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SEMIREGULAR HERMITE TETRAHEDRAL FINITE ELEMENTS*

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Abstract. Tetrahedral finite C^0-elements of the Hermite type satisfying the maximum angle condition are presented and the corresponding finite element interpolation theorems in the maximum norm are proved.

Keywords: tetrahedral finite elements of the Hermite type, maximum and minimum angle conditions, finite element interpolation theorems

MSC 2000: 65N30

1. Introduction

The problem of finite element interpolation theorems under the maximum angle condition was studied in several papers (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], pp. 209–213, [11], pp. 391–396, [12]); however, almost all results concern only triangular and tetrahedral finite elements of the Lagrange type. The exception are [11] and [12] (the remark on triangular finite elements of Hermite type in [1], p. 222 is not sufficiently general—see [12], Remark 5.3).

This paper is a generalization of some theorems from [12] and [13] to the three-dimensional case of tetrahedral finite elements. The case of tetrahedral finite elements of the Hermite type is rather different from the case of triangles: In the two-dimensional case the cubic element has nine parameters fixed—they are the function values and the first partial derivatives at the vertices; these nine parameters guarantee the C^0-continuity. The tenth parameter can be chosen relatively freely, because it has no influence on the C^0-continuity of the element. Thus, in [12]

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various possibilities for the tenth parameter were examined. In the three-dimensional case *all* twenty parameters of the cubic polynomial on a tetrahedron are necessary for guaranteeing the C^0-continuity and sixteen parameters are fixed (the function values and first partial derivatives at the vertices). (This follows from the fact that if two tetrahedra have a common face we need for guaranteeing the C^0-continuity ten parameters on the face—and only nine of them are obtained as linear combinations of the parameters prescribed at the vertices of the common face.) Thus the problem how to choose the remaining four parameters on a semiregular tetrahedron is more complicated than in the two-dimensional case (also because of greater complexity of a three-dimensional simplex). However, it can be expected that the three-dimensional case is in a certain way a generalization of the two-dimensional one. This expectation is confirmed in this paper.

We start with the notion "a semiregular tetrahedron." Its definition is a generalization of the two-dimensional case: A tetrahedron is semiregular iff the maximum angle made by two arbitrary faces is less than or equal to $\omega_0 < \pi$. There are three basic types of semiregular tetrahedra (see Figs. 1–3).

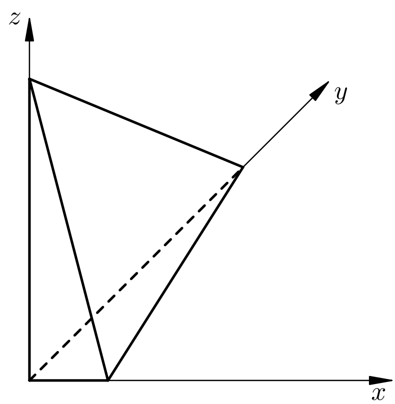

Fig. 1. Semiregular tetrahedron of type $K1$.

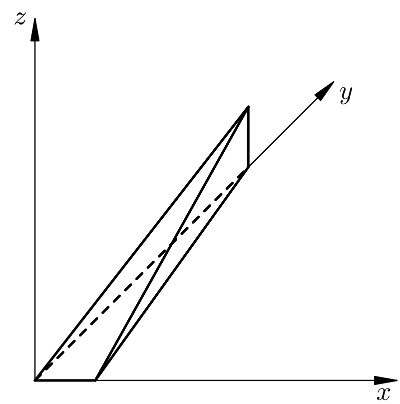

Fig. 2. Semiregular tetrahedron of type $K2$.

We say that a tetrahedron is *regular* iff it is semiregular and has regular triangular faces.

A tetrahedron which is not semiregular is called irregular. Such a tetrahedron can have regular triangular faces (see Fig. 4, where the tetrahedron has vertices $[0,0,0]$, $[1,0,0]$, $[0,1,0]$ and $[\frac{1}{3},\frac{1}{3},\varepsilon]$; here ε can be arbitrarily small).

It is interesting that in this paper only known results from two dimensions are directly applied to Hermite tetrahedra.

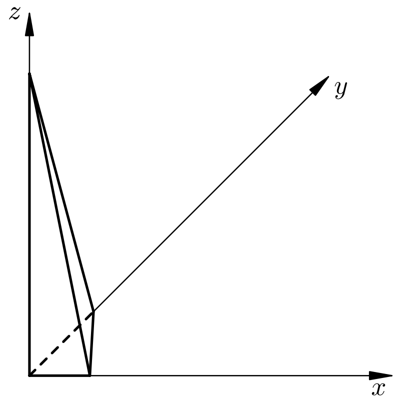

Fig. 3. Semiregular tetrahedron of type $K3$.

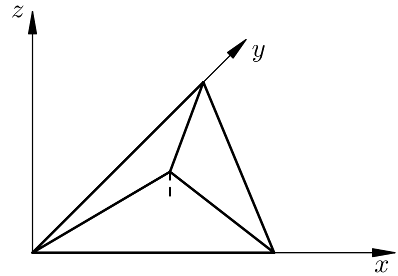

Fig. 4. An irregular tetrahedron.

2. Basic estimates

In [12], the following theorem was proved; this theorem will be generalized in this section to the three-dimensional case.

Theorem 2.1. *Let $\overline{T}$ be a closed triangle with the interior T and vertices P_1, P_2, P_3. Let*

$$a = \operatorname{dist}(P_2, P_3), \quad b = \operatorname{dist}(P_1, P_3), \quad c = \operatorname{dist}(P_1, P_2)$$

and let α, β and γ be the angles at P_1, P_2 and P_3, respectively. Let the vertices be denoted in such a way that

$$a \leqslant b \leqslant c, \quad \alpha \leqslant \beta \leqslant \gamma. \tag{2.1}$$

Let $\varphi \in C^1(\overline{T})$ and let φ have bounded classical derivatives in the interior T of $\overline{T}$,

$$|D^i\varphi(P)| \leqslant M_4 \ \ \forall |i| = 4 \ \ \forall P \in T, \tag{2.2}$$

$$D^i\varphi(P_j) = 0 \ \ \forall |i| \leqslant 1 \ \ (j = 1, 2, 3), \quad \frac{\partial\varphi}{\partial n_a}(Q_1) = 0, \tag{2.3}$$

where Q_1 is the midpoint of the side P_2P_3 and n_a the unit normal to P_2P_3 and where the following multiindex notation for derivatives is used:

$$i = (i_1, i_2), \ \ |i| = i_1 + i_2 \ \ (i_1 \geqslant 0, \ i_2 \geqslant 0), \ \ D^i u = \frac{\partial^{|i|} u}{\partial x^{i_1} \partial y^{i_2}}.$$

Then

$$(2.4)\qquad |\varphi(P)| \leqslant \frac{1}{96}\left(1+8\left(\frac{a}{c}\right)^3\right) M_4\, c^4 \;\; \forall P \in \overline{T},$$

$$(2.5)\qquad \left|\frac{\partial \varphi}{\partial x_j}(P)\right| \leqslant \frac{4}{15}\left(1+5\left(\frac{a}{c}\right)^2\right)\frac{1}{\sin\beta} M_4 c^3 \;\; \forall P \in \overline{T} \quad (j=1,2).$$

