Pavel Krejčí

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RESONANCE IN PREISACH SYSTEMS*

Pavel Krejčí, Praha and Berlin

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Abstract. This paper deals with the asymptotic behavior as $t \to \infty$ of solutions u to the forced Preisach oscillator equation $\ddot{w}(t) + u(t) = \psi(t)$, $w = u + \mathcal{P}[u]$, where $\mathcal{P}$ is a Preisach hysteresis operator, $\psi \in L^\infty(0,\infty)$ is a given function and $t \geqslant 0$ is the time variable. We establish an explicit asymptotic relation between the Preisach measure and the function ψ (or, in a more physical terminology, a balance condition between the hysteresis dissipation and the external forcing) which guarantees that every solution remains bounded for all times. Examples show that this condition is qualitatively optimal. Moreover, if the Preisach measure does not identically vanish in any neighbourhood of the origin in the Preisach half-plane and $\lim\limits_{t\to\infty} \psi(t) = 0$, then every bounded solution also asymptotically vanishes as $t \to \infty$.

Keywords: Preisach model, hysteresis, forced oscillations, asymptotic behavior

MSC 2000: 34D40, 47H30

Introduction

Time evolution in systems with hysteresis represents one of the typical issues that arise naturally in mathematical models of elastoplasticity, friction modeling, ferromagnetism, phase transitions and many others, and which are described by (ordinary or partial) differential equations containing hysteresis operators. Many different problems in this area have recently been studied, see for example [1], [2], [3], [5], [6], [8], [14], [18]. There are, however, relatively few publications devoted to the asymptotic behavior of oscillating systems, where rate-independent hysteresis is the only source of energy dissipation. We consider here the following problem: is the hysteretic dissipation strong enough to control the amplitude of forced oscillations with

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a bounded forcing at resonance? The answer is known in cases where all hysteresis loops are convex, like e.g. in classical strain-stress laws of elastoplasticity. Then it is possible to derive a second order energy inequality which gives a higher order a priori bound for the solution. For instance, in the case of quasilinear hyperbolic PDEs with convex hysteretic constitutive laws, positive results have been obtained on global existence, uniqueness and asymptotic behavior of solutions, see [14]. In ferromagnetism, the situation is more complicated. Even if we consider the simple one-dimensional Preisach model, it turns out that, because of the effect of saturation, only small amplitude loops are convex, see Fig. 1. The corresponding uniaxial hyperbolic Maxwell equations in a ferromagnetic medium have been solved only if the data are so small that the solutions do not leave the convexity domain, see [12], [14].

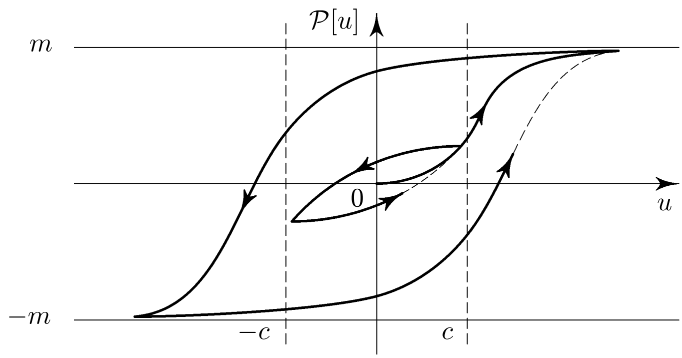

Figure 1. A hysteresis diagram with saturation limit m and convexity limit c.

The aim of this paper is to make a first step towards the investigation of large amplitude oscillations in Preisach systems outside the convexity domain. It is natural to expect that, due to the nonconvexity, large amplitude solutions to the Maxwell equations would exhibit shocks. This makes the analysis extremely complicated and even local existence of solutions for large data has not been proved yet.

As a model example, we therefore propose to study a simple hysteretic oscillator governed by a second order ODE of the form

$$\ddot{w}(t) + u(t) = \psi(t), \qquad w = u + \mathcal{P}[u], \tag{0.1}$$

where $\mathcal{P}$ is a Preisach operator (see Definition 1.5 below), $\psi \in L^\infty(0,\infty)$ is a given function and $t \geqslant 0$ is the time variable.

The paper is divided into five sections. In Section 1 we give the definition and recall some important properties of the Preisach operator. The main results are listed in Section 2. In Theorem 2.2 we establish an asymptotic condition between ψ

and the Preisach measure which is sufficient for the boundedness of every solution u of Eq. (0.1). Theorem 2.3 states that every bounded solution tends to 0 as $t \to \infty$ provided $\lim\limits_{t\to\infty} \psi(t) = 0$ and the operator $\mathcal{P}$ does not degenerate to 0 in any neighbourhood of the origin. Proposition 2.4 says that the conditions in Theorem 2.2 are, at least qualitatively, optimal, and in Proposition 2.5 we show that the precise bound for the decay rate of $\psi(t)$ such that every solution to Eq. (0.1) remains bounded for each choice of the data independently of the Preisach operator is $t^{-1/2}$. Sections 3 and 4 are devoted to the proofs of Theorems 2.2 and 2.3, respectively. In Section 5 we prove Propositions 2.4 and 2.5.

This paper is a detailed version of the contribution [13] presented at the 2nd International Symposium on Hysteresis Modelling and Micromagnetics in Perugia, June 1999.

1. The Preisach operator

We do not give an exhaustive list of publications devoted to the investigation of mathematical properties of the Preisach model introduced in [16], see e.g. [7], [9], [10], [15], [17], [18]. The approach we use here is based on an equivalent formulation (see Proposition 1.6 below) which relates the Preisach operator to variational inequalities and makes the analysis more transparent.

In what follows, we denote by C^0 the space of continuous functions $u\colon\ [0,\infty[\ \to \mathbb{R}$, endowed with the system of seminorms

$$\|u\|_{[0,t]} := \max_{0\leqslant s\leqslant t} |u(s)| \qquad \text{for } u \in C^0 \quad \text{and} \quad t \geqslant 0. \tag{1.1}$$

The basic concept in the Preisach model is the delayed switching element or *relay* with values $+1$ or -1, depending on two real parameters v (interaction field) and r (critical field of coercivity). We consider the parameter space $\mathbb{R}^2_+ := \{(r,v) \in \mathbb{R}^2\,;\ r > 0\}$ (the Preisach half-plane). The relay can be described by an operator $\mathcal{R}_{r,v}\colon\ \exp(\mathbb{R}^2_+) \times C^0 \to BV_{\mathrm{loc}}(0,\infty)$ which maps an initial magnetization distribution (represented by a subset $\Omega^0 \subset \mathbb{R}^2_+$ such that the relays are initially set to $+1$ on Ω^0 and to -1 on its complement), and a continuous input $u(t)$ (time-dependent magnetic field) into a piecewise constant output $m_{r,v}(t) = \mathcal{R}_{r,v}[\Omega^0, u](t)$ (time-dependent magnetization). It is formally defined as follows (see Fig. 2).

Let $(r,v) \in \mathbb{R}^2_+$, $\Omega^0 \subset \mathbb{R}^2_+$, $u \in C^0$ and $t \geqslant 0$ be given. We define sets

$$\begin{aligned}
A^-(0) &:= \{(r,v) \in \mathbb{R}^2_+\,;\ v - u(0) \geqslant r\},\\
A^0(0) &:= \{(r,v) \in \mathbb{R}^2_+\,;\ |v - u(0)| < r\},\\
A^+(0) &:= \{(r,v) \in \mathbb{R}^2_+\,;\ v - u(0) \leqslant -r\},\\
S(t) &:= \{\tau \in [0,t];\ |u(\tau) - v| = r\},
\end{aligned}$$

and put $\tau_t := \max S(t)$ provided $S(t) \neq \emptyset$. The value of $m_{r,v}(t)$ is given by the formula

$$m_{r,v}(0) := \begin{cases} +1 & \text{if } (r,v) \in A^+(0) \cup (A^0(0) \cap \Omega^0),\\ -1 & \text{if } (r,v) \in A^-(0) \cup (A^0(0) \setminus \Omega^0),\end{cases} \tag{1.2}$$

$$m_{r,v}(t) := \begin{cases} m_{r,v}(0) & \text{if } S(t) = \emptyset,\\ \frac{1}{r}\big(u(\tau_t) - v\big) & \text{if } S(t) \neq \emptyset.\end{cases} \tag{1.3}$$

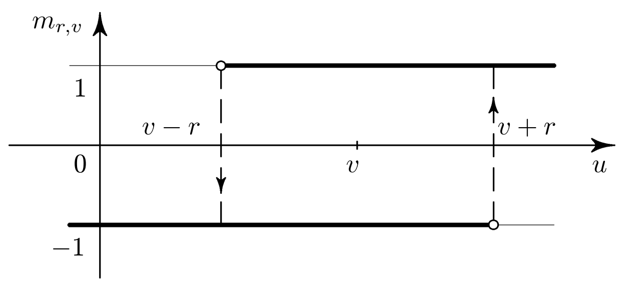

Figure 2. A diagram of the relay with thresholds $v+r$, $v-r$.

At each time $t \geqslant 0$, the half-plane $\mathbb{R}^2_+$ is split into the '+1'-region and the '−1'-region. Instead of considering the (discontinuous) evolution of each individual relay, it is more convenient to describe directly the (continuous) evolution of the interface between the two regions. With this intention (see Lemma 1.3 below), we introduce the so-called *play operator* as the solution operator of a particular evolution variational inequality.

Definition 1.1. Let Λ_M for $M > 0$ denote the set of functions $\lambda \in W^{1,\infty}(0,\infty)$ such that $|\lambda'(r)| \leqslant 1$ for a.e. $r > 0$, $\lambda(r) = 0$ for $r \geqslant M$, and let $\Lambda := \cup_{M>0}\Lambda_M$. For a given initial configuration $\lambda^0 \in \Lambda$, a given number $r > 0$ and a given input function $u \in C^0$ we define the value of the play operator $p_r\colon\ \Lambda \times C^0 \mapsto C^0 \cap BV_{\mathrm{loc}}(0,\infty)$ with threshold r as the solution $p_r[\lambda^0, u](t) := \xi_r(t)$ of the variational inequality written in the form of a Stieltjes integral

$$
(1.4) \qquad \begin{cases} u(t) - \xi_r(t) \in [-r, r] \ \ \forall t > 0, \\ \displaystyle\int_0^t \big(u(\tau) - \xi_r(\tau) - x(\tau)\big)\, \mathrm{d}\xi_r(\tau) \geqslant 0 \ \ \forall t > 0, \ \forall x \in C^0\colon \ \|x\|_{[0,t]} \leqslant r, \\ \xi_r(0) = \max\{u(0) - r, \min\{u(0) + r, \lambda^0(r)\}\} \end{cases}
$$

The output $\xi_r = p_r[\lambda^0, u]$ of the play operator admits an 'explicit' representation in each interval of monotonicity $[t_0, t_1]$ of the input function u. For $t \in [t_0, t_1]$ we have

$$
(1.5) \qquad \xi_r(t) = \begin{cases} \max\{\xi_r(t_0), u(t) - r\} & \text{if } u \text{ increases}, \\ \min\{\xi_r(t_0), u(t) + r\} & \text{if } u \text{ decreases}, \end{cases}
$$

see Fig. 3. Formula 1.5 has been traditionally used as an alternative definition of the play (cf. [10], [18], [6]) in the space of piecewise monotone continuous functions, which is then extended to C^0 by a standard density and continuity argument based on Ineq. (1.6) below.

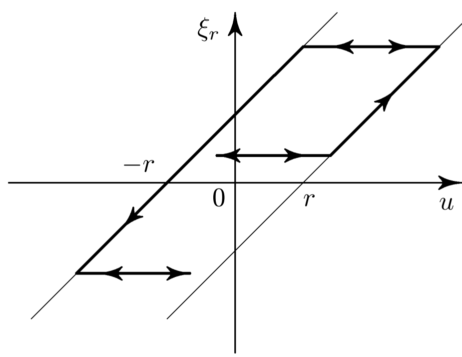

Figure 3. A diagram of the play operator.

We recall the following properties of the play (see [6], [10], [14], [18]).

Lemma 1.2.

(i) *For arbitrary $M > 0$, $\lambda^0 \in \Lambda_M$, $r > 0$, $u \in C^0$ and $t \geqslant 0$ put $\lambda_t(r) := p_r[\lambda^0, u](t)$, $M_t := \max\{M, \|u\|_{[0,t]}\}$. Then $\lambda_t \in \Lambda_{M_t}$, $\lambda_t(0+) = u(t)$.*

(ii) *For every $\lambda^0 \in \Lambda$, $r > 0$, $u, v \in C^0$ and $t \geqslant 0$ we have*

$$
(1.6) \qquad \big|p_r[\lambda^0, u](t) - p_r[\lambda^0, v](t)\big| \leqslant \|u - v\|_{[0,t]}.
$$

(iii) *For every* $\lambda^0 \in \Lambda$, $r > 0$ *and* $u \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ *we have* $\xi_r = p_r[\lambda^0, u] \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ *and the identities*

$$\dot\xi_r \dot u = \dot\xi_r^2, \quad \dot\xi_r(u - \xi_r) = r|\dot\xi_r| \tag{1.7}$$

hold for a.e. $t > 0$.

The relation between the two-parametric system $\mathcal{R}_{r,v}$ of relays and the one-parametric system p_r of plays is given in Lemma 1.3 below which was proved in [11].

Lemma 1.3. *Let* $\lambda^0 \in \Lambda$ *and* $u \in C^0$ *be given. Put* $\Omega^0 := \{(r,v) \in \mathbb{R}^2_+\,;\ v < \lambda^0(r)\}$. *Then for every* $t \geqslant 0$ *and* $(r,v) \in \mathbb{R}^2_+$, $v \neq p_r(\lambda^0,u)(t)$ *we have*

$$\mathcal{R}_{r,v}[\Omega^0, u](t) = \begin{cases} +1 & \text{if } v < p_r(\lambda^0,u)(t), \\ -1 & \text{if } v > p_r(\lambda^0,u)(t). \end{cases} \tag{1.8}$$

Fig. 4 provides an illustration to Lemma 1.3. For a fixed $t^* \geqslant 0$, the curve $v = \lambda_{t^*}(r) = p_r(\lambda^0,u)(t^*)$ determines the position at time t^* of the interface in the (r,v)-plane between the region below, where all switches $m_{r,v}$ are $+1$, and the region above, where all switches are -1. The function $\lambda_{t^*} \in \Lambda$ thus describes the *memory state* of the relay system at time t^* and the set Λ represents the *state space* for the Preisach model.

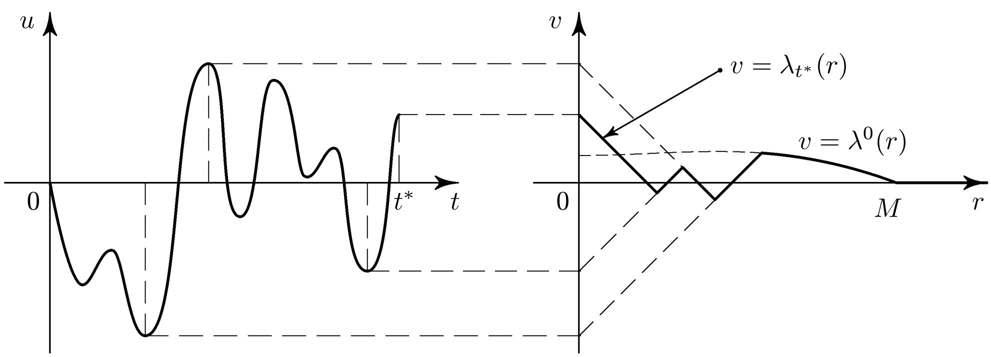

Figure 4. The memory curve $v = \lambda_{t^*}(r)$.

The Preisach operator will be defined under the following hypothesis.

Hypothesis 1.4. *Let* $\mu\colon \mathbb{R}^2_+ \to \mathbb{R}$ *be a measurable function which does not identically vanish and satisfies*

(i) $\mu(r,v) = \mu(r,-v)$ *for a.e.* $(r,v) \in \mathbb{R}^2_+$,

(ii) *there exists a function* $\beta \in L^1(0,\infty)$ *such that* $0 \leqslant \mu(r,v) \leqslant \beta(r)$ *for a.e.* $(r,v) \in \mathbb{R}^2_+$. *We denote* $b := \int_0^\infty \beta(r)\,\mathrm{d}r$.

(iii) $M > 0$ *and* $\lambda^0 \in \Lambda_M$ *are given,* $\Omega^0 = \{(r,v) \in \mathbb{R}^2_+; v < \lambda^0(r)\}$.

For $(r,v) \in \mathbb{R}^2_+$ *we put*

$$
\begin{cases} g(r,v) := \displaystyle\int_0^v \mu(r,z)\,\mathrm{d}z, \\ G(r,v) := \displaystyle\int_0^v z\mu(r,z)\,\mathrm{d}z = vg(r,v) - \int_0^v g(r,z)\,\mathrm{d}z. \end{cases} \tag{1.9}
$$

Definition 1.5. Let Hypothesis 1.4 hold and let $\mathcal{R}_{r,v}$ be the relay operator. For $u \in C^0$ and $t \geqslant 0$ we define the value of the Preisach operator $\mathcal{P}\colon C^0 \to C^0$ by the formula

$$
\mathcal{P}[u](t) := \lim_{K\to\infty} \frac{1}{2} \int_0^K \int_{-K}^K \mathcal{R}_{r,v}[\Omega^0,u](t)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r. \tag{1.10}
$$

The above definition is meaningful thanks to the symmetry condition (i) in Hypothesis 1.4. The general theory remains valid also for non-symmetric functions μ under some additional restrictions, see e.g. [6], [15], [18], [14]. Here, the symmetry assumption enables us to avoid unnecessary technical complications.

We list without proof some properties of the Preisach operator that are needed in the sequel. An interested reader can consult e.g. Section II.3 of [14]. In fact, the statements (i)–(iii) follow immediately from Lemmas 1.2 and 1.3, the proof of (iv) is however rather complicated, cf. also [6], [7].

Proposition 1.6. *Let Hypothesis* 1.4 *hold. Then*

(i) *for every* $u \in C^0$, $t \geqslant 0$ *and* $\lambda^0 \in \Lambda$ *we have*

$$
\mathcal{P}[u](t) = \int_0^\infty g\big(r, p_r[\lambda^0,u](t)\big)\,\mathrm{d}r, \tag{1.11}
$$

(ii) *for every* $u,v \in C^0$ *and* $t \geqslant 0$ *we have*

$$
|\mathcal{P}[u](t) - \mathcal{P}[v](t)| \leqslant b\|u-v\|_{[0,t]}, \tag{1.12}
$$

(iii) *for every* $u \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ *we have* $w := u + \mathcal{P}[u] \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ *and*

$$
\dot{u}^2(t) \leqslant \dot{u}(t)\dot{w}(t) \leqslant \dot{w}^2(t) \leqslant (1+b)^2\dot{u}^2(t) \quad \textit{for a.e. } t > 0, \tag{1.13}
$$

(iv) *for every* $w \in C^0$ *there exists a unique* $u \in C^0$ *such that* $w = u + \mathcal{P}[u]$ *and for every* $u,v \in C^0$, $t \geqslant 0$ *we have*

$$
|u(t) - v(t)| \leqslant 2\|(u + \mathcal{P}[u]) - (v + \mathcal{P}[v])\|_{[0,t]}. \tag{1.14}
$$

In the analysis of Eq. (0.1), the hysteresis energy dissipation plays a central role. As a consequence of Lemma 1.2, we have the following result (for more details, see Section II.4 of [14].

Proposition 1.7. *Let Hypothesis* 1.4 *hold. We introduce the potential energy operator* $\mathcal{V}$ *and the dissipation operator* $\mathcal{D}$ *by formulas*

$$\mathcal{V}[u](t) := \frac{1}{2}\, u^2(t) + \int_0^\infty G\big(r, p_r[\lambda^0, u](t)\big)\,\mathrm{d}r, \tag{1.15}$$

$$\mathcal{D}[u](t) := \int_0^\infty rg\big(r, p_r[\lambda^0, u](t)\big)\,\mathrm{d}r \tag{1.16}$$

for $u \in C^0$ *and* $t \geqslant 0$. *For* $u \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ *put* $w := u + \mathcal{P}[u]$. *Then we have* $w, \mathcal{V}[u], \mathcal{D}[u] \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ *and for a.e.* $t > 0$ *the following identity holds:*

$$\dot{w}(t)u(t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}[u](t) = \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}[u](t)\right|. \tag{1.17}$$

2. Main results

Eq. (0.1) can be written as a first order system

$$\begin{cases} \dot{w}(t) = v(t), \\ \dot{v}(t) = -(I+\mathcal{P})^{-1}[w](t) + \psi(t), \end{cases} \tag{2.1}$$

where I is the identity and $(I+\mathcal{P})^{-1}$ is the inverse operator to $I+\mathcal{P}$. The Lipschitz continuity of $(I+\mathcal{P})^{-1}$ in Proposition 1.6 (iv) ensures that system (2.1), and therefore also Eq. (0.1), admit a unique global solution for arbitrary initial data $w(0), \dot{w}(0)$. There is a one-to-one correspondence between the initial values of $w(0)$ and $u(0)$. Indeed, by Eqs. (1.4), (1.11) we have

$$w(0) = u(0) + \int_0^\infty g\big(r, \max\{u(0) - r, \min\{u(0) + r, \lambda^0(r)\}\}\big)\,\mathrm{d}r,$$

where the right-hand side of the identity is an increasing continuous function of $u(0)$.

Every solution u of Eq. (0.1) satisfies the following rough estimate.

Lemma 2.1. *Let $\psi \in L^\infty(0,\infty)$ be given. For $t \geqslant 0$ put*

$$\varrho(t) := \sup\text{ess}\,\{|\psi(s)|;\ s \geqslant t\}, \quad R(t) := \int_0^t \varrho(\tau)\,\mathrm{d}\tau.$$

For a solution u to Eq. (0.1) *we define the total energy functional*

$$E(t) := \frac{1}{2}\dot{w}^2(t) + \mathcal{V}[u](t) \quad \text{for } t \geqslant 0.$$

Then for every $t \geqslant 0$ we have

$$E(t) \leqslant \left(\sqrt{E(0)} + \frac{R(t)}{\sqrt{2}}\right)^2, \quad |u(t)| \leqslant R(t) + \sqrt{2E(0)}.$$

Proof. Multiplying Eq. (0.1) by $\dot{w}(t)$ and using Proposition 1.7, we obtain

$$\dot{E}(t) + \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}[u](t)\right| = \psi(t)\dot{w}(t) \qquad \text{a.e.,} \tag{2.2}$$

where $|\psi(t)\dot{w}(t)| \leqslant \varrho(t)\sqrt{2E(t)}$, hence $\dot{E}(t) \leqslant \varrho(t)\sqrt{2E(t)}$ a.e. This yields

$$\sqrt{E(t)} - \sqrt{E(0)} \leqslant \frac{R(t)}{\sqrt{2}} \ \ \forall t \geqslant 0.$$

By Eq. (1.15) we have $|u(t)| \leqslant \sqrt{2\mathcal{V}[u](t)} \leqslant \sqrt{2E(t)}$, and the assertion follows. □

We now state the main results of this paper.

Theorem 2.2 (Boundedness). *Let Hypothesis* 1.4 *be fulfilled and let $\psi \in L^\infty(0,\infty)$ be given. For $x > 0$ put*

$$\Phi(x) := \frac{1}{x}\int_0^x \int_0^{x-r} r\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r, \tag{2.3}$$

and assume that the functions ϱ, R from Lemma 2.1 *satisfy the implication*

$$\lim_{t\to\infty} R(t) = \infty \ \Rightarrow \ \kappa := \limsup_{t\to\infty} \frac{\varrho(t)}{\Phi(R(t))} < \frac{1}{1+b}. \tag{2.4}$$

Then every solution u of Eq. (0.1) *is bounded in* $[0,\infty[$.

Theorem 2.3 (Asymptotic decay). *Let Hypothesis* 1.4 *be fulfilled and let $\psi \in L^\infty(0,\infty)$ be given. Assume moreover that $\lim\limits_{t\to\infty}\varrho(t) = 0$ and $\Phi(x) > 0$ for every*

$x > 0$. *Then every bounded solution* u *of Eq.* (0.1) *asymptotically vanishes, that is,* $\lim_{t\to\infty} u(t) = 0$.

The statement of Theorem 2.2 is trivial if $R(t)$ is bounded; the boundedness of u then immediately follows from Lemma 2.1. On the other hand, condition (2.4) is automatically satisfied if $\lim_{x\to\infty} \Phi(x) = \infty$. In this case, according to Theorem 2.2, resonance will never occur. The intermediate cases are more interesting. Let us introduce a family of triangles

$$\Theta(x) := \{(r,v) \in\,]0,\infty[^2;\ r+v \leqslant x\} \quad \text{for } x > 0. \tag{2.5}$$

The 'initial magnetization curve' φ is given by the formula

$$\varphi(x) = \int_0^x g(r, x-r)\,\mathrm{d}r = \iint_{\Theta(x)} \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r,$$

and for every $x > y > 0$ we have

$$\Phi(x) = \frac{1}{x}\iint_{\Theta(x)} r\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \leqslant \varphi(x) - \varphi(y) + \frac{y}{x}\,\varphi(y).$$

In particular, if φ is bounded, that is, if the medium admits a finite saturation limit, then $\lim_{x\to\infty} \Phi(x) = 0$ and condition (2.4) represents an actual restriction on the decay of the function ϱ, cf. Proposition 2.4 below which shows that condition (2.4) in Theorem 2.2 is (with a small gap) substantial.

Proposition 2.4 (Optimality I). *Let Hypothesis* 1.4 *hold. Assume moreover that the function* μ *is of the form*

$$\mu(r,v) = \alpha(r+|v|) \quad \textit{for } (r,v) \in\,]0,\infty[\, \times \mathbb{R}, \tag{2.6}$$

where α *is a bounded nonnegative function which is nonincreasing in an interval* $[\bar{x},\infty[$ *and such that the function*

$$\Phi(x) = \frac{1}{x}\iint_{\Theta(x)} r\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = \frac{1}{2x}\int_0^x s^2\alpha(s)\,\mathrm{d}s$$

is nonincreasing in $[\bar{x},\infty[$ *and* $\lim_{x\to\infty}\Phi(x) = 0$. *Then there exists a function* ψ *such that the functions* ϱ, R *from Lemma* 2.1 *satisfy the condition*

$$\kappa := \limsup_{t\to\infty} \frac{\varrho(t)}{\Phi(R(t))} \leqslant 4\pi(1+b), \tag{2.7}$$

and Eq. (0.1) *admits at least one unbounded solution.*

We will see below (for instance in the proof of Lemma 3.3) the physical meaning of conditions (2.4), (2.7). They can be interpreted as a balance between the forcing (represented by ϱ) and the hysteresis dissipation (represented by Φ).

We have explicit estimates for $\varrho(t)$ if μ is e.g. of the form (2.6) with $\alpha(s) = a(1+s)^{-2-\varepsilon}$, where $a > 0$ and $\varepsilon \in]0,1[$ are given constants. Then $\Phi(x)$ is of the order $x^{-\varepsilon}$ as $x \to \infty$, and condition (2.4) is satisfied provided $\varrho(t)$ decays at least like $c(1+t)^{-p}$ for $p = \varepsilon/(1+\varepsilon)$ and $c > 0$ sufficiently small.

The following result (Proposition 2.5) characterizes the asymptotic behavior of solutions to Eq. (0.1) independently of the operator $\mathcal{P} \neq 0$. It is interesting to compare it with the case $\mathcal{P} \equiv 0$, where every solution is bounded if R is bounded, and every solution is unbounded if e.g. $\psi(t) = \varrho_k \cos t$ for $t \in [2k\pi, 2(k+1)\pi[$, $k = 0, 1, \ldots$, and $\lim\limits_{t\to\infty} R(t) = 2\pi \sum\limits_{k=0}^{\infty} \varrho_k = \infty$. We see that there is a sharp qualitative jump when passing from $\mathcal{P} \neq 0$ to $\mathcal{P} \equiv 0$.

Proposition 2.5 (Optimality II). *Let Hypothesis* 1.4 *hold. If* $\lim\limits_{t\to\infty} \sqrt{t}\varrho(t) = 0$, *then every solution u to Eq.* (0.1) *is bounded. Conversely, for every $\varepsilon > 0$ there exists a Preisach operator satisfying Hypothesis* 1.4, *an initial datum $x_0 > 0$ and a function $\psi \in L^\infty(0,\infty)$ such that* $\limsup\limits_{t\to\infty} \sqrt{t}\varrho(t) < \varepsilon$ *and the solution u to Eq.* (0.1) *with initial conditions $u(0) = -x_0$, $\dot{w}(0) = 0$ is unbounded.*

The next section is devoted to the proof of Theorem 2.2.

We start with some auxiliary results on the energy balance in intervals of monotonicity.

Lemma 3.1. *Let u be a solution of Eq.* (0.1) *and let u be monotone in an interval* $[t_0, t_1]$. *For $r > 0$ and $t \in [t_0, t_1]$ put $\lambda_t(r) := p_r[\lambda^0, u](t)$, and for $\lambda \in \Lambda$ and $v \in \mathbb{R}$ put $Q_\lambda(v) := \min\{r \geqslant 0;\ |v - \lambda(r)| = r\}$. Then we have*

$$\frac{1}{2}\big(\dot{w}^2(t_1) - \dot{w}^2(t_0) + u^2(t_1) - u^2(t_0)\big) + \int_0^{Q_{\lambda_{t_0}}(u(t_1))} \int_{\lambda_{t_0}(r)}^{u(t_1)-r} (r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = \int_{t_0}^{t_1} \psi(t)\dot{w}(t)\,\mathrm{d}t \tag{3.1}$$

if u is nondecreasing in $[t_0, t_1]$, *and*

$$\frac{1}{2}\big(\dot{w}^2(t_1) - \dot{w}^2(t_0) + u^2(t_1) - u^2(t_0)\big) + \int_0^{Q_{\lambda_{t_0}}(u(t_1))} \int_{u(t_1)+r}^{\lambda_{t_0}(r)} (r-v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = \int_{t_0}^{t_1} \psi(t)\dot{w}(t)\,\mathrm{d}t \tag{3.2}$$

if u is nonincreasing in $[t_0, t_1]$.

P r o o f. Integrating Eq. (2.2) from t_0 to t_1 we obtain

$$\frac{1}{2}\big(\dot{w}^2(t_1) - \dot{w}^2(t_0)\big) + \mathcal{V}[u](t_1) - \mathcal{V}[u](t_0) + \operatorname*{Var}_{[t_0,t_1]} \mathcal{D}[u] = \int_{t_0}^{t_1} \psi(t)\dot{w}(t)\,\mathrm{d}t. \tag{3.3}$$

The function $t \mapsto \mathcal{D}[u](t)$ is monotone in the interval $[t_0, t_1]$, hence $\operatorname{Var}_{[t_0,t_1]} \mathcal{D}[u] = |\mathcal{D}[u](t_1) - \mathcal{D}[u](t_0)|$. If u is nondecreasing in $[t_0, t_1]$, then formula (1.5) yields

$$\lambda_{t_1}(r) = \begin{cases} u(t_1) - r & \text{for } r < Q_{\lambda_{t_0}}(u(t_1)), \\ \lambda_{t_0}(r) & \text{for } r \geqslant Q_{\lambda_{t_0}}(u(t_1)), \end{cases} \tag{3.4}$$

hence

$$\begin{aligned} \mathcal{V}[u](t_1) - \mathcal{V}[u](t_0) + \operatorname*{Var}_{[t_0,t_1]} \mathcal{D}[u] &= \mathcal{V}[u](t_1) + \mathcal{D}[u](t_1) - \mathcal{V}[u](t_0) - \mathcal{D}[u](t_0) \\ &= \int_0^{Q_{\lambda_{t_0}}(u(t_1))} \big(G(r, u(t_1) - r) - G(r, \lambda_{t_0}(r)) + rg(r, u(t_1) - r) \\ &\quad - rg(r, \lambda_{t_0}(r))\big)\,\mathrm{d}r + \frac{1}{2}(u^2(t_1) - u^2(t_0)) \\ &= \int_0^{Q_{\lambda_{t_0}}(u(t_1))} \int_{\lambda_{t_0}(r)}^{u(t_1)-r} (r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r + \frac{1}{2}(u^2(t_1) - u^2(t_0)), \end{aligned} \tag{3.5}$$

and identity (3.1) follows easily. If u is nonincreasing in $[t_0, t_1]$, then similarly

$$\lambda_{t_1}(r) = \begin{cases} u(t_1) + r & \text{for } r < Q_{\lambda_{t_0}}(u(t_1)), \\ \lambda_{t_0}(r) & \text{for } r \geqslant Q_{\lambda_{t_0}}(u(t_1)), \end{cases} \tag{3.6}$$

hence

$$\begin{aligned} \mathcal{V}[u](t_1) - \mathcal{V}[u](t_0) + \operatorname*{Var}_{[t_0,t_1]} \mathcal{D}[u] &= \mathcal{V}[u](t_1) - \mathcal{D}[u](t_1) - \mathcal{V}[u](t_0) + \mathcal{D}[u](t_0) \\ &= \int_0^{Q_{\lambda_{t_0}}(u(t_1))} \big(G(r, u(t_1) + r) - G(r, \lambda_{t_0}(r)) - rg(r, u(t_1) + r) \\ &\quad + rg(r, \lambda_{t_0}(r))\big) \, \mathrm{d}r + \frac{1}{2}(u^2(t_1) - u^2(t_0)) \\ &= \int_0^{Q_{\lambda_{t_0}}(u(t_1))} \int_{u(t_1)+r}^{\lambda_{t_0}(r)} (r - v)\mu(r, v) \, \mathrm{d}v \, \mathrm{d}r + \frac{1}{2}(u^2(t_1) - u^2(t_0)) \end{aligned} \tag{3.7}$$

analogously as above, and the proof is complete. $\square$

Lemma 3.2. *Let $[t_0, t_1] \subset [0, \infty[$ be an interval such that u is monotone in $[t_0, t_1]$ and $u(t) \neq 0$ for $t \in]t_0, t_1[$. Then the following implications hold:*

(i) *If $\dot u(t) \cdot u(t) \geqslant 0$ for $t \in]t_0, t_1[$ and $|u(t_1)| > 2(1 + b)\varrho(t_0)$, then $\dot u(t) \neq 0$ for $t \in [t_0, t_1[$.*

(ii) *If $\dot u(t) \cdot u(t) \leqslant 0$ for $t \in]t_0, t_1[$ and $|u(t_0)| > 2(1 + b)\varrho(t_0)$, then $\dot u(t) \neq 0$ for $t \in]t_0, t_1]$.*

In both cases (i), (ii) *we moreover have $t_1 - t_0 < \frac{1}{2}T$, where $T = 2\pi(1 + b)$.*

Proof. (i) Assume for example that u is nonnegative and nondecreasing in $[t_0, t_1]$ (the other case is analogous). For $t \in [t_0, t_1]$ we have by Lemma 3.1 and Ineq. (1.13)

$$\begin{aligned} \dot w^2(t_1) - \dot w^2(t) + u^2(t_1) - u^2(t) + 2 \int_0^{Q_{\lambda_t}(u(t_1))} \int_{\lambda_t(r)}^{u(t_1)-r} (r + v)\mu(r, v) \, \mathrm{d}v \, \mathrm{d}r \\ \leqslant 2\varrho(t_0)(1 + b) \, (u(t_1) - u(t)). \end{aligned} \tag{3.8}$$

For the sake of simplicity, put $c := \varrho(t_0) \, (1 + b)$. For every $v \geqslant \lambda_t(r)$ we have $r + v \geqslant r + \lambda_t(r) \geqslant u(t) \geqslant 0$, and Ineqs. (3.8), (1.13) yield for $t \in [t_0, t_1]$

$$(1 + b)^2 \dot u^2(t) \geqslant \dot w^2(t) \geqslant (u(t_1) - c)^2 - (u(t) - c)^2. \tag{3.9}$$

Note that $\ddot w(t_1) < 0$; especially, u is strictly increasing in a left neighbourhood of t_1 and $u(t) < u(t_1)$ for all $t \in [t_0, t_1[$. From Ineq. (3.9) we thus obtain

$$(1 + b)^2 \dot u^2(t) \geqslant \dot w^2(t) \geqslant (u(t_1) - u(t))(u(t_1) + u(t) - 2c) > 0 \tag{3.10}$$

for all $t \in [t_0, t_1[$. On the other hand, Ineq. (3.9) yields

$$(3.11) \qquad t_1 - t_0 \leqslant (1+b) \int_{u(t_0)}^{u(t_1)} \frac{\mathrm{d}u}{\sqrt{(u(t_1)-c)^2 - (u-c)^2}}$$
$$= (1+b) \int_{\frac{u(t_0)-c}{u(t_1)-c}}^{1} \frac{\mathrm{d}v}{\sqrt{1-v^2}} < (1+b) \int_{-1}^{1} \frac{\mathrm{d}v}{\sqrt{1-v^2}} = \frac{T}{2}.$$

(ii) Consider again only one case, assuming e.g. that u is nonnegative and nonincreasing in $[t_0, t_1]$. Analogously as before we have for all $t \in [t_0, t_1]$

$$(3.12) \quad \dot{w}^2(t) - \dot{w}^2(t_0) + u^2(t) - u^2(t_0) + 2 \int_0^{Q_{\lambda_{t_0}}(u(t))} \int_{u(t)+r}^{\lambda_{t_0}(r)} (r-v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r$$
$$\geqslant -2c(u(t_0) - u(t)).$$

For $v \geqslant u(t) + r$ we have $r - v \leqslant -u(t) \leqslant 0$, hence

$$(3.13) \qquad (1+b)^2 \dot{u}^2(t) \geqslant \dot{w}^2(t) \geqslant (u(t_0)-c)^2 - (u(t)-c)^2 \quad \forall t \in [t_0, t_1],$$

and we argue as in the case (i). The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Let Hypothesis* 1.4 *hold and let* ϱ, Φ *be the functions from Theorem* 2.2. *Let* u *be a solution of Eq.* (0.1). *Let* T *be as above and assume that there exist* $t_1 > T$, $\delta > 0$ *such that*
(i) $\varrho(t_1 - T) \leqslant \delta$,
(ii) $|u(t_1)| = \|u\|_{[0,t_1]} > \max\{4\delta(1+b), M\}$.
Then there exists $t_0 \in\,]t_1 - T, t_1[$ *such that* u *is strictly monotone in* $[t_0, t_1]$, $\dot{u}(t_0) = 0$, $u(t_0) \cdot u(t_1) < 0$, $|u(t_0)| \geqslant |u(t_1)| - 2\delta(1+b)$, $\Phi(|u(t_0)|) \leqslant \delta(1+b)$.

P r o o f. It suffices to assume that $u_1 := u(t_1) > 0$; the other case is then obtained by symmetry. We then have $\dot{u}(t_1) \geqslant 0$, $\dot{w}(t_1) \geqslant 0$ and $\ddot{w}(t_1) = \psi(t_1) - u(t_1) < 0$, hence u is increasing in a left neighbourhood of t_1. Put

$$(3.14) \qquad t_0 := \min\{\tau \in [t_1 - T, t_1[;\ u \text{ increases in } [\tau, t_1]\}.$$

As in Ineq. (3.8), we have for all $t \in [t_0, t_1]$

$$(3.15) \quad \dot{w}^2(t_1) - \dot{w}^2(t) + u_1^2 - u^2(t) + 2 \int_0^{Q_{\lambda_t}(u_1)} \int_{\lambda_t(r)}^{u_1 - r} (r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r$$
$$\leqslant 2\delta(1+b)(u_1 - u(t)).$$

It follows from Lemma 1.2 (i) that $\lambda_t(r) = 0$ for every $r \geqslant u_1$ and $t \in [0, t_1]$, hence for $r \leqslant u_1$ and $t \in [t_0, t_1]$ we have $Q_{\lambda_t}(u_1) = u_1$, $|\lambda_t(r)| \leqslant u_1 - r$, and

$$(3.16) \qquad \int_{\lambda_t(r)}^{u_1-r} v\mu(r,v)\,\mathrm{d}v = \int_{\lambda_t(r)}^{|\lambda_t(r)|} v\mu(r,v)\,\mathrm{d}v + \int_{|\lambda_t(r)|}^{u_1-r} v\mu(r,v)\,\mathrm{d}v \geqslant 0.$$

In order to simplify the presentation, put again $c := \delta(1+b)$. Combining Ineqs. (3.15) and (3.16), we obtain for all $t \in [t_0, t_1]$

$$(3.17) \quad \dot{w}^2(t_1) - \dot{w}^2(t) + u_1^2 - u^2(t) + 2\int_0^{u_1}\int_{\lambda_t(r)}^{u_1-r} r\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \leqslant 2c(u_1 - u(t)).$$

Let $s_0 \in [t_0, t_1]$ be arbitrarily chosen and such that

$$(3.18) \qquad u(s_0) \geqslant -u_1 + 2c.$$

Ineq. (3.17) and Proposition 1.6 (iii) entail for $t \in [s_0, t_1]$

$$(3.19) \qquad (1+b)^2\dot{u}^2(t) \geqslant \dot{w}^2(t) \geqslant (u_1 - c)^2 - (u(t) - c)^2,$$

consequently

$$(3.20) \qquad t_1 - s_0 \leqslant (1+b)\int_{u(s_0)}^{u_1} \frac{\mathrm{d}u}{\sqrt{(u_1-c)^2 - (u-c)^2}}.$$

By hypothesis (3.18) we have $|u - c| < |u_1 - c|$ for all $u \in\,]u(s_0), u_1[$, hence the integral in (3.20) is meaningful. It can be estimated similarly as in Eq. (3.11) and we obtain $t_1 - s_0 < \frac{1}{2}T$. Moreover, from Ineq. (3.19) we infer that $\dot{u}(t) > 0$ for all $t \in\,]s_0, t_1[$, hence $t_0 \leqslant s_0$ and

$$(3.21) \qquad -u_1 \leqslant u(t_0) \leqslant -u_1 + 2c < -2c.$$

Lemma 3.2 implies that $s_0 - t_0 < \frac{1}{2}T$, hence $t_1 - t_0 < T$ and $\dot{u}(t_0) = 0$.

To complete the proof, we go back to Ineq. (3.17) for $t = t_0$. We have $\lambda_{t_0}(r) \leqslant u(t_0) + r$ for all $r \in [0, u_1]$ and $|u(t_0)| \leqslant u_1$, hence

$$(3.22) \qquad u_1^2 - u^2(t_0) + 2\int_0^{|u(t_0)|}\int_{-|u(t_0)|+r}^{|u(t_0)|-r} r\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \leqslant 2c(u_1 + |u(t_0)|),$$

where

$$(3.23) \qquad \begin{aligned}\int_0^{|u(t_0)|}\int_{-|u(t_0)|+r}^{|u(t_0)|-r} r\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r &= 2\iint_{\Theta(|u(t_0)|)} r\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \\ &= 2|u(t_0)|\Phi(|u(t_0)|).\end{aligned}$$

We therefore have $\Phi(|u(t_0)|) \leqslant c$ and Lemma 3.3 is proved. □

We now pass to the proof of Theorem 2.2.

P r o o f of Theorem 2.2. According to Lemma 2.1, it suffices to consider the case

$$\lim_{t\to\infty} R(t) = \infty. \tag{3.24}$$

Assume that there exists an unbounded solution u to Eq. (0.1). For every $n \in \mathbb{N}$ put $t_n := \min\{t \geqslant 0;\ |u(t)| \geqslant n\}$. By Lemma 2.1 we have $t_n \to \infty$ as $n \to \infty$ and for each n sufficiently large, say

$$n > n_0 \geqslant \max\{|u(0)|, M, 4\varrho(0)(1+b)\}, \tag{3.25}$$

we can apply Lemma 3.3 at the point t_n with $\delta = \varrho(t_n - T)$. Putting $t_n^0 := \max\{t \in]t_n - T, t_n[;\ \dot{u}(t) = 0\}$, we obtain from Lemma 3.3 that

$$|u(t_n^0)| \geqslant n - 2(1+b)\varrho(t_n - T), \tag{3.26}$$

$$\Phi(|u(t_n^0)|) \leqslant (1+b)\varrho(t_n - T). \tag{3.27}$$

We distinguish three cases:

(a) $\lim\limits_{t\to\infty} \varrho(t) = \varrho_0 > 0$. Then we have

$$\begin{aligned}\frac{1}{1+b} &= \varrho_0 \lim_{n\to\infty} \frac{1}{(1+b)\varrho(t_n - T)} \leqslant \varrho_0 \limsup_{n\to\infty} \frac{1}{\Phi(|u(t_n^0)|)}\\ &\leqslant \varrho_0 \limsup_{t\to\infty} \frac{1}{\Phi(R(t))} = \kappa,\end{aligned}$$

which contradicts the assumption (2.4).

(b) $\lim\limits_{t\to\infty} \varrho(t) = 0$, $\liminf\limits_{x\to\infty} \Phi(x) > 0$. The contradiction now follows directly from Ineq. (3.27).

(c) $\lim\limits_{t\to\infty} \varrho(t) = 0$, $\liminf\limits_{x\to\infty} \Phi(x) = 0$.

Let $\bar{x} > 0$ be fixed in such a way that $\Phi(\bar{x}) > 0$. For $x \geqslant \bar{x}$ put

$$\hat{\Phi}(x) := \min\{\Phi(y);\ \bar{x} \leqslant y \leqslant x\}.$$

Then $\hat{\Phi}$ is nonincreasing in $[\bar{x}, \infty[$, $\lim\limits_{x\to\infty} \hat{\Psi}(x) = 0$. We first check that we have

$$\limsup_{t\to\infty} \frac{\varrho(t)}{\hat{\Phi}(R(t))} = \kappa. \tag{3.28}$$

It is clear that $\hat{\Phi}(x) \leqslant \Phi(x)$ for every $x \geqslant \bar{x}$; it therefore suffices to prove that

$$\limsup_{t\to\infty} \frac{\varrho(t)}{\hat{\Phi}(R(t))} \leqslant \kappa. \tag{3.29}$$

Let $\varepsilon > 0$ be given. We find $t_0 > 0$ such that $R(t_0) > \bar{x}$ and $\varrho(t)/\Phi(R(t)) \leqslant \kappa + \varepsilon$ for $t \geqslant t_0$. Put $t_1 := \min\{\tau \geqslant t_0\,;\ \Phi(R(\tau)) \leqslant \hat{\Phi}(R(t_0))\}$. Then $\Phi(R(t_1)) = \hat{\Phi}(R(t_1))$ and for every $t > t_1$ there exists $\hat{t} \in [t_1, t]$ such that $\hat{\Phi}(R(t)) = \hat{\Phi}(R(\hat{t})) = \Phi(R(\hat{t}))$, hence

$$\frac{\varrho(t)}{\hat{\Phi}(R(t))} = \frac{\varrho(t)}{\Phi(R(\hat{t}))} \leqslant \frac{\varrho(\hat{t})}{\Phi(R(\hat{t}))} \leqslant \kappa + \varepsilon,$$

hence Eq. (3.28) holds.

We are now ready to complete the proof in the case (c). By Lemma 2.1 and Ineq. (3.26) we have $R(t_n^0) \geqslant |u(t_n^0)| - \sqrt{2E(0)} \geqslant n - 2(1+b)\varrho(0) - \sqrt{2E(0)}$. From the inequality $R(t_n^0) - R(t_n - T) \leqslant \int_{t_n - T}^{t_n} \varrho(t)\,\mathrm{d}t$ it follows that

$$R(t_n - T) \geqslant n - c_0, \quad \text{where } c_0 := (T + 2(1+b))\varrho(0) + \sqrt{2E(0)}, \tag{3.30}$$

hence

$$\hat{\Phi}(R(t_n - T)) \leqslant \hat{\Phi}(n - c_0) \qquad \text{for } n > \max\{n_0, c_0 + \bar{x}\}. \tag{3.31}$$

On the other hand, the function $x \mapsto x\Phi(x)$ is nondecreasing by definition. This implies in particular that

$$\Phi(|u(t_n^0)|) \geqslant \frac{n - c_0}{|u(t_n^0)|}\,\Phi(n - c_0) \geqslant \left(1 - \frac{c_0}{n}\right)\Phi(n - c_0). \tag{3.32}$$

Combining the inequalities (3.27), (3.31) and (3.32) we obtain

$$\left(1 - \frac{c_0}{n}\right)\Phi(n - c_0) \leqslant (1+b)\frac{\varrho(t_n - T)}{\hat{\Phi}(R(t_n - T))}\,\hat{\Phi}(n - c_0).$$

Using the assumption (2.4) and Eq. (3.28) we infer from the above inequality that for n sufficiently large we have $\Phi(n - c_0) < \hat{\Phi}(n - c_0)$, which contradicts the definition of $\hat{\Phi}$. Theorem 2.2 is proved. $\square$

4. Asymptotic decay

The objective of this section is to prove Theorem 2.3. Throughout the section we assume that the hypotheses of Theorem 2.3 are fulfilled and that u is a solution of Eq. (0.1).

We begin with some auxiliary results on the local behavior of solutions.

Lemma 4.1. *Let u be monotone in an interval $]t_0, t_1[$, $0 \leqslant t_0 < t_1 \leqslant \infty$, $\dot u(t_0) = \dot u(t_1-) = 0$, and let $\delta \geqslant \varrho(t_0)$ be a constant. Put $u_0 := u(t_0)$, $u_1 := u(t_1-)$. Then we have*

(i) $|u_0| + |u_1| \leqslant 2\delta(1+b)$ *if* $u_0 \cdot u_1 \geqslant 0$,

(ii) $\left.\begin{array}{l} |u_1| \leqslant (1+b)(|u_0| + 2\delta) \\ |u_0| \leqslant (1+b)(|u_1| + 2\delta) \end{array}\right\}$ *if* $u_0 \cdot u_1 < 0$

(iii) *if* $\min\{|u_0|, |u_1|\} > 2\delta(1+b)$, *then* $t_1 - t_0 < T$.

Proof. Parts (i) and (iii) follow from Lemma 3.2. To prove the inequalities in (ii), assume for instance that u is nondecreasing in $]t_0, t_1[$ (the other case is again obtained by symmetry), $u_0 < 0$, $u_1 > 0$. By Lemma (3.1) and Eq. (1.13) we have

$$\begin{aligned} -2c(u_1 - u_0) \leqslant u_1^2 - u_0^2 + 2\int_0^{Q_{\lambda_{t_0}}(u_1)} \int_{\lambda_{t_0}(r)}^{u_1 - r} (r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \\ \leqslant 2c(u_1 - u_0), \end{aligned} \tag{4.1}$$

where $c := \delta(1+b)$ and

$$\int_0^{Q_{\lambda_{t_0}}(u_1)} \int_{\lambda_{t_0}(r)}^{u_1 - r} (r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = I_1 + I_2$$

with

$$\begin{aligned} I_1 &:= \int_0^{Q_{\lambda_{t_0}}(0)} \int_{\lambda_{t_0}(r)}^{-r} (r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r, \\ I_2 &:= \int_0^{Q_{\lambda_{t_0}}(u_1)} \int_{\max\{-r,\, \lambda_{t_0}(r)\}}^{u_1 - r} (r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r. \end{aligned}$$

Since $\mu(r,v) \leqslant \beta(r)$ a.e., the integrals I_1, I_2 fulfil the conditions

$$\begin{aligned} 0 \geqslant I_1 &\geqslant \int_0^\infty \int_{u_0 - r}^{-r} (r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \geqslant -\frac{b}{2}\,u_0^2, \\ 0 \leqslant I_2 &\leqslant \int_0^\infty \int_{-r}^{u_1 - r} (r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \leqslant \frac{b}{2}\,u_1^2. \end{aligned}$$

Combining the above inequalities with Ineq. (4.1), we obtain

$$\begin{aligned} u_1^2 - 2\delta(1+b)u_1 \leqslant (1+b)(u_0^2 + 2\,\delta\,|u_0|), \\ u_0^2 - 2\delta(1+b)|u_0| \leqslant (1+b)(u_1^2 + 2\delta u_1), \end{aligned}$$

and the assertion follows easily. □

Lemma 4.2. *Let u be monotone in $[t_0, t_1]$, $\dot{u}(t_0) = \dot{u}(t_1) = 0$, $u(t_0) \cdot u(t_1) < 0$, $|u(t_0)| + |u(t_1)| > 2\delta(1 + b)$ and let $\Phi(|u(t_0)|) > \delta(1 + b)$ for some $\delta \geqslant \varrho(t_0)$. Then $Q_{\lambda_{t_0}}(u(t_1)) > |u(t_1)|$. If moreover $|\lambda_{t_0}(r) - u(t_0)| = r$ for all $r \in [0, |u(t_0)|]$, then $Q_{\lambda_{t_0}}(u(t_1)) < |u(t_0)|$ and the inequality*

$$(|u(t_0)| + \delta)^2 - (|u(t_1)| + \delta)^2 \geqslant 4|u(t_1)|\left(\frac{\Phi(|u(t_1)|)}{1 + b} - \delta\right) \tag{4.2}$$

holds.

P r o o f. We may again assume that u is nondecreasing in $[t_0, t_1]$. Put $u_1 := u(t_1) > 0$, $u_0 := u(t_0) < 0$, $q := Q_{\lambda_{t_0}}(u_1)$ and assume that $q \leqslant u_1$. Then $\lambda_{t_0}(q) = u_1 - q \geqslant 0$ and

$$q - \frac{u_1 + |u_0|}{2} = \frac{1}{2}(q + \lambda_{t_0}(0) - \lambda_{t_0}(q)) \geqslant 0,$$

hence $|u_0| \leqslant u_1$. Moreover, for $r \in [0, q]$ we have

$$u_1 - r - |\lambda_{t_0}(r)| = |\lambda_{t_0}(q)| - |\lambda_{t_0}(r)| + q - r \geqslant 0.$$

In Ineq. (4.1) we now estimate the integral using the fact that

$$\begin{aligned}\int_0^q \int_{\lambda_{t_0}(r)}^{u_1 - r} v\mu(r, v)\,\mathrm{d}v\,\mathrm{d}r = &\int_0^q \int_{\lambda_{t_0}(r)}^{|\lambda_{t_0}(r)|} v\mu(r, v)\,\mathrm{d}v\,\mathrm{d}r \\ &+ \int_0^q \int_{|\lambda_{t_0}(r)|}^{u_1 - r} v\mu(r, v)\,\mathrm{d}v\,\mathrm{d}r \geqslant 0\end{aligned}$$

and

$$\begin{aligned}\int_0^q \int_{\lambda_{t_0}(r)}^{u_1 - r} r\mu(r, v)\,\mathrm{d}v\,\mathrm{d}r &\geqslant \int_0^q \int_{\lambda_{t_0}(r)}^{|\lambda_{t_0}(r)|} r\mu(r, v)\,\mathrm{d}v\,\mathrm{d}r \\ &\geqslant \int_0^{|u_0|} \int_{u_0 + r}^{|u_0| - r} r\mu(r, v)\,\mathrm{d}v\,\mathrm{d}r = 2|u_0|\Phi(|u_0|).\end{aligned}$$

Consequently, Ineq. (4.1) yields

$$u_1^2 - u_0^2 + 4|u_0|\Phi(|u_0|) \leqslant 2\delta(1 + b)(u_1 + |u_0|). \tag{4.3}$$

By hypothesis, we have $\Phi(u_0) > \delta(1 + b)$, and from Ineq. (4.3) it follows that $u_1^2 - u_0^2 - 2\delta(1 + b)(u_1 - |u_0|) < 0$, which is a contradiction.

We therefore have $Q_{\lambda_{t_0}}(u_1) > u_1$. Assume now that $|\lambda_{t_0}(r) - u_0| = r$ for all $r \in [0, |u_0|]$. Eq. (3.6) yields $\lambda_{t_0}(r) = u_0 + r$ for $r \in [0, |u_0|]$. Let us assume that for some $t \in [t_0, t_1]$ we have $u(t) = |u_0|$. Then $|\lambda_{t_0}(r) - u(t)| = |2u_0 + r| = 2|u_0| - r$ for $r \in [0, |u_0|]$.

By the definition of $Q_{\lambda_{t_0}}$ in Lemma 3.1 we obtain $Q_{\lambda_{t_0}}(u(t)) = |u_0|$ and

$$\dot{w}^2(t) + 2\int_0^{|u_0|}\int_{-|u_0|+r}^{|u_0|-r}(r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \leqslant 4\delta(1+b)|u_0|, \tag{4.4}$$

where

$$\int_0^{|u_0|}\int_{-|u_0|+r}^{|u_0|-r}(r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = 2\iint_{\Theta(|u_0|)} r\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = 2|u_0|\Phi(|u_0|).$$

From Ineq. (4.4) we thus conclude that $\Phi(|u_0|) \leqslant \delta(1+b)$, which is a contradiction.

We therefore have $u(t) < |u_0|$ for all $t \in [t_0, t_1]$. This yields in particular $Q_{\lambda_{t_0}}(u_1) = \frac{1}{2}(u_1 + |u_0|) \in \,]u_1, |u_0|[$.

It remains to check that Ineq. (4.2) holds. From Lemma 3.1 we obtain

$$u_1^2 - u_0^2 + 2\int_0^{(u_1+|u_0|)/2}\int_{u_0+r}^{u_1-r}(r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \leqslant 2\delta(1+b)(u_1+|u_0|), \tag{4.5}$$

where (see Fig. 5)

$$\begin{aligned}\int_0^{(u_1+|u_0|)/2}\int_{u_0+r}^{u_1-r}(r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r &= \int_0^{u_1}\int_{-u_1+r}^{u_1-r}(r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \\ &+ \int_0^{(u_1+|u_0|)/2}\int_{u_0+r}^{\min\{-u_1+r,u_1-r\}}(r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r.\end{aligned} \tag{4.6}$$

The first integral on the right-hand side of Eq. (4.6) can be rewritten as

$$\int_0^{u_1}\int_{-u_1+r}^{u_1-r}(r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = 2\iint_{\Theta(u_1)} r\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = 2u_1\Phi(u_1). \tag{4.7}$$

The second integral in Eq. (4.6) contains a negative contribution corresponding to the shaded area in Fig. 5:

$$\begin{aligned}&\int_0^{(u_1+|u_0|)/2}\int_{u_0+r}^{\min\{-u_1+r,u_1-r\}}(r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \\ &\geqslant \int_0^{|u_0|/2}\int_{u_0+r}^{\min\{-u_1+r,-r\}}(r+v)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \\ &\geqslant \int_0^{|u_0|/2}\int_{u_0+r}^{\min\{-u_1+r,-r\}}(r+v)\beta(r)\,\mathrm{d}v\,\mathrm{d}r \\ &\geqslant -\frac{1}{2}\int_0^{u_1/2}\left((|u_0|-2r)^2-(u_1-2r)^2\right)\beta(r)\,\mathrm{d}r - \frac{1}{2}\int_{u_1/2}^{|u_0|/2}(|u_0|-2r)^2\beta(r)\,\mathrm{d}r \\ &\geqslant -\frac{1}{2}\left(u_0^2-u_1^2\right)\int_0^{|u_0|/2}\beta(r)\,\mathrm{d}r \geqslant -\frac{b}{2}\left(u_0^2-u_1^2\right).\end{aligned} \tag{4.8}$$

Combining Ineqs. (4.5)–(4.8) we obtain

$$u_1^2 - u_0^2 + \frac{4u_1\Phi(u_1)}{1+b} \leqslant 2\delta(u_1 + |u_0|),$$

and Ineq. (4.2) follows easily. Lemma 4.2 is proved. □

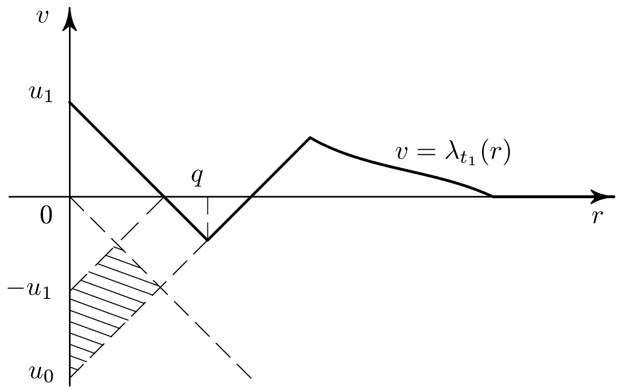

Figure 5. Memory curve at time t_1, with $q = Q_{\lambda_{t_0}}(u_1) = \frac{1}{2}(u_1 + |u_0|)$.

Theorems 2.2 and 2.3 are complementary in the sense that Theorem 2.2 refers to the asymptotics of the function Φ at infinity, while in the proof of Theorem 2.3 we will rather make use of the properties of Φ in a neighbourhood of 0. We obviously have

$$\frac{\Phi(x)}{x} \leqslant \frac{1}{x^2}\int_0^x r(x-r)\beta(r)\,\mathrm{d}r \leqslant \frac{1}{4}\int_0^x \beta(r)\,\mathrm{d}r, \tag{4.9}$$

hence in particular $\lim_{x\to 0+} \Phi(x)/x = 0$. Assuming that the hypotheses of Theorem 2.3 hold, we can therefore find $\bar{x} > 0$ such that

$$0 < \Phi(x) < x \ \ \forall x \in [0, \bar{x}] \tag{4.10}$$

and define an auxiliary function $\overline{\Phi}$ by the formula

$$\overline{\Phi}(x) := \begin{cases} \inf\{\Phi(y);\ x < y \leqslant \bar{x}\} & \text{for } \ x \in [0, \bar{x}], \\ \Phi(\bar{x}) & \text{for } \ x > \bar{x}. \end{cases} \tag{4.11}$$

Then $\overline{\Phi}\colon\ [0,\infty[\ \to [0, \Phi(\bar{x})]$ is positive, continuous and nondecreasing in $]0,\infty[$, $\overline{\Phi}(0) = 0$. Let us denote by $\overline{\Phi}^{-1}\colon\ [0, \Phi(\bar{x})[\ \to [0,\infty[$ the right-continuous inverse to $\overline{\Phi}$, that is,

$$\overline{\Phi}(x)^{-1} := \inf\{y > 0; \overline{\Phi}(y) > x\} \qquad \text{for } x \in [0, \Phi(\bar{x})[. \tag{4.12}$$

We obviously have $\overline{\Phi}^{-1}(0) = 0$, $\overline{\Phi}^{-1}(x) > x$ for all $x \in \,]0, \Phi(\bar{x})[$. We further introduce other auxiliary functions

$$\Delta(\delta) := (1+b)\big(2\delta + (1+b)\big(2\delta + \overline{\Phi}^{-1}(4\delta(1+b)^2)\big)\big), \tag{4.13}$$

$$\Gamma(\delta) := \overline{\Phi}^{-1}(4\delta(1+b)^2) \tag{4.14}$$

defined for $\delta \in [0, \bar{\delta}[$, $\bar{\delta} := \Phi(\bar{x})/(4(1+b)^2)$. Both Δ and Γ are increasing and right-continuous in their domain of definition, $\Delta(0) = \Gamma(0) = 0$. The following lemma illustrates the meaning of the functions Δ and Γ.

Lemma 4.3. *Let $\delta \in \,]0, \bar{\delta}[$ be given such that $\Delta(\delta) \leqslant \bar{x}$, and assume that there exists $t^* \geqslant 0$ such that*

$$\varrho(t^*) \leqslant \delta, \quad \dot{u}(t^*) = 0, \quad |u(t^*)| \leqslant \Gamma(\delta). \tag{4.15}$$

Then $|u(t)| < \Delta(\delta)$ for all $t \geqslant t^*$.

P r o o f. Assume that there exists $t > t^*$ such that $|u(t)| \geqslant \Delta(\delta)$. Put

$$\tau := \min\{t \geqslant t^*; |u(t)| \geqslant \Delta(\delta)\}.$$

Then $\tau > t^*$ and $|u(t)| < \Delta(\delta)$ for $t \in [t^*, \tau[$. Put

$$t_2 := \min\{t \geqslant \tau;\ \dot{u}(t) = 0\}.$$

By Lemma 3.2, the definition of t_2 is meaningful. Indeed, we put $t_2 := \tau$ if $\dot{u}(\tau) = 0$; otherwise we have $\dot{u}(t) \cdot u(t) > 0$ and $|u(t)| > \Delta(\delta)$ in a right neighbourhood of τ. From Lemma 3.2 it follows that we have $t_2 \in [\tau, \tau + \frac{1}{2}T]$ and

$$|u(t_2)| \geqslant \Delta(\delta), \quad \dot{u}(t_2) = 0, \quad u(t_2) \cdot u(\tau) > 0.$$

We now continue backwards, putting

$$t_1 := \min\{t \in [t^*, t_2[; u \text{ is monotone in } [t, t_2]\}.$$

Then $\dot{u}(t_1) = 0$ and from Lemma 4.1 we obtain $u(t_1) \cdot u(t_2) < 0$, hence $t_1 < \tau$ and $|u(t_1)| < \Delta(\delta)$. On the other hand, Lemma 4.1 yields

$$|u(t_1)| \geqslant \frac{\Delta(\delta)}{1+b} - 2\delta > \Gamma(\delta) \geqslant |u(t^*)|. \tag{4.16}$$

We therefore have $t_1 > t^*$, and we may put

$$t_0 := \min\{t \in [t^*, t_1[;\ u \text{ is monotone in } [t, t_1]\}.$$

Then $\dot{u}(t_0) = 0$ and

$$\Delta(\delta) > |u(t_0)| \geqslant \frac{|u(t_1)|}{1+b} - 2\delta \geqslant \Gamma(\delta).$$

This yields $\Phi(|u(t_0)|) \geqslant \overline{\Phi}(\Gamma(\delta)) = 4\delta(1+b)^2 > \delta(1+b)$. From Lemma 4.2 we conclude that $Q_{\lambda_{t_0}}(u(t_1)) > |u(t_1)|$. By Eqs. (3.4), (3.6) we therefore have $|\lambda_{t_1}(r) - u(t_1)| = r$ for all $r \in [0, |u(t_1)|]$. Furthermore, from Ineq. (4.16) it follows that $\Phi(|u(t_1)|) > \delta(1+b)$. Applying again Lemma 4.2 in the interval $[t_1, t_2]$, we obtain $|u(t_2)| < |u(t_1)|$, which is a contradiction. Lemma 4.3 is proved. □

We are now ready to pass to the proof of Theorem 2.3.

P r o o f of Theorem 2.3. Let $\varepsilon > 0$ be given. The proof consists in finding $t^* \geqslant 0$ such that

$$|u(t)| \leqslant \varepsilon \ \ \forall t \geqslant t^*. \tag{4.17}$$

Let $\bar{u} \geqslant \bar{x}$ be an upper bound for the solution u, that is,

$$|u(t)| \leqslant \bar{u} \ \ \forall t \geqslant 0. \tag{4.18}$$

We fix a $\delta \in]0, \bar{\delta}[$ such that

$$\Delta(\delta) < \min\{\varepsilon, \bar{x}\} \tag{4.19}$$

$$2\delta(1+b) < \min\{\Phi(x);\ \bar{x} \leqslant x \leqslant \bar{u}\}. \tag{4.20}$$

Let $\bar{t} \geqslant 0$ be arbitrarily chosen such that $\varrho(\bar{t}) \leqslant \delta$. If $|u(t)| < \varepsilon$ for all $t > \bar{t} + \frac{1}{2}T$, then condition (4.17) holds for $t^* = \bar{t} + \frac{1}{2}T$ and we are done. If this is not the case, then there exists $\hat{t} > \bar{t} + \frac{1}{2}T$ such that $|u(\hat{t})| \geqslant \varepsilon$. We distinguish three cases.

(i) $u(\hat{t}) \cdot \dot{u}(\hat{t}) > 0$. Then $|u|$ is increasing in a neighbourhood of $\hat{t}$ and we put

$$s_0 := \sup\{t > \hat{t};\ |u| \text{ is increasing in } [\hat{t}, t]\}. \tag{4.21}$$

We are in the situation of Lemma 3.2 (i) with $t_0 = \hat{t}$ and any $t_1 \in]\hat{t}, s_0[$, and we obtain $s_0 \leqslant \hat{t} + \frac{1}{2}T$, $\dot{u}(s_0) = 0$.

(ii) $u(\hat{t}) \cdot \dot{u}(\hat{t}) < 0$. Then $|u|$ is decreasing in a neighbourhood of $\hat{t}$ and we put

$$s_0 := \inf\{t \in [\bar{t}, \hat{t}];\ |u| \text{ is decreasing in } [t, \hat{t}]\}. \tag{4.22}$$

Using Lemma 3.2 (ii) with $t_1 = \hat{t}$ and any $t_0 \in \,]s_0, \hat{t}[$, we obtain $s_0 \geqslant \hat{t} - \frac{1}{2}T$, $\dot{u}(s_0) = 0$.

(iii) $\dot{u}(\hat{t}) = 0$. Then we simply put $s_0 := \hat{t}$.

In all cases (i)–(iii) we have

$$s_0 > \bar{t}, \;\; |u(s_0)| \geqslant \varepsilon, \;\; \dot{u}(s_0) = 0, \;\; \varrho(s_0) \leqslant \delta. \tag{4.23}$$

We define recurrently a sequence $\{s_k\,;\; k = 0, 1, 2, \ldots\}$ by the formula

$$|u(s_k)| > \Gamma(\delta) \Rightarrow s_{k+1} := \sup\{t \geqslant s_k\,;\; u \text{ is monotone in } [s_k, t]\}, \tag{4.24}$$

and put

$$n := \inf\{k \in \mathbb{N};\;\; |u(s_k)| \leqslant \Gamma(\delta)\}. \tag{4.25}$$

Lemma 4.1 entails that

$$|u(s_1)| \geqslant \frac{\varepsilon}{1+b} - 2\delta > \Gamma(\delta),$$

hence, in particular, $n \geqslant 2$. Furthermore, for $k = 0, 1, \ldots, n-1$ we have

$$|u(s_k)| > \Gamma(\delta) > 4\delta(1+b)^2,$$

and from Lemma 4.1 it follows that

$$|u(s_n)| \geqslant \frac{|u(s_{n-1})|}{1+b} - 2\delta > 2\delta(1+b).$$

Again, Lemma 4.1 enables us to conclude that

$$s_k - s_{k-1} < T, \quad \dot{u}(s_k) = 0, \quad u(s_k) \cdot u(s_{k-1}) < 0 \;\; \forall k = 1, \ldots, n.$$

By Ineq. (4.20) we further have

$$\Phi(|u(s_k|) \geqslant 2\delta(1+b) \qquad \text{for } k = 0, 1, \ldots, n-1.$$

Using Lemma 4.2 successively in the intervals $[s_{k-1}, s_k]$, $k = 1, \ldots, n$, we obtain $|u(s_1)| > |u(s_2)| > \ldots > |u(s_n)|$, and

$$(|u(s_{k-1})| + \delta)^2 - (|u(s_k)| + \delta)^2 \geqslant 4|u(s_k)| \left(\frac{\Phi(|u(s_k)|)}{1+b} - \delta \right) \geqslant 16\,\delta^2(1+b)^2$$

for $k = 2, \ldots, n-1$. Summing up the above inequalities, we obtain

$$16\,\delta^2(1+b)^2(n-2) \leqslant (|u(s_1)| + \delta)^2 - (|u(s_{n-1})| + \delta)^2 \leqslant (\bar{u} + \delta)^2. \tag{4.26}$$

It now suffices to put $t^* := s_n$. We have $t^* \leqslant \hat{t} + T(n + \frac{1}{2})$, where n satisfies Ineq. (4.26). Lemma 4.3 and Ineq. (4.19) now complete the proof. $\square$

E x a m p l e 4.4. Let us consider free oscillations described by Eq. (0.1) with $\psi \equiv 0$. Repeating the argument of the proof of Theorem 2.3 above with $\delta = 0$, we construct a sequence $0 < t_1 < t_2 < \ldots$ such that $t_{k+1} - t_k < T$, $u(t_k) \cdot u(t_{k+1}) < 0$, $\dot{u}(t_k) = 0$, u is strictly monotone in $[t_k, t_{k+1}]$ and the inequality

$$u(t_k)^2 - u(t_{k+1})^2 \geqslant \frac{4|u(t_{k+1})|\Phi(|u(t_{k+1})|)}{1+b} \tag{4.27}$$

holds for all $k \in \mathbb{N}$. Assuming that $\Phi(x) > 0$ for all $x > 0$ as in Theorem 2.3, we estimate the decay rate of u by introducing auxiliary functions

$$f(x) := \left(x^2 + \frac{4x\Phi(x)}{1+b}\right)^{1/2}, \quad g(x) := x - f^{-1}(x) \quad \text{for } x \geqslant 0, \tag{4.28}$$

where f^{-1} is the inverse to f. We have $g(0) = 0$, $\lim\limits_{x \to 0+} g(x)/x = 0$, $g(x) > 0$ for $x > 0$. For $k \in \mathbb{N}$ put $x_k := |u(t_k)|$. Ineq. (4.27) can be written in the form

$$x_k - x_{k+1} \geqslant g(x_k) \quad \text{for } k \in \mathbb{N}. \tag{4.29}$$

Let us define functions

$$\bar{g}(x) := \min\{g(y);\ x \leqslant y \leqslant x_1\}, \quad \gamma(x) := \int_x^{x_1} \frac{\mathrm{d}y}{\bar{g}(y)} \quad \text{for } x \in\,]0, x_1].$$

Then γ is decreasing in $]0, x_1]$, $\gamma(0+) = +\infty$, and for all $k \in \mathbb{N}$ we have by (4.29)

$$\gamma(x_{k+1}) - \gamma(x_k) = \int_{x_{k+1}}^{x_k} \frac{\mathrm{d}y}{\bar{g}(y)} \geqslant \frac{x_k - x_{k+1}}{\bar{g}(x_k)} \geqslant 1,$$

hence $\gamma(x_n) \geqslant n - 1$ for all $n \in \mathbb{N}$. For each $t \in [t_n, t_{n+1}]$ we have $t < t_1 + nT$ and $|u(t)| \leqslant x_n$, and we conclude that $\gamma(|u(t)|) \geqslant (t - t_1 - T)/T$ for all $t > t_1 + T$. This enables us to estimate the decay rate of u by the formula

$$|u(t)| \leqslant \gamma^{-1}\left(\frac{t - t_1 - T}{T}\right) \qquad \text{for } t > t_1 + T,$$

where γ^{-1} is the inverse function to γ.

As a typical special case, assume that $\mu(r, v) \geqslant \alpha_0 > 0$ for $r + |v| \leqslant x_1$. Then $\Phi(x) \geqslant (\alpha_0/6)x^2$ for $x \leqslant x_1$, and using the fact that $\lim\limits_{x \to 0+} \Phi(x)/x = 0$, we obtain

$$\liminf_{x \to 0+} \frac{g(x)}{x^2} = \liminf_{y \to 0+} \frac{f(y) - y}{f^2(y)} = \liminf_{y \to 0+} \frac{\Phi(y)}{(1+b)y^2} \geqslant \frac{\alpha_0}{6(1+b)}.$$

Therefore there exists $c > 0$ such that $g(x) \geqslant cx^2$ for $x \in [0, x_1]$, hence

$$\gamma(x) \leqslant \frac{1}{cx} - \frac{1}{cx_1} \quad \text{for } x \in\,]0, x_1].$$

This yields $\gamma^{-1}(\tau) \leqslant x_1/(1 + cx_1\tau)$ for $\tau \geqslant 0$, hence the decay rate of $|u(t)|$ is at least of the order $1/t$, cf. also Example III.2.6 of [14].

In this section we prove Propositions 2.4 and 2.5. It is easy to see that conditions (i), (ii) of Hypothesis 1.4 are automatically satisfied in the situation of Proposition 2.4. Indeed, we may take $\beta(r) = \alpha(r)$ for $r > \bar{x}$, $\beta(r) = A := \sup\{\alpha(s);\ s \leqslant \bar{x}\}$ for $r \leqslant \bar{x}$, and we obtain

$$\int_{\bar{x}}^{\infty} \beta(r)\,\mathrm{d}r = 4\int_{\bar{x}}^{\infty} \frac{\Phi(r)}{r^2}\,\mathrm{d}r - \frac{2\Phi(\bar{x})}{\bar{x}} \leqslant \frac{2\Phi(\bar{x})}{\bar{x}} < \infty.$$

Hypothesis 1.4 (ii) thus holds with $b = A\bar{x} + 2\Phi(\bar{x})/\bar{x}$.

P r o o f of Proposition 2.4. We consider Eq. (0.1) with initial conditions $\lambda^0 \equiv 0$, $\dot{w}(0) = 0$, $u(0) = -x_0$ for some $x_0 > \max\{\bar{x}, 4\Phi(\bar{x})(1+b)\}$, and with a right-hand side ψ of the form

$$\psi(t) = \varrho(t)\operatorname{sign}\big(\dot{u}(t)\big), \tag{5.1}$$

where ϱ is a positive nonincreasing function.

We construct simultaneously the function ϱ and the solution u by induction in the following way.

Let $n \in \mathbb{N}$ be given and let us assume that there exists a sequence $0 = t_0 < t_1 < \ldots < t_n$ such that $t_k - t_{k-1} \leqslant T = 2\pi(1+b)$ and the function $(-1)^{k+1}u$ is increasing in $[t_{k-1}, t_k]$ for $k = 1, \ldots, n$, $\dot{u}(t_k) = 0$ for $k = 0, \ldots, n$, the sequence $\{x_k := (-1)^{k+1}u(t_k);\ k = 0, \ldots, n\}$ is increasing, and

$$\varrho(t) = \delta_{k-1} := 2\Phi(x_{k-1}) \quad \text{for } t \in [t_{k-1}, t_k[,\ \ k = 1, \ldots, n. \tag{5.2}$$

The induction step consists in putting

$$\varrho(t) = \delta_n := 2\Phi(x_n) \quad \text{for} \quad t \in [t_n, t_{n+1}[, \tag{5.3}$$

where t_{n+1} is to be found in such a way that the above properties hold for $k = 0, \ldots, n+1$. It suffices to assume that n is even; the opposite case is obtained by symmetry. We then have $u(t_n) = -x_n$, $\dot{w}(t_n) = 0$, $\ddot{w}(t_n+) = x_n + 2\Phi(x_n) > 0$, hence the solution u is increasing in a right neighbourhood of t_n. Put

$$t_{n+1} := \sup\{t > t_n;\ \dot{u}(t) \geqslant 0 \text{ in } [t_n, t]\}. \tag{5.4}$$

Let $t \in [t_n, t_{n+1}[$ be such that

$$u(t) < x_n. \tag{5.5}$$

Analogously as in the identity (3.7), we have

$$w(t) - w(t_n) = u(t) - u(t_n) + \int_0^{Q_{\lambda_{t_n}}(u(t))} \int_{-x_n+r}^{u(t)-r} \mu(r,v)\,\mathrm{d}v\,\mathrm{d}r,$$

and from Lemma 3.1 it follows that

$$\begin{aligned}(5.6)\qquad \dot w^2(t) + u^2(t) - x_n^2 + 2\int_0^{Q_{\lambda_{t_n}}(u(t))} \int_{-x_n+r}^{u(t)-r} (r+v-\delta_n)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r \\ = 2\,\delta_n(u(t)+x_n).\end{aligned}$$

By induction hypothesis, we have $\lambda_{t_n}(r) = \min\{0, -x_n + r\}$ for all $r \geqslant 0$, hence $Q_{\lambda_{t_n}}(u(t)) = \frac{1}{2}(u(t)+x_n)$. The function

$$f(x) := x^2 - x_n^2 + 2\int_0^{x+x_n/2} \int_{-x_n+r}^{x-r} (r+v-\delta_n)\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r - 2\,\delta_n(x+x_n)$$

satisfies $f(-x_n) = 0$, $f(x_n) = -4\Phi(x_n)(2\int_0^{x_n} s\alpha(s)\,\mathrm{d}s + x_n) < 0$ and $f'(x) = 2(x-\delta_n)\big(1+\int_0^{(x+x_n)/2} \alpha(r+|x-r|)\,\mathrm{d}r\big)$ a.e., hence $f(x) < 0$ for all $x \in \,]-x_n, x_n]$. From Eq. (5.6) it follows that

$$(5.7)\qquad \dot w^2(t) = -f(u(t)) > 0,$$

and we conclude that there exists $\bar t < t_{n+1}$ such that $u(\bar t) = x_n$. At each point $\hat t \in [\bar t, t_{n+1}[$ we can apply Lemma 3.3, which yields $\hat t - t_n < T$, hence $t_{n+1} - t_n \leqslant T$, $x_{n+1} := u(t_{n+1}) > x_n$, and the induction step is complete.

We now estimate the difference $x_{n+1} - x_n$. For all $n \in \mathbb{N} \cup \{0\}$ and $r \geqslant 0$ we have $\lambda_{t_n}(r) = (-1)^{n+1}\max\{0, x_n - r\}$. Formula (5.6) for $t = t_{n+1}$ has the form

$$\begin{aligned}(5.8)\qquad x_{n+1}^2 - x_n^2 &+ 2\int_0^{x_n}\int_{-x_n+r}^{0} (r+v-\delta_n)\alpha(r-v)\,\mathrm{d}v\,\mathrm{d}r \\ &+ 2\int_0^{x_{n+1}}\int_0^{x_{n+1}-r} (r+v-\delta_n)\alpha(r+v)\,\mathrm{d}v\,\mathrm{d}r \\ &= 2\,\delta_n(x_{n+1}+x_n),\end{aligned}$$

that is,

$$\begin{aligned}(5.9)\qquad & x_{n+1}^2 - x_n^2 + 4x_{n+1}\Phi(x_{n+1}) \\ &= 2\delta_n\left(x_{n+1} + x_n + \int_0^{x_n} \alpha(s)s\,\mathrm{d}s + \int_0^{x_{n+1}} \alpha(s)s\,\mathrm{d}s\right) \\ &\geqslant 4\Phi(x_n)(x_{n+1} + x_n + 2\Phi(x_n) + 2\Phi(x_{n+1})).\end{aligned}$$

This implies that $x_{n+1}^2 \geqslant (x_n + 2\Phi(x_n))^2$, hence

$$x_{n+1} - x_n \geqslant 2\Phi(x_n), \tag{5.10}$$

and, in particular, $\lim\limits_{n\to\infty} x_n = \infty$.

On the other hand, for every n we have $R(t_n) = \sum\limits_{k=1}^{n} (t_k - t_{k-1})\delta_{k-1} \leqslant 2T \sum\limits_{k=0}^{n-1} \Phi(x_k)$. From Ineq. (5.10) it therefore follows that

$$R(t_n) \leqslant T(x_n - x_0). \tag{5.11}$$

For n sufficiently large and $t \in [t_n, t_{n+1}[$ we thus obtain $R(t) \leqslant R(t_n) + (t - t_n)\delta_n \leqslant T\big(x_n + 2\Phi(x_n) - x_0\big)$ and

$$\frac{\varrho(t)}{\Phi(R(t))} \leqslant \frac{2\Phi(x_n)}{\Phi(T(x_n + 2\Phi(x_n) - x_0))} \leqslant 2T. \tag{5.12}$$

Proposition 2.4 is proved. □

P r o o f of Proposition 2.5. We have $\liminf\limits_{x\to\infty} x\Phi(x) > 0$ and $\lim\limits_{t\to\infty} R(t)/\sqrt{t} = 0$, hence either R is bounded and the assertion follows from Lemma 2.1, or R is unbounded and we just check that condition (2.4) of Theorem 2.2 is fulfilled. Indeed, we have

$$\limsup_{t\to\infty} \frac{\varrho(t)}{\Phi(R(t))} = \limsup_{t\to\infty} \frac{R(t)\varrho(t)}{R(t)\Phi(R(t))} = 0,$$

hence every solution is bounded according to Theorem 2.2.

Conversely, let $\varepsilon > 0$ be given. For some $\eta > 0$ (to be specified later) we consider the function μ in the form

$$\mu(r, v) = \begin{cases} 3\eta & \text{if } \ r + |v| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \geqslant 1$ we have $\Phi(x) = \eta/(2x)$, hence the assumptions of Proposition 2.4 are fulfilled with $b = 3\eta$. We choose $x_0 > 1$ and define the solution $u(t)$ to Eq. (0.1) according to the construction in the proof of Proposition 2.4. By Ineq. (5.9), the sequence of local maxima and minima $x_n = |u(t_n)|$ satisfies the inequality $x_{n+1}^2 - x_n^2 \geqslant 4x_n\Phi(x_n) = 2\eta$, hence

$$x_n \geqslant \sqrt{x_0^2 + 2\,\eta\, n} \quad \forall n \geqslant 0,$$

and we have $\varrho(t) = 2\Phi(x_n) = \eta/x_n$ for $t \in [t_n, t_{n+1}[$, $t_{n+1} - t_n \leqslant T = 2\pi(1+3\eta)$ for $n \geqslant 0$. On the other hand, Lemma 2.1 yields $R(t_n) \geqslant x_n - \sqrt{2E(0)}$, where

$$E(0) = \frac{1}{2}x_0^2 + \iint_{\Theta(x_0)} v\mu(r,v)\,\mathrm{d}v\,\mathrm{d}r = \frac{1}{2}\big(x_0^2 + \eta\big),$$

hence

$$R(t) \geqslant R(t_n) \geqslant \sqrt{x_0^2 + 2\eta n} - \sqrt{x_0^2 + \eta} \quad \text{for } t \in [t_n, t_{n+1}[,\ \ n \geqslant 0.$$

By Ineq. 5.12, we have

$$\limsup_{t\to\infty} R(t)\varrho(t) = \frac{\eta}{2} \limsup_{t\to\infty} \frac{\varrho(t)}{\Phi(R(t))} \leqslant 2\pi\eta(1+3\eta).$$

Note that for $t \in [t_n, t_{n+1}[$ we have

$$\sqrt{t}\varrho(t) \leqslant \sqrt{t_{n+1}}\varrho(t) \leqslant \frac{\sqrt{2\pi(n+1)(1+3\eta)}}{\sqrt{x_0^2 + 2\eta n} - \sqrt{x_0^2+\eta}} R(t)\varrho(t).$$

This enables us to conclude that

$$\limsup_{t\to\infty} \sqrt{t}\varrho(t) \leqslant 2(\pi(1+3\eta))^{3/2}\sqrt{\eta},$$

and for η sufficiently small we obtain the assertion. $\square$

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Author's address: Pavel Krejčí, Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstr. 39, 10117 Berlin, Germany. *On leave from*: Matematický ústav AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `krejci@math.cas.cz`, `krejci@wias-berlin.de`.