Miloslav Feistauer; Karel Najzar; Veronika Sobotíková

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ON THE FINITE ELEMENT ANALYSIS OF PROBLEMS WITH NONLINEAR NEWTON BOUNDARY CONDITIONS IN NONPOLYGONAL DOMAINS*

Miloslav Feistauer, Karel Najzar
and Veronika Sobotíková, Praha

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Abstract. The paper is concerned with the study of an elliptic boundary value problem with a nonlinear Newton boundary condition considered in a two-dimensional nonpolygonal domain with a curved boundary. The existence and uniqueness of the solution of the continuous problem is a consequence of the monotone operator theory. The main attention is paid to the effect of the basic finite element variational crimes: approximation of the curved boundary by a polygonal one and the evaluation of integrals by numerical quadratures. With the aid of some important properties of Zlámal's ideal triangulation and interpolation, the convergence of the method is analyzed.

Keywords: elliptic equation, nonlinear Newton boundary condition, monotone operator method, finite element approximation, approximation of a curved boundary, numerical integration, ideal triangulation, ideal interpolation, convergence of the finite element method

MSC 2000: 65N30, 65N15

Introduction

A number of problems of technology and science are described by partial differential equations equipped with nonlinear Newton boundary conditions. Let us mention, e.g., radiation and heat transfer problems ([1], [19], [21]), modelling of electrolysis of aluminium with turbulent flow at the boundary ([6], [22]) and some

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problems of elasticity ([15]). In some cases the boundary value problem with a nonlinear Newton boundary condition is reformulated as a nonlinear boundary integral equation and solved numerically with the aid of the boundary element method (see, e.g., [16], [17]). Another quite natural possibility is to introduce the concept of a weak solution and apply the finite element method to the numerical solution of this problem.

In the analysis of the finite element discrete problem with a nonlinear Newton boundary condition one meets a number of obstacles, particularly in the very topical case when the nonlinearity is unbounded and has a polynomial behaviour. The first results for a problem of this type were obtained in [6], where the existence and uniqueness of the solution of the continuous problem was proved with the aid of the monotone operator theory, and the convergence of the approximate solutions to the exact one was established under the assumption that all integrals appearing in the discrete problem were evaluated exactly. In [8], the convergence of the finite element method was proved in the case that both the volume and boundary integrals were calculated with the aid of quadrature formulae. In the analysis of the boundary terms we were not successful in applying the well-known Ciarlet–Raviart theory ([3], [4]) of the finite element numerical integration because of the nonlinearity on the boundary. The convergence analysis was obtained with the aid of a suitable modification of results from [27]. Furthermore, the work [9] is concerned with the derivation of error estimates. They were obtained thanks to the uniform monotonicity of the problem which we had derived in [9]. (In the previous papers only strict monotonicity was established.) However, in contrast to standard nonlinear situations treated, e.g., in [2], [11], [12], [18], [28], where strong monotonicity was used, here we do not get an optimal $O(h)$ error estimate for linear finite elements. The order of convergence is reduced due to the fact that only uniform monotonicity with growth of degree $t^{2+\alpha}, \alpha > 0$, holds now, and due to the nonlinearity in the boundary integrals. Moreover, the application of numerical integration in the nonlinear boundary integral can also lead to further reduction of the rate of convergence. The theoretically established decrease of the order of convergence caused by the nonlinearity in the Newton boundary condition was confirmed with the aid of numerical experiments in [10].

In the above mentioned papers [6], [8], [9], [10], the domain was assumed to be polygonal. In practice one meets, of course, problems in nonpolygonal domains with piecewise curved boundaries. Then such a domain is approximated by a polygonal one over which the finite element discretization is applied. By G. Strang ([25]), the approximation of the boundary and the use of numerical integration represent the basic *finite element variational crimes.* They were analyzed in a number of works. As a fundamental literature we mention [3] and [4]. The finite element variational crimes in the approximations of nonlinear elliptic problems were investigated in [12], [13], [7], [11], [24], [28] and [29]. The main tools were Zlámal's concepts of ideal triangulation and ideal interpolation ([30]).

Here we will be concerned with the finite element analysis of the boundary value problem with nonlinear Newton boundary conditions considered in a general two-dimensional nonpolygonal domain. With the aid of the above mentioned Zlámal's techniques, we establish the convergence of the approximate finite element solutions to the exact one, taking into account the effect of numerical integration and approximation of the curved boundary. The contents of the paper is as follows: In Section 1, the continuous problem is formulated and the concept of a weak solution is introduced. In Section 2, the problem is discretized by the finite element method (with the approximation of the boundary and the use of quadrature formulae for the evaluation of the integrals appearing in the weak formulation). Section 3 is devoted to the definition and important properties of the ideal triangulation and the associated ideal interpolation. Finally, in Section 4, the convergence of the method is established.