Let $\overline{K1}$ be a tetrahedron with one short side P_2P_3 (see Fig. 5; we first consider tetrahedra with three edges perpendicular to one another; the general case is mentioned in the text connected with Fig. 9). The symbol $K1$ will denote its interior and $\partial K1$ its boundary. We will consider a function $\varphi \in C^4(\overline{K1})$ with the following properties (we have now $\alpha = (\alpha_1, \alpha_2, \alpha_3)$):

$$(2.6)\qquad D^\alpha \varphi(P_i) = 0 \quad |\alpha| \leqslant 1 \quad (i=1,\dots,4),$$

$$(2.7)\qquad \frac{\partial \varphi}{\partial n_1}(Q_{23}) = \frac{\partial \varphi}{\partial n_2}(Q_{14}) = \frac{\partial \varphi}{\partial n_3}(Q_{14}) = \frac{\partial \varphi}{\partial n_4}(Q_{23}) = 0,$$

$$(2.8)\qquad |D^\alpha \varphi(P)| \leqslant M_4 \;\; \forall |\alpha| = 4 \;\; \forall P \in \overline{K1},$$

where $P_1,\dots,P_4$ are the vertices of the tetrahedron $\overline{K1}$ (see Fig. 5) and Q_{ij} is the midpoint of the edge P_iP_j. The symbol ϱ_i will denote the plane containing the triangular face $\overline{T}_i$ opposite to the vertex P_i. The symbol n_i appearing in (2.7) denotes the unit normal to the boundary ∂T_i of the triangle $\overline{T}_i$ (in Fig. 5 all these normals are outward; of course, in a tetragonalization each normal common to more tetrahedra will be outward for some tetrahedra and inward for other tetrahedra); this normal lies, of course, in the plane ϱ_i. (As to weakening inequality (2.8) see Remark 2.5.)

The symbols α_i $(i=2,3,4)$ will denote the three angles at the vertex P_1 lying in triangular faces $\overline{T}_2$, $\overline{T}_3$, $\overline{T}_4$. Similarly, β_i $(i=1,3,4)$ denote the angles at the vertex P_2, γ_i $(i=1,2,4)$ denote the angles at the vertex P_3 and δ_i $(i=1,2,3)$ the angles at the vertex P_4. The symbol ω_{ij} will denote the acute angle made by the planes ϱ_i and ϱ_j.

In each plane ϱ_i we can choose a Cartesian coordinate system x_i, y_i. The axis z_i belonging to this system is oriented in the direction of the normal to the triangular face $\overline{T}_i$, so that it is not necessary to choose a special symbol for the normals to the triangular faces.

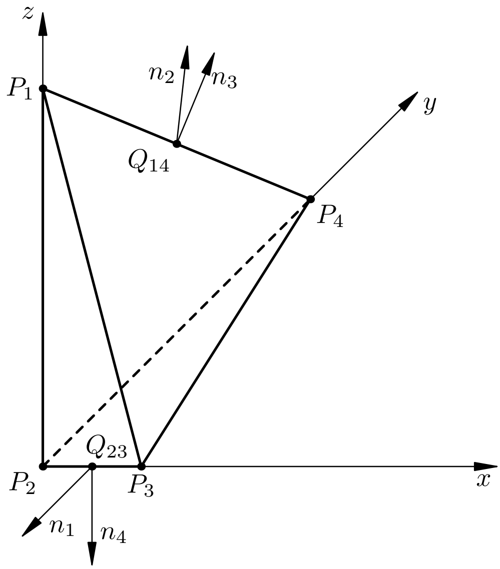

Fig. 5. Choosing the normals in the case of type $K1$.

Theorem 2.1 and assumptions (2.6)–(2.8) imply the following estimates:

$$|\varphi(P)| \leqslant CM_4 h^4 \qquad \forall P \in \partial K1, \tag{2.9}$$

$$\left|\frac{\partial\varphi}{\partial x_1}(P)\right|, \left|\frac{\partial\varphi}{\partial y_1}(P)\right| \leqslant \frac{CM_4}{\sin\gamma_1} h^3 \qquad \forall P \in \overline{T}_1, \tag{2.10}$$

$$\left|\frac{\partial\varphi}{\partial x_2}(P)\right|, \left|\frac{\partial\varphi}{\partial y_2}(P)\right| \leqslant \frac{CM_4}{\sin(\max(\alpha_2,\delta_2))} h^3 \quad \forall P \in \overline{T}_2, \tag{2.11}$$

$$\left|\frac{\partial\varphi}{\partial x_3}(P)\right|, \left|\frac{\partial\varphi}{\partial y_3}(P)\right| \leqslant \frac{CM_4}{\sin(\max(\alpha_3,\delta_3))} h^3 \quad \forall P \in \overline{T}_3, \tag{2.12}$$

$$\left|\frac{\partial\varphi}{\partial x_4}(P)\right|, \left|\frac{\partial\varphi}{\partial y_4}(P)\right| \leqslant \frac{CM_4}{\sin\gamma_4} h^3 \qquad \forall P \in \overline{T}_4, \tag{2.13}$$

where h is the length of the largest edge of the tetrahedron $\overline{K1}$ and C is a generic constant.

Now we estimate the derivatives $\frac{\partial\varphi}{\partial z_i}$ at the vertices of $\overline{K1}$ and at the midpoints Q_{ij} of the edges. Assumptions (2.6) and (2.7) imply

$$\frac{\partial\varphi}{\partial z_j}(P_i) = 0 \quad (i = 1,\ldots,4;\ j = 1,\ldots,4,\ j \neq i), \tag{2.14}$$

$$\frac{\partial\varphi}{\partial z_1}(Q_{23}) = \frac{\partial\varphi}{\partial z_4}(Q_{23}) = 0, \quad \frac{\partial\varphi}{\partial z_2}(Q_{14}) = \frac{\partial\varphi}{\partial z_3}(Q_{14}) = 0. \tag{2.15}$$

As $\frac{\partial\varphi}{\partial z_1}$ is a linear combination of the derivatives $\frac{\partial\varphi}{\partial x_3}$ and $\frac{\partial\varphi}{\partial y_3}$, estimates (2.12) imply

$$\left|\frac{\partial\varphi}{\partial z_1}(Q_{24})\right| \leqslant \frac{CM_4}{\sin(\max(\alpha_3,\delta_3))}h^3. \tag{2.16}$$

Similarly, estimates (2.10) give

$$\left|\frac{\partial\varphi}{\partial z_3}(Q_{24})\right| \leqslant \frac{CM_4}{\sin\gamma_1}h^3; \tag{2.17}$$

estimates (2.13) yield

$$\left|\frac{\partial\varphi}{\partial z_3}(Q_{12})\right| \leqslant \frac{CM_4}{\sin\gamma_4}h^3 \tag{2.18}$$

and estimates (2.12) imply

$$\left|\frac{\partial\varphi}{\partial z_4}(Q_{12})\right| \leqslant \frac{CM_4}{\sin(\max(\alpha_3,\delta_3))}h^3. \tag{2.19}$$

