty,$$

where  $C_6$  does not depend on  $\delta$ . □

*Proof of Theorem 2.1.* By Lemma 3.4, there exist a functional  $\mathcal{G} \in (\tilde{X}_h)'$  and a subsequence  $\{\mathcal{G}_m\} \subset \{\mathcal{G}_k\}$  such that

$$(3.13) \quad \mathcal{G}_m \rightarrow \mathcal{G} \quad \text{in } (\tilde{X}_h)'.$$

Choose an arbitrary  $v \in \mathbf{K}_h$ . Since  $v_N \leq 0$  on  $\Gamma_C$ , we have

$$\begin{aligned} \Phi_{\delta_m}(u_m, v - u_m) &= \delta_m^{-1} \int_{\Gamma_C} \Pi^h([u_{mN}]_+(v_N - u_{mN})) \, ds \\ &\leq -\delta_m^{-1} \int_{\Gamma_C} \Pi^h([u_{mN}]_+ u_{mN}) \, ds \leq 0. \end{aligned}$$

As a consequence, we may write

$$(3.14) \quad a(u_m, v - u_m) + j_{\delta_m}(u_m, v) - j_{\delta_m}(u_m, u_m) \geq L(v - u_m).$$

Passing to the limit with  $m \rightarrow \infty$  and using (3.10), we obtain

$$a(u_m, v - u_m) \rightarrow a(u, v - u), \quad L(v - u_m) \rightarrow L(v - u).$$

Next, we have

$$\begin{aligned} j_{\delta_m}(u_m, v) - j_{\delta_m}(u_m, u_m) &= \int_{\Gamma_C} \Pi^h(G_m \mathcal{F}(u_{mT})(|v_T| - |u_{mT}|)) \, ds \\ &= \int_{\Gamma_C} \Pi^h(G_m(\mathcal{F}(u_{mT}) - \mathcal{F}(u_T))(|v_T| - |u_{mT}|)) \, ds \\ &\quad + \int_{\Gamma_C} \Pi^h(G_m \mathcal{F}(u_T)(|v_T| - |u_T|)) \, ds \\ &\quad + \int_{\Gamma_C} \Pi^h(G_m \mathcal{F}(u_T)(|u_T| - |u_{mT}|)) \, ds \\ &= J_1 + J_2 + J_3. \end{aligned}$$

For any  $\varphi \in C(\overline{\Gamma_C})$  we may write

$$\int_{\Gamma_C} \Pi^h(G_m \varphi) \, ds = \int_{\Gamma_C} \Pi^h(G_m \Pi^h \varphi) \, ds = \langle \mathcal{G}_m, \Pi^h \varphi \rangle.$$

Therefore,  $J_1$  can be estimated as

$$\begin{aligned} |J_1| &= |\langle \mathcal{G}_m, \Pi^h((\mathcal{F}(u_{mT}) - \mathcal{F}(u_T))(|v_T| - |u_{mT}|)) \rangle| \\ &\leq C \|\Pi^h((\mathcal{F}(u_{mT}) - \mathcal{F}(u_T))(|v_T| - |u_{mT}|))\|_{\infty} \\ &\leq CC_L \| |u_{mT}| - |u_T| \|_{\infty} \| |v_T| - |u_{mT}| \|_{\infty} \rightarrow 0, \end{aligned}$$

using also Lemma 3.4.

On the basis of (3.13) we obtain

$$J_2 = \langle \mathcal{G}_m, \Pi^h(\mathcal{F}(u_T)(|v_T| - |u_T|)) \rangle \rightarrow \langle \mathcal{G}, \Pi^h(\mathcal{F}(u_T)(|v_T| - |u_T|)) \rangle.$$

Finally,

$$|J_3| = \langle \mathcal{G}_m, \Pi^h(\mathcal{F}(u_T)(|u_T| - |u_{mT}|)) \rangle \leq C \|\mathcal{F}\|_{\infty} \| |u_T| - |u_{mT}| \|_{\infty} \rightarrow 0.$$

Employing these results in the limiting process of (3.14), we arrive at

$$(3.15) \quad a(u, v - u) + \langle \mathcal{G}, \Pi^h(\mathcal{F}(u_T)(|v_T| - |u_T|)) \rangle \geq L(v - u).$$

Lemma 3.3 yields the estimate

$$\int_{\Gamma_C} \Pi^h([u_{mN}]_+^2) ds \leq C\delta_m.$$

Passing to the limit, we obtain

$$\int_{\Gamma_C} \Pi^h([u_N]_+^2) ds = 0,$$

so that  $[u_N]_+ = 0$  at all nodes of the triangulation of  $\overline{\Gamma_C}$ . Since  $u_N \in X_h|_{\Gamma_C}$ , we have  $u_N \leq 0$  everywhere on  $\Gamma_C$  and  $u \in \mathbf{K}_h$  follows.

