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# LINEAR SCHEME FOR FINITE ELEMENT SOLUTION OF NONLINEAR PARABOLIC-ELLIPTIC PROBLEMS WITH NONHOMOGENEOUS DIRICHLET BOUNDARY CONDITION

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*Abstract.* The computation of nonlinear quasistationary two-dimensional magnetic fields leads to a nonlinear second order parabolic-elliptic initial-boundary value problem. Such a problem with a nonhomogeneous Dirichlet boundary condition on a part $\Gamma_1$ of the boundary is studied in this paper. The problem is discretized in space by the finite element method with linear functions on triangular elements and in time by the implicit-explicit method (the left-hand side by the implicit Euler method and the right-hand side by the explicit Euler method). The scheme we get is linear. The strong convergence of the method is proved under the assumptions that the boundary $\partial\Omega$ is piecewise of class $C^3$ and the initial condition belongs to $L_2$ only. Strong monotonicity and Lipschitz continuity of the form $a(v,w)$ is not an assumption, but a property of this form following from its physical background.

*Keywords*: finite element method, parabolic-elliptic problems, two-dimensional electromagnetic field

*MSC 2000*: 65N30, 65M60

## 1. Introduction

For two media the computation of a nonlinear quasistationary two-dimensional electromagnetic field leads to the following nonlinear parabolic-elliptic initial-boundary value problem. Given a two-dimensional bounded domain $\Omega$ and its subdomains $\Omega_E$, $\Omega_P$ with $\overline{\Omega} = \overline{\Omega}_E \cup \overline{\Omega}_P$, $\Omega_E \cap \Omega_P = \emptyset$, $\operatorname{meas} \Omega_P > 0$ and such that $\Gamma = \partial\Omega$, $\partial\Omega_P$, $\partial\Omega_E$ are Lipschitz continuous and piecewise of class $C^3$, find a function $u\colon \overline{\Omega} \times (0,T) \to \mathbb{R}^1$ such that its restrictions $u_M := u|_{\Omega_M}$ $(M = E, P)$

satisfy the equations

$$\sigma \frac{\partial u_P}{\partial t} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\nu_P(|\operatorname{grad} u_P|)\frac{\partial u_P}{\partial x_i}\right) + f_P \quad \text{in} \quad \Omega_P \times (0,T), \tag{1}$$

$$0 = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\nu_E(|\operatorname{grad} u_E|)\frac{\partial u_E}{\partial x_i}\right) + f_E \quad \text{in} \quad \Omega_E \times (0,T), \tag{2}$$

where $0 < T < \infty$ and $\sigma = \sigma(x) > 0$, $\nu_M = \nu_M(s)$, $s = |\operatorname{grad} u_M|$, $f_M = f_M(x,t)$ are given functions. Further, $u$ should satisfy boundary conditions on $\partial\Omega$:

$$\begin{aligned} u &= \bar{u} \quad &&\text{on} \quad \Gamma_1 \times (0,T), \\ \nu\frac{\partial u}{\partial n} &= q \quad &&\text{on} \quad \Gamma_2 \times (0,T), \end{aligned}$$

where $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\operatorname{meas}_1 \Gamma_1 > 0$ and $n$ is the unit outward normal to $\partial\Omega$. The initial condition is prescribed on $\Omega_P$ only:

$$u_P(x,0) = u_0^P(x) \ \ \forall x \in \Omega_P.$$

On $\partial\Omega_P \cap \partial\Omega_E$ the function $u$ has to satisfy for $t \in (0,T)$ the so-called transition conditions

$$[u]_E^P = \left[\nu\frac{\partial u}{\partial n^*}\right]_E^P = 0,$$

where $n^*$ denotes the unit normal to $\partial\Omega_E \cap \partial\Omega_P$ oriented in a unique way and $[f]_E^P$ has the following meaning:

$$[f]_E^P := \lim_{B\to A} f(B) - \lim_{C\to A} f(C)$$

for arbitrary points $A \in \partial\Omega_P \cap \partial\Omega_E$, $B \in \Omega_P$, $C \in \Omega_E$.

We assume that the function $\bar{u}$ is so smooth that there exists a function $z$ such that

$$z \in H^1(\Omega), \ \ z_P \in H^2(\Omega_P), \ \ z_E \in H^2(\Omega_E), \ \ \operatorname{tr}(z) = \bar{u} \text{ on } \Gamma_1, \tag{3}$$

where $H^k(\Omega)$ $(k = 0,1,2,\dots)$ denotes the Sobolev space $W_2^k(\Omega)$ and $\operatorname{tr}(v)$ is the trace of the function $v \in H^1(\Omega)$ on the boundary $\partial\Omega$ (see [12], Theorem $\mathcal{P}$.73).

The function $\sigma$ has the meaning of electrical conductivity, $\nu = 1/\mu$ is the magnetic reluctivity, $f = J_{e3}$ with $J_{e3}$ the $x_3$-component of the density of the external current and $u$ is the $x_3$-component of the magnetic vector potential, $u = A_3$. In engineering applications $\sigma$ is considered piecewise constant and is equal to zero in the nonconductive parts of a machine and greater than zero in the conductive parts. We will consider $\sigma = 1$ in (1) for simplicity.

We can derive equations (1), (2) from Maxwell's equations (see [4]).

## 2. Formulation of the problem

Using Green's theorem we can reformulate the initial-boundary value problem in the following way:

Problem 2.1. Let a form $a(v, w)$ be given by the relation

$$a(v,w) = \sum_{M=E,P} a_M(v,w), \quad a_M(v,w) = \int_{\Omega_M} \nu_M(|\operatorname{grad} v|) \sum_{i=1}^{2} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}x, \tag{4}$$

where $\nu_M(s) \in C^1([0,\infty))$ $(M = E, P)$ are functions satisfying

$$0 < \gamma_M \leqslant \frac{\mathrm{d}}{\mathrm{d}s} [s\nu_M(s)] \leqslant \beta_M \ \ \forall s \in [0,\infty), \tag{5}$$

where $\beta_M > \gamma_M > 0$ are constants. Let $z$ satisfy (3) and let $u_0^P, f$ be given functions such that

$$\begin{gathered} u_0^P \in L_2(\Omega_P), \\ f_M \in L_2(I, W^1_\infty(\widetilde{\Omega}_M)), \ \ \dot{f}_M \in L_2(I, W^1_\infty(\widetilde{\Omega}_M)) \ \ (M = E, P), \end{gathered} \tag{6}$$

where $I = (0,T)$, $T > 0$, $\dot{f}_M$ denotes the strong derivative with respect to the time $t$ of the abstract function $f_M \equiv f_M(t)$ and $\widetilde{\Omega}_M$ will be specified later (see (31)).

Find an abstract function $u\colon\ I \to H^1(\Omega)$ with the properties

$$u \in L_\infty(I, H^1(\Omega)), \ \ u_P \in C(\bar{I}, L_2(\Omega_P)) \cap L_\infty(I, H^1(\Omega_P)), \tag{7}$$

$$\dot{u}_P \in L_2(I, V_P^*), \tag{8}$$

$$u_P(0) = u_0^P \in L_2(\Omega_P), \tag{9}$$

$$\operatorname{tr}(u(t)) = \operatorname{tr} z \quad \text{in} \quad L_2(\Gamma_1) \ \ \forall t \in I - E \quad (\operatorname{meas}_1 E = 0), \tag{10}$$

$$\begin{aligned} &\int_0^t \{\langle \dot{u}_P(\tau), v_P(\tau)\rangle_P + a(u(\tau), v(\tau))\} \, \mathrm{d}\tau \\ &\qquad = \int_0^t (f(\tau), v(\tau)) \, \mathrm{d}\tau \ \ \forall v \in L_2(I, V) \ \ \forall t \in \bar{I}, \end{aligned} \tag{11}$$

where we set

$$\int_0^t (f(\tau), v(\tau))\,\mathrm{d}\tau = \sum_{M=E,P} \int_0^t (f_M(\tau), v_M(\tau))_M\,\mathrm{d}\tau \tag{12}$$

and

$$V = \{v \in H^1(\Omega)\colon\ \mathrm{tr}(v) = 0 \quad \text{on} \quad \Gamma_1\},$$
$$V_P = \{v_P \in H^1(\Omega_P)\colon\ \mathrm{tr}(v_P) = 0 \quad \text{on} \quad \Gamma_1 \cap \partial\Omega_P\};$$

$(\cdot,\cdot)$ and $(\cdot,\cdot)_M$ $(M = E, P)$ denote the scalar products in the spaces $L_2(\Omega)$ and $L_2(\Omega_M)$, respectively. The symbol $V_P^*$ denotes the dual space of $V_P$ and $\langle\cdot,\cdot\rangle_P$ is the duality between $V_P^*$ and $V_P$.

Remark 2.2. For greater simplicity we consider only a homogeneous Neumann boundary condition on $\Gamma_2$. The case of a nonhomogeneous one is similar to [6].

We will define a discrete problem where the nonlinearity is removed. To this end we add to both sides of (11) the bilinear form

$$l(v,w) = \sum_{M=E,P} l_M(v,w), \quad l_M(v,w) = \Theta_M \int_{\Omega_M} \sum_{i=1}^{2} \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\,\mathrm{d}x \tag{13}$$

where $\Theta_M$ $(M = E, P)$ are positive constants satisfying the condition

$$\Theta_M > \frac{1}{2}\beta_M \tag{14}$$

and $\beta_M$ are constants from (5). Then we can write relation (11) in the form

$$\begin{aligned} &\int_0^t \{\langle \dot{u}_P(\tau), v_P(\tau)\rangle_P + l(u(\tau), v(\tau))\}\,\mathrm{d}\tau \\ &\qquad = \int_0^t \{\,\mathrm{d}(u(\tau), v(\tau)) + (f(\tau), v(\tau))\}\,\mathrm{d}\tau \ \ \forall v \in L_2(I,V) \ \ \forall t \in \bar{I} \end{aligned} \tag{15}$$

where $d(v,w)$ is defined by

$$d(v,w) = l(v,w) - a(v,w). \tag{16}$$

It can be shown that the form $a(v,w)\colon\ H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}^1$ has a potential $J(v)$, i.e. that there exists a functional $J(v)\colon\ H^1(\Omega) \to \mathbb{R}^1$ which is G-differentiable at arbitrary $v \in H^1(\Omega)$ and satisfies

$$a(v,w) = J'(v,w) \ \ \forall v,w \in H^1(\Omega), \tag{17}$$

where $J'(v,w)$ is the Gâteaux derivative of $J(v)$ at $w \in H^1(\Omega)$. Further, $J(v)$ is twice G-differentiable at arbitrary $v \in H^1(\Omega)$ and has the following properties:

$$J(0) = 0, \quad J'(0,w) = 0 \ \ \forall w \in H^1(\Omega), \tag{18}$$

$$|J''(v,w,z)| \leqslant \beta |w|_1 |z|_1 \ \ \forall v,w,z \in H^1(\Omega), \tag{19}$$

$$J''(v,w,w) \geqslant \gamma |w|_1^2 \ \ \forall v,w \in H^1(\Omega), \tag{20}$$

where $\gamma \leqslant \beta$ are positive constants not depending on $v$, $w$, $z$ and $|\cdot|_1$ is a seminorm in $H^1(\Omega)$; $J''$ denotes the second Gâteaux derivative of $J$. Zlámal proved in [15] that the form $a(v,w)$ appearing in variational problems which correspond to nonlinear quasistationary electromagnetic fields has a potential $J(v)$ with all the above presented properties that have the following consequences.

**Lemma 2.3.** *Let conditions* (17)–(20) *be satisfied. Then we have for all* $v,w,z \in H^1(\Omega)$

$$a(v, v-w) - a(w, v-w) \geqslant \gamma |v-w|_1^2, \tag{21}$$

$$|a(v,w) - a(z,w)| \leqslant \beta |v-z|_1 \, |w|_1, \tag{22}$$

$$\frac{1}{2}\gamma |v|_1^2 \leqslant J(v) \leqslant \frac{1}{2}\beta |v|_1^2, \tag{23}$$

$$a(v, v-w) \geqslant J(v) - J(w) + \frac{1}{2}\gamma |v-w|_1^2, \tag{24}$$

$$a(v, w-v) + J(v) - J(w) \geqslant -\frac{1}{2}\beta |v-w|_1^2. \tag{25}$$

Proof. For the proof see [10], p. 12. □

Let us define the functional

$$J(v) = \sum_{M=E,P} J_M(v), \quad J_M(v) = \int_{\Omega_M} F_M(|\operatorname{grad} v|) \, \mathrm{d}x$$

where

$$F_M(y) = \int_0^y s\nu_M(s) \, \mathrm{d}s \quad (M = E, P).$$

In [15] it is shown that $J(v)$ satisfies estimates (19), (20) with $\gamma = \min(\gamma_E, \gamma_P)$, $\beta = \max(\beta_E, \beta_P)$, where $\gamma_M < \beta_M$ $(M = E, P)$ are positive constants from (5).

Remark 2.4. According to definition (4) of the forms $a_M(v,w)$ $(M = E, P)$ all relations (21)–(25) are also true for the forms $a_M(v,w)$. In particular, the forms $a_M(v,w)$ are Lipschitz continuous:

$$|a_M(v,w) - a_M(z,w)| \leqslant \beta_M |v-z|_{1,M} |w|_{1,M} \ \ \forall v,w,z \in H^1(\Omega_M).$$

**Lemma 2.5.** *Let*

$$d_M(v,w) = l_M(v,w) - a_M(v,w). \tag{26}$$

*We have for all* $v, w, z \in H^1(\Omega_M)$

$$|d_M(v,w) - d_M(z,w)| \leqslant \tau_M |v-z|_{1,M} |w|_{1,M} \quad (M = E, P) \tag{27}$$

*where* $\tau_M$ *is a constant independent of* $v, w, z$ *and such that*

$$0 < \tau_M < \Theta_M. \tag{28}$$

Proof. We follow ideas from [15]. We estimate the functionals

$$L_M(v) = \int_{\Omega_M} \left\{ \frac{1}{2} \Theta_M |\operatorname{grad} v|^2 - F_M(|\operatorname{grad} v|) \right\} \mathrm{d}x \quad (M = E, P).$$

With regard to [1, Chap. 2], (13) and (4) we get

$$\begin{aligned} L'_M(v,w) &= \frac{\mathrm{d}}{\mathrm{d}\vartheta} L_M(v + \vartheta w)\big|_{\vartheta=0} \\ &= \int_{\Omega_M} \left\{ \Theta_M \sum_{i=1}^{2} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} - \nu_M(|\operatorname{grad} v|) \sum_{i=1}^{2} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \right\} \mathrm{d}x \\ &= l_M(v,w) - a_M(v,w). \end{aligned} \tag{29}$$

Thus $L_M(v)$ is the potential of $d_M(v,w)$.

Further, we see that

$$\begin{aligned} L''_M(v,w,z) &= \frac{\mathrm{d}}{\mathrm{d}\vartheta} L'_M(v + \vartheta z, w)\big|_{\vartheta=0} \\ &= \int_{\Omega_M} \left\{ \Theta_M \sum_{i=1}^{2} \frac{\partial z}{\partial x_i} \frac{\partial w}{\partial x_i} - \nu_M(\eta) \sum_{i=1}^{2} \frac{\partial z}{\partial x_i} \frac{\partial w}{\partial x_i} \right. \\ &\qquad \left. - \nu'_M(\eta)\eta^{-1} \sum_{i=1}^{2} \frac{\partial v}{\partial x_i} \frac{\partial z}{\partial x_i} \sum_{j=1}^{2} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_j} \right\} \mathrm{d}x \\ &= \int_{\Omega_M} (\operatorname{grad} z)^T \mathbf{D_M} \operatorname{grad} w \, \mathrm{d}x, \end{aligned}$$

where $\nu'_M(s) = \mathrm{d}\nu_M(s)/\mathrm{d}s$, $\eta = |\operatorname{grad} v|$ and the matrix $\mathbf{D_M}$ has the form

$$\mathbf{D_M} = \begin{pmatrix} \Theta_M - \alpha_M - \delta_M \left(\frac{\partial v}{\partial x_1}\right)^2 & -\delta_M \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2} \\ -\delta_M \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2} & \Theta_M - \alpha_M - \delta_M \left(\frac{\partial v}{\partial x_2}\right)^2 \end{pmatrix}$$

where $\alpha_M = \nu_M(\eta)$, $\delta_M = \eta^{-1}\nu'_M(\eta)$. The eigenvalues $\mu_{1,2}$ of $\mathbf{D_M}$ are of the form

$$\mu_i = \Theta_M - \alpha_M - \frac{1}{2}(\delta_M \pm |\delta_M|)\eta^2 \quad (i = 1, 2).$$

It can be shown that

$$\mu_{1,2} = \begin{cases} \Theta_M - \alpha_M = \Theta_M - \nu_M(\eta), \\ \Theta_M - \alpha_M - \delta_M\eta^2 = \Theta_M - [\eta\nu_M(\eta)]'. \end{cases} \tag{30}$$

Integration of (5) over $[0, t]$ $(t > 0)$ and the continuity of $\nu_M(s)$ yield

$$\gamma_M \leqslant \nu_M(\eta) \leqslant \beta_M \ \ \forall \eta \in [0, \infty).$$

This together with (5) and (30) implies

$$\gamma_M \leqslant \alpha_M + \frac{1}{2}(\delta_M \pm |\delta_M|)\eta^2 \leqslant \beta_M.$$

As we assume condition (14), we can prove that

$$|\mu_i| \leqslant \Theta_M - \varrho_M,$$

where $\varrho_M = \min(\gamma_M, 2(\Theta_M - \frac{1}{2}\beta_M))$, $0 < \varrho_M < \Theta_M$. Hence

$$|L''_M(v, w, z)| \leqslant \tau_M |z|_{1,M} |w|_{1,M}$$

where $0 < \tau_M = \Theta_M - \varrho_M < \Theta_M$ $(M = E, P)$. Using Taylor's theorem in the form

$$L'_M(\omega + \psi, \varphi) = L'_M(\omega, \varphi) + L''_M(\omega + \vartheta\psi, \varphi, \psi)$$

where $0 < \vartheta < 1$ and $\omega$, $\varphi$, $\psi$ are arbitrary functions from $H^1(\Omega_M)$, by (29) we obtain relation (27). $\square$

R e m a r k 2.6. Lipschitz continuity (27) and its discrete form (42) with the constant $\tau_M$ satisfying (28) will play an essential role in Theorem 4.14.