Let s_1 be a direction parallel to the plane ϱ_1 and perpendicular to the edge P_3P_4 and s_2 a direction parallel to the plane ϱ_2 and perpendicular also to the edge P_3P_4. These two directions make an angle ω_{12} (which is the angle made by the planes ϱ_1 and ϱ_2). According to (2.10) and (2.11), we have

$$\left|\frac{\partial\varphi}{\partial s_1}(Q_{34})\right| \leqslant \frac{CM_4}{\sin\gamma_1}h^3, \quad \left|\frac{\partial\varphi}{\partial s_2}(Q_{34})\right| \leqslant \frac{CM_4}{\sin(\max(\alpha_2,\delta_2))}h^3,$$

hence (see Fig. 6a; in Fig. 6b, (ξ,η) is the plane orthogonal to P_3P_4 which passes through the point Q_{34})

$$\left|\frac{\partial\varphi}{\partial z_1}(Q_{34})\right| \leqslant \frac{CM_4}{\sin\omega_{12}}\left(\frac{1}{\sin\gamma_1}+\frac{1}{\sin(\max(\alpha_2,\delta_2))}\right)h^3. \tag{2.20}$$

Similarly we obtain

$$\left|\frac{\partial\varphi}{\partial z_2}(Q_{34})\right| \leqslant \frac{CM_4}{\sin\omega_{12}}\left(\frac{1}{\sin\gamma_1}+\frac{1}{\sin(\max(\alpha_2,\delta_2))}\right)h^3, \tag{2.21}$$

$$\left|\frac{\partial\varphi}{\partial z_2}(Q_{13})\right| \leqslant \frac{CM_4}{\sin\omega_{24}}\left(\frac{1}{\sin(\max(\alpha_2,\delta_2))}+\frac{1}{\sin\gamma_4}\right)h^3, \tag{2.22}$$

$$\left|\frac{\partial\varphi}{\partial z_4}(Q_{13})\right| \leqslant \frac{CM_4}{\sin\omega_{24}}\left(\frac{1}{\sin(\max(\alpha_2,\delta_2))}+\frac{1}{\sin\gamma_4}\right)h^3. \tag{2.23}$$

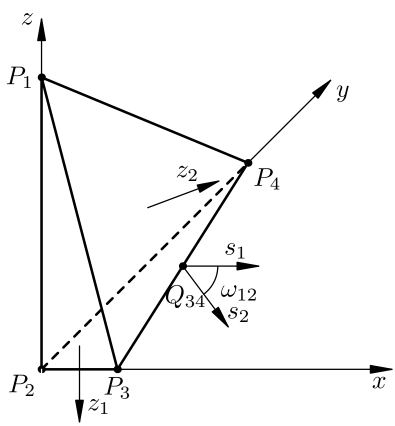

Fig. 6a. To the proof of (2.20).

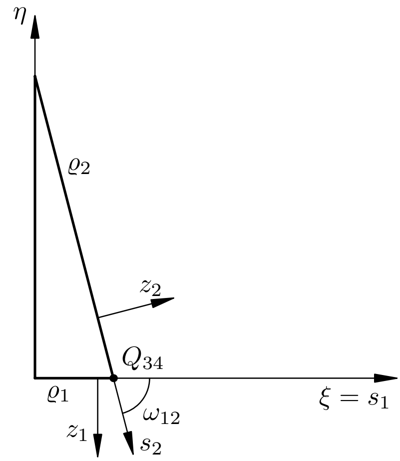

Fig. 6b. To the proof of (2.20).

Let us set

$$\omega = \min(\omega_{12}, \omega_{24}), \quad \sigma = \min(\alpha_2, \alpha_3, \gamma_1, \gamma_4, \delta_2, \delta_3). \tag{2.24}$$

Then we can write relations (2.15)–(2.23) in a unique form

$$\left|\frac{\partial \varphi}{\partial z_i}(Q_{jk})\right| \leqslant \frac{CM_4}{\sin\omega \sin\sigma} h^3 \quad (i = 1, \ldots, 4;\ j \neq i,\ k \neq i,\ k \neq j). \tag{2.25}$$

Now we come to the crucial point of our considerations which consists of several applications of the following Lemma 2.2. Although this lemma was proved in [13], we reproduce briefly its proof because of the importance of the lemma in this paper.

Lemma 2.2. *Let a function $\psi(x_i, y_i)$ be continuous on a closed triangle $\overline{T}_i$ and have derivatives of the third order in its interior T_i bounded by a constant K_3. Further, let $\psi(P_j) = \eta_j$, $\psi(Q_{km}) = \zeta_{km}$, P_j being the vertices of $\overline{T}_i$ and Q_{km} the midpoints of its sides. Then we have on $\overline{T}_i$*

$$|\psi(x_i, y_i)| \leqslant 6\eta + \tfrac{1}{3} K_3 h^3, \quad \eta = \max(|\eta_j|, |\zeta_{km}|). \tag{2.26}$$

P r o o f is based on the following three lemmas:

Lemma A. *Let $f \in C^1(\overline{T})$. Let s_1, s_2 be two directions making an angle ω. Let $\frac{\partial f}{\partial s_1}(P) = k_1$, $\frac{\partial f}{\partial s_2}(P) = k_2$, P being a point in the (x_i, y_i)-plane. If $0 < \omega < \frac{1}{3}\pi$ then*

$$\left|\frac{\partial f}{\partial t}(P)\right| \leqslant \frac{2\sqrt{3}}{3} \max|k_j|,$$

where t is any direction lying inside the acute angle formed by s_1 and s_2.

Lemma B. *Let $g(0) = \eta_1$, $g(l/2) = \eta_2$, $g(l) = \eta_3$ and $|g^{(3)}(s)| \leqslant N_3$ in $(0,l)$. Then for $s \in [0,l]$*

$$|g(s)| \leqslant \frac{5}{4}\max|\eta_j| + \frac{\sqrt{3}}{6^3}N_3 l^3,$$
$$|g'(s)| \leqslant \frac{8}{l}\max|\eta_j| + \frac{1}{4}N_3 l^2.$$

Lemma C. *Let $g(0) = \eta_1$, $g'(0) = k_1$, $g(l) = \eta_2$ and $|g^{(3)}(s)| \leqslant N_3$ in $(0,l)$. Then for $s \in [0,l]$*

$$|g(s)| \leqslant \max|\eta_j| + \frac{l}{4}|k_1| + \frac{2}{81}N_3 l^3.$$

Lemmas B and C are simple results of the interpolation theory in one variable. As to Lemma A, let $\omega = \beta - \alpha$, where α and β are the angles made by s_1 and s_2, respectively, with the positive direction of the x-axis ($\beta > \alpha$). We have

$$k_1 = \frac{\partial f}{\partial x}(P)\cos\alpha + \frac{\partial f}{\partial y}(P)\sin\alpha, \quad k_2 = \frac{\partial f}{\partial x}(P)\cos\beta + \frac{\partial f}{\partial y}(P)\sin\beta.$$

Solving these two equations with respect to $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ we obtain

$$\frac{\partial f}{\partial x}(P) = \frac{k_1\sin\beta - k_2\sin\alpha}{\sin\omega}, \quad \frac{\partial f}{\partial y}(P) = \frac{-k_1\cos\beta + k_2\cos\alpha}{\sin\omega}.$$

Hence,

$$\left|\frac{\partial f}{\partial t}(P)\right| = \frac{k_1\sin(\omega-\varepsilon) + k_2\sin\varepsilon}{\sin\omega} \leqslant \max_{j=1,2}|k_j|\frac{\sin(\omega-\varepsilon)+\sin\varepsilon}{\sin\omega},$$

where ε is the angle which is made by the direction t with the direction s_1. It is easy to see that

$$\max(\sin(\omega-\varepsilon)+\sin\varepsilon) = 2\sin\frac{\omega}{2}.$$

As $0 < \omega \leqslant \frac{1}{3}\pi$, the assertion of Lemma A follows.

We sketch the proof of Lemma 2.2 only in the case that ψ has bounded derivatives of the third order on $\overline{T}_i$. (For more details see [13].)

Let us denote the sides of $\overline{T}_i$ by $a_i \leqslant b_i \leqslant c_i$. By the second part of Lemma B we have (with $N_3 = 2\sqrt{2}K_3$)

$$\left|\frac{\partial\psi}{\partial s_1}(P_1)\right| \leqslant \frac{8}{b_i}\eta + \frac{\sqrt{2}}{2}K_3c_i^2, \quad \left|\frac{\partial\psi}{\partial s_2}(P_1)\right| \leqslant \frac{8}{c_i}\eta + \frac{\sqrt{2}}{2}K_3c_i^2, \tag{2.27}$$

s_1 and s_2 being now the directions of the sides b_i and c_i, respectively, and P_1 the vertex lying opposite the side a_i. By Lemma A we obtain from inequalities (2.27) that

$$\left|\frac{\partial \psi}{\partial s}(P_1)\right| \leqslant \frac{16\sqrt{3}}{3}\frac{\eta}{b_i} + \frac{\sqrt{6}}{3}K_3 c_i^2, \tag{2.28}$$

s being any direction lying in the angle made by the sides b_i and c_i. Let $P \neq P_1$ be an arbitrary point of $\overline{T}_i$ and P' the point on the side a_i which lies on the line going through P_1 and P. By the first part of Lemma B we obtain

$$|\psi(P')| \leqslant \frac{5}{4}\eta + \frac{2\sqrt{6}}{6^3}K_3 c_i^3.$$

This inequality, the assumption $|\psi(P_1)| \leqslant \eta$ and (2.28) imply, according to Lemma C,

$$|\psi(P)| \leqslant \frac{5}{4}\eta + \frac{2\sqrt{6}}{6^3}K_3 c_i^3 + \frac{4\sqrt{3}}{3}\frac{c_i}{b_i}\eta + \frac{\sqrt{6}}{12}K_3 c_i^3 + \frac{\sqrt{2}}{20}K_3 c_i^3 \leqslant 6\eta + \frac{1}{3}K_3 c_i^3$$

because $c_i/b_i < 2$. □

Lemma 2.2 and relations (2.8), (2.14), (2.25) imply the following estimates (we set $\psi = \frac{\partial \varphi}{\partial z_i}$):

$$\left|\frac{\partial \varphi}{\partial z_i}(P)\right| \leqslant \frac{CM_4 h^3}{\sin\omega \sin\sigma} \quad \forall P \in \overline{T}_i \quad (i = 1, \dots, 4). \tag{2.29}$$

Using the angle σ, we can write estimates (2.10)–(2.13) for the derivatives with respect to x_i and y_i also in a concise form:

$$\left|\frac{\partial \varphi}{\partial x_i}(P)\right| \leqslant \frac{CM_4 h^3}{\sin\sigma} \quad \forall P \in \overline{T}_i \quad (i = 1, \dots, 4), \tag{2.30}$$

$$\left|\frac{\partial \varphi}{\partial y_i}(P)\right| \leqslant \frac{CM_4 h^3}{\sin\sigma} \quad \forall P \in \overline{T}_i \quad (i = 1, \dots, 4). \tag{2.31}$$

As $x_i y_i z_i$ are Cartesian coordinate systems, we have by (2.29)–(2.31) in the global Cartesian coordinate system x, y, z

$$\left|\frac{\partial \varphi}{\partial x}(P)\right|, \left|\frac{\partial \varphi}{\partial y}(P)\right|, \left|\frac{\partial \varphi}{\partial z}(P)\right| \leqslant \frac{CM_4 h^3}{\sin\omega \sin\sigma} \quad \forall P \in \overline{T}_i \quad (i = 1, \dots, 4). \tag{2.32}$$

Let $A \in \overline{K1}$ ($A \neq P_1$) be an arbitrary fixed point and $\varrho(A)$ the plane passing through the point A and parallel to the (x, y)-plane. Let $\overline{T} = \varrho(A) \cap \overline{K1}$. At every point $P \in \partial T$ estimates (2.32) are satisfied. Hence, using again Lemma 2.2,

$$\left|\frac{\partial \varphi}{\partial x}(A)\right|, \left|\frac{\partial \varphi}{\partial y}(A)\right|, \left|\frac{\partial \varphi}{\partial z}(A)\right| \leqslant \frac{CM_4 h^3}{\sin\omega \sin\sigma} \quad \forall A \in \overline{K1}. \tag{2.33}$$

In estimating the function values we will use the following lemma which is proved in [13, Lemma 5]:

Lemma 2.3. *Let* $g(0) = \eta_0$, $g(l) = \eta_1$, $g'(0) = k_0$, $g'(l) = k_1$ *and* $|g^{(4)}(s)| \leqslant K_4$ *in* $(0, l)$. *Then for* $s \in [0, l]$

$$|g(s)| \leqslant \max|\eta_j| + \frac{4l}{27}(|k_0| + |k_1|) + \frac{K_4}{16 \cdot 24} l^4,$$
$$|g'(s)| \leqslant \frac{3}{2l}(|\eta_0| + |\eta_1|) + \frac{1}{3}(|k_0| + |k_1|) + \frac{K_4}{24} l^3.$$

Let $A \in \overline{K1}$ be an arbitrary fixed point for which $A \neq P_1$. Let $B \in \overline{T}_1$ be the point lying on the line passing through P_1 and A. Let us set $l = \operatorname{dist}(P_1, B)$ and consider the function $g = \varphi\big|_{P_1 B}$. Then, using the first part of Lemma 2.3, we obtain from relations (2.6), (2.8), (2.9), (2.10) and (2.29) (for $i = 1$)

$$|\varphi(A)| \leqslant \frac{C M_4 h^4}{\sin\omega \sin\sigma} \quad \forall A \in \overline{K1}. \tag{2.34}$$

This result together with (2.33) is sufficient for obtaining all results introduced in Section 3. Nevertheless, to satisfy the law of mathematical elegance we derive (at least in the case of $\overline{K1}$) an estimate for the function values independent of the geometry of the tetrahedron considered.

The transformation

$$x = h_x \xi, \quad y = h_y \eta, \quad z = h_z \zeta \tag{2.35}$$

with $h_x = \overline{P_2 P_3}$, $h_y = \overline{P_2 P_4}$, $h_z = \overline{P_1 P_2}$ maps one-to-one the tetrahedron $\overline{K1}$ lying in the Cartesian coordinate system (x, y, z) onto the reference tetrahedron $\overline{K}_0$ lying in the Cartesian system (ξ, η, ζ) and having the vertices

$$P_1^* = [0,0,1], \ \ P_2^* = [0,0,0], \ \ P_3^* = [1,0,0], \ \ P_4^* = [0,1,0].$$

In the case considered both the coordinate systems are identical; however, this does not violate the universality of the idea.

Let us define a function

$$\widetilde{\varphi}(\xi, \eta, \zeta) = \frac{1}{M_4 h^4} \varphi(h_x \xi, h_y \eta, h_z \zeta). \tag{2.36}$$

As $\varphi \in C^4(\overline{K1})$, we have $\widetilde{\varphi} \in C^4(\overline{K}_0)$ and

$$D^\alpha \widetilde{\varphi}(P_i^*) = 0 \quad |\alpha| \leqslant 1 \quad (i = 1, \ldots, 4), \tag{2.37}$$

$$|D^\alpha \widetilde{\varphi}(\xi, \eta, \zeta)| \leqslant 1 \;\; \forall\, |\alpha| = 4 \;\; \forall\, [\xi, \eta, \zeta] \in \overline{K}_0, \tag{2.38}$$

$$\frac{\partial \widetilde{\varphi}}{\partial \eta}(Q_{23}^*) = 0, \;\; \frac{\partial \widetilde{\varphi}}{\partial \zeta}(Q_{23}^*) = 0, \tag{2.39}$$

where Q_{23}^* corresponds to Q_{23} in transformation (2.35).

It remains to estimate $\left|\frac{\partial \widetilde{\varphi}}{\partial \nu_2}(Q_{14}^*)\right|$ and $\left|\frac{\partial \widetilde{\varphi}}{\partial \nu_3}(Q_{14}^*)\right|$, where Q_{14}^* corresponds to Q_{14} and ν_i is the unit normal to ∂T_i^* of the triangle $\overline{T}_i^*$ which lies in the plane ϱ_i^*. ($\overline{T}_i^*$ corresponds to $\overline{T}_i$ and ϱ_i^* to ϱ_i in (2.35).) Let s_1 be the direction of $\overrightarrow{P_1 P_4}$ and s_2, s_3 two mutually orthogonal directions which are orthogonal to s_1. Then we have by (2.7)

$$\frac{\partial \varphi}{\partial s_2}(Q_{14}) = \frac{\partial \varphi}{\partial s_3}(Q_{14}) = 0. \tag{2.40}$$

According to the second part of Lemma 2.3, we can write using (2.6) and (2.8)

$$\left|\frac{\partial \varphi}{\partial s_1}(Q_{14})\right| \leqslant C M_4 h^3. \tag{2.41}$$

Relations (2.40) and (2.41) imply

$$\left|\frac{\partial \varphi}{\partial x}(Q_{14})\right|, \left|\frac{\partial \varphi}{\partial y}(Q_{14})\right|, \left|\frac{\partial \varphi}{\partial z}(Q_{14})\right| \leqslant C M_4 h^3. \tag{2.42}$$

Hence by (2.36) and (2.42)

$$\left|\frac{\partial \widetilde{\varphi}}{\partial \xi}(Q_{14}^*)\right| = \left|\frac{1}{M_4 h^4}\frac{\partial \varphi}{\partial x}(Q_{14}) h_x\right| \leqslant C. \tag{2.43}$$

Similarly,

$$\left|\frac{\partial \widetilde{\varphi}}{\partial \eta}(Q_{14}^*)\right| \leqslant C, \quad \left|\frac{\partial \widetilde{\varphi}}{\partial \zeta}(Q_{14}^*)\right| \leqslant C. \tag{2.44}$$

It is necessary to estimate $\frac{\partial \widetilde{\varphi}}{\partial \xi_i}$, $\frac{\partial \widetilde{\varphi}}{\partial \eta_i}$ on $\overline{T}_i^*$ for $i = 1, \ldots, 4$. Theorem 2.1 implies

$$|D_1^\alpha \widetilde{\varphi}(\xi_1, \eta_1)| \leqslant C_1^* \quad |\alpha| \leqslant 1, \;\; \forall\, [\xi_1, \eta_1] \in \overline{T}_1^*, \tag{2.45}$$

$$|D_4^\alpha \widetilde{\varphi}(\xi_4, \eta_4)| \leqslant C_4^* \quad |\alpha| \leqslant 1, \;\; \forall\, [\xi_4, \eta_4] \in \overline{T}_4^*, \tag{2.46}$$

where

$$D_i^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_i^{\alpha_1} \partial y_i^{\alpha_2}}, \quad \alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2. \tag{2.47}$$

Repeating the considerations of the proof of [12, Theorem 2.1], we obtain on the base of (2.37), (2.43) and (2.44)

$$|D_2^\alpha \widetilde{\varphi}(\xi_2, \eta_2)| \leqslant C_2^* \quad |\alpha| \leqslant 1, \ \forall [\xi_2, \eta_2] \in \overline{T}_2^*, \tag{2.48}$$

$$|D_3^\alpha \widetilde{\varphi}(\xi_3, \eta_3)| \leqslant C_3^* \quad |\alpha| \leqslant 1, \ \forall [\xi_3, \eta_3] \in \overline{T}_3^*. \tag{2.49}$$

Relations (2.37), (2.38) and (2.45)–(2.49) yield again by means of Lemma 2.2

$$\left| \frac{\partial \widetilde{\varphi}}{\partial \zeta}(\xi, \eta, 0) \right| \leqslant C^* \ \ \forall [\xi, \eta] \in \overline{T}_1^*. \tag{2.50}$$

Let us choose an arbitrary fixed point $A^* \in \overline{K}_0$ $(A^* \neq P_1^*)$. Let $B^* \in \overline{T}_1^*$ be a point lying on the line passing through P_1^* and A^*. Let us consider the function $g = \widetilde{\varphi}\big|_{P_1^* B^*}$. Then using the first part of Lemma 2.3, we obtain by means of (2.37) (for $i = 1$), (2.45) and (2.50)

$$|\widetilde{\varphi}(A^*)| \leqslant C(\overline{K}_0) \ \ \forall A^* \in \overline{K}_0, \tag{2.51}$$

where the constant $C(\overline{K}_0)$ depends on the tetrahedron $\overline{K}_0$ only. Relations (2.36) and (2.51) yield

$$|\varphi(A)| \leqslant C(\overline{K}_0) M_4 h^4 \ \ \forall A \in \overline{K1}, \tag{2.52}$$

which was to be proved.

The second group of semiregular tetrahedral finite elements are tetrahedra with *two* short edges (which cannot have a common vertex). A typical representative, which will be denoted $\overline{K2}$, can be obtained from $\overline{K1}$ by contracting the edge P_1P_4 (and appropriately dilating the edges P_1P_2 and P_1P_3)—see Fig. 7. The definition of the nodal points and of the parameters prescribed at them is in the case of $\overline{K2}$ the same as in the case of $\overline{K1}$. Parameters at Q_{ij} are prescribed as couples on both short edges, at which faces making small angles meet. Hence, estimates (2.33) and (2.34) can be obtained in the same way as in the case of $\overline{K1}$.

It remains to analyze the case of tetrahedra with *three* short edges. The corresponding representative tetrahedron, which will be denoted by the symbol $\overline{K3}$, can be obtained from $\overline{K1}$ by contracting edges P_2P_4 and P_3P_4—see Fig. 8.

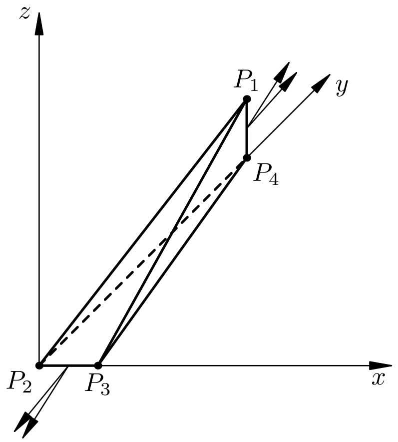

Fig. 7. Tetrahedron of type $K2$ with normals.

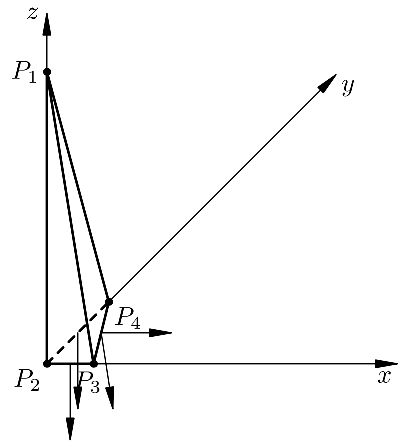

Fig. 8. Tetrahedron of type $K3$ with normals.

On edges, the nodal points Q_{ij} and parameters defined at them are prescribed in a way different from the cases of $\overline{K1}$ and $\overline{K2}$ (cf. (2.7)):

$$\frac{\partial\varphi}{\partial n_1}(Q_{34}) = \frac{\partial\varphi}{\partial n_2}(Q_{34}) = \frac{\partial\varphi}{\partial n_3}(Q_{24}) = \frac{\partial\varphi}{\partial n_4}(Q_{23}) = 0. \tag{2.53}$$

The way how to derive estimates (2.33) and (2.34) is only a simple modification of the way used in the case of $\overline{K1}$; thus we omit it. Hence, we arrive at the following theorem:

Theorem 2.4. *Let $\overline{K}$ be one of the tetrahedra $\overline{K1}$, $\overline{K2}$, $\overline{K3}$. Let $\varphi \in C^4(\overline{K})$ and let*

$$|D^\alpha\varphi(P)| \leqslant M_4 \;\; \forall\,|\alpha| = 4 \;\; \forall P \in \overline{K}, \tag{2.54}$$

$$D^\alpha\varphi(P_j) = 0 \;\; \forall\,|\alpha| \leqslant 1 \quad (j = 1, \ldots, 4), \tag{2.55}$$

where $P_1, \ldots, P_4$ are vertices of $\overline{K}$ in the order which is indicated in Fig. 5. Let in the cases of $\overline{K1}$ and $\overline{K2}$ the remaining four conditions be of the form (2.7) and in the case of $\overline{K3}$ of the form (2.53). Then estimates of the type (2.33), (2.34) hold, i.e.,

$$\left|\frac{\partial\varphi}{\partial x}(A)\right|, \left|\frac{\partial\varphi}{\overline{y}}(A)\right|, \left|\frac{\partial\varphi}{\partial z}(A)\right| \leqslant \frac{CM_4h^3}{\sin\omega\sin\sigma} \;\; \forall A \in \overline{K}, \tag{2.56}$$

$$|\varphi(A)| \leqslant \frac{CM_4h^4}{\sin\omega\sin\sigma} \;\; \forall A \in \overline{K}, \tag{2.57}$$

where in the cases of $\overline{K1}$ *and* $\overline{K2}$ *the angles* ω *and* σ *are defined in relations* (2.24) *and in the case of* $\overline{K3}$ *we have*

$$\omega = \min(\omega_{12}, \omega_{23}, \omega_{24}), \quad \sigma = \min(\gamma_1, \gamma_2, \gamma_4, \delta_2, \delta_3). \tag{2.58}$$

R e m a r k 2.5. Assumption (2.54) can be weakened to the form

$$|D^\alpha \varphi(P)| \leqslant M_4 \ \ \forall |\alpha| = 4, \ \ \forall P \in K, \tag{2.59}$$

where K is the interior of $\overline{K}$. In the case (2.59) we can use the trick with an inscribed tetrahedron $\overline{K}' \subset K$ in the same way as in [13]. Considerations connected with it are cumbersome; thus we omitted it. Similar remark concerns (2.8).

R e m a r k 2.6. Similarly to the case of $\overline{K1}$ estimate (2.57) can be improved to the form (2.52). However, this improvement has no influence on the results introduced in Section 3.

3. Applications of basic estimates

Theorem 3.1. *A polynomial* $p(x, y, z)$ *of degree not greater than three in three variables is uniquely determined by its twenty values which have in the cases of* $\overline{K1}$ *and* $\overline{K2}$ *the form*

$$D^\alpha p(P_j) \quad |\alpha| \leqslant 1 \quad (j = 1, \dots, 4), \tag{3.1}$$

$$\frac{\partial p}{\partial n_1}(Q_{23}), \ \frac{\partial p}{\partial n_2}(Q_{14}), \ \frac{\partial p}{\partial n_3}(Q_{14}), \ \frac{\partial p}{\partial n_4}(Q_{23}), \tag{3.2}$$

where the meaning of the symbols P_i, Q_{jk} *and* n_i *is the same as in Section* 2. *In the case of* $\overline{K3}$ *the twenty values have the form* (3.1) *and* (3.3), *where*

$$\frac{\partial p}{\partial n_1}(Q_{34}), \ \frac{\partial p}{\partial n_2}(Q_{34}), \ \frac{\partial p}{\partial n_3}(Q_{24}), \ \frac{\partial p}{\partial n_4}(Q_{23}). \tag{3.3}$$

P r o o f. It is sufficient to prove the uniqueness. In the cases of $\overline{K1}$ and $\overline{K2}$, let us assume that the values (3.1), (3.2) are equal to zero. Setting $\varphi(x, y, z) = p(x, y, z)$ in Theorem 2.4, we have $M_4 = 0$ and estimate (2.57) implies $p(x, y, z) \equiv 0$. The case of $\overline{K3}$ can be treated in the same way. □

Theorem 3.2. *Let $u \in C^4(\overline{K})$ and let $|D^\alpha u(P)| \leqslant M_4$ for all $|\alpha| = 4$ and all $P \in \overline{K}$. Let $p(x, y, z)$ be a polynomial of degree not greater than three which satisfies for $\overline{K} = \overline{K1}$ or $\overline{K} = \overline{K2}$ the relations*

$$D^\alpha p(P_j) = D^\alpha u(P_j) \quad |\alpha| \leqslant 1 \quad (j = 1, \dots, 4), \tag{3.4}$$

$$\begin{aligned} &\frac{\partial p}{\partial n_1}(Q_{23}) = \frac{\partial u}{\partial n_1}(Q_{23}), \quad \frac{\partial p}{\partial n_2}(Q_{14}) = \frac{\partial u}{\partial n_2}(Q_{14}), \\ &\frac{\partial p}{\partial n_3}(Q_{14}) = \frac{\partial u}{\partial n_3}(Q_{14}), \quad \frac{\partial p}{\partial n_4}(Q_{23}) = \frac{\partial u}{\partial n_4}(Q_{23}). \end{aligned} \tag{3.5}$$

Then the function

$$\varphi(x, y, z) \equiv u(x, y, z) - p(x, y, z) \tag{3.6}$$

satisfies relations (2.56)–(2.57). *The modification of assumption* (3.5) *in the case of $\overline{K3}$ is obvious.*

P r o o f. It follows from the assumptions of Theorem 3.2 that function (3.6) satisfies all conditions of Theorem 2.4. □

Of course, Theorems 3.1 and 3.2 hold not only for tetrahedra $\overline{K1}$, $\overline{K2}$ and $\overline{K3}$ but also for tetrahedra arising from $\overline{K1}$, $\overline{K2}$ and $\overline{K3}$ by deformation (see, e.g., Fig. 9 where $\operatorname{dist}(P_1, (y, z)) > 0$ and possibly $\operatorname{dist}(P_1, (x, z)) > 0$). The proof of Theorem 2.4 is in this case without any change, assuming that the resulting tetrahedra remain semiregular.

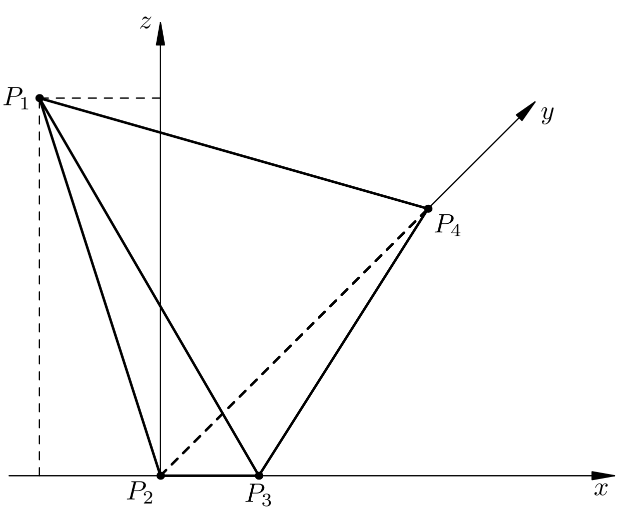

Fig. 9. A general case of type $K1$.

The tetrahedra used most frequently are flat tetrahedra which arise from $\overline{K1}$ by means of deformation (see Fig. 10). However, in such tetragonalizations the finite element method cannot be used. The explanation is clear from Fig. 11: on a common face of two tetrahedra the tenth parameters (i.e., normal derivatives) are situated on different sides of the face; thus the continuity of the global function is not guaranteed.

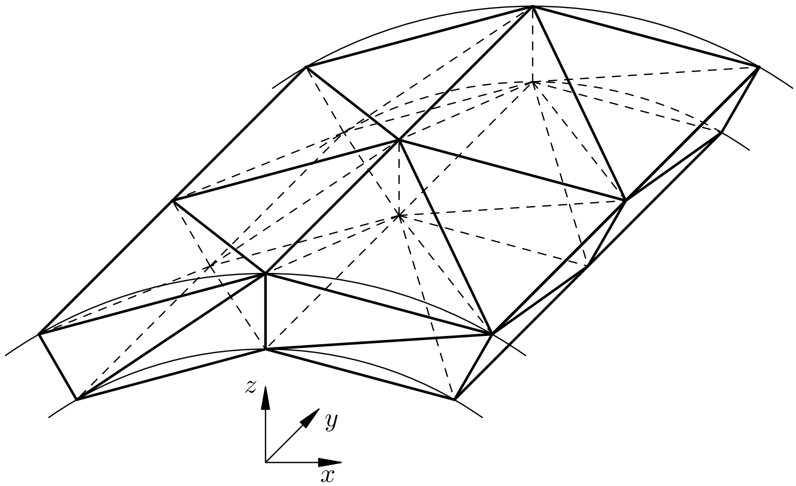

Fig. 10. The use of semiregular tetrahedral elements.

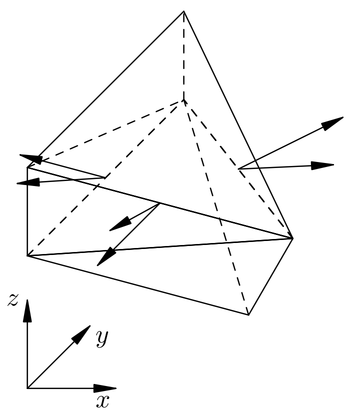

Fig. 11. Disaster.

It seems that we have got into a blind alley. However, there is a remedy having at least three variants which we will introduce. The first is sketched in Fig. 12. Instead of prescribing two normal derivatives at the point Q_{12} in the directions n_{123} and n_{126} we prescribe only one in the normal direction n_{123} which lies in the plane $P_1P_2P_3$.

The twentieth parameter in the tetrahedron $P_1P_2P_3P_6$ will be prescribed at the point Q_{26} as the normal derivative in the direction n^*_{126} which lies in the plane $P_1P_2P_6$. This is the only parameter prescribed at the point Q_{26}. Instead of the derivative $\frac{\partial\varphi}{\partial n_{264}}(Q_{26})$ we prescribe the normal derivative $\frac{\partial\varphi}{\partial n^*_{264}}(Q_{46})$. If the tetragonalization has only one layer (see Fig. 10) then we can prescribe at the point Q_{46} the derivative $\frac{\partial\varphi}{\partial n_{456}}(Q_{46})$ (in this case we obtain at this point a better accuracy—see later); however for the symmetry reason we can prescribe the derivative $\frac{\partial\varphi}{\partial n^*_{456}}(Q_{45})$. Doing this we obtain a piecewise polynomial function which is continuous in the polyhedron with six vertices $P_1, \ldots, P_6$ which consists of three tetrahedra $P_1P_2P_3P_6$, $P_1P_2P_6P_4$, $P_2P_6P_4P_5$ with disjoint interiors.

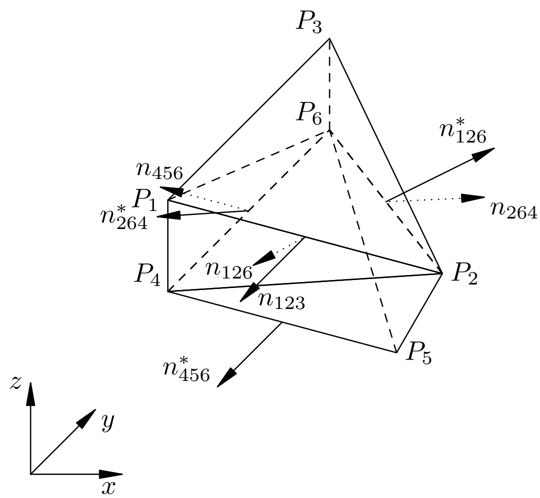

Fig. 12. The remedy.

Of course, we must pay something for this change: now the semiregular (flat) tetrahedron cannot have the short edge arbitrarily small; we must assume that (see Fig. 13)

$$\operatorname{dist}(P_2, P_3) = O(h^{1+\varepsilon}), \quad 0 < \varepsilon < 3$$

with

$$\sin\omega_{23} = \frac{O(h^{1+\varepsilon})}{O(h)} \geqslant Ch^{\varepsilon} \quad (0 < \varepsilon < 3). \tag{3.7}$$

In applications we usually take (because of error estimates and a sufficient semiregularity) $\varepsilon = 1$.

Inspecting the proof of Theorem 2.4 we see that at the point Q_{14} (a critical point—this notion is used for a point in which the optimum estimate (2.25) does not hold)

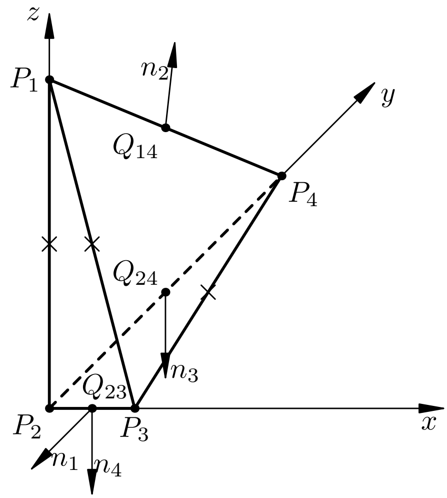

Fig. 13. Concerning the estimates (3.9), (3.10).

we have

$$\left|\frac{\partial\varphi}{\partial z_2}(Q_{14})\right|, \left|\frac{\partial\varphi}{\partial z_3}(Q_{14})\right| \leqslant C^* h^{3-\varepsilon}. \tag{3.8}$$

The other relations and estimates from (2.14), (2.15) and (2.25) remain without any change. Thus, instead of estimates (2.56), (2.57) we derive

$$\left|\frac{\partial\varphi}{\partial x}(A)\right|, \left|\frac{\partial\varphi}{\partial y}(A)\right|, \left|\frac{\partial\varphi}{\partial z}(A)\right| \leqslant \frac{CM_4 h^{3-\varepsilon}}{\sin\omega\sin\sigma} \ \ \forall A \in \overline{K1}, \tag{3.9}$$

$$|\varphi(A)| \leqslant \frac{CM_4 h^{4-\varepsilon}}{\sin\omega\sin\sigma} \ \ \forall A \in \overline{K1}, \tag{3.10}$$

where $0 < \varepsilon < 3$.

The second variant: Instead of conditions

$$\frac{\partial\varphi}{\partial n_2}(Q_{14}) = \frac{\partial\varphi}{\partial n_3}(Q_{24}) = 0$$

we prescribe

$$\varphi(S_2) = \varphi(S_3) = 0$$

where S_2 and S_3 are the centers of gravity of the triangular faces T_2 and T_3, respectively. We have again one critical point Q_{14} at which (3.8) holds with a presumably greater constant C^*. (This fact follows from [13, Theorem 2].) Thus we arrive again at estimates (3.9), (3.10).

The third variant (the case of the classical parameters for the cubic Hermite tetrahedral finite element): Instead of four normal derivatives (see Fig. 13) we prescribe

$$\varphi(S_i) = 0 \quad (i = 1, \ldots, 4)$$

where S_i is the center of gravity of the triangular face T_i $(i = 1, \ldots, 4)$. In this case all six midpoints Q_{ij} of the edges P_iP_j are critical points (this follows from the estimates for gradients in the case of regular triangular cubic elements with $\sin\alpha \geqslant Ch$—see [13, Theorem 2]); we have instead of (2.15), (2.25)

$$\left|\frac{\partial\varphi}{\partial z_i}(Q_{jk})\right| \leqslant C^* h^{3-\varepsilon} \quad (i = 1, \ldots, 4;\ j \neq i,\ k \neq i,\ k \neq j).$$

Estimates (3.9), (3.10) again hold; only numerical experiments will show whether the third variant is worse (because of six critical points instead of one; this fact follows again from [13, Theorem 2]).

Using estimates (3.9), (3.10), we can prove a general convergence theorem of the finite element method for a finite element procedure using Hermite tetrahedral finite C^0-elements just described (this means, to prove the convergence of the finite element method without any rate of convergence under the assumptions guaranteeing the unique existence of the solution of the given variational problem only). We restrict ourselves, for simplicity, to the linear problem corresponding to a mixed boundary value problem of the Poisson equation with the homogeneous Dirichlet boundary condition on Γ_1 in a bounded polyhedral domain Ω (without use of numerical integration), where $\Gamma_1 \subset \partial\Omega$, $\mathrm{meas}_2\,\Gamma_1 > 0$ (we assume that Γ_1 is a union of polygons which can lie in different faces of $\overline{\Omega}$): Find $u \in V$ such that

$$a(u, v) = L(v) \ \ \forall v \in V,$$

where

$$V = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_1\},$$

$$a(u, v) = \iiint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$L(v) = \iiint_\Omega vf\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iint_{\Gamma_2} vq\,\mathrm{d}\sigma \quad (\Gamma_2 = \partial\Omega - \Gamma_1)$$

with $f \in L_2(\Omega)$ and $q \in L_2(\Gamma_2)$.

We divide the given polyhedral domain $\overline{\Omega}$ (in this case usually narrow) into semiregular tetrahedra in such a way that each two tetrahedra are either disjoint, or have a common vertex, or a common edge, or a common face and

$$\omega \geqslant \omega_0 > 0, \quad \sigma \geqslant \sigma_0 > 0 \tag{3.11}$$

and consider a sequence $\{\mathcal{D}_h\}$ of such divisions, where every member satisfies (3.11) and assumption (3.7) and where $h \to 0$ (with h being the length of the largest edge in the given division). We define on every division $\mathcal{D}_h$ the finite dimensional space

$$V_h = \{v \in C^0(\overline{\Omega})\colon\ v(x,y,z) = p(x,y,z)\ \ \forall (x,y,z) \in \overline{K} \subset \mathcal{D}_h,\ \ v = 0 \text{ on } \Gamma_1\},$$

where $p(x,y,z)$ is the polynomial from the remedy. We look for a $u_h \in V_h$ such that

$$a(u_h, v) = L(v)\ \ \forall v \in V_h.$$

The theorem on the convergence of Galerkin's method says that if

$$\operatorname{dist}(V_h, v) = \inf_{w \in V_h} \|w - v\|_{1,\Omega} \to 0\ \ \forall v \in V \tag{3.12}$$

then

$$\|u - u_h\|_{1,\Omega} \to 0. \tag{3.13}$$

It is sufficient to prove (3.12) for all $v \in C^\infty(\overline{\Omega}) \cap V$ because $C^\infty(\overline{\Omega}) \cap V$ is dense in V.

Let $v_0 \in C^\infty(\overline{\Omega}) \cap V$ be an arbitrary fixed function. It satisfies the relation

$$|D^\alpha v_0(x,y,z)| \leqslant M_4(v_0) \quad [x,y,z] \in \overline{\Omega},\ \ |\alpha| = 4,$$

where $M_4(v_0)$ is a constant depending on v_0. We construct a function $w_h \in V_h$ which on tetrahedra $\overline{K} \in \mathcal{D}_h$ is equal to our polynomials determined by parameters equal to the corresponding parameters of the function $v_0(x,y,z)$. We have

$$\|w_h - v_0\|_{1,\Omega}^2 = \sum_{K \subset \mathcal{D}_h} \iiint_K \sum_{|\alpha| \leqslant 1} [D^\alpha (w_h - v_0)]^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Hence, according to estimates (3.9), (3.10) and assumptions (3.7) and (3.11),

$$\|w_h - v_0\|_{1,\Omega} \leqslant \frac{C}{\sin\omega_0 \sin\sigma_0} M_4(v_0) h^{3-\varepsilon}.$$

Thus $\|w_h - v_0\|_{1,\Omega} \to 0$ for $h \to 0$ and as

$$\operatorname{dist}(V_h, v_0) \leqslant \|w_h - v_0\|_{1,\Omega},$$

relation (3.12) follows and (3.13) holds.

If $u \in C^1(\overline{\Omega})$ and $|D^\alpha u(x,y,z)| \leqslant M_4(u)$ for all $[x,y,z] \in \Omega$ with all $|\alpha| = 4$ then we can prove in a similar way that

$$\|u_h - u\|_{1,\Omega} \leqslant \frac{C}{\sin\omega_0 \sin\sigma_0} M_4(u) h^{3-\varepsilon}. \tag{3.14}$$

This means that the maximum rate of convergence in the case of our finite elements is $\mathcal{O}(h^{3-\varepsilon})$.

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