Let us set

$$v = u_m \pm \mathbf{R}(\psi\mathbf{n}),$$

where  $\psi = \Pi^h\varphi$  and  $\varphi \in C_0(\overline{\Gamma_C})$  as in the proof of Lemma 3.4. The definition of  $\Phi_\delta$  and (3.11) imply that

$$\Phi_{\delta_m}(u_m, \mathbf{R}(\psi\mathbf{n})) = \langle \mathcal{G}_m, \psi \rangle = L(\mathbf{R}(\psi\mathbf{n})) - a(u_m, \mathbf{R}(\psi\mathbf{n})).$$

Passing to the limit and using the definition (2.1), (2.2), we obtain

$$(3.16) \quad \langle \mathcal{G}, \psi \rangle = L(\mathbf{R}(\psi\mathbf{n})) - a(u, \mathbf{R}(\psi\mathbf{n})) = - \langle t_N^h(u), \psi \rangle.$$

If we set

$$\psi = \Pi^h(\mathcal{F}(u_T)(|v_T| - |u_T|)),$$

the inequality (3.15) can be rewritten as

$$a(u, v - u) - \langle t_N^h(u), \Pi^h(\mathcal{F}(u_T)(|v_T| - |u_T|)) \rangle \geq L(v - u).$$

Thus  $u$  is a solution of the problem (2.3). The estimate

$$\|u\|_{1,\Omega} \leq \|L\|_{-1}/C_0$$

is an immediate consequence of (3.10) and (3.9).

From (3.16) we deduce

$$\begin{aligned} |\langle t_N^h(u), \psi \rangle| &\leq (\|L\|_{-1} + C_1\|u\|_{1,\Omega})\|\mathbf{R}(\psi\mathbf{n})\|_{1,\Omega} \\ &\leq (1 + C_1C_0^{-1})\|L\|_{-1}Ch_0^{-1/2}\|\psi\|_{0,\Gamma_C} \end{aligned}$$

as in the proof of Lemma (2.4). Consequently,

$$\|t_N^h(u)\|_* \leq Mh_0^{-1/2}\|L\|_{-1}$$

follows. □

**Acknowledgement.** The valuable comments of the reviewer are gratefully acknowledged.

#### *References*

- [1] *P. G. Ciarlet*: Basic Error Estimates for Elliptic Problems. In: Handbook of Numerical Analysis, vol. II (P. G. Ciarlet, J. L. Lions, eds.). North-Holland, Amsterdam, 1991.
- [2] *Ch. Eck*: Existenz und Regularität der Lösungen für Kontaktprobleme mit Reibung. Dissertation Thesis. Univ. Stuttgart, 1996.
- [3] *Ch. Eck, J. Jarušek*: Existence results for the static contact problem with Coulomb friction. *Math. Models Methods Appl. Sci.* 8 (1998), 445–468.
- [4] *J. Franců*: Monotone operators. A survey directed to applications to differential equations. *Appl. Math.* 35 (1990), 257–301.
- [5] *J. Haslinger*: Least square method for solving contact problems with friction obeying the Coulomb law. *Appl. Math.* 29 (1984), 212–224.
- [6] *J. Haslinger*: Approximation of the Signorini problem with friction, obeying Coulomb law. *Math. Methods Appl. Sci.* 5 (1983), 422–437.
- [7] *C. Licht, E. Pratt and M. Raous*: Remarks on a numerical method for unilateral contact including friction. In: *Unilateral Problems in Structural Analysis*. Birkhäuser, Basel, 1991, pp. 129–144.

*Author's address:* Ivan Hlaváček, Matematický ústav AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic.