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A REMARK ON THE LOCAL LIPSCHITZ CONTINUITY OF VECTOR HYSTERESIS OPERATORS

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Abstract. It is known that the vector stop operator with a convex closed characteristic $Z$ of class $C^1$ is locally Lipschitz continuous in the space of absolutely continuous functions if the unit outward normal mapping $n$ is Lipschitz continuous on the boundary $\partial Z$ of $Z$. We prove that in the regular case, this condition is also necessary.

Keywords: variational inequality, hysteresis operators

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1. Introduction

Mathematical models of multidimensional hysteresis phenomena in elastoplasticity or ferromagnetism are often based on the variational inequality (see e.g. [1], [2], [4], [5], [10], [12], [15], [16])

$$\begin{cases}
\langle \dot{u}(t) - \dot{x}(t), x(t) - w \rangle \geq 0 \ \forall w \in Z, \\
x(t) \in Z \ \forall t \in [0, T], \\
x(0) = x^0 \in Z,
\end{cases}$$

where $u \in W^{1,1}(0, T; X)$ is a given function, $X$ a Hilbert space endowed with a scalar product $\langle \cdot, \cdot \rangle$, $Z \subset X$ is a convex closed set, $t \in [0, T]$ is the time variable and the dot denotes the derivative with respect to $t$.

The existence of a unique solution $x \in W^{1,1}(0, T; X)$ to problem (1.1) is a special case of classical results for evolution variational inequalities, cf. e.g. [3], [10].
In stochastics, inequality (1.1) is known as a special case of the *Skorokhod problem* ([8], [9]). In the theory of hysteresis operators, the solution mapping

(1.2) \[ \mathcal{S}: Z \times W^{1,1}(0, T; X) \rightarrow W^{1,1}(0, T; X): (x^0, u) \mapsto x \]

is called the *stop operator with characteristic* Z and its properties have been systematically studied (see [11], [16], [12], [13]) together with its extension to the space \( C([0, T]; X) \) of continuous functions. The dynamics described by the operator \( \mathcal{S} \) is a special case of a *sweeping process*, see [14].

Analytical properties of the stop in the space \( W^{1,1}(0, T; X) \) endowed with the norm

(1.3) \[ |u|_{1,1} := |u(0)| + \int_0^T |\dot{u}(t)| \, dt \]

depend substantially on the geometry of the characteristic Z. The operator \( \mathcal{S}: Z \times W^{1,1}(0, T; X) \rightarrow W^{1,1}(0, T; X) \) is always continuous, see Theorem I.3.12 of [12]. It was conjectured without proof in [11] that this mapping is Lipschitz continuous if Z is a polyhedron and locally Lipschitz continuous if the boundary \( \partial Z \) of Z is smooth. These statements have been rigorously proved only recently in [7] and [6], respectively. In [6], it was shown that the Lipschitz continuity of the mapping

(1.4) \[ n: \partial Z \rightarrow \partial B_1(0) \]

(by \( B_r(z) \) we denote the ball centered at \( z \in X \) with radius \( r > 0 \)), which with each \( x \in \partial Z \) associates the unit outward normal \( n(x) \) to Z at the point \( x \), is sufficient for the local Lipschitz continuity of the stop. Another proof which also yields an explicit upper bound for the Lipschitz coefficient (optimal if Z is a ball) can be found in [13] as a generalization of the technique used in [5] for the ball.

Example 3.2 of [6] shows that the stop is not necessarily locally Lipschitz continuous if the mapping \( n \) is only 1/2-Hölder continuous. The aim of this paper is to fill the gap and to prove that the local Lipschitz continuity cannot be expected if \( \partial Z \) is of class \( C^1 \) and the ratio \( |n(x) - n(y)| / |x - y|, x, y \in \partial Z \), is unbounded.

Let us note that this is not just an academic question. A precise upper bound for the Lipschitz coefficient of the stop has been substantially exploited in [5] for proving the well-posedness of constitutive laws of elastoplasticity with nonlinear kinematic hardening.
2. Main result

We consider the simplest case \( X = \mathbb{R}^2 \) and fix a convex closed set \( Z \subset X \) of class \( C^1 \) in such a way that there exists a point \( x^* \in \partial Z \) for which we have

\[
(2.1) \quad \lim_{x \to x^*} \frac{\|n(x) - n(x^*)\|}{\|x - x^*\|} = +\infty.
\]

By shifting and rotating the coordinate system we may assume that \( x^* = 0 \) and that there exists \( \varepsilon > 0 \) such that

\[
(2.2) \quad Z \cap ([-\varepsilon, \varepsilon]^2) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in [-\varepsilon, \varepsilon]^2; \ b \geq G(a) \right\},
\]

where \( G: [-\varepsilon, \varepsilon] \to \mathbb{R}^+ \) is a convex function, \( G(0) = 0 \), and its derivative \( g = G' \) is continuous, increasing, \( g(0) = 0 \) and \( \lim_{a \to 0^+} g(a)/a = +\infty \) (see Fig. 1).

We make the following simplifying assumptions.

**Hypothesis 2.1.**

(i) \( G: [-\varepsilon, \varepsilon] \to \mathbb{R}^+ \) is convex and even, \( G(0) = 0 \),
(ii) \( g = G' \) is increasing and concave in \([0, \varepsilon]\), \( g(0) = 0 \), \( g'(0^+) = +\infty \).

The rest of this paper is devoted to the proof of the following result.

**Theorem 2.2.** Let \( Z \subset \mathbb{R}^2 \) be a convex closed set satisfying condition (2.2) and Hypothesis 2.1. Then for every \( R > 0 \) there exists a function \( u \in W^{1,1}(0, 1; \mathbb{R}^2) \) such that \( |u|_{1,1} \leq 1 \), and initial conditions \( x^0, y^0 \in Z \) such that the functions \( x = \mathcal{S}(x^0, u), y = \mathcal{S}(y^0, u) \), where \( \mathcal{S} \) is the stop operator (1.2), satisfy the inequality

\[
(2.3) \quad \int_0^1 |\dot{x}(t) - \dot{y}(t)| \, dt \geq R |x^0 - y^0|.
\]
3. Proof of Theorem 2.2

We follow the construction from Example 3.2 of [6]. Taking a smaller \( \varepsilon > 0 \) if necessary, we may assume that

\[
\varepsilon < \frac{1}{2\sqrt{2}}, \quad g(\varepsilon) < \frac{1}{\sqrt{2}}.
\]

We fix some \( a_0 \in ]0, \varepsilon[ \) (arbitrary, for the moment) and construct a sequence \( \{a_k; k \in \mathbb{N} \cup \{0\}\} \) by induction in the following way. Let \( a_0 > a_1 > \ldots > a_k > 0 \) be already given and let us consider the differential equation

\[
\dot{r}_k = \frac{1 - g(a_k - t) g(r_k)}{1 + g^2(r_k)}, \quad r_k(0) = 0,
\]

in the domain \((t, r_k) \in \mathcal{D}_k := [0, a_k] \times [0, a_k]\). The function

\[
F : (t, r_k) \mapsto \frac{1 - g(a_k - t) g(r_k)}{1 + g^2(r_k)}
\]

is continuous in \( \mathcal{D}_k \) and \( 0 < F(t, r_k) < 1 \) whenever \((t, r_k) \in \mathcal{D}_k, r_k > 0 \). Moreover, the function \( r_k \mapsto F(t, r_k) \) is decreasing in \([0, a_k]\) for every \( t \in [0, a_k] \); problem (3.2) therefore admits in \( \mathcal{D}_k \) a unique maximal solution \( r_k : [0, a_k] \to [0, a_k], 0 < \dot{r}_k(t) < 1 \) for all \( t \in ]0, a_k[\). Putting

\[
a_{k+1} := r_k(a_k)
\]

we thus have \( 0 < a_{k+1} < a_k \) and the induction step is complete. By construction, we moreover have for every \( k \in \mathbb{N} \cup \{0\}\)

\[
a_{k+1} \geq a_k \frac{1 - g^2(a_k)}{1 + g^2(a_k)} \geq a_k (1 - 2g^2(a_k)).
\]

For \( k \in \mathbb{N} \cup \{0\} \) put

\[
t_0 := 0, \quad t_{k+1} := t_k + a_k, \quad T := \sum_{k=0}^{\infty} a_k \leq \infty.
\]

We choose two points \( x^0, y^0 \in Z \) in the form

\[
x^0 := \begin{pmatrix} -a_0 \\ G(a_0) \end{pmatrix}, \quad y^0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
and define functions $\varpi, \varphi, \varrho: [0, T[ \to \mathbb{R}^2$ by the formulas
\begin{align}
\varpi(0) := 0, \quad \varphi(0) := x^0, \quad \varrho(0) := y^0,
\end{align}
\begin{align}
\bar{\varpi}(t) := \begin{cases} 
\varpi(t_j) + \left( G(t_{j+1} - t) - G(a_j) \right) & \text{for } t \in ]t_j, t_{j+1}[, \ j \text{ even}, \\
\varphi(t_j) + \left( G(t_{j+1} - t) - G(a_j) \right) & \text{for } t \in ]t_j, t_{j+1}[, \ j \text{ odd},
\end{cases}
\end{align}
\begin{align}
\varphi(t) := \begin{cases} 
\varphi(t_j) + \bar{\varpi}(t) - \varpi(t_j) & \text{for } t \in ]t_j, t_{j+1}[, \ j \text{ even}, \\
\left( -r_j(t - t_j) \right) & \text{for } t \in ]t_j, t_{j+1}[, \ j \text{ odd},
\end{cases}
\end{align}
\begin{align}
\bar{\varrho}(t) := \begin{cases} 
\left( r_j(t - t_j) \right) & \text{for } t \in ]t_j, t_{j+1}[, \ j \text{ even}, \\
\varrho(t_j) + \bar{\varpi}(t) - \varpi(t_j) & \text{for } t \in ]t_j, t_{j+1}[, \ j \text{ odd},
\end{cases}
\end{align}
where $r_j: [0, a_j] \to [0, a_{j+1}]$ is the solution of equation (3.2) for $j \in \mathbb{N} \cup \{0\}$.

Let us check by induction that we have
\begin{align}
\varpi = \mathcal{I}(x^0, \varpi), \quad \varrho = \mathcal{I}(y^0, \varpi) \quad \text{in } [0, T[.
\end{align}
Assume that identities (3.11) hold for $t \in [0, t_k]$, and let for instance $k$ be even, $k \geq 0$ (the case ‘$k$ odd’ is analogous). For $k \geq 2$ we have
\begin{align}
\varphi(t_k) = \begin{pmatrix} -r_{k-1}(t_k - t_{k-1}) \\ G(r_{k-1}(t_k - t_{k-1})) \end{pmatrix} = \begin{pmatrix} -a_k \\ G(a_k) \end{pmatrix},
\end{align}
\begin{align}
\varrho(t_k) = \begin{pmatrix} r_{k-2}(t_{k-1} - t_{k-2}) \\ G(r_{k-2}(t_{k-1} - t_{k-2})) \end{pmatrix} - \begin{pmatrix} a_{k-1} \\ G(a_{k-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{align}
for $k = 0$ the above identities hold by the choice (3.6), (3.7) of initial conditions. For $t \in ]t_k, t_{k+1}]$ we have by definition
\[ \bar{\varphi}(t) := \bar{\varphi}(t_k) + \bar{\varpi}(t) - \bar{\varpi}(t_k) = \begin{pmatrix} t - t_{k+1} \\ G(t_{k+1} - t) \end{pmatrix}, \quad \bar{\varrho}(t) := \begin{pmatrix} r_k(t - t_k) \\ G(r_k(t - t_k)) \end{pmatrix}. \]
In particular, both $\varphi, \varrho$ are absolutely continuous in $[0, t_{k+1}]$ and $\varphi(t), \varrho(t)$ belong to $Z$ for all $t \in [t_k, t_{k+1}]$. Since $\mathcal{I}(t) = \mathcal{N}(t)$ for all $t \in ]t_k, t_{k+1}[$, the function $\varpi$ is
automatically a solution of problem (1.1) in \([0, t_{k+1}]\). The same argument applies to \(g\) provided we check that the inequality

\[(3.14) \quad \langle \dot{\pi}(t) - \dot{g}(t), g(t) - w \rangle \geq 0 \quad \forall w \in Z\]

holds in \([t_k, t_{k+1}]\).

Equation (3.2) yields

\[(3.15) \quad \dot{r}_k(t - t_k) = \frac{1 - g(t_{k+1} - t) g(r_k(t - t_k))}{1 + g^2(r_k(t - t_k))} \quad \text{for } t \in ]t_k, t_{k+1}[,
]

hence

\[(3.16) \quad \dot{\pi}(t) - \dot{g}(t) = \frac{g(t_{k+1} - t) + g(r_k(t - t_k))}{\sqrt{1 + g^2(r_k(t - t_k))}} n(g(t)),
\]

where

\[(3.17) \quad n(g(t)) := \frac{1}{\sqrt{1 + g^2(r_k(t - t_k))}} \left( g(r_k(t - t_k)) \right)
\]
is the unit outward normal to \(Z\) at the point \(g(t)\) and inequality (3.14) follows from the convexity of \(Z\). We have thus proved that identities (3.11) are fulfilled.

An elementary computation yields for all \(j \in \mathbb{N} \cup \{0\}\)

\[(3.18) \quad \int_{t_j}^{t_{j+1}} |\dot{\pi}(t)| \, dt = \int_{t_j}^{t_{j+1}} \sqrt{1 + g^2(t_{j+1} - t)} \, dt
\]

\[= \int_0^{a_j} \sqrt{1 + g^2(s)} \, ds \leq \sqrt{2} \, a_j,
\]

\[(3.19) \quad \int_{t_j}^{t_{j+1}} |\dot{\pi}(t) - \dot{g}(t)| \, dt = \int_{t_j}^{t_{j+1}} \frac{g(t_{j+1} - t) + g(r_j(t - t_j))}{\sqrt{1 + g^2(r_j(t - t_j))}} \, dt
\]

\[\geq \frac{1}{\sqrt{2}} \int_{t_j}^{t_{j+1}} g(t_{j+1} - t) \, dt = \frac{1}{\sqrt{2}} \, G(a_j).
\]

The proof of Theorem 2.2 consists in choosing an appropriate value of \(a_0\) in the above construction and putting

\[(3.20) \quad u(t) := \begin{cases} \pi(t) & \text{for } t \in [0, t_n], \\ \pi(t_n) & \text{for } t \in ]t_n, 1[, \end{cases}
\]

with some \(n\) depending on \(a_0\) such that \(t_n < 1\). More precisely, we choose \(n\) to be the integer part of \(1/(\sqrt{2} \, a_0)\),

\[(3.21) \quad n := \left\lfloor \frac{1}{\sqrt{2} \, a_0} \right\rfloor,
\]

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and, according to assumption (3.1), we have

\begin{equation}
\frac{1}{2\sqrt{2}} \leq n a_0 \leq \frac{1}{\sqrt{2}}.
\end{equation}

Definition (3.5) yields

\[ t_n = \sum_{k=0}^{n-1} a_k \leq n a_0 \leq \frac{1}{\sqrt{2}} < 1, \]

hence formula (3.20) is meaningful. Inequality (3.18) yields

\begin{equation}
\left| u \right|_{1,1} = \int_0^1 |\dot{u}(t)| \, dt = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\hat{u}(t)| \, dt \leq \sqrt{2} \sum_{k=0}^{n-1} a_k \leq 1.
\end{equation}

Let now \( R > 0 \) be given. The proof will be complete if we check that inequality (2.3) holds for a suitable choice of \( a_0 \).

Let us first estimate the integral \( \int_0^1 |\dot{x}(t) - \dot{y}(t)| \, dt \) from below. We obviously have \( x = \mathfrak{x}, y = \mathfrak{y} \) in \([0, t_n], \dot{x} = \dot{y} = 0 \) in \([t_n, 1], \) consequently

\begin{equation}
\int_0^1 |\dot{x}(t) - \dot{y}(t)| \, dt = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\dot{x}(t) - \dot{y}(t)| \, dt \geq \frac{1}{\sqrt{2}} \sum_{k=0}^{n-1} G(a_k)
\end{equation}

according to inequality (3.19).

We define auxiliary functions

\begin{equation}
\varphi(s) := 2 s g^2(s) \quad \Phi(s) := \int_s^\varepsilon \frac{dr}{\varphi(r)} \quad \text{for} \ s \in [0, \varepsilon].
\end{equation}

Then \( \Phi' = -1/\varphi, \ \Phi(\varepsilon) = 0, \ \Phi(0+) = +\infty, \ \varphi(0) = 0 \) and Hypothesis 2.1 (ii) entails \( \lim_{s \to 0^+} \varphi'(s) = 0 \). Inequality (3.4) can be written in the form

\begin{equation}
a_{k+1} \geq a_k - \varphi(a_k),
\end{equation}

which implies that

\begin{equation}
\Phi(a_{k+1}) - \Phi(a_k) = \int_{a_{k+1}}^{a_k} \frac{dr}{\varphi(r)} \leq \frac{a_k - a_{k+1}}{\varphi(a_k + 1)} \leq \frac{\varphi(a_k)}{\varphi(a_k - \varphi(a_k))}
\end{equation}

for \( k \in \mathbb{N} \cup \{0\} \). Note that

\begin{equation}
\lim_{s \to 0^+} \frac{\varphi(s) - \varphi(s - \varphi(s))}{\varphi(s)} = \lim_{s \to 0^+} \frac{1}{\varphi(s)} \int_{s-\varphi(s)}^{s} \varphi'(r) \, dr = 0,
\end{equation}
hence

\[ (3.29) \quad \lim_{s \to 0^+} \frac{\varphi(s)}{\varphi(s - \varphi(s))} = 1. \]

Consequently, we can put

\[ (3.30) \quad \alpha := \sup_{s \in (0, \varepsilon]} \frac{\varphi(s)}{\varphi(s - \varphi(s))} < \infty \]

and from inequality (3.27) it follows that

\[ (3.31) \quad \Phi(a_{k+1}) - \Phi(a_k) \leq \alpha \quad \forall k \in \mathbb{N} \cup \{0\}. \]

Let \( \Phi^{-1} : \mathbb{R}^+ \to (0, \varepsilon] \) be the inverse function to \( \Phi \). Summing up the above inequalities over \( k \), we obtain

\[ (3.32) \quad a_k \geq \Phi^{-1}(\Phi(a_0) + \alpha k) \quad \forall k \in \mathbb{N} \cup \{0\}. \]

Combining relations (3.32) and (3.22), we have

\[ (3.33) \quad \sum_{k=0}^{n-1} G(a_k) \geq \sum_{k=0}^{n-1} G(\Phi^{-1}(\Phi(a_0) + \alpha k)) \geq \int_0^n G(\Phi^{-1}(\Phi(a_0) + \alpha x)) \, dx \]

\[ \geq \int_0^{\frac{1}{2\sqrt{2}a_0}} G(\Phi^{-1}(\Phi(a_0) + \alpha x)) \, dx. \]

The estimates (3.33) and (3.24) together with the elementary inequality \(|x^0 - y^0| = \sqrt{a_0^2 + G^2(a_0)} \leq 2a_0\) show that Theorem 2.2 will be proved if

\[ (3.34) \quad \limsup_{s \to 0^+} \frac{1}{s} \int_0^{\frac{1}{2\sqrt{2}a_0}} G(\Phi^{-1}(\Phi(s) + \alpha x)) \, dx = \infty, \]

that is,

\[ (3.35) \quad \limsup_{s \to 0^+} \frac{1}{s} \int_{\Phi(s)}^{\Phi(s) + \frac{\beta}{s}} G(\Phi^{-1}(y)) \, dy = \infty \quad \text{with} \quad \beta = \frac{\alpha}{2\sqrt{2}}. \]

By Hypothesis 2.1 (ii), we have \( 2G(z) \geq zg(z) \) and \( g(z) \leq g(s) \) for \( 0 < z < s < \varepsilon \), hence

\[ (3.36) \quad \frac{1}{s} \int_{\Phi(s)}^{\Phi(s) + \frac{\beta}{s}} G(\Phi^{-1}(y)) \, dy = \frac{1}{2s} \int_{\Phi^{-1}(\Phi(s) + \frac{\beta}{s})}^{s} \frac{G(z)}{zg^2(z)} \, dz \]

\[ \geq \frac{1}{4g(s)} \left( 1 - \frac{1}{s} \Phi^{-1}\left( \Phi(s) + \frac{\beta}{s} \right) \right). \]
Let us define an auxiliary function \( \psi(v) := 1/\Phi^{-1}(v) \) for \( v > 0 \). Then \( \psi(0) = 1/\varepsilon \), \( \lim_{v \to +\infty} \psi(v) = +\infty \), \( \psi \) is increasing in \( \mathbb{R}^+ \) and satisfies the differential equation

(3.37) \[ \psi'(v) = 2\psi(v) g^2\left(\frac{1}{\psi(v)}\right). \]

By the change of variables \( s = 1/\psi(v) \) we obtain

(3.38) \[ \frac{1}{s} \Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right) = \frac{\psi(v)}{\psi(v + \beta \psi(v))}. \]

According to the Mean Value Theorem, for all \( v > 0 \) we have

(3.39) \[ \frac{\psi(v + \beta \psi(v))}{\psi(v)} = 1 + \beta \psi'(m(v)) \]

for some \( m(v) \in [v, v + \beta \psi(v)] \). Using Eq. (3.37) and the fact that the function \( s \mapsto g(s)/s \) is decreasing, we obtain

(3.40) \[ \frac{\psi(v + \beta \psi(v))}{\psi(v)} = 1 + 2\beta \psi(m(v)) g^2\left(\frac{1}{\psi(m(v))}\right) \]

\[ \geq 1 + 2\beta \frac{\psi^2(v) g^2\left(\frac{1}{\psi(v)}\right)}{\psi(m(v))} \]

\[ \geq 1 + 2\beta \frac{\psi^2(v) g^2\left(\frac{1}{\psi(v)}\right)}{\psi(v + \beta \psi(v))}, \]

hence

(3.41) \[ \frac{\psi(v + \beta \psi(v))}{\psi(v)} \geq \frac{1}{2} + \left(\frac{1}{4} + 2\beta g^2\left(\frac{1}{\psi(v)}\right)\right)^{1/2} \quad \forall v > 0. \]

In terms of \( s = 1/\psi(v) \), the above inequality reads

(3.42) \[ \frac{1}{s} \Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right) \leq \left(\frac{1}{2} + \left(\frac{1}{4} + 2\beta g^2(s)\right)^{1/2}\right)^{-1} \quad \forall s \in [0, \varepsilon], \]

and we conclude that for all \( s \in [0, \varepsilon] \) we have

(3.43) \[ \frac{1}{g(s)} \left(1 - \frac{1}{s} \Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right)\right) \geq 2\beta \frac{g(s)}{s} \left(\frac{1}{2} + \left(\frac{1}{4} + 2\beta g^2(s)\right)^{1/2}\right)^{-2}. \]
Taking into account estimates (3.36) and (3.43), we see that relation (3.35) is fulfilled provided

\[ \lim_{s \to 0^+} \sup \frac{g(s)}{s} \left( \frac{1}{2} + \left( \frac{1}{4} + 2\beta \frac{g^2(s)}{s} \right)^{1/2} \right)^{-2} = +\infty. \]

We distinguish two cases.

A. \( \exists \gamma > 0: \lim_{s \to 0^+} \frac{g^2(s)}{s} \geq \gamma. \)

The function \( x \mapsto x \left( \frac{1}{2} + \left( \frac{1}{4} + x \right)^{1/2} \right)^{-2} \) is increasing for \( x > 0 \), hence

\[ \lim_{s \to 0^+} \sup \frac{g^2(s)}{s} \left( \frac{1}{2} + \left( \frac{1}{4} + 2\beta \frac{g^2(s)}{s} \right)^{1/2} \right)^{-2} \geq \gamma \left( \frac{1}{2} + \left( \frac{1}{4} + 2\beta \gamma \right)^{1/2} \right)^{-2} > 0 \]

and \( \lim_{s \to 0^+} 1/g(s) = +\infty \), which yields the assertion.

B. \( \lim_{s \to 0^+} \frac{g^2(s)}{s} = 0. \)

Then

\[ \lim_{s \to 0^+} \frac{g^2(s)}{s} \left( \frac{1}{2} + \left( \frac{1}{4} + 2\beta \frac{g^2(s)}{s} \right)^{1/2} \right)^{-2} = 1 \]

and \( \lim_{s \to 0^+} g(s)/s = +\infty \), with the same conclusion as above.

Theorem 2.2 is proved. \( \square \)

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