## 3. Discrete problem

Let us approximate the domain $\Omega$ by a domain $\Omega_h$ with a polygonal boundary $\partial\Omega_h$ the vertices of which lie on $\partial\Omega$. Let $\mathcal{T}_h$ be a triangulation of $\Omega_h$. This triangulation consists of two subtriangulations $\mathcal{T}_{hE}$ and $\mathcal{T}_{hP}$ such that $\mathcal{T}_h = \mathcal{T}_{hE} \cup \mathcal{T}_{hP}$, $\mathcal{T}_{hE} \cap \mathcal{T}_{hP} = \emptyset$, $\mathcal{T}_{hE}$ and $\mathcal{T}_{hP}$ are triangulations of $\Omega_{hE}$ and $\Omega_{hP}$, respectively, where $\Omega_{hM}$ is a polygonal approximation of $\Omega_M$ $(M = E, P)$. We assume that the points forming the set $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ are nodal points of $\mathcal{T}_h$. With every triangulation $\mathcal{T}_h$ we associate three parameters $h, \bar{h}$ and $\vartheta_h$ defined by

$$h = \max_{T\in\mathcal{T}_h} h_T, \quad \bar{h} = \min_{T\in\mathcal{T}_h} h_T, \quad \vartheta_h = \min_{T\in\mathcal{T}_h} \vartheta_T$$

where $h_T$ and $\vartheta_T$ are the length of the greatest side and the smallest angle, respectively, of the triangle $\overline{T} \in \mathcal{T}_h$. We restrict ourselves to triangulations satisfying the conditions

$$\begin{aligned} \vartheta_h \geqslant \vartheta_0 > 0 \ \ \forall h \in (0, h_0) \quad \vartheta_0 = \text{const}, \\ \bar{h}/h \geqslant C_0 > 0 \ \ \forall h \in (0, h_0) \quad C_0 = \text{const}\,. \end{aligned}$$

The bounded domains $\widetilde{\Omega}, \widetilde{\Omega}_E, \widetilde{\Omega}_P$ appearing in (6) satisfy

$$\widetilde{\Omega} \supset \bar{\Omega} \cup \bar{\Omega}_h, \quad \widetilde{\Omega}_M \supset \bar{\Omega}_M \cup \bar{\Omega}_{hM} \ \ \forall h \in (0, h_0). \tag{31}$$

For all $v, w \in H^1(\Omega_h)$ we define forms

$$\begin{aligned} a_h(v,w) &= \sum_{M=E,P} a_{hM}(v,w), \\ a_{hM}(v,w) &= \int_{\Omega_{hM}} \nu_M(|\operatorname{grad} v|) \sum_{i=1}^{2} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \,\mathrm{d}x. \end{aligned} \tag{32}$$

We again have

$$a_h(v,w) = J_h'(v,w) \ \ \forall v, w \in H^1(\Omega_h)$$

where

$$J_h(v) = \sum_{M=E,P} J_{hM}(v), \quad J_{hM}(v) = \int_{\Omega_{hM}} F_M(|\operatorname{grad} v|) \,\mathrm{d}x. \tag{33}$$

**Lemma 3.1.** *For all $v, w, z \in H^1(\Omega_h)$ we have*

$$a_h(v, v-w) - a_h(w, v-w) \geqslant \gamma |v-w|^2_{1,\Omega_h}, \tag{34}$$

$$|a_h(v,w) - a_h(z,w)| \leqslant \beta |v-z|_{1,\Omega_h} |w|_{1,\Omega_h}, \tag{35}$$

$$\frac{1}{2}\gamma |v|^2_{1,\Omega_h} \leqslant J_h(v) \leqslant \frac{1}{2}\beta |v|^2_{1,\Omega_h}, \tag{36}$$

$$a_h(v, v-w) \geqslant J_h(v) - J_h(w) + \frac{1}{2}\gamma |v-w|^2_{1,\Omega_h}, \tag{37}$$

$$a_h(v, w-v) + J_h(v) - J_h(w) \geqslant -\frac{1}{2}\beta |v-w|^2_{1,\Omega_h}. \tag{38}$$

P r o o f. See [11], Lemma 2.1. □

R e m a r k 3.2. Similarly to (38) we can derive the relation

$$a_h(v, w-v) + J_h(v) - J_h(w) \geqslant -\frac{1}{2}\big(\beta_P |v-w|^2_{1,P_h} + \beta_E |v-w|^2_{1,E_h}\big), \tag{39}$$

which we will use in a priori estimates.

We can also define for all $v, w \in H^1(\Omega_h)$ forms $l_h(v,w)$ by

$$l_h(v,w) = \sum_{M=E,P} l_{hM}(v,w), \quad l_{hM}(v,w) = \Theta_M \int_{\Omega_{hM}} \sum_{i=1}^{2} \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\,\mathrm{d}x, \tag{40}$$

where the constants $\Theta_M$ $(M = E, P)$ satisfy condition (14). Further, we define

$$d_{hM}(v,w) = l_{hM}(v,w) - a_{hM}(v,w) \tag{41}$$

and introduce the following lemma similar to Lemma 2.5.

**Lemma 3.3.** *For all $v, w, z \in H^1(\Omega_{hM})$ we have*

$$|d_{hM}(v,w) - d_{hM}(z,w)| \leqslant \tau_M |v-z|_{1,M_h} |w|_{1,M_h} \quad (M = E, P), \tag{42}$$

*where $\tau_M$ is a constant independent of $v$, $w$, $z$ and satisfying* (28), *i.e.* $\tau_M < \Theta_M$.

P r o o f. Inspecting the proof of Lemma 2.5 we see that the functional

$$L_{hM}(v) = \frac{1}{2} l_{hM}(v,v) - J_{hM}(v) \quad (M = E, P)$$

where $l_{hM}(v,w)$ and $J_{hM}(v)$ are given by (40) and (33), respectively, is the potential of the form $d_{hM}(v,w)$ defined by (41). Thus in the same way we can derive an estimate

$$|L''_{hM}(v,w,z)| \leqslant \tau_M |z|_{1,M_h} |w|_{1,M_h}$$

with $0 < \tau_M < \Theta_M$ $(M = E, P)$ and using Taylor's theorem we prove (42). □

Let us define finite dimensional subspaces of $H^1(\Omega_h) \cap C(\bar{\Omega}_h)$

$$
\begin{aligned}
X_h &= \{v \in C(\bar{\Omega}_h) \colon \ v|_T \text{ is linear for all } T \in \mathcal{T}_h\},\\
V_h &= \{v \in X_h \colon \ v(P_i) = 0 \ \ \forall P_i \in \bar{\Gamma}_1\},\\
W_h &= \{v \in X_h \colon \ v(P_i) = z(P_i) \ \ \forall P_i \in \bar{\Gamma}_1\}.
\end{aligned}
$$

According to (32) and (40), we have for all $v, w \in X_h$

$$
\begin{aligned}
a_h(v,w) &= \sum_{M=E,P} \sum_{T \in \mathcal{T}_{hM}} \sum_{i=1}^{2} \nu_M(|\operatorname{grad} v|_T) \frac{\partial v}{\partial x_i}\Big|_T \frac{\partial w}{\partial x_i}\Big|_T \operatorname{meas} T,\\
l_h(v,w) &= \sum_{M=E,P} \sum_{T \in \mathcal{T}_{hM}} \sum_{i=1}^{2} \Theta_M \frac{\partial v}{\partial x_i}\Big|_T \frac{\partial w}{\partial x_i}\Big|_T \operatorname{meas} T.
\end{aligned}
$$

As the derivatives are constant on triangles no numerical integration is needed for the computation of the forms $a_h(v,w)$ and $l_h(v,w)$. Also the term

$$
(v,w)_{\Omega_{hP}} = \sum_{T \in \mathcal{T}_{hP}} (v,w)_T \ \ \forall v, w \in X_h
$$

can be computed exactly without the use of numerical integration. Thus only the last term on the right-hand side of (15) will be approximated by means of a quadrature formula on a triangle. The symbol $(f_M(t_i), w_M)^I_{M_h}$, where $w \in X_h$, will denote this approximation of $(f_M(t_i), w_M)_{M_h}$.

**Lemma 3.4.** *Let $g \in W^1_\infty(\widetilde{\Omega})$ and let the quadrature formula on a triangle used for the computation of $(g,v)^I_{\Omega_h}$ be of degree of precision $d = 1$. Then*

$$
|(g,v)_{\Omega_h} - (g,v)^I_{\Omega_h}| \leqslant Ch \|g\|_{1,\infty,\widetilde{\Omega}} \|v\|_{1,\Omega_h} \ \ \forall v \in X_h \tag{43}
$$

*where the constant $C$ does not depend on $h$, $v$ and $g$.*

P r o o f. Lemma 3.4 is a consequence of [2, Theorem 4.1.5]. □

Let $\{h_n\}_{n=1}^\infty$ be a sequence such that $h_n > 0$, $h_n > h_{n+1}$, $\lim\limits_{n\to\infty} h_n = 0$ and let $\{\Omega_{h_n}\}_{n=1}^\infty$ and $\{\mathcal{T}_{h_n}\}_{n=1}^\infty \subset \{\mathcal{T}_h\}$ be the corresponding sequences of polygonal domains and triangulations, respectively. Let $\{\Delta t_n\}_{n=1}^\infty$ be a sequence independent of $\{h_n\}_{n=1}^\infty$ with the properties $\Delta t_n > 0$, $\lim\limits_{n\to\infty} \Delta t_n = 0$, $r_n := T/\Delta t_n =$ integer.

In order to simplify the notation we will write $\Omega_n$, $X_n$, $V_n$ and $a_n(v,w)$, $(v,w)_n$ instead of $\Omega_{h_n}$, $X_{h_n}$, $V_{h_n}$ and $a_{h_n}(v,w)$, $(v,w)_{\Omega_{h_n}}$, etc.

We discretize Problem 2.1 in space by the finite element method with linear functions on triangular elements. The discretization in time is carried out by applying the implicit Euler method to the left-hand side and the explicit Euler method to the right-hand side of (15). (Let us note that the idea of implicit-explicit methods goes back to [5], [3].) We get a scheme which is linear:

$$\Delta t_n^{-1}(\Delta U_{nP}^i, v_P)_{P_n} + l_n(U_n^i, v) = d_n(U_n^{i-1}, v) + (f(t_{i-1}), v)_n^I \ \ \forall v \in V_n, \tag{44}$$

where $\Delta U_{nP}^i = U_{nP}^i - U_{nP}^{i-1}$, the forms $l_n(v,w), d_n(v,w)$ are defined by (40) and

$$d_n(v,w) = l_n(v,w) - a_n(v,w). \tag{45}$$

We set

$$(f(t_{i-1}), v)_n^I := \sum_{M=E,P} (f_M(t_{i-1}), v_M)_{M_n}^I \ \ \forall v \in V_n. \tag{46}$$

The scheme (44) cannot be used for $i=1$ as the initial value $u_0^P$ is known on $\Omega_P$ only. Therefore, $U_n^1$ is defined as follows:

$$\Delta t_n^{-1}(\Delta U_{nP}^1, v_P)_{P_n} + a_n(U_n^1, v) = (f(t_1), v)_n^I \ \ \forall v \in V_n, \tag{47}$$

where $U_{nP}^0 = u_{0n}^P \in L_2(\Omega_{nP})$ and $u_{0n}^P$ satisfy (52). Let us note that (47) is a nonlinear scheme considered (for arbitrary $i \geqslant 1$) in [14], [11].

The following discrete problem approximates Problem 2.1.

P r o b l e m 3.5. Let $n$ be a given integer and let $r_n = T/\Delta t_n$. Let

$$t_i = i\Delta t_n \quad (i = 1, \dots, r_n). \tag{48}$$

Let the forms $a_n(v,w)$ and $l_n(v,w)$ be given by (32) and (40), respectively. Find $U_n^i \in W_n$, $(i = 1, \dots, r_n)$ such that

$$\Delta t_n^{-1}(\Delta U_{nP}^1, v_P)_{P_n} + a_n(U_n^1, v) = (f(t_1), v)_n^I \ \ \forall v \in V_n, \tag{49}$$

$$U_{nP}^0 = u_{0n}^P \in L_2(\Omega_{nP}), \tag{50}$$

$$\Delta t_n^{-1}(\Delta U_{nP}^i, v_P)_{P_n} + l_n(U_n^i, v) = d_n(U_n^{i-1}, v) + (f(t_{i-1}), v)_n^I \ \ \forall v \in V_n, \ \ i \geqslant 2, \tag{51}$$

where $d_n(v,w)$ is defined by (45) and $\{u_{0n}^P\}_{n=1}^\infty$, $u_{0n}^P \in L_2(\Omega_{nP})$ is a sequence satisfying the relation

$$\lim_{n\to\infty} \|u_{0n}^P - \mathring{u}_0^P\|_{0,P_n} = 0, \tag{52}$$

where $\mathring{u}_0^P \in L_2(\widetilde{\Omega})$ is the extension of $u_0^P \in L_2(\Omega_P)$ by zero.

**Theorem 3.6.** *The solution $U_n^i$ $(i=1,\dots,r_n)$ of Problem* 3.5 *exists and is unique.*

P r o o f. For the proof see [15], pp. 430–431. Let us note that the existence and uniqueness of $U_n^i$ $(i \geqslant 2)$ follow from the fact that the quadratic form $b_n(v,v)$, where

$$b_n(v,w) = (v_P, w_P)_{P_n} + \Delta t_n l_n(v,w),$$

is bounded from below by $C\|v\|^2_{1,\Omega_n}$ ($C$ is a positive constant independent of $n$) which is a consequence of the inequality (20) of [8], i.e.

$$\|v\|^2_{1,\Omega_h} \leqslant C(\|v\|^2_{0,P_h} + |v|^2_{1,\Omega_h}) \ \ \forall v \in X_h. \tag{53}$$

$\square$

Now we will extend the approximate solution of Problem 3.5 to the whole interval $[0,T]$. For this purpose we introduce some auxiliary definitions and lemmas.

**Definition 3.7.** A triangle $\overline{T} \in \mathcal{T}_h$ is called a boundary triangle if it has two vertices lying on $\partial\Omega$ (or $\partial\Omega_E \cap \partial\Omega_P$). Let $P_1$, $P_2$, $P_3$ be the vertices of a boundary triangle $\overline{T}$, $P_1$ lying in $\Omega_M$ $(M = E, P)$. The curved triangle $\overline{T}^{\mathrm{id}}$ with two straight sides $P_1P_2$, $P_1P_3$ and one curved side which is formed by the part of $\partial\Omega$ (or $\partial\Omega_E \cap \partial\Omega_P$) lying between the points $P_2$, $P_3$ is called the ideal triangle. (The triangle $\overline{T}$ is an approximation of $\overline{T}^{\mathrm{id}}$.) The ideal triangulation $\mathcal{T}_h^{\mathrm{id}}$ of the domain $\Omega$ corresponding to $\mathcal{T}_h$ is the triangulation of $\Omega$ in which we replace all boundary triangles in $\mathcal{T}_h$ by their ideal triangles.

**Definition 3.8.** Let $w \in X_h$. The function $\overline{w}\colon \overline{\Omega}_h \cup \overline{\Omega} \to \mathbb{R}^1$ is called the natural extension of $w$ if $\overline{w} = w$ on $\overline{\Omega}_h$ and

$$\overline{w}|_{\overline{T}^{\mathrm{id}}} = p|_{\overline{T}^{\mathrm{id}}} \quad \text{on} \quad \overline{T}^{\mathrm{id}} \supset \overline{T}$$

where $p$ is the linear polynomial satisfying $p|_{\overline{T}} = w|_{\overline{T}}$ and $\overline{T}^{\mathrm{id}}$ denotes the ideal triangle.

**Lemma 3.9.** *Let $\overline{T}^{\mathrm{id}}$ be an ideal triangle with vertices $P_i^T$ $(i=1,2,3)$, $P_2^T, P_3^T \in \partial\Omega$. Let $\overline{T}_0$ be the closed triangle which lies in the $\xi,\eta$-plane and has vertices $P_1^* = (0,0)$, $P_2^* = (1,0)$, $P_3^* = (0,1)$. There exists a transformation*

$$x = x^{\mathrm{id}}(\xi,\eta), \quad y = y^{\mathrm{id}}(\xi,\eta) \tag{54}$$

*which maps one-to-one the reference triangle $\overline{T}_0$ onto the ideal triangle $\overline{T}^{\mathrm{id}}$ in such a way that $P_i^* \leftrightarrow P_i^T$, $(i=1,2,3)$, $P_1^*P_j^* \leftrightarrow P_1^TP_j^T$, $(j=2,3)$, $P_2^*P_3^* \leftrightarrow \Sigma_T^h$ and*

$T_0 \leftrightarrow T^{\mathrm{id}}$, *where* $\Sigma_T^h$ *is the curved side of the ideal triangle* $\overline{T}^{\mathrm{id}}$. *Let* $p(\xi,\eta)$ *be a linear polynomial and let the mapping*

$$\xi = \xi^{\mathrm{id}}(x,y), \quad \eta = \eta^{\mathrm{id}}(x,y)$$

*be inverse to transformation* (54). *Then the function*

$$\widetilde{w}(x,y) = p(\xi^{\mathrm{id}}(x,y), \eta^{\mathrm{id}}(x,y)) \tag{55}$$

*has the following properties:*

(1) $\widetilde{w}(x,y)$ *is linear along the segments* $P_1^T P_2^T$, $P_1^T P_3^T$;
(2) $\widetilde{w}(P_i^T) = p(P_i^*)$, $i = 1,2,3$;
(3) *if* $\widetilde{w}(P_2^T) = \widetilde{w}(P_3^T) = 0$, *then* $\widetilde{w}(P) = 0 \ \forall P \in \Sigma_T^h$;
(4) *let the boundary* $\partial\Omega$ *be piecewise of class* $C^3$, *let* $u \in H^2(T^{\mathrm{id}})$ *and let* $\widetilde{w}$ *be uniquely determined by the conditions* $\widetilde{w}(P_i^T) = u(P_i^T)$, $(i = 1,2,3)$. *Then we have*

$$\|\widetilde{w} - u\|_{k,T^{\mathrm{id}}} \leqslant C h_T^{2-k} \|u\|_{2,T^{\mathrm{id}}} \quad (k = 0,1), \tag{56}$$

*where the constant* $C$ *does not depend on* $h_T$ *and* $u$.

P r o o f. The proof follows from [13] and [9]. $\square$

**Definition 3.10.** Let $\mathcal{T}_h^{\mathrm{id}}$ be the ideal triangulation of $\overline{\Omega}$ corresponding to the given triangulation $\mathcal{T}_h$. Let $w \in X_h$. The function $\widehat{w} \in H^1(\Omega)$ is said to be associated with $w$ if

(1) $\widehat{w} \in C(\overline{\Omega})$;
(2) $\widehat{w}(P_i) = w(P_i) \ \forall P_i$;
(3) $\widehat{w}$ is linear on each triangle $\overline{T} \in \mathcal{T}_h \cap \mathcal{T}_h^{\mathrm{id}}$ and on each ideal triangle $\overline{T}^{\mathrm{id}} \in \mathcal{T}_h^{\mathrm{id}}$ lying along $\Gamma_2$ (i.e. $\widehat{w} = \overline{w}$ on $\overline{T}^{\mathrm{id}} \supset \overline{T}$ and $\widehat{w}|_{\overline{T}^{\mathrm{id}}}$ is the restriction of $w|_{\overline{T}}$ to $\overline{T}^{\mathrm{id}} \subset \overline{T}$);
(4) if $\overline{T}^{\mathrm{id}} \in \mathcal{T}_h^{\mathrm{id}}$ lies along $\Gamma_1$ and $\overline{T} \in \mathcal{T}_h$ is its approximation then $\widehat{w} = \widetilde{w}$ on $\overline{T}^{\mathrm{id}}$, where $\widetilde{w}$ is given by (55).

R e m a r k 3.11. Using the rule "first indices, then bars, tildes, dots and hats" for a function $w \in X_n$ the symbol $\overline{w}_M$ denotes the natural extension of $w_M$ from $\overline{\Omega}_{nM}$ onto $\overline{\Omega}_{nM} \cup \overline{\Omega}_M$ and $\hat{w}_M$ denotes the function from $H^1(\Omega_M) \cap C(\overline{\Omega}_M)$ associated with $w_M$.

It should be noted that $(\widehat{w})_M = \widehat{w}_M$ for all $w \in X_n$ while $(\overline{w})_M \neq \overline{w}_M$.

Using the solution of Problem 3.5 we define the finite element Rothe functions

$$\widehat{U}_n(t) = \widehat{U}_n^{i-1} + (\Delta\widehat{U}_n^i/\Delta t_n)(t - t_{i-1}), \quad t \in [t_{i-1}, t_i], \quad (i = 2, \dots, r_n) \tag{57}$$

where $t_i$ are given by (48) and functions $\widehat{U}_n^i \in H^1(\Omega)$ are associated with $U_n^i$. On the interval $[0, \Delta t_n]$ we set

$$\widehat{U}_{nP}(t) = \widehat{U}_{nP}^0 + (\Delta \widehat{U}_{nP}^1/\Delta t_n)t, \quad t \in [0, \Delta t_n].$$

As Theorem 4.14 holds for the associated functions $\widehat{U}_{nP}$, $\widehat{Z}_n$, we need $\widehat{U}_{nP}^0 = \widehat{u}_{0n}^P$. For that reason we assume that

$$U_{nP}^0 = u_{0n}^P \in X_n. \tag{58}$$

If we use (57) we can also define

$$\widehat{Z}_n(t) = \widehat{U}_n^1, \ t \in [0, \Delta t_n], \quad \widehat{Z}_n(t) = \widehat{U}_n(t), \ t \in [\Delta t_n, T]. \tag{59}$$

## 4. Existence, uniqueness, convergence

Let $\{s_j^P\}$, $s_j^P \in C_0^\infty(\Omega_P)$, be a sequence satisfying

$$\lim_{j\to\infty} \|z_P + s_j^P - u_0^P\|_{0,P} = 0. \tag{60}$$

For every pair $j, n$ we define the following auxiliary discrete problem.

P r o b l e m 4.1. Let $a_n(v,w)$, $l_n(v,w)$, $d_n(v,w)$ and $(f(t_{i-1}),v)_n^I$ be the same as in Problem 3.5 and let $r_n = T/\Delta t_n$. Find $S_{jn}^i \in W_n$, $i = 1, \dots, r_n$ ($j, n$ fixed), such that

$$\Delta t_n^{-1}(\Delta S_{jnP}^1, v_P)_{P_n} + a_n(S_{jn}^1, v) = (f(t_1), v)_n^I \ \ \forall v \in V_n, \tag{61}$$

$$S_{jn}^0 = I_n(z + \overset{\circ}{s}{}_j^P) \in W_n, \tag{62}$$

$$\Delta t_n^{-1}(\Delta S_{jnP}^i, v_P)_{P_n} + l_n(S_{jn}^i, v) = d_n(S_{jn}^{i-1}, v) + (f(t_{i-1}), v)_n^I \ \ \forall v \in V_n \ \ (i \geqslant 2) \tag{63}$$

where $\Delta S_{jnP}^i = S_{jnP}^i - S_{jnP}^{i-1}$, $\overset{\circ}{s}{}_j^P \in C_0^\infty(\widetilde{\Omega})$ is the extension of $s_j^P \in C_0^\infty(\Omega_P)$ by zero and $I_n w \in X_n$ is the interpolant of a function $w \in C(\overline{\Omega}_n)$.

**Theorem 4.2.** *The solution $S_{jn}^i$ ($i = 1, \dots, r_n$) of Problem 4.1 exists and is unique.*

P r o o f. This theorem can be proved in the same way as Theorem 3.6. □

**Lemma 4.3.** *Let $(f(t_{i-1}),v)_n^I$ be computed by means of a quadrature formula of degree of precision $d=1$. Then we have*

$$\sum_{i=1}^m \|\Delta S^i_{jnP}/\Delta t_n\|^2_{0,P_n}\Delta t_n + \sum_{i=1}^m \|\Delta S^i_{jn}\|^2_{1,\Omega_n} + \|S^m_{jn}\|^2_{1,\Omega_n} \leqslant C(j) \tag{64}$$
$$\forall m,n\ \ (1\leqslant m\leqslant r_n)$$

*where the constant $C(j)$ does not depend on $m$ and $n$.*

Proof. In what follows the symbols $C$, $C(j)$ will denote positive constants independent of $h_n$ and $\Delta t_n$ with generally different values at any two different places.

A) First we prove that $\|S^1_{jn}\|_{1,\Omega_n}\leqslant C(j)$. Choosing $v=\Delta S^1_{jn}\in V_n$ in (61), using (37) and then (36) together with the discrete form of Friedrichs' inequality [12, (29.1)] we get

$$\begin{aligned}\|\Delta S^1_{jnP}/\Delta t_n\|^2_{0,P_n}\Delta t_n + \frac{\gamma}{2}C\|\Delta S^1_{jn}\|^2_{1,\Omega_n} + \frac{\gamma}{2}|S^1_{jn}|^2_{1,\Omega_n}&\\ \leqslant \frac{\beta}{2}|S^0_{jn}|^2_{1,\Omega_n} + (f(t_1),\Delta S^1_{jn})^I_n.&\end{aligned} \tag{65}$$

Let us add the term $\frac{\gamma}{2}\|S^1_{jn}\|^2_{0,\Gamma_{1n}}$ to both sides of (65). With regard to $S^1_{jn}\in W_n$ we have

$$\|S^1_{jn}\|^2_{0,\Gamma_{1n}} = \|I_n z\|^2_{0,\Gamma_{1n}} \leqslant \|I_n z\|^2_{0,\partial\Omega_n}.$$

Applying the discrete forms of Friedrichs' and trace inequalities [12, (29.5), (29.2)] and the above relation to (65) we obtain

$$\begin{aligned}&\|\Delta S^1_{jnP}/\Delta t_n\|^2_{0,P_n}\Delta t_n + \|\Delta S^1_{jn}\|^2_{1,\Omega_n} + \|S^1_{jn}\|^2_{1,\Omega_n}\\ &\qquad\leqslant C\left\{\|S^0_{jn}\|^2_{1,\Omega_n} + \|I_n z\|^2_{1,\Omega_n} + (f(t_1),\Delta S^1_{jn})^I_n\right\}.\end{aligned} \tag{66}$$

The finite element interpolation theorem for linear polynomials on a triangle and relations (62) yield

$$\begin{aligned}&C\|S^0_{jn}\|^2_{1,\Omega_n}\\ &\quad\leqslant C\sum_{M=E,P}\left\{\|z^C_M + (\overset{\circ}{s}{}^P_j)_M\|_{1,M_n} + \|I_{nM}(z_M + (\overset{\circ}{s}{}^P_j)_M) - (z^C_M + (\overset{\circ}{s}{}^P_j)_M)\|_{1,M_n}\right\}^2\\ &\quad\leqslant C\sum_{M=E,P}\|z^C_M + (\overset{\circ}{s}{}^P_j)_M\|^2_{2,\widetilde\Omega_M} \leqslant C\left\{\|z_E\|^2_{2,E} + \|z_P\|^2_{2,P} + \|s^P_j\|^2_{2,P}\right\} \leqslant C(j),\end{aligned} \tag{67}$$

where $z^C_M\in H^2(\widetilde\Omega_M)$ is the Calderon extension of $z_M\in H^2(\Omega_M)$ $(M=E,P)$. Similarly to the above,

$$\begin{aligned}C\|I_n z\|^2_{1,\Omega_n} &\leqslant C\sum_{M=E,P}\left\{\|I_{nM}(z_M) - z^C_M\|_{1,M_n} + \|z^C_M\|_{1,M_n}\right\}^2\\ &\leqslant C\sum_{M=E,P}\|z^C_M\|^2_{2,\widetilde\Omega_M} \leqslant C\left\{\|z_E\|^2_{2,E} + \|z_P\|^2_{2,P}\right\} \leqslant C.\end{aligned} \tag{68}$$

Lemma 3.4, assumptions (6), (46), (67) and the inequality

$$|ab| \leqslant \varepsilon a^2/2 + b^2/(2\varepsilon) \quad a \geqslant 0,\, b \geqslant 0,\, \varepsilon > 0 \tag{69}$$

with various values of $\varepsilon$ imply

$$\begin{aligned} C|(f(t_1), S^0_{jn})^I_n| &\leqslant C\bigl\{|(f(t_1), S^0_{jn})_n| \\ &\quad + |(f(t_1), S^0_{jn})^I_n - (f(t_1), S^0_{jn})_n|\bigr\} \\ &\leqslant C \sum_{M=E,P} \|f_M(t_1)\|_{1,\infty,\tilde{\Omega}_M} \|S^0_{jnM}\|_{1,M_n} \\ &\leqslant C(j)\bigl\{\|f_E\|_{AC(\bar{I},W^1_\infty(\tilde{\Omega}_E))} + \|f_P\|_{AC(\bar{I},W^1_\infty(\tilde{\Omega}_P))}\bigr\}, \end{aligned} \tag{70}$$

$$\begin{aligned} C|(f(t_1), S^1_{jn})^I_n| &\leqslant C \sum_{M=E,P} \|f_M(t_1)\|_{1,\infty,\tilde{\Omega}_M} \|S^1_{jnM}\|_{1,M_n} \\ &\leqslant C\bigl\{\|f_E\|^2_{AC(\bar{I},W^1_\infty(\tilde{\Omega}_E))} + \|f_P\|^2_{AC(\bar{I},W^1_\infty(\tilde{\Omega}_P))}\bigr\} \\ &\quad + \frac{1}{2}\|S^1_{jn}\|^2_{1,\Omega_n}. \end{aligned} \tag{71}$$

Using inequalities (67), (68), (70) and (71) we obtain from (66)

$$\|\Delta S^1_{jnP}/\Delta t_n\|^2_{0,P_n}\Delta t_n + \|\Delta S^1_{jn}\|^2_{1,\Omega_n} + \|S^1_{jn}\|^2_{1,\Omega_n} \leqslant C(j), \tag{72}$$

which gives

$$\|S^1_{jn}\|_{1,\Omega_n} \leqslant C(j). \tag{73}$$

B) Now we prove the inequality

$$\begin{aligned} &\sum_{i=2}^{m} \|\Delta S^i_{jnP}/\Delta t_n\|^2_{0,P_n}\,\Delta t_n + \sum_{i=2}^{m} \|\Delta S^i_{jn}\|^2_{1,\Omega_n} + \frac{1}{2}\|S^m_{jn}\|^2_{1,\Omega_n} \\ &\qquad \leqslant C(j) + \frac{1}{2}\Delta t_n \sum_{i=2}^{m-1} \|S^i_{jn}\|^2_{1,\Omega_n} \quad (2 \leqslant m \leqslant r_n). \end{aligned} \tag{74}$$

Let us choose $v = \Delta S^i_{jn} \in V_n$ in (63). After summing from $i = 2$ to $i = m$ we obtain

$$\begin{aligned} \sum_{i=2}^{m} \|\Delta S^i_{jnP}/\Delta t_n\|^2_{0,P_n}\Delta t_n + \sum_{i=2}^{m} a_n(S^{i-1}_{jn}, \Delta S^i_{jn}) + \sum_{i=2}^{m} l_n(\Delta S^i_{jn}, \Delta S^i_{jn}) \\ = \sum_{i=2}^{m} (f(t_{i-1}), \Delta S^i_{jn})^I_n. \end{aligned} \tag{75}$$

We set $\kappa_M = \Theta_M - \frac{1}{2}\beta_M$, $(M = E, P)$. We have $\kappa_M > 0$ owing to (14), hence $\kappa = \min(\kappa_E, \kappa_P) > 0$. It follows from (39) that

$$\begin{aligned}\sum_{i=2}^{m} l_n(\Delta S_{jn}^i, \Delta S_{jn}^i) &+ \sum_{i=2}^{m} a_n(S_{jn}^{i-1}, \Delta S_{jn}^i) \\ &\geqslant \sum_{i=2}^{m} \bigl\{\kappa |\Delta S_{jn}^i|_{1,\Omega_n}^2 + J_n(S_{jn}^i) - J_n(S_{jn}^{i-1})\bigr\} \\ &= J_n(S_{jn}^m) - J_n(S_{jn}^1) + \kappa \sum_{i=2}^{m} |\Delta S_{jn}^i|_{1,\Omega_n}^2.\end{aligned}$$

Applying this relation to (75) and using the discrete form of Friedrichs' inequality [12, (29.1)] and (36) we obtain

$$\begin{aligned}\sum_{i=2}^{m} \|\Delta S_{jnP}^i/\Delta t_n\|_{0,P_n}^2 \Delta t_n &+ \frac{\gamma}{2} |S_{jn}^m|_{1,\Omega_n}^2 + \kappa C \sum_{i=2}^{m} \|\Delta S_{jn}^i\|_{1,\Omega_n}^2 \\ &\leqslant \frac{\beta}{2} |S_{jn}^1|_{1,\Omega_n}^2 + \sum_{i=2}^{m} (f(t_{i-1}), \Delta S_{jn}^i)_n^I.\end{aligned} \tag{76}$$

Let us add the term $\frac{\gamma}{2}\|S_{jn}^m\|_{0,\Gamma_{1n}}^2$ to both sides of (76). Similarly as in part A using the discrete forms of Friedrichs' and trace inequalities [12, (29.5), (29.2)] we come to the inequality

$$\begin{aligned}\sum_{i=2}^{m} \|\Delta S_{jnP}^i/\Delta t_n\|_{0,P_n}^2 \Delta t_n &+ \sum_{i=2}^{m} \|\Delta S_{jn}^i\|_{1,\Omega_n}^2 + \|S_{jn}^m\|_{1,\Omega_n}^2 \\ &\leqslant C\biggl\{\|S_{jn}^1\|_{1,\Omega_n}^2 + \|I_n z\|_{1,\Omega_n}^2 + \sum_{i=2}^{m} (f(t_{i-1}), \Delta S_{jn}^i)_n^I\biggr\}.\end{aligned} \tag{77}$$

Summation by parts gives

$$\begin{aligned}\sum_{i=2}^{m} (f(t_{i-1}), \Delta S_{jn}^i)_n^I &= (f(t_{m-1}), S_{jn}^m)_n^I - (f(t_1), S_{jn}^1)_n^I \\ &\quad - \sum_{i=2}^{m-1} (f(t_i) - f(t_{i-1}), S_{jn}^i)_n^I.\end{aligned} \tag{78}$$

From Lemma 3.4, assumptions (6), (46), (73) and inequality (69) it follows that

$$\begin{aligned}C|(f(t_{m-1}), S_{jn}^m)_n^I| &\leqslant C \sum_{M=E,P} \|f_M(t_{m-1})\|_{1,\infty,\tilde{\Omega}_M} \|S_{jnM}^m\|_{1,M_n} \\ &\leqslant C\bigl\{\|f_E\|_{AC(\bar{I}, W_\infty^1(\tilde{\Omega}_E))}^2 + \|f_P\|_{AC(\bar{I}, W_\infty^1(\tilde{\Omega}_P))}^2\bigr\} \\ &\quad + \frac{1}{2}\|S_{jn}^m\|_{1,\Omega_n}^2,\end{aligned} \tag{79}$$

$$
\begin{aligned}
(80) \qquad C|(f(t_1), S^1_{jn})^I_n| &\leqslant C \sum_{M=E,P} \|f_M(t_1)\|_{1,\infty,\tilde\Omega_M} \|S^1_{jnM}\|_{1,M_n} \\
&\leqslant C(j)\bigl\{\|f_E\|_{AC(\bar I, W^1_\infty(\tilde\Omega_E))} + \|f_P\|_{AC(\bar I, W^1_\infty(\tilde\Omega_P))}\bigr\}.
\end{aligned}
$$

By analogy,

$$
\begin{aligned}
(81) \quad & C\Bigl|\sum_{i=2}^{m-1}(f(t_i) - f(t_{i-1}), S^i_{jn})^I_n\Bigr| \\
&\leqslant C \sum_{M=E,P}\sum_{i=2}^{m-1} \|f_M(t_i) - f_M(t_{i-1})\|_{1,\infty,\tilde\Omega_M} \|S^i_{jnM}\|_{1,M_n} \\
&= C \sum_{M=E,P}\sum_{i=2}^{m-1} \Bigl\|\int_{t_{i-1}}^{t_i} \dot f_M(t)\,\mathrm{d}t\Bigr\|_{1,\infty,\tilde\Omega_M} \|S^i_{jnM}\|_{1,M_n} \\
&\leqslant C \sum_{M=E,P}\sum_{i=2}^{m-1} \Bigl\{\Delta t_n \int_{t_{i-1}}^{t_i} \|\dot f_M(t)\|^2_{1,\infty,\tilde\Omega_M}\,\mathrm{d}t\Bigr\}^{1/2} \|S^i_{jnM}\|_{1,M_n} \\
&\leqslant C \sum_{M=E,P} \sqrt{\Delta t_n}\, \|\dot f_M\|_{L_2(I, W^1_\infty(\tilde\Omega_M))} \Bigl\{\sum_{i=2}^{m-1} \|S^i_{jnM}\|^2_{1,M_n}\Bigr\}^{1/2} \\
&\leqslant C\bigl\{\|\dot f_E\|^2_{L_2(I, W^1_\infty(\tilde\Omega_E))} + \|\dot f_P\|^2_{L_2(I, W^1_\infty(\tilde\Omega_P))}\bigr\} \\
&\quad + \frac{1}{2}\Delta t_n \sum_{i=2}^{m-1} \|S^i_{jn}\|^2_{1,\Omega_n}.
\end{aligned}
$$

Relations (77), (73), (68) and (78)–(81) imply (74).

By the discrete form of Gronwall's lemma (see [12], Theorem $\mathcal{P}.134$) we obtain from (74)

$$
\sum_{i=2}^{m} \|\Delta S^i_{jnP}/\Delta t_n\|^2_{0,P_n}\Delta t_n + \sum_{i=2}^{m} \|\Delta S^i_{jn}\|^2_{1,\Omega_n} + \|S^m_{jn}\|^2_{1,\Omega_n} \leqslant C(j) \quad (2 \leqslant m \leqslant r_n).
$$

Finally, (64) is a consequence of this estimate and (72) where the constant $C(j)$ depends on $(\|z_E\|_{2,E} + \|z_P\|_{2,P} + \|s^P_j\|_{2,P})$. $\square$

The norms $\|\cdot\|_{0,P_n}$, $\|\cdot\|_{1,\Omega_n}$ in inequality (64) depend on $n$. In order to obtain a priori estimates introduced in Corollary 4.4, where the norms $\|\cdot\|_{0,P}$, $\|\cdot\|_1$ appear, we must consider the functions $\widehat S^i_{jnP}$, $\widehat S^i_{jn}$ associated with $S^i_{jnP}$, $S^i_{jn}$ and apply Lemma 48.5 of [12].

**Corollary 4.4.** *Under the assumption of Lemma 4.3 we have for all $n \geqslant n_0$*

$$\sum_{i=1}^{m} \|\Delta \widehat{S}^i_{jnP}/\Delta t_n\|^2_{0,P} \Delta t_n + \sum_{i=1}^{m} \|\Delta \widehat{S}^i_{jn}\|^2_1 + \|\widehat{S}^m_{jn}\|^2_1 \leqslant C(j) \ \forall m, n \ (1 \leqslant m \leqslant r_n), \tag{82}$$

*where the constant $C(j)$ does not depend on $m$ and $n$.*

For every pair $j, n$ let us define the finite element Rothe functions

$$\widehat{S}_{jn}(t) = \widehat{S}^{i-1}_{jn} + (\Delta \widehat{S}^i_{jn}/\Delta t_n)\,(t - t_{i-1}), \quad t \in [t_{i-1}, t_i]\,(i = 1, \ldots, r_n), \tag{83}$$

$$\widehat{S}_{jnP}(t) = \widehat{S}^{i-1}_{jnP} + (\Delta \widehat{S}^i_{jnP}/\Delta t_n)(t - t_{i-1}), \ t \in [t_{i-1}, t_i]\,(i = 1, \ldots, r_n) \tag{84}$$

and the step-functions

$$\widetilde{S}_{jn}(t) = \widehat{S}^{i-1}_{jn}, \quad t \in [t_{i-1}, t_i) \ (i = 1, \ldots, r_n), \quad \widetilde{S}_{jn}(T) = \widehat{S}^{r_n-1}_{jn}, \tag{85}$$

where $t_i$ are given by (48).

**Corollary 4.5.** *The finite element Rothe functions $\widehat{S}_{jn}(t)$, $\widehat{S}_{jnP}(t)$ and the step-functions $\widetilde{S}_{jn}(t)$ satisfy the relations*

$$\|\widehat{S}_{jn}(t)\|_1 \leqslant C(j) \ \forall t \in \bar{I} \ \forall n, \tag{86}$$

$$\|\widetilde{S}_{jn}(t)\|_1 \leqslant C(j) \ \forall t \in \bar{I} \ \forall n, \tag{87}$$

$$\|\widetilde{S}_{jn} - \widehat{S}_{jn}\|^2_{L_2(I, H^1(\Omega))} \leqslant C(j)\Delta t_n \ \forall n, \tag{88}$$

$$\int_0^T \|\,\mathrm{d}\widehat{S}_{jnP}(t)/\,\mathrm{d}t\|^2_{0,P}\,\mathrm{d}t \leqslant C(j) \ \forall n. \tag{89}$$

P r o o f. All the relations follow immediately from Corollary 4.4, (67), [12, Lemma 48.5] and the definition of functions (83)–(85). $\square$

**Lemma 4.6.** *For a fixed $j$ we have*

$$\|\widehat{S}^0_{jn} - (z + \overset{\circ}{s}{}^P_j)\|_k \leqslant C h^{2-k}_n \{\|z_E\|_{2,E} + \|z_P\|_{2,P} + \|s^P_j\|_{2,P}\} \to 0 \quad \textit{if} \quad n \to \infty \ (k = 0, 1), \tag{90}$$

$$\|\widehat{S}^0_{jnP} - (z_P + s^P_j)\|_{k,P} \leqslant C h^{2-k}_n \{\|z_P\|_{2,P} + \|s^P_j\|_{2,P}\} \to 0 \quad \textit{if} \quad n \to \infty \ (k = 0, 1). \tag{91}$$

P r o o f. Lemma 4.6 is a consequence of (56), the finite element interpolation theorem for linear polynomials on a triangle, assumption (3) and (62). $\square$

**Lemma 4.7.** *Let $\widetilde{f}_{nM}$ $(M = E, P)$ be step-functions defined by*

$$\widetilde{f}_{nM}(t) = f_M(t_{i-1}), \quad t \in [t_{i-1}, t_i) \ (i = 1, \dots, r_n), \quad \widetilde{f}_{nM}(T) = f_M(t_{r_n - 1}) \tag{92}$$

*where $t_i$ are given by (48) and the functions $f_M$ $(M = E, P)$ satisfy assumptions (6). Then we have*

$$\widetilde{f}_{nM} \to f_M \quad \text{in} \quad L_2(I, W^1_\infty(\widetilde{\Omega}_M)) \ (M = E, P). \tag{93}$$

P r o o f. The proof follows the same lines as in [12], p. 360. □

**Lemma 4.8.** *Let $j$ be fixed. Then there exist a subsequence $\{\widehat{S}_{jk}\}$ of the sequence $\{\widehat{S}_{jn}\}$ and a function $u_j$ such that*

$$u_j \in L_\infty(I, H^1(\Omega)), \tag{94}$$

$$u_{jP} \in AC(\bar{I}, L_2(\Omega_P)) \cap L_\infty(I, H^1(\Omega_P)), \tag{95}$$

$$\dot{u}_{jP} \in L_2(I, L_2(\Omega_P)), \tag{96}$$

$$\widehat{S}_{jkP} \to u_{jP} \quad \text{in} \quad C(\bar{I}, L_2(\Omega_P)), \tag{97}$$

$$\widehat{S}_{jk} \rightharpoonup u_j \quad \text{weakly in} \quad L_2(I, H^1(\Omega)), \tag{98}$$

$$\widetilde{S}_{jk} \rightharpoonup u_j \quad \text{weakly in} \quad L_2(I, H^1(\Omega)), \tag{99}$$

$$\mathrm{d}\widehat{S}_{jkP}/\mathrm{d}t \rightharpoonup \dot{u}_{jP} \quad \text{weakly in} \quad L_2(I, L_2(\Omega_P)), \tag{100}$$

$$u_{jP}(0) = z_P + s_j^P \quad \text{in} \quad C(\bar{I}, L_2(\Omega_P)). \tag{101}$$

P r o o f. A) Relation (86) yields $\|\widehat{S}_{jn}\|_{L_2(I, H^1(\Omega))} \leqslant C(j)$. According to Theorem $\mathcal{P}$.132 of [12], there exist a subsequence of the sequence $\{\widehat{S}_{jn}\}_{n=1}^\infty$ (let us denote it again $\{\widehat{S}_{jk}\}$) and a function $u_j \in L_2(I, H^1(\Omega))$ such that

$$\widehat{S}_{jk} \rightharpoonup u_j \quad \text{weakly in} \quad L_2(I, H^1(\Omega)), \tag{102}$$

which is (98).

As the norm $\|\cdot\|_1$ is weakly lower semicontinuous on $H^1(\Omega)$ (see [7], p. 183) relations (86) and (102) imply

$$\|u_j(t)\|_1 \leqslant \liminf_{k \to \infty} \|\widehat{S}_{jk}(t)\|_1 \leqslant C(j) \ \ \forall t \in \bar{I}.$$

Thus $u_j \in L_\infty(I, H^1(\Omega))$.

Relation (99) is a consequence of (88) and (102).

B) From (89) it follows that

$$\|\widehat{S}_{jnP}(t'') - \widehat{S}_{jnP}(t')\|_{0,P} = \left\| \int_{t'}^{t''} (\mathrm{d}\widehat{S}_{jnP}(t)/\mathrm{d}t)\,\mathrm{d}t \right\|_{0,P}$$
$$\leqslant C(j)|t''-t'|^{1/2} \quad \forall t', t'' \in \bar{I} = [0,T].$$

Hence the functions $\widehat{S}_{jnP}(t)$ $(n = 1, 2, \ldots)$ are (for fixed $j$) equicontinuous on $\bar{I}$ in the norm $\|\cdot\|_{0,P}$. Relation (86) and Rellich's theorem [12, Theorem $\mathcal{P}$.65] imply that the sequence $\{\widehat{S}_{jnP}(t)\}$ is relatively compact in $L_2(\Omega_P)$ for every $t \in \bar{I}$. According to the generalization of the Arzelà-Ascoli theorem [12, Theorem $\mathcal{P}$.101], there exist a subsequence $\{\widehat{S}_{jkP}\}$ of the sequence $\{\widehat{S}_{jnP}\}$ and a function $w \in C(\bar{I}, L_2(\Omega_P))$ such that

$$\widehat{S}_{jkP} \to w \quad \text{in} \quad C(\bar{I}, L_2(\Omega_P)).$$

This relation yields that

$$\text{(103)} \quad \int_0^T \|\widehat{S}_{jkP}(t) - w(t)\|_{0,P}^2 \,\mathrm{d}t \leqslant T \max_{t\in\bar{I}} \|\widehat{S}_{jkP}(t) - w(t)\|_{0,P}^2 \to 0 \quad \text{for} \quad k \to \infty.$$

As the form

$$\int_0^T (z(t), v(t))\,\mathrm{d}t, \quad v \in L_2(I, H^1(\Omega))$$

is a linear bounded functional on $L_2(I, H^1(\Omega))$ for every fixed $z \in L_2(I, L_2(\Omega))$, relation (102) implies

$$\widehat{S}_{jk} \rightharpoonup u_j \quad \text{weakly in} \quad L_2(I, L_2(\Omega)).$$

With regard to this result and (103) we obtain $w(t) = u_{jP}(t)$ in $L_2(\Omega_P)$ and (97) holds.

C) For every $t \in \bar{I}$ and for every $k$ we have (for a fixed $j$)

$$\text{(104)} \quad (\widehat{S}_{jkP}(t), v_P)_P - (\widehat{S}^0_{jkP}, v_P)_P = \int_0^t \big((\mathrm{d}\widehat{S}_{jkP}(\tau)/\,\mathrm{d}\tau), v_P\big)_P \,\mathrm{d}\tau \;\; \forall v_P \in L_2(\Omega_P).$$

According to (89) and Theorem $\mathcal{P}$.132 of [12], we can extract a subsequence (we will denote it again $\{\,\mathrm{d}\widehat{S}_{jkP}/\mathrm{d}t\}$) of the sequence $\{\,\mathrm{d}\widehat{S}_{jkP}/\mathrm{d}t\}$ such that

$$\text{(105)} \qquad \mathrm{d}\widehat{S}_{jkP}/\mathrm{d}t \rightharpoonup g_{jP} \quad \text{weakly in} \quad L_2(I, L_2(\Omega_P)).$$

Passing to the limit for $k \to \infty$ in (104) and using (97), (91) and (105) we obtain

$$(u_{jP}(t), v_P)_P - (z_P + s_j^P, v_P)_P = \int_0^t (g_{jP}(\tau), v_P)_P \,\mathrm{d}\tau \;\; \forall v_P \in L_2(\Omega_P).$$

Using [12, Corollary $\mathcal{P}$.112(b)] we can write this relation in the form

$$\left(u_{jP}(t) - z_P - s_j^P - \int_0^t g_{jP}(\tau)\,\mathrm{d}\tau, v_P\right)_P = 0 \ \forall v_P \in L_2(\Omega_P).$$

From the last relation we get

$$u_{jP}(t) = z_P + s_j^P + \int_0^t g_{jP}(\tau)\,\mathrm{d}\tau.$$

Thus according to [12, Theorem $\mathcal{P}$.113], $u_{jP} \in AC(\bar{I}, L_2(\Omega_P))$, $u_{jP}(t)$ satisfies the initial condition (101) and we have

$$\dot{u}_{jP}(t) = g_{jP}(t) \quad \text{a.e. in } I. \tag{106}$$

By (105) we have

$$\begin{aligned} &\int_0^T \big((\,\mathrm{d}\widehat{S}_{jkP}(t)/\,\mathrm{d}t), v_P(t)\big)_P\,\mathrm{d}t \\ &\qquad \to \int_0^T (g_{jP}(t), v_P(t))_P\,\mathrm{d}t \ \ \forall v_P \in L_2(I, L_2(\Omega_P)). \end{aligned} \tag{107}$$

Relations (106) and (107) imply (100).

Relation (106) gives $\dot{u}_{jP} \in L_2(I, L_2(\Omega_P))$, which is (96). □

In the next lemma and remarks we introduce some relations which we will need in the proof of Theorem 4.12.

**Lemma 4.9.** *Let $\overline{w}_M$, $\bar{v}_n$, $\bar{v}_{nM}$ be the natural extensions of $w_M$, $v_n$, $v_{nM}$, respectively. Let $\hat{v}_n$, $\hat{v}_{nM}$, $\hat{w}_M$ be functions associated with $v_n$, $v_{nM}$, $w_M$, respectively. Then we have*

$$\|\overline{w}_M\|^2_{k,\varepsilon_{nM}} \leqslant Ch_n \|w_M\|^2_{k,M_n} \ \ \forall w \in X_n \ \ (\varepsilon = \tau, \omega) \ \ (k = 0, 1), \tag{108}$$

*where the constant $C$ does not depend on $n$ and $w$ and $\varepsilon_{nM} = \tau_{nM}$, $\omega_{nM}$ are defined by*

$$\omega_{nM} = \Omega_M - \overline{\Omega}_{nM}, \quad \tau_{nM} = \Omega_{nM} - \overline{\Omega}_M \quad (M = E, P). \tag{109}$$

*Further, we have*

$$
\begin{aligned}
&(110) \quad \lim_{n\to\infty} \|\bar{v}_n - v\|_1 = 0, \quad \lim_{n\to\infty} \|\bar{v}_{nM} - v_M\|_{1,M} = 0 \ \ \forall v \in V,\\
&(111) \quad \lim_{n\to\infty} \|\widehat{v}_n - v\|_1 = 0 \ \ \forall v \in V,\\
&(112) \quad \|\bar{v}_{nM}\|_{1,M} \leqslant K(v) \ \ \forall v \in V \ \ (M = E, P),\\
&(113) \quad \|\widehat{v}_n\|_1 \leqslant K(v) \ \ \forall v \in V,\\
&(114) \quad \|\bar{v}_{nM} - \widehat{v}_{nM}\|_{1,M} \leqslant K(v)h_n \ \ \forall v \in V \ \ (M = E, P),\\
&(115) \quad \|\widehat{w}_M - \overline{w}_M\|_{k,M} \leqslant Ch_n\|w_M\|_{k,M_n} \ \ \forall w \in X_n \ \ (M = E, P) \ \ (k = 0, 1),
\end{aligned}
$$

*where $K(v)$ is a constant depending only on $v$ and $\Omega$.*

P r o o f. See [12, Lemma 28.8] or [11], pp. 359–360 and [11], pp. 361–362. □

R e m a r k 4.10. We will also use the following notation:

$$
\begin{aligned}
a_{\varepsilon_{nM}}(v, w) &= \int_{\varepsilon_{nM}} \nu_M(|\mathrm{grad}\, v|) \sum_{i=1}^{2} \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\,\mathrm{d}x,\\
l_{\varepsilon_{nM}}(v, w) &= \Theta_M \int_{\varepsilon_{nM}} \sum_{i=1}^{2} \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\,\mathrm{d}x,
\end{aligned}
$$

where $\varepsilon_{nM} = \tau_{nM}, \omega_{nM}$, $(M = E, P)$ are defined by (109).

The forms $a_{\varepsilon_{nM}}(v, w)$ and $l_{\varepsilon_{nM}}(v, w)$ are bounded and we have

$$
\begin{aligned}
|a_{\varepsilon_{nM}}(v, w)| &\leqslant \beta_M |v|_{1,\varepsilon_{nM}} |w|_{1,\varepsilon_{nM}} \ \ \forall v, w \in H^1(\widetilde{\Omega}),\\
|l_{\varepsilon_{nM}}(v, w)| &\leqslant \Theta_M |v|_{1,\varepsilon_{nM}} |w|_{1,\varepsilon_{nM}} \ \ \forall v, w \in H^1(\widetilde{\Omega}).
\end{aligned}
$$

P r o o f. The proof is similar to that in [11], p. 362. □

R e m a r k 4.11. Let us define an auxiliary finite element Rothe function

$$
(116) \quad \overline{S}_{jnP}(t) = \overline{S}^{i-1}_{jnP} + (\Delta\overline{S}^{i}_{jnP}/\Delta t_n)(t - t_{i-1}), \quad t \in [t_{i-1}, t_i] \quad (i = 1, \ldots, r_n)
$$

and step-functions

$$\begin{aligned}
(117)\qquad \widetilde{s}_{jn}(t) &= \overline{S}_{jn}^{i-1}, \quad t \in [t_{i-1}, t_i) \quad (i = 1, \dots, r_n),\\
\widetilde{s}_{jn}(T) &= \overline{S}_{jn}^{r_n-1},\\
(118)\qquad \widetilde{s}_{jnM}(t) &= \overline{S}_{jnM}^{i-1}, \quad t \in [t_{i-1}, t_i) \quad (i = 1, \dots, r_n),\\
\widetilde{s}_{jnM}(T) &= \overline{S}_{jnM}^{r_n-1} \ (M = E, P),\\
(119)\qquad \widetilde{S}_{jnM}(t) &= \widehat{S}_{jnM}^{i-1}, \quad t \in [t_{i-1}, t_i) \quad (i = 1, \dots, r_n),\\
\widetilde{S}_{jnM}(T) &= \widehat{S}_{jnM}^{r_n-1} \quad (M = E, P),\\
(120)\qquad \widetilde{f}_n(t) &= f(t_{i-1}), \quad t \in [t_{i-1}, t_i) \quad (i = 1, \dots, r_n),\\
\widetilde{f}_n(T) &= f(t_{r_n-1}),\\
(121)\qquad \widetilde{\varphi}_n(t) &= \varphi(t_i), \quad t \in (t_{i-1}, t_i] \quad (i = 1, \dots, r_n),\\
\widetilde{\varphi}_n(0) &= \varphi(t_1),
\end{aligned}$$

where $t_i$ are defined by (48), $\overline{S}_{jn}^{i-1}$, $\overline{S}_{jnM}^{i-1}$ denote natural extensions of $S_{jn}^{i-1}$, $S_{jnM}^{i-1}$, the symbol $\widehat{S}_{jnM}^{i-1}$ denotes the function associated with $S_{jnM}^{i-1}$ and $\varphi \in C^\infty(\bar{I})$.

**Theorem 4.12.** *The function $u_j$ from Lemma* 4.8 *and the strong derivative $\dot{u}_{jP}$ of $u_{jP}$ form the unique pair satisfying the relations*

$$\begin{aligned}
(122)\qquad \int_0^t (\dot{u}_{jP}(\tau), v_P(\tau))_P \,\mathrm{d}\tau + \int_0^t a(u_j(\tau), v(\tau)) \,\mathrm{d}\tau \\
= \int_0^t (f(\tau), v(\tau)) \,\mathrm{d}\tau \ \ \forall v \in L_2(I, V) \ \ \forall t \in \bar{I},
\end{aligned}$$

$$(123)\qquad u_{jP}(0) = z_P + s_j^P,$$

$$(124)\qquad \operatorname{tr}(u_j(t)) = \operatorname{tr} z \quad \textit{in} \quad L_2(\Gamma_1) \ \ \forall t \in I - E_j,$$

*where* $\operatorname{meas}_1 E_j = 0$ *and we have*

$$\begin{aligned}
(125)\qquad \widehat{S}_{jnP} &\to u_{jP} && \textit{in} \quad C(\bar{I}, L_2(\Omega_P)),\\
(126)\qquad \widehat{S}_{jn} &\to u_j && \textit{in} \quad L_2(I, H^1(\Omega)),\\
(127)\qquad \widetilde{S}_{jn} &\to u_j && \textit{in} \quad L_2(I, H^1(\Omega)),\\
(128)\qquad \mathrm{d}\widehat{S}_{jnP}/\mathrm{d}t &\rightharpoonup \dot{u}_{jP} && \textit{weakly in} \quad L_2(I, L_2(\Omega_P)).
\end{aligned}$$

Proof. A) Let $w \in V$ be an arbitrary, but fixed function. Let $\{w_n\}$, where $w_n \in V_n$, be such a sequence that

$$(129)\qquad \lim_{n\to\infty} \|w_n - w^C\|_{1,\Omega_n} = 0, \quad \lim_{n\to\infty} \|w_{nM} - w^C\|_{1,M_n} = 0 \quad (M = E, P),$$

where $w^C$ is the Calderon extension of $w$. The existence of $\{w_n\}$ is guaranteed by Theorem 31.4 of [12]. We consider a function $\varphi \in C^\infty(\bar{I})$. Let us set $v = w_n\varphi(t_i)$, $w_n \in V_n$ in (63), $v = w_n\varphi(t_1)$ in (61) and let us multiply these relations by $\Delta t_n$. After summing up from $i = 1$ to $i = r_n$ we get

$$
\begin{aligned}
(130)\qquad \sum_{i=1}^{r_n} (\Delta S^i_{jnP}, w_{nP})_{P_n}\varphi(t_i) &+ \Delta t_n a_n(S^1_{jn}, w_n)\varphi(t_1) \\
&+ \Delta t_n \sum_{i=2}^{r_n} a_n(S^{i-1}_{jn}, w_n)\varphi(t_i) \\
= &- \Delta t_n \sum_{i=2}^{r_n} l_n(\Delta S^i_{jn}, w_n)\varphi(t_i) + \Delta t_n (f(t_1), w_n)^I_n \varphi(t_1) \\
&+ \Delta t_n \sum_{i=2}^{r_n} (f(t_{i-1}), w_n)^I_n \varphi(t_i).
\end{aligned}
$$

Let us use the auxiliary functions (116), (117), (120) and (121). Then we can write (130) in the form

$$
\begin{aligned}
(131)\qquad \int_0^T (\,\mathrm{d}\overline{S}_{jnP}(t)/\,\mathrm{d}t, w_{nP})_{P_n}\widetilde{\varphi}_n(t)\,\mathrm{d}t &+ \int_0^T a_n(\widetilde{s}_{jn}(t), w_n)\widetilde{\varphi}_n(t)\,\mathrm{d}t \\
&+ \Delta t_n[a_n(\overline{S}^1_{jn}, w_n) - a_n(\overline{S}^0_{jn}, w_n)]\varphi(\Delta t_n) \\
= &- \Delta t_n \sum_{i=2}^{r_n} l_n(\Delta \overline{S}^i_{jn}, w_n)\varphi(t_i) + \int_0^T (\widetilde{f}_n(t), w_n)^I_n \widetilde{\varphi}_n(t)\,\mathrm{d}t \\
&+ \Delta t_n (f(\Delta t_n), w_n)^I_n \varphi(\Delta t_n) - \Delta t_n (f(0), w_n)^I_n \varphi(\Delta t_n).
\end{aligned}
$$

Let us use the functions given by (84), (85), (92), (116)–(120). As we have

$$
\begin{aligned}
a(\widetilde{S}_{jk}(t), \widehat{w}_k) &= \sum_{M=E,P} a_M(\widetilde{S}_{jkM}(t), \widehat{w}_{kM}), \\
a_k(\widetilde{s}_{jk}(t), \overline{w}_k) &= \sum_{M=E,P} \{a_M(\widetilde{s}_{jkM}(t), \overline{w}_{kM}) + a_{\tau_{kM}}(\widetilde{s}_{jkM}(t), \overline{w}_{kM}) \\
&\qquad - a_{\omega_{kM}}(\widetilde{s}_{jkM}(t), \overline{w}_{kM})\},
\end{aligned}
$$

where $\{k\}$ is the subsequence of the sequence $\{n\}$ appearing in Lemma 4.8 and the sets $\varepsilon_{kM}$, ($\varepsilon = \tau, \omega$, $M = E, P$) are defined in (109), we can write relation (131) for a subsequence $\{k\}$ of $\{n\}$ in the following way:

$$\begin{aligned}(132)\quad &\int_0^T (\mathrm{d}\widehat{S}_{jkP}(t)/\,\mathrm{d}t, \widehat{w}_{kP})_P \widetilde{\varphi}_k(t)\,\mathrm{d}t + \sum_{m=1}^{3} A_{kP}^m + \int_0^T a(\widetilde{S}_{jk}(t), \widehat{w}_k)\widetilde{\varphi}_k(t)\,\mathrm{d}t \\ &+ \sum_{m=1}^{3}\sum_{M=E,P} B_{kM}^m + R_k^1 + \sum_{m=1}^{3}\sum_{M=E,P} C_{kM}^m - R_k^2 - \sum_{m=1}^{3}\sum_{M=E,P} D_{kM}^m \\ = &- R_k^3 - \sum_{m=1}^{3}\sum_{M=E,P} E_{kM}^m + \int_0^T (\widetilde{f}_k(t), \widehat{w}_k)\widetilde{\varphi}_k(t)\,\mathrm{d}t \\ &+ \sum_{m=1}^{3}\sum_{M=E,P} F_{kM}^m + R_k^4 + \sum_{m=1}^{3}\sum_{M=E,P} G_{kM}^m - R_k^5 - \sum_{m=1}^{3}\sum_{M=E,P} H_{kM}^m\end{aligned}$$

where we define

$$\begin{aligned}A_{kP}^1 &= \int_0^T (\,\mathrm{d}\widehat{S}_{jkP}(t)/\mathrm{d}t, \overline{w}_{kP} - \widehat{w}_{kP})_P \widetilde{\varphi}_k(t)\,\mathrm{d}t,\\ A_{kP}^2 &= \int_0^T (\,\mathrm{d}\overline{S}_{jkP}(t)/\mathrm{d}t - \,\mathrm{d}\widehat{S}_{jkP}(t)/\mathrm{d}t, \overline{w}_{kP})_P \widetilde{\varphi}_k(t)\,\mathrm{d}t,\\ A_{kP}^3 &= \int_0^T [(\,\mathrm{d}S_{jkP}(t)/\mathrm{d}t, w_{kP})_{\tau_{kP}} - (\mathrm{d}\overline{S}_{jkP}(t)/\mathrm{d}t, \overline{w}_{kP})_{\omega_{kP}}]\widetilde{\varphi}_k(t)\,\mathrm{d}t,\\ B_{kM}^1 &= \int_0^T a_M(\widetilde{S}_{jkM}(t), \overline{w}_{kM} - \widehat{w}_{kM})\widetilde{\varphi}_k(t)\,\mathrm{d}t,\\ B_{kM}^2 &= \int_0^T [a_M(\widetilde{s}_{jkM}(t), \overline{w}_{kM}) - a_M(\widetilde{S}_{jkM}(t), \overline{w}_{kM})]\widetilde{\varphi}_k(t)\,\mathrm{d}t,\\ B_{kM}^3 &= \int_0^T [a_{\tau_{kM}}(\widetilde{s}_{jkM}(t), w_{kM}) - a_{\omega_{kM}}(\widetilde{s}_{jkM}(t), \overline{w}_{kM})]\widetilde{\varphi}_k(t)\,\mathrm{d}t,\\ C_{kM}^1 &= \Delta t_k a_M(\widehat{S}_{jkM}^1, \overline{w}_{kM} - \widehat{w}_{kM})\varphi(\Delta t_k),\\ C_{kM}^2 &= \Delta t_k [a_M(\overline{S}_{jkM}^1, \overline{w}_{kM}) - a_M(\widehat{S}_{jkM}^1, \overline{w}_{kM})]\varphi(\Delta t_k),\\ C_{kM}^3 &= \Delta t_k [a_{\tau_{kM}}(S_{jkM}^1, w_{kM}) - a_{\omega_{kM}}(\overline{S}_{jkM}^1, \overline{w}_{kM})]\varphi(\Delta t_k),\\ D_{kM}^1 &= \Delta t_k a_M(\widehat{S}_{jkM}^0, \overline{w}_{kM} - \widehat{w}_{kM})\varphi(\Delta t_k),\\ D_{kM}^2 &= \Delta t_k [a_M(\overline{S}_{jkM}^0, \overline{w}_{kM}) - a_M(\widehat{S}_{jkM}^0, \overline{w}_{kM})]\varphi(\Delta t_k),\\ D_{kM}^3 &= \Delta t_k [a_{\tau_{kM}}(S_{jkM}^0, w_{kM}) - a_{\omega_{kM}}(\overline{S}_{jkM}^0, \overline{w}_{kM})]\varphi(\Delta t_k),\\ E_{kM}^1 &= \Delta t_k \sum_{i=2}^{r_k} l_M(\Delta\widehat{S}_{jkM}^i, \overline{w}_{kM} - \widehat{w}_{kM})\varphi(t_i),\\ E_{kM}^2 &= \Delta t_k \sum_{i=2}^{r_k} l_M(\Delta(\overline{S}_{jkM}^i - \widehat{S}_{jkM}^i), \overline{w}_{kM})\varphi(t_i),\end{aligned}$$

$$
\begin{aligned}
E^3_{kM} &= \Delta t_k \sum_{i=2}^{r_k} [l_{\tau_{kM}}(\Delta S^i_{jkM}, w_{kM}) - l_{\omega_{kM}}(\Delta \overline{S}^i_{jkM}, \overline{w}_{kM})]\varphi(t_i),\\
F^1_{kM} &= \int_0^T (\widetilde{f}_{kM}(t), \overline{w}_{kM} - \widehat{w}_{kM})_M \widetilde{\varphi}_k(t)\,\mathrm{d}t,\\
F^2_{kM} &= \int_0^T [(\widetilde{f}_{kM}(t), w_{kM})_{\tau_{kM}} - (\widetilde{f}_{kM}(t), \overline{w}_{kM})_{\omega_{kM}}]\widetilde{\varphi}_k(t)\,\mathrm{d}t,\\
F^3_{kM} &= \int_0^T [(\widetilde{f}_{kM}(t), w_{kM})^I_{M_k} - (\widetilde{f}_{kM}(t), w_{kM})_{M_k}]\widetilde{\varphi}_k(t)\,\mathrm{d}t,\\
G^1_{kM} &= \Delta t_k (f_M(\Delta t_k), \overline{w}_{kM} - \widehat{w}_{kM})_M \varphi(\Delta t_k),\\
G^2_{kM} &= \Delta t_k [(f_M(\Delta t_k), w_{kM})_{\tau_{kM}} - (f_M(\Delta t_k), \overline{w}_{kM})_{\omega_{kM}}]\varphi(\Delta t_k),\\
G^3_{kM} &= \Delta t_k [(f_M(\Delta t_k), w_{kM})^I_{M_k} - (f_M(\Delta t_k), w_{kM})_{M_k}]\varphi(\Delta t_k),\\
H^1_{kM} &= \Delta t_k (f_M(0), \overline{w}_{kM} - \widehat{w}_{kM})_M \varphi(\Delta t_k),\\
H^2_{kM} &= \Delta t_k [(f_M(0), w_{kM})_{\tau_{kM}} - (f_M(0), \overline{w}_{kM})_{\omega_{kM}}]\varphi(\Delta t_k),\\
H^3_{kM} &= \Delta t_k [(f_M(0), w_{kM})^I_{M_k} - (f_M(0), w_{kM})_{M_k}]\varphi(\Delta t_k),\\
R^1_k &= \Delta t_k\, a(\widehat{S}^1_{jk}, \widehat{w}_k)\varphi(\Delta t_k),\\
R^2_k &= \Delta t_k\, a(\widehat{S}^0_{jk}, \widehat{w}_k)\varphi(\Delta t_k),\\
R^3_k &= \Delta t_k \sum_{i=2}^{r_k} l(\Delta \widehat{S}^i_{jk}, \widehat{w}_k)\varphi(t_i),\\
R^4_k &= \Delta t_k\, (f(\Delta t_k), \widehat{w}_k)\varphi(\Delta t_k),\\
R^5_k &= \Delta t_k (f(0), \widehat{w}_k)\varphi(\Delta t_k).
\end{aligned}
$$

B) Let the symbol $H^*(\Omega)$ denote the dual space of $H^1(\Omega)$. For every $k \in \{k\}$ and every $t \in I$ we can define $\chi_{jk}(t) \in H^*(\Omega)$ by the relation

$$\langle \chi_{jk}(t), w\rangle := a(\widetilde{S}_{jk}(t), w) \ \ \forall w \in H^1(\Omega). \tag{133}$$

From (85) it follows that

$$\|\chi_{jk}(t)\|_* \equiv \sup_{\|w\|_1=1} \langle \chi_{jk}(t), w\rangle = \sup_{\|w\|_1=1} a(\widehat{S}^{i-1}_{jk}, w), \quad t \in [t_{i-1}, t_i) \ \ (i = 1, \dots, r_n).$$

By (22) (with $z = 0$) and (82) from Corollary 4.4 we get $\|\chi_{jk}(t)\|_* \leqslant \beta C(j)\ \forall t \in I$. Thus $\chi_{jk} \in L_\infty(I, H^*(\Omega))$ and we have

$$\|\chi_{jk}\|_{L_2(I,H^*(\Omega))} \leqslant \sqrt{T}\beta C(j), \tag{134}$$

where $T$ is the length of the interval $I$. This result, the reflexivity of $H^1(\Omega)$ and Theorems $\mathcal{P}.133$, $\mathcal{P}.125$, $\mathcal{P}.132$ of [12] imply the existence of a subsequence of $\{k\}$

(we denote it again by $\{k\}$) and an abstract function $\chi_j \in L_2(I, H^*(\Omega))$ such that

$$\chi_{jk} \rightharpoonup \chi_j \quad \text{weakly in} \quad L_2(I, H^*(\Omega)). \tag{135}$$

It is not difficult to prove that (details are omitted)

$$\lim_{k\to\infty} \int_0^T (\mathrm{d}\widehat{S}_{jkP}(t)/\,\mathrm{d}t, \widehat{w}_{kP})_P \widetilde{\varphi}_k(t)\,\mathrm{d}t = \int_0^T (\dot{u}_{jP}(t), w_P)_P \varphi(t)\,\mathrm{d}t, \tag{136}$$

$$\lim_{k\to\infty} \int_0^T a(\widetilde{S}_{jk}(t), \widehat{w}_k) \widetilde{\varphi}_k(t)\,\mathrm{d}t = \int_0^T \langle \chi_j(t), w\rangle \varphi(t)\,\mathrm{d}t, \tag{137}$$

$$\lim_{k\to\infty} \int_0^T (\widetilde{f}_k(t), \widehat{w}_k) \widetilde{\varphi}_k(t)\,\mathrm{d}t = \int_0^T (f(t), w)\varphi(t)\,\mathrm{d}t, \tag{138}$$

$$\lim_{k\to\infty} A_{kP}^m = 0, \qquad \lim_{k\to\infty} B_{kM}^m = 0 \qquad (m = 1,2,3; M = E, P), \tag{139}$$

$$\lim_{k\to\infty} C_{kM}^m = 0, \qquad \lim_{k\to\infty} D_{kM}^m = 0 \qquad (m = 1,2,3; M = E, P), \tag{140}$$

$$\lim_{k\to\infty} E_{kM}^m = 0, \qquad \lim_{k\to\infty} F_{kM}^m = 0 \qquad (m = 1,2,3; M = E, P), \tag{141}$$

$$\lim_{k\to\infty} G_{kM}^m = 0, \qquad \lim_{k\to\infty} H_{kM}^m = 0 \qquad (m = 1,2,3; M = E, P), \tag{142}$$

$$\lim_{k\to\infty} R_k^m = 0 \qquad (m = 1, \dots, 5), \tag{143}$$

where $\{k\}$ is the same subsequence of $\{n\}$ as in (135). We show (136)–(138) only; the other relations can be proved using techniques from the proofs of [11, Theorem 3.8] and [12, Theorem 46.4], where relations similar to (139)–(143) have been proved.

Let us express the term on the left-hand side of (136) in the form

$$\int_0^T \big(\mathrm{d}\widehat{S}_{jkP}(t)/\mathrm{d}t, \widehat{w}_{kP}\big)_P \widetilde{\varphi}_k(t)\,\mathrm{d}t = \int_0^T \big(\mathrm{d}\widehat{S}_{jkP}(t)/\,\mathrm{d}t, \widehat{w}_{kP} - w_P\big)_P \widetilde{\varphi}_k(t)\,\mathrm{d}t$$
$$+ \int_0^T \big(\mathrm{d}\widehat{S}_{jkP}(t)/\mathrm{d}t, w_P\big)_P (\widetilde{\varphi}_k(t) - \varphi(t))\,\mathrm{d}t + \int_0^T \big(\mathrm{d}\widehat{S}_{jkP}(t)/\mathrm{d}t, w_P\big)_P \varphi(t)\,\mathrm{d}t.$$

Relation (136) follows from (100), (89), (111) and [12, Lemma 46.2].

Similarly, using (133) let us write

$$\int_0^T a\big(\widetilde{S}_{jk}(t), \widehat{w}_k\big) \widetilde{\varphi}_k(t)\,\mathrm{d}t = \int_0^T \langle \chi_{jk}(t), \widehat{w}_k\rangle \widetilde{\varphi}_k(t)\,\mathrm{d}t =$$
$$= \int_0^T \langle \chi_{jk}(t), \widehat{w}_k - w\rangle \widetilde{\varphi}_k(t)\,\mathrm{d}t + \int_0^T \langle \chi_{jk}(t), w\rangle (\widetilde{\varphi}_k(t) - \varphi(t))\,\mathrm{d}t$$
$$+ \int_0^T \langle \chi_{jk}(t), w\rangle \varphi(t)\,\mathrm{d}t.$$

Relations (134), (111), [12, Lemma 46.2] and (135) imply (137).

Finally, we can write

$$\begin{aligned}\int_0^T (\widetilde{f}_k(t),\widehat{w}_k)\widetilde{\varphi}_k(t)\,\mathrm{d}t = \sum_{M=E,P}\Big\{&\int_0^T \big(\widetilde{f}_{kM}(t),\widehat{w}_{kM}-w_M\big)_M\widetilde{\varphi}_k(t)\,\mathrm{d}t\\ &+\int_0^T \big(\widetilde{f}_{kM}(t),w_M\big)_M(\widetilde{\varphi}_k(t)-\varphi(t))\,\mathrm{d}t\\ &+\int_0^T \big(\widetilde{f}_{kM}(t),w_M)_M\varphi(t)\,\mathrm{d}t\Big\}.\end{aligned}$$

Relations (111), (93), [12, Lemma 46.2] and (12) yield (138).

Passing to the limit for $k\to\infty$ in (132) and using (136)–(143) we obtain

$$\begin{aligned}\int_0^T (\dot u_{jP}(t),w_P)_P\varphi(t)\,\mathrm{d}t + \int_0^T \langle\chi_j(t),w\rangle\varphi(t)\,\mathrm{d}t&\\ = \int_0^T (f(t),w)\varphi(t)\,\mathrm{d}t\ \ \forall w\in V\ \ \forall\varphi\in C^\infty(\bar I).&\end{aligned}\tag{144}$$

C) Restricting (144) to $\varphi\in C_0^\infty(I)$ we get

$$\int_0^T \{(\dot u_{jP}(t),w_P)_P+\langle\chi_j(t),w\rangle-(f(t),w)\}\varphi(t)\,\mathrm{d}t = 0\ \ \forall w\in V\ \ \forall\varphi\in C_0^\infty(I).$$

Hence by [12, Lemma $\mathcal{P}$.128]

$$(\dot u_{jP}(t),w_P)_P+\langle\chi_j(t),w\rangle = (f(t),w)\ \ \forall t\in I-E_w\ \ \forall w\in V,$$

where $\operatorname{meas}_1 E_w=0$. Let us integrate this relation over the interval $[t',t'']\subset\bar I$. We obtain

$$\begin{aligned}\int_{t'}^{t''}\{(\dot u_{jP}(t),w_P)_P+\langle\chi_j(t),w\rangle\}\,\mathrm{d}t&\\ = \int_{t'}^{t''}(f(t),w)\,\mathrm{d}t\ \ \forall t'<t''\in[0,T]\ \ \forall w\in V.&\end{aligned}\tag{145}$$

Let us choose $v\in L_2(I,V)$ and $t\in I$ arbitrarily. Let $\{z_n\}\subset L_2(I,V)$ be a sequence of step-functions such that

$$z_n\to v\quad\text{in}\quad L_2(I,V).\tag{146}$$

The existence of the sequence $\{z_n\}$ with the property (146) is guaranteed by [12, Theorem $\mathcal{P}$.118]. Then according to (145), we can write

$$\int_0^t \{(\dot{u}_{jP}(\tau), z_{nP}(\tau))_P + \langle \chi_j(\tau), z_n(\tau)\rangle\}\,\mathrm{d}\tau = \int_0^t (f(\tau), z_n(\tau))\,\mathrm{d}\tau. \tag{147}$$

Passing to the limit for $n \to \infty$ we conclude

$$\begin{aligned}\int_0^t \{(\dot{u}_{jP}(\tau), v_P(\tau))_P + \langle \chi_j(\tau), v(\tau)\rangle\}\,d\tau \\ = \int_0^t (f(\tau), v(\tau))\,\mathrm{d}\tau \ \ \forall v \in L_2(I,V) \ \ \forall t \in \bar{I}.\end{aligned} \tag{148}$$

D) In this part we prove that

$$\widehat{S}_{jk} \to u_j, \quad \widetilde{S}_{jk} \to u_j \quad \text{in} \quad L_2(I, H^1(\Omega)). \tag{149}$$

The strong monotonicity (21) of $a(v,w)$ gives

$$\begin{aligned}\int_0^T a\big(\widetilde{S}_{jk}(t), \widetilde{S}_{jk}(t) - u_j(t)\big)\,\mathrm{d}t - \int_0^T a\big(u_j(t), \widetilde{S}_{jk}(t) - u_j(t)\big)\,\mathrm{d}t \\ \geqslant \gamma \int_0^T \big|\widetilde{S}_{jk}(t) - u_j(t)\big|_1^2\,\mathrm{d}t.\end{aligned} \tag{150}$$

From (99) it follows that

$$\lim_{k\to\infty} \int_0^T a\big(u_j(t), \widetilde{S}_{jk}(t) - u_j(t)\big)\,\mathrm{d}t = 0. \tag{151}$$

According to (133), (135), we have

$$\lim_{k\to\infty} \int_0^T a\big(\widetilde{S}_{jk}(t), u_j(t)\big)\,\mathrm{d}t = \int_0^T \langle \chi_j(t), u_j(t)\rangle\,\mathrm{d}t. \tag{152}$$

We will prove that

$$\begin{aligned}\limsup_{k\to\infty} \int_0^T a\big(\widetilde{S}_{jk}(t), \widetilde{S}_{jk}(t)\big)\,\mathrm{d}t \\ \leqslant \int_0^T \langle \chi_j(t), z + \overset{\circ}{s}{}_j^P\rangle\,\mathrm{d}t + \int_0^T (f(t), u_j(t) - z - \overset{\circ}{s}{}_j^P)\,\mathrm{d}t \\ - \frac{1}{2}\big\|u_{jP}(T) - z_P - s_j^P\big\|_{0,P}^2.\end{aligned} \tag{153}$$

We consider (61) and (63) only for $\{k\} \subset \{n\}$ and multiply these relations by $\Delta t_k$. We choose $v = R^1_{jk} \in V_k$ in (61) and $v = R^{i-1}_{jk} \in V_k$ in (63), where

$$R^{i-1}_{jk} = S^{i-1}_{jk} - S^0_{jk}, \quad R^{i-1}_{jkP} = S^{i-1}_{jkP} - S^0_{jkP}, \quad i \geqslant 2.$$

Summing up from $i = 1$ to $i = r_k$ we obtain

$$\begin{aligned}(\Delta S^1_{jkP}, R^1_{jkP})_{P_k} + \sum_{i=2}^{r_k} (\Delta S^i_{jkP}, R^{i-1}_{jkP})_{P_k} + \Delta t_k a_k(S^1_{jk}, R^1_{jk}) \\ + \Delta t_k \sum_{i=2}^{r_k} a_k(S^{i-1}_{jk}, R^{i-1}_{jk}) \\ = \Delta t_k (f(t_1), R^1_{jk})^I_k + \Delta t_k \sum_{i=2}^{r_k} (f(t_{i-1}), R^{i-1}_{jk})^I_k \\ - \Delta t_k \sum_{i=2}^{r_k} l_k(\Delta S^i_{jk}, R^{i-1}_{jk}).\end{aligned}$$

As $\widehat{R}^1_{jk} = \Delta \widehat{S}^1_{jk} \in V$, $\widehat{R}^{i-1}_{jk} = \widehat{S}^{i-1}_{jk} - \widehat{S}^0_{jk} \in V$ the previous relation yields

$$\begin{aligned}(154)\quad \int_0^T (\widetilde{S}_{jk}(t), \widetilde{S}_{jk}(t))\,\mathrm{d}t = -(\Delta \widehat{S}^1_{jkP}, \Delta \widehat{S}^1_{jkP})_P \\ - (\Delta \widehat{S}^1_{jkP}, \Delta(\overline{S}^1_{jkP} - \widehat{S}^1_{jkP}))_P - (\Delta(\overline{S}^1_{jkP} - \widehat{S}^1_{jkP}), \Delta \overline{S}^1_{jkP})_P \\ - \{(\Delta S^1_{jkP}, \Delta S^1_{jkP})_{\tau_{kP}} - (\Delta \overline{S}^1_{jkP}, \Delta \overline{S}^1_{jkP})_{\omega_{kP}}\} \\ - \sum_{i=2}^{r_k} \big(\Delta \widehat{S}^i_{jkP}, \widehat{R}^{i-1}_{jkP}\big)_P - \sum_{i=2}^{r_k} \big(\Delta \widehat{S}^i_{jkP}, \overline{R}^{i-1}_{jkP} - \widehat{R}^{i-1}_{jkP}\big)_P \\ - \sum_{i=2}^{r_k} \big(\Delta(\overline{S}^i_{jkP} - \widehat{S}^i_{jkP}), \overline{R}^{i-1}_{jkP}\big)_P - \sum_{i=2}^{r_k} \{(\Delta S^i_{jkP}, R^{i-1}_{jkP})_{\tau_{kP}} - (\Delta \overline{S}^i_{jkP}, \overline{R}^{i-1}_{jkP})_{\omega_{kP}}\} \\ + \int_0^T a(\widetilde{S}_{jk}(t), \widehat{S}^0_{jk})\,\mathrm{d}t - \Delta t_k a(\widehat{S}^1_{jk}, \widehat{R}^1_{jk}) \\ - \Delta t_k \sum_{M=E,P} a_M(\widehat{S}^1_{jkM}, \overline{R}^1_{jkM} - \widehat{R}^1_{jkM}) - \\ - \Delta t_k \sum_{M=E,P} \{a_M(\overline{S}^1_{jkM}, \overline{R}^1_{jkM}) - a_M(\widehat{S}^1_{jkM}, \overline{R}^1_{jkM})\} \\ - \Delta t_k \sum_{M=E,P} \{a_{\tau_{kM}}(S^1_{jkM}, R^1_{jkM}) - a_{\omega_{kM}}(\overline{S}^1_{jkM}, \overline{R}^1_{jkM})\} \\ - \Delta t_k \sum_{i=2}^{r_k} \sum_{M=E,P} a_M(\widehat{S}^{i-1}_{jkM}, \overline{R}^{i-1}_{jkM} - \widehat{R}^{i-1}_{jkM})\end{aligned}$$

$$
\begin{aligned}
&- \Delta t_k \sum_{i=2}^{r_k} \sum_{M=E,P} \{a_M(\overline{S}^{i-1}_{jkM}, \overline{R}^{i-1}_{jkM}) - a_M(\widehat{S}^{i-1}_{jkM}, \overline{R}^{i-1}_{jkM})\} \\
&- \Delta t_k \sum_{i=2}^{r_k} \sum_{M=E,P} \{a_{\tau_{kM}}(S^{i-1}_{jkM}, R^{i-1}_{jkM}) - a_{\omega_{kM}}(\overline{S}^{i-1}_{jkM}, \overline{R}^{i-1}_{jkM})\} \\
&+ \int_0^T (\widetilde{f}_k(t), \widetilde{S}_{jk}(t))\,\mathrm{d}t - \int_0^T (\widetilde{f}_k(t), \widehat{S}^0_{jk})\,\mathrm{d}t \\
&+ \Delta t_k \sum_{i=2}^{r_k} \sum_{M=E,P} (f_M(t_{i-1}), \overline{R}^{i-1}_{jkM} - \widehat{R}^{i-1}_{jkM})_M \\
&+ \Delta t_k \sum_{i=2}^{r_k} \sum_{M=E,P} \big\{\big(f_M(t_{i-1}), R^{i-1}_{jkM}\big)_{\tau_{kM}} - \big(f_M(t_{i-1}), \overline{R}^{i-1}_{jkM}\big)_{\omega_{kM}}\big\} \\
&+ \Delta t_k \sum_{i=2}^{r_k} \sum_{M=E,P} \big\{\big(f_M(t_{i-1}), R^{i-1}_{jkM}\big)^I_{M_k} - (f_M(t_{i-1}), R^{i-1}_{jkM})_{M_k}\big\} \\
&+ \Delta t_k \big(f(t_1), \widehat{R}^1_{jk}\big) + \Delta t_k \sum_{M=E,P} \big(f_M(t_1), \overline{R}^1_{jkM} - \widehat{R}^1_{jkM}\big)_M \\
&+ \Delta t_k \sum_{M=E,P} \big\{\big(f_M(t_1), R^1_{jkM}\big)_{\tau_{kM}} - (f_M(t_1), \overline{R}^1_{jkM})_{\omega_{kM}}\big\} \\
&+ \Delta t_k \sum_{M=E,P} \big\{\big(f_M(t_1), R^1_{jkM}\big)^I_{M_k} - (f_M(t_1), R^1_{jkM})_{M_k}\big\} \\
&- \Delta t_k \sum_{i=2}^{r_k} l(\Delta \widehat{S}^i_{jk}, \widehat{R}^{i-1}_{jk}) - \Delta t_k \sum_{i=2}^{r_k} \sum_{M=E,P} l_M(\Delta \widehat{S}^i_{jkM}, \overline{R}^{i-1}_{jkM} - \widehat{R}^{i-1}_{jkM}) \\
&- \Delta t_k \sum_{i=2}^{r_k} \sum_{M=E,P} l_M(\Delta(\overline{S}^i_{jkM} - \widehat{S}^i_{jkM}), \overline{R}^{i-1}_{jkM}) \\
&- \Delta t_k \sum_{i=2}^{r_k} \sum_{M=E,P} \{l_{\tau_{kM}}(\Delta S^i_{jkM}, R^{i-1}_{jkM}) - l_{\omega_{kM}}(\Delta \overline{S}^i_{jkM}, \overline{R}^{i-1}_{jkM})\}.
\end{aligned}
$$

Now we estimate the first and the fifth term on the right-hand side of (154). By relation (84) we can write

$$
\begin{aligned}
(155)\qquad & -(\Delta \widehat{S}^1_{jkP}, \Delta \widehat{S}^1_{jkP})_P - \sum_{i=2}^{r_k} \Big(\Delta \widehat{S}^i_{jkP}, \widehat{R}^{i-1}_{jkP}\Big)_P \\
&= -\big\|\Delta \widehat{S}^1_{jkP}\big\|^2_{0,P} - \sum_{i=2}^{r_k} \Big(\widehat{R}^i_{jkP} - \widehat{R}^{i-1}_{jkP}, \widehat{R}^{i-1}_{jkP}\Big)_P
\end{aligned}
$$

$$
\begin{aligned}
&= -\big\|\Delta\widehat{S}^1_{jkP}\big\|^2_{0,P}\\
&\quad -\sum_{i=2}^{r_k}\Big(\widehat{R}^i_{jkP}-\widehat{R}^{i-1}_{jkP},\frac{1}{2}(\widehat{R}^{i-1}_{jkP}-\widehat{R}^i_{jkP})+\frac{1}{2}(\widehat{R}^{i-1}_{jkP}+\widehat{R}^i_{jkP})\Big)_P\\
&= -\big\|\Delta\widehat{S}^1_{jkP}\big\|^2_{0,P}+\frac{1}{2}\sum_{i=2}^{r_k}\big\|\Delta\widehat{S}^i_{jkP}\big\|^2_{0,P}\\
&\quad -\frac{1}{2}\sum_{i=2}^{r_k}\big(\big\|\widehat{R}^i_{jkP}\big\|^2_{0,P}-\big\|\widehat{R}^{i-1}_{jkP}\big\|^2_{0,P}\big)\\
&\leqslant \frac{1}{2}\Delta t_k\int_0^T\big\|\,\mathrm{d}\widehat{S}_{jkP}(t)/\mathrm{d}t\big\|^2_{0,P}\,\mathrm{d}t-\frac{1}{2}\big\|\widehat{S}_{jkP}(T)-\widehat{S}^0_{jkP}\big\|^2_{0,P}.
\end{aligned}
$$

Relation (89) implies

$$
\lim_{k\to\infty}\Delta t_k\int_0^T\big\|\,\mathrm{d}\widehat{S}_{jkP}(t)/\,\mathrm{d}t\big\|^2_{0,P}\,\mathrm{d}t\leqslant C(j)\lim_{k\to\infty}\Delta t_k=0. \tag{156}
$$

According to (97) and (91) we have

$$
\lim_{k\to\infty}\|\widehat{S}_{jkP}(T)-\widehat{S}^0_{jkP}\|_{0,P}=\|u_{jP}(T)-z_P-s^P_j\|_{0,P}. \tag{157}
$$

Thus relations (155)–(157) give

$$
\begin{aligned}
&\limsup_{k\to\infty}\Big\{-\big(\Delta\widehat{S}^1_{jkP},\Delta\widehat{S}^1_{jkP}\big)_P-\sum_{i=2}^{r_k}\big(\Delta\widehat{S}^i_{jkP},\widehat{R}^{i-1}_{jkP}\big)_P\Big\}\\
&\qquad\leqslant -\frac{1}{2}\|u_{jP}(T)-z_P-s^P_j\|^2_{0,P}.
\end{aligned} \tag{158}
$$

Using (90), (133) and (135) we find

$$
\lim_{k\to\infty}\int_0^T a\big(\widetilde{S}_{jk}(t),\widehat{S}^0_{jk}\big)\,\mathrm{d}t=\int_0^T\big\langle\chi_j(t),z+\overset{\circ}{s}{}^P_j\big\rangle\,\mathrm{d}t. \tag{159}
$$

Further, taking into account (12), (93), (99) and (90) we obtain

$$
\lim_{k\to\infty}\int_0^T\big(\widetilde{f}_k(t),\widetilde{S}_{jk}(t)\big)\,\mathrm{d}t=\int_0^T(f(t),u_j(t))\,\mathrm{d}t, \tag{160}
$$

$$
\lim_{k\to\infty}\int_0^T\big(\widetilde{f}_k(t),\widehat{S}^0_{jk}\big)\,\mathrm{d}t=\int_0^T(f(t),z+\overset{\circ}{s}{}^P_j)\,\mathrm{d}t. \tag{161}
$$

It is not difficult (only technical) to prove that the remaining terms on the right-hand side of (154) tend to zero with $k \to \infty$ ($j$ is fixed). (The proof is a simple modification of considerations introduced in part C of the proof of Theorem 3.8 in [11]. Thus summarizing we see that all results (158)–(161) and relation (154) imply (153).

In part E we will prove that

$$\int_0^T \langle \chi_j(t), u_j(t)\rangle \,\mathrm{d}t = \int_0^T \langle \chi_j(t), z + \overset{\circ}{s}{}_j^P\rangle \,\mathrm{d}t + \int_0^T \bigl(f(t), u_j(t) - z - \overset{\circ}{s}{}_j^P\bigr)\,\mathrm{d}t - \frac{1}{2}\|u_{jP}(T) - z_P - s_j^P\|_{0,P}^2. \tag{162}$$

Comparing this result with (153) we obtain

$$\limsup_{k\to\infty} \int_0^T a\bigl(\widetilde{S}_{jk}(t), \widetilde{S}_{jk}(t)\bigr)\,\mathrm{d}t \leqslant \int_0^T \langle \chi_j(t), u_j(t)\rangle \,\mathrm{d}t. \tag{163}$$

Relations (152) and (163) yield

$$\limsup_{k\to\infty} \int_0^T a\bigl(\widetilde{S}_{jk}(t), \widetilde{S}_{jk}(t) - u_j(t)\bigr)\,\mathrm{d}t \leqslant 0.$$

This relation together with (150) and (151) implies

$$\limsup_{k\to\infty} \int_0^T |\widetilde{S}_{jk}(t) - u_j(t)|_1^2 \,\mathrm{d}t \leqslant 0.$$

Therefore, we have

$$\lim_{k\to\infty} \int_0^T |\widetilde{S}_{jk}(t) - u_j(t)|_1^2 \,\mathrm{d}t = 0. \tag{164}$$

By (97) we get

$$\int_0^T \|\widehat{S}_{jkP}(t) - u_{jP}(t)\|_{0,P}^2 \,\mathrm{d}t \leqslant T \max_{t\in\bar{I}} \|\widehat{S}_{jkP}(t) - u_{jP}(t)\|_{0,P}^2 \to 0 \ \text{ for } \ k \to \infty. \tag{165}$$

According to (88), it follows that

$$\lim_{k\to\infty} \|\widetilde{S}_{jkP} - \widehat{S}_{jkP}\|_{L_2(I,L_2(\Omega_P))}^2 = 0. \tag{166}$$

As

$$\|\widetilde{S}_{jkP} - u_{jP}\|_{L_2(I,L_2(\Omega_P))} \leqslant \|\widetilde{S}_{jkP} - \widehat{S}_{jkP}\|_{L_2(I,L_2(\Omega_P))} + \|\widehat{S}_{jkP} - u_{jP}\|_{L_2(I,L_2(\Omega_P))},$$

relations (165) and (166) yield

$$\lim_{k\to\infty} \|\widetilde{S}_{jkP} - u_{jP}\|_{L_2(I,L_2(\Omega_P))} = 0. \tag{167}$$

Now we use the continuous form of inequality (53) (see [8, (1)]):

$$\|u\|_1^2 \leqslant C(\|u\|_{0,P}^2 + |u|_1^2) \;\; \forall u \in H^1(\Omega). \tag{168}$$

Thus relations (164), (167) and (168) give

$$\lim_{k\to\infty} \|\widetilde{S}_{jk} - u_j\|_{L_2(I,H^1(\Omega))} = 0$$

and relation $(149)_2$ holds. Moreover, relations $(149)_2$ and (88) imply $(149)_1$.

E) Now we prove relation (162) used in part D. Let us define

$$\widehat{Q}_{jk}^{i-1} := \widehat{S}_{jk}^{i-1} - \widehat{I}_k(z + \overset{\circ}{s}{}_j^P) \in V, \tag{169}$$

where $I_k(w) \in X_h$ is the interpolant of $w$ and $\widehat{I}_k(w)$ denotes the function associated with $I_k(w)$. By (62), (67), [12, Lemma 48.5] and a priori estimates (82) we obtain

$$\|\widehat{Q}_{jk}^{i-1}\|_1 \leqslant \|\widehat{S}_{jk}^{i-1}\|_1 + \|\widehat{S}_{jk}^0\|_1 \leqslant C(j).$$

Hence for the step-functions

$$\widetilde{Q}_{jk}(t) = \widehat{Q}_{jk}^{i-1}, \quad t \in [t_{i-1}, t_i) \quad (i = 1, \dots, r_k) \tag{170}$$

we have

$$\|\widetilde{Q}_{jk}\|_{L_2(I,V)} \leqslant C(j)\sqrt{T} \quad (k = 1, \dots, r_k). \tag{171}$$

According to (171) and the reflexivity of $L_2(I,V)$ (which follows from [12, Theorem $\mathcal{P}$.125] and Theorem $\mathcal{P}$.132 of [12], there exist a subsequence of the sequence $\{k\}$ (we denote it again by $\{k\}$) and a function $w_j \in L_2(I,V)$ such that

$$\widetilde{Q}_{jk} \rightharpoonup w_j \quad \text{in} \quad L_2(I,V) \subset L_2(I,H^1(\Omega)). \tag{172}$$

Now let $\{\Phi\}$ be a set of all linear functionals on $L_2(I,H^1(\Omega))$. Using (99) we have for an arbitrary functional $\Phi \in \{\Phi\}$

$$\lim_{k\to\infty} \Phi(\widetilde{S}_{jk}) = \Phi(u_j). \tag{173}$$

As by (90) and (62)

$$\lim_{k\to\infty} \big\| z + \overset{\circ}{s}{}_j^P - \widehat{I}_k(z + \overset{\circ}{s}{}_j^P) \big\|_1 = 0,$$

we have

$$\lim_{k\to\infty} \Phi(\widehat{I}_k(z + \overset{\circ}{s}{}_j^P)) = \Phi(z + \overset{\circ}{s}{}_j^P) \ \ \forall \Phi \in \{\Phi\}. \tag{174}$$

Relations (85), (169) and (170) imply

$$\widetilde{Q}_{jk}(t) = \widetilde{S}_{jk}(t) - \widehat{I}_k(z + \overset{\circ}{s}{}_j^P). \tag{175}$$

Let us choose $\Phi \in \{\Phi\}$ arbitrarily. The linearity of the functional $\Phi$ and relations (175), (173), (174) give

$$\lim_{k\to\infty} \Phi(\widetilde{Q}_{jk}) = \lim_{k\to\infty} \Phi(\widetilde{S}_{jk}) - \lim_{k\to\infty} \Phi(\widehat{I}_k(z + \overset{\circ}{s}{}_j^P)) = \Phi(u_j - z - \overset{\circ}{s}{}_j^P). \tag{176}$$

On the other hand, relation (172) implies

$$\lim_{k\to\infty} \Phi(\widetilde{Q}_{jk}) = \Phi(w_j). \tag{177}$$

Hence relations (176) and (177) and the uniqueness of the weak limit yield $w_j = u_j - z - \overset{\circ}{s}{}_j^P$ in $L_2(I, H^1(\Omega))$. Thus we have

$$w_j(t) = u_j(t) - z - \overset{\circ}{s}{}_j^P \quad \text{in} \quad H^1(\Omega) \ \ \forall t \in I - E_j. \tag{178}$$

Let us set now $v = w_j \in L_2(I, V)$ in relation (148):

$$\int_0^t \{(\dot{u}_{jP}(\tau), w_{jP}(\tau))_P + \langle \chi_j(\tau), w_j(\tau)\rangle\}\, \mathrm{d}\tau = \int_0^t (f(\tau), w_j(\tau))\, \mathrm{d}\tau. \tag{179}$$

As the equality in $L_2(\Omega)$ means the equality almost everywhere, it follows from the properties of the Lebesgue integral and (178), (179) that we have

$$\begin{aligned} \int_0^t \big(\dot{u}_{jP}(\tau), u_{jP}(\tau) - z_P - s_j^P\big)_P \, \mathrm{d}\tau + \int_0^t \big\langle \chi_j(\tau), u_j(\tau) - z - \overset{\circ}{s}{}_j^P \big\rangle \, \mathrm{d}\tau \\ = \int_0^t \big(f(\tau), u_j(\tau) - z - \overset{\circ}{s}{}_j^P\big)\, \mathrm{d}\tau. \end{aligned} \tag{180}$$

Let us set $t = T$ in (180). For the validity of (162) it remains to prove

$$\int_0^T \big(\dot{u}_{jP}(t), u_{jP}(t) - z_P - s_j^P\big)_P \, \mathrm{d}t = \frac{1}{2} \big\| u_{jP}(T) - z_P - s_j^P \big\|_{0,P}^2. \tag{181}$$

We see that

$$(182)\qquad \int_0^T \big(\dot u_{jP}(t), u_{jP}(t)\big)_P \,\mathrm{d}t = \frac{1}{2}\big(u_{jP}(T), u_{jP}(T)\big)_P - \frac{1}{2}(u_{jP}(0), u_{jP}(0))_P.$$

According to [12, Theorem $\mathcal{P}$.112(b)], we have

$$(183)\qquad \int_0^T \big(\dot u_{jP}(t), z_P + s_j^P\big)_P \,\mathrm{d}t = \bigg(\int_0^T \dot u_{jP}(t)\,\mathrm{d}t, z_P + s_j^P\bigg)_P = \big(u_{jP}(T), z_P + s_j^P\big)_P - \big(u_{jP}(0), z_P + s_j^P\big)_P.$$

As by (101) we have $u_{jP}(0) = z_P + s_j^P$, subtracting (183) from (182) we obtain (181).

F) Relation (123) is relation (101) from Lemma 4.8.

G) Now we prove relation (124). By virtue of (178), relation

$$(184)\qquad w_j(t) = u_j(t) - z - \overset{\circ}{s}{}_j^P \quad \forall t \in I - E_j$$

is satisfied almost everywhere in $\Omega$. Then the both sides of (184) are equal from the point of view of the space $H^1(\Omega)$. As $w_j \in L_2(I, V)$ we have $\mathrm{tr}(w_j(t)) = 0$ on $\Gamma_1$. This relation and (184) imply (124).

H) Using the Lipschitz continuity (22) of the form $a(v, w)$ we can write

$$\bigg|\int_0^t \big\{a\big(u_j(\tau), v(\tau)\big) - a\big(\widetilde S_{jk}(\tau), v(\tau)\big)\big\}\,\mathrm{d}\tau\bigg| \leqslant \beta \|u_j - \widetilde S_{jk}\|_{L_2(I,H^1(\Omega))}\,\|v\|_{L_2(I,V)} \quad \forall v \in L_2(I, V)\ \ \forall t \in \bar I.$$

Passing to the limit for $k \to \infty$ and taking into account (133), (135) and $(149)_2$ we find

$$\int_0^t a\big(u_j(\tau), v(\tau)\big)\,\mathrm{d}\tau = \int_0^t \langle \chi_j(\tau), v(\tau)\rangle\,\mathrm{d}\tau \quad \forall v \in L_2(I, V)\ \ \forall t \in \bar I.$$

Combining this result with (148) we obtain (122).

I) Now we prove the uniqueness of the solution of problem (122)–(124). Let us assume that there exist two functions $u_j^1, u_j^2$ satisfying together with their strong derivatives $\dot u_j^1, \dot u_j^2$ relations (122)–(124). Then we have

$$(185)\qquad \int_0^t \big\{\big(\dot u_{jP}^1(\tau) - \dot u_{jP}^2(\tau), v_P(\tau)\big)_P + a\big(u_j^1(\tau), v(\tau)\big) - a\big(u_j^2(\tau), v(\tau)\big)\big\}\,\mathrm{d}\tau = 0 \quad \forall v \in L_2(I, V)\ \ \forall t \in \bar I,$$

$$(186)\qquad u_{jP}^1(0) - u_{jP}^2(0) = 0.$$

Choosing $w = u_j^1 - u_j^2$ and using (124) we obtain

$$\operatorname{tr}(w(t)) = 0 \quad \text{in} \quad L_2(\Gamma_1) \ \forall t \in I - E_j. \tag{187}$$

By virtue of the equality $\operatorname{meas}_1 E_j = 0$, relation (187) and the fact that $u_j^i \in L_2(I, H^1(\Omega))$ $(i = 1, 2)$, we have $w \in L_2(I, V)$. Thus we can set $v = w$ in (185). Using the strong monotonicity (21) of $a(v, w)$, (186) and Friedrichs' inequality [12, Theorem $\mathcal{P}$.84] we obtain after integrating (185)

$$\frac{1}{2}\|u_{jP}^1(t) - u_{jP}^2(t)\|_{0,P}^2 + C\int_0^t \|u_j^1(\tau) - u_j^2(\tau)\|_1^2 \,\mathrm{d}\tau \leqslant 0 \ \ \forall t \in \bar{I}.$$

This inequality implies $u_{jP}^1(t) = u_{jP}^2(t)$ in $L_2(\Omega_P)$ $\forall t \in \bar{I}$ and

$$\int_{t'}^{t''} \|u_j^1(t) - u_j^2(t)\|_1^2 \,\mathrm{d}t = 0 \ \ \forall t', t'' \in \bar{I}.$$

Hence $\|u_j^1(t) - u_j^2(t)\|_1 = 0$ for almost all $t \in I$.

J) It remains to prove (125)–(128). Till now we have proved (97), (100) and (149), where $\{k\}$ is a subsequence of the sequence $\{n\}$. However, the uniqueness of the solution $u_j$ of the variational problem (122)–(124) implies that $\{k\} \equiv \{n\}$ (for details see [10], p. 26). Thus relations (125)–(128) hold. □

**Theorem 4.13.** *The solution of Problem* 2.1 *exists and is unique and we have*

$$\begin{aligned}
&(188) && u_{jP} \to u_P && \textit{in} \quad C(\bar{I}, L_2(\Omega_P)),\\
&(189) && u_j \to u && \textit{in} \quad L_2(I, H^1(\Omega)),\\
&(190) && \dot{u}_{jP} \rightharpoonup \dot{u}_P && \textit{weakly in} \quad L_2(I, V_P^*).
\end{aligned}$$

P r o o f. With small modifications we can follow the proof of [11, Theorem 3.10], only we consider the space $L_2(I, H^1(\Omega))$ instead of $L_2(I, V)$. We show just the differences.

Relation (189) gives $u \in L_2(I, H^1(\Omega))$. However, we need to prove that $u \in L_\infty(I, H^1(\Omega))$. From (189) it also follows that

$$u_j \rightharpoonup u \quad \text{in} \quad L_2(I, H^1(\Omega)). \tag{191}$$

As the norm $\|\cdot\|_1$ is weakly lower semicontinuous on $H^1(\Omega)$, similarly as in the proof of Lemma 4.8, owing to (191) and (94) we get

$$\|u(t)\|_1 \leqslant \liminf_{j\to\infty} \|u_j(t)\|_1 \leqslant C \ \ \forall t \in \bar{I}.$$

Thus $u \in L_\infty(I, H^1(\Omega))$. This fact together with (188) gives (7).

Now we prove (10). We can write

$$\int_0^T \|z - u(t)\|_{0,\Gamma_1}\,\mathrm{d}t \leqslant \int_0^T \|z - u_j(t)\|_{0,\Gamma_1}\,\mathrm{d}t + \int_0^T \|u_j(t) - u(t)\|_{0,\Gamma_1}\,\mathrm{d}t. \tag{192}$$

By virtue of (124), we have

$$\int_0^T \|z - u_j(t)\|_{0,\Gamma_1}\,\mathrm{d}t = 0.$$

Relation (189) and [12, Theorem $\mathcal{P}$.73(b)] imply

$$\int_0^T \|u_j(t) - u(t)\|_{0,\Gamma_1}\,\mathrm{d}t \leqslant C \int_0^T \|u_j(t) - u(t)\|_1\,\mathrm{d}t \to 0.$$

These results and (192) prove (10). □

Using Theorems 4.12, 4.13 and Lemma 3.3 which is essential we can prove the main result of this paper.

**Theorem 4.14.** *Let* (52) *and* (58) *be satisfied. Then we have*

$$\widehat{U}_{nP} \to u_P \quad \text{in} \quad C(\bar{I}, L_2(\Omega_P)), \tag{193}$$

$$\widehat{Z}_n \to u \quad \text{in} \quad L_2(I, H^1(\Omega)) \tag{194}$$

*where the functions* $\widehat{U}_{nP}(t)$ *and* $\widehat{Z}_n(t)$ *are given by relations* (57)–(59) *and* $u$ *is the solution of Problem* 2.1.

P r o o f. A) The idea of the proof is the same as that of the proof of Theorem 3.11 in [11]. We derive only relations (204) and (205) which correspond to similar ones in the proof mentioned, but their proofs are completely different. Let us set

$$R^i_{jn} = S^i_{jn} - U^i_n \quad (i = 1, \ldots, r_n), \qquad R^i_{jnP} = S^i_{jnP} - U^i_{nP} \quad (i = 0, 1, \ldots, r_n). \tag{195}$$

We have $R^i_{jn} \in V_n$ $(i = 1, \ldots, r_n)$.

Subtracting (49) from (61) and (51) from (63) and multiplying by $\Delta t_n$ we obtain

$$(\Delta R^1_{jnP}, v_P)_{P_n} + \Delta t_n \{a_n(S^1_{jn}, v) - a_n(U^1_n, v)\} = 0 \;\; \forall v \in V_n, \tag{196}$$

$$(\Delta R^i_{jnP}, v_P)_{P_n} + \Delta t_n l_n(R^i_{jn}, v) - \Delta t_n \left\{d_n(S^{i-1}_{jn}, v) - d_n(U^{i-1}_n, v)\right\} = 0, \quad v \in V_n \;\; (i = 2, \ldots, r_n). \tag{197}$$

It is easy to derive the identity

$$(198)\qquad (\Delta R^i_{jnP}, R^i_{jnP})_{P_n} = \frac{1}{2}\|\Delta R^i_{jnP}\|^2_{0,P_n} + \frac{1}{2}\|R^i_{jnP}\|^2_{0,P_n} - \frac{1}{2}\|R^{i-1}_{jnP}\|^2_{0,P_n}.$$

Let us set $v = R^1_{jn}$ in (196). Using the strong monotonicity (34) of the form $a_n(v,w)$, the discrete form of Friedrichs' inequality [12, (29.1)] and identity (198) with $i=1$ we find

$$(199)\qquad \|R^1_{jnP}\|^2_{0,P_n} + C\Delta t_n\|R^1_{jn}\|^2_{1,\Omega_n} \leqslant \|R^0_{jnP}\|^2_{0,P_n}.$$

Let us choose $v = R^i_{jn}$ in (197) and sum from $i=2$ to $i = m \leqslant r_n$. Owing to (198) we have

$$(200)\qquad \begin{aligned}\sum_{i=2}^m (\Delta R^i_{jnP}, R^i_{jnP})_{P_n} &= \frac{1}{2}\sum_{i=2}^m \|\Delta R^i_{jnP}\|^2_{0,P_n} + \frac{1}{2}\|R^m_{jnP}\|^2_{0,P_n} - \frac{1}{2}\|R^1_{jnP}\|^2_{0,P_n}\\ &\geqslant \frac{1}{2}\|R^m_{jnP}\|^2_{0,P_n} - \|R^1_{jnP}\|^2_{0,P_n}.\end{aligned}$$

Applying Lemma 3.3, (40), inequality (69) and the discrete form of Friedrichs' inequality [12, (29.1)] we obtain the bound

$$(201)\qquad \begin{aligned}&\Delta t_n \sum_{i=2}^m \bigl\{l_n(R^i_{jn}, R^i_{jn}) - \bigl(d_n(S^{i-1}_{jn}, R^i_{jn}) - d_n(U^{i-1}_n, R^i_{jn})\bigr)\bigr\}\\ &\geqslant \Delta t_n \sum_{i=2}^m \sum_{M=E,P} \bigl\{\Theta_M |R^i_{jnM}|^2_{1,M_n} - \tau_M |R^{i-1}_{jnM}|_{1,M_n}|R^i_{jnM}|_{1,M_n}\bigr\}\\ &\geqslant \Delta t_n \sum_{i=2}^m \sum_{M=E,P} \Bigl\{\Theta_M |R^i_{jnM}|^2_{1,M_n} - \frac{1}{2}\tau_M(|R^{i-1}_{jnM}|^2_{1,M_n} + |R^i_{jnM}|^2_{1,M_n})\Bigr\}\\ &= \frac{1}{2}\Delta t_n \sum_{M=E,P}\Bigl\{\tau_M |R^m_{jnM}|^2_{1,M_n} - \tau_M|R^1_{jnM}|^2_{1,M_n} + 2\varrho_M \sum_{i=2}^m |R^i_{jnM}|^2_{1,M_n}\Bigr\}\\ &\geqslant \Delta t_n \varrho C \sum_{i=2}^m \|R^i_{jn}\|^2_{1,\Omega_n} - \frac{1}{2}\Delta t_n \tau \|R^1_{jn}\|^2_{1,\Omega_n},\end{aligned}$$

where $\tau = \max(\tau_E, \tau_P) > 0$, $\varrho = \min(\varrho_E, \varrho_P) > 0$ and $0 < \tau_M = \Theta_M - \varrho_M < \Theta_M$, $\varrho_M = \min(\gamma_M, 2(\Theta_M - \frac{1}{2}\beta_M))$ $(M = E, P)$.

Thus using (200), (201) and (199) we have

$$\begin{aligned}&\|R^m_{jnP}\|^2_{0,P_n} + C\Delta t_n \sum_{i=2}^m \|R^i_{jn}\|^2_{1,\Omega_n}\\ &\qquad\leqslant \|R^1_{jnP}\|^2_{0,P_n} + \tau\Delta t_n\|R^1_{jn}\|^2_{1,\Omega_n} \leqslant \|R^0_{jnP}\|^2_{0,P_n}\quad (m = 2,\dots,r_n).\end{aligned}$$

This relation together with (199), (195) gives

$$\|R^m_{jnP}\|^2_{0,P_n} \leqslant \|R^0_{jnP}\|^2_{0,P_n} \quad (0 \leqslant m \leqslant r_n), \tag{202}$$

$$C\Delta t_n \sum_{i=1}^{m} \|S^i_{jn} - U^i_n\|^2_{1,\Omega_n} \leqslant \|R^0_{jnP}\|^2_{0,P_n} \quad (1 \leqslant m \leqslant r_n). \tag{203}$$

As by (195), (50) and (62) we see that

$$R^0_{jnP} = I_{nP}(z_P + s^P_j) - u^P_{0n},$$

relations (202), (203) and [12, Lemma 48.5] imply

$$\|\widehat{R}^m_{jnP}\|_{0,P} \leqslant C\|I_{nP}(z_P + s^P_j) - u^P_{0n}\|_{0,P_n} \quad (0 \leqslant m \leqslant r_n), \tag{204}$$

$$C\Delta t_n \sum_{i=1}^{r_n} \|\widehat{S}^i_{jn} - \widehat{U}^i_n\|^2_1 \leqslant C^*\|I_{nP}(z_P + s^P_j) - u^P_{0n}\|^2_{0,P_n}. \tag{205}$$

Relations (204) and (205) are analogous to [11, (3.91)] and the last relation in [11], p. 375, respectively. With only small modifications we can now follow the proof of Theorem 3.11 of [11]. □

R e m a r k. This paper is a generalization of the results of [15] and [11]: a linear scheme of [15] is generalized to the case of a domain with a nonpolygonal boundary and a nonhomogeneous Dirichlet boundary condition is taken into account.

As some theorems and lemmas from [15], [11] and [12] are applied in the proofs of the theorems mentioned above it would be suitable to present them. However, to keep the extent of the paper within reasonable limits only the appropriate references were given.

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