1. Formulation of the problem. Weak solution

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ and let $\partial\Omega$ be piecewise of class C^3. By $\overline{\Omega}$ we denote the closure of Ω.

We consider the following *boundary value problem*: Find $u\colon \overline{\Omega} \to \mathbb{R}$ such that

$$-\triangle u = f \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial n} + \kappa |u|^{\alpha} u = \varphi \quad \text{on } \partial\Omega, \tag{1.2}$$

where $f\colon \Omega \to \mathbb{R}$ and $\varphi\colon \partial\Omega \to \mathbb{R}$ are given functions, n is the unit outward normal and $\kappa > 0$, $\alpha \geqslant 0$ are given constants. The *classical solution* of the above problem can be defined as a function $u \in C^2(\overline{\Omega})$ satisfying (1.1) and (1.2).

In what follows we will work with the well-known Lebesgue and Sobolev spaces $L^p(\Omega)$, $L^p(\partial\Omega)$, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, $W^{k,p}(\partial\Omega)$. (See, e.g., [20].) By $\|\cdot\|_{k,p,\Omega}$ and $\|\cdot\|_{k,p,\partial\Omega}$ we denote the standard norms in $W^{k,p}(\Omega)$ and $W^{k,p}(\partial\Omega)$, respectively. Then $\|\cdot\|_{0,p,\Omega}$ and $\|\cdot\|_{0,p,\partial\Omega}$ mean, of course, the norms in $L^p(\Omega)$ and $L^p(\partial\Omega)$. The symbol $|\cdot|_{k,p,\Omega}$ denotes the seminorm in $W^{k,p}(\Omega)$. (Similar notation will be used for the Lebesgue and Sobolev spaces over other sets.)

Let us assume that

$$f \in L^2(\Omega), \quad \varphi \in L^2(\partial\Omega).$$

In the usual way we can introduce a weak formulation of problem (1.1)–(1.2). To this end, we define the following forms:

$$
\begin{aligned}
(1.3)\qquad\qquad b(u,v) &= \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x,\\
d(u,v) &= \kappa\int_{\partial\Omega}|u|^\alpha uv\,\mathrm{d}S,\\
a(u,v) &= b(u,v)+d(u,v),\\
L^\Omega(v) &= \int_\Omega fv\,\mathrm{d}x,\quad L^\Gamma(v)=\int_{\partial\Omega}\varphi v\,\mathrm{d}S,\\
L(v) &= L^\Omega(v)+L^\Gamma(v),\\
&\quad u,v\in H^1(\Omega).
\end{aligned}
$$

Defintion 1. We say that a function $u\colon\Omega\to\mathbb{R}$ is a *weak solution of problem* (1.1)–(1.2), if

$$
\begin{aligned}
(1.4)\qquad\qquad &\text{a) } u\in H^1(\Omega),\\
&\text{b) } a(u,v)=L(v)\ \ \forall v\in H^1(\Omega).
\end{aligned}
$$

It was shown in [8] that

Theorem 1.1. *Problem* (1.4) *has exactly one solution.*

2. Finite element discretization

We consider a system $\{\Omega_h\}_{h\in(0,h_0)}$, $0<h_0<1$, of polygonal approximations of Ω. Let Ω^* be such a domain with Lipschitz-continuous boundary that $\Omega\cup\Omega_h\subset\Omega^*$ for every $h\in(0,h_0)$.

Let $\mathcal{T}_h$, $h\in(0,h_0)$, be triangulations of domains Ω_h with the following properties:

(T1) Any triangulation is formed by a finite number of closed triangles T.

(T2) $\overline{\Omega}_h=\bigcup\limits_{T\in\mathcal{T}_h}T$.

(T3) If $T_i,T_j\in\mathcal{T}_h$, $T_i\neq T_j$, then either $T_i\cap T_j=\emptyset$ or $T_i\cap T_j$ is a common vertex or $T_i\cap T_j$ is a common side of T_i and T_j.

(T4) If $T\in\mathcal{T}_h$, then at most two vertices of T lie on $\partial\Omega$.

We denote by σ_h the set of all vertices of $\mathcal{T}_h$. Let

(T5) $\sigma_h\subset\overline{\Omega}$, $\sigma_h\cap\partial\Omega_h\subset\partial\Omega$.

(T6) The points from $\partial\Omega$, where the condition of C^3-smoothness of $\partial\Omega$ is not satisfied, are elements of σ_h.

We will denote by h_T and ϑ_T the length of the maximal side and the magnitude of the minimal angle of $T \in \mathcal{T}_h$, respectively. We set

$$\widetilde{h} = \max_{T \in \mathcal{T}_h} h_T, \quad \vartheta_h = \min_{T \in \mathcal{T}_h} \vartheta_T. \tag{2.1}$$

We assume the index h to be chosen in such a way that $h = \widetilde{h}$. Considering the numbers ϑ_h, we will suppose the existence of such a constant $\vartheta_0 > 0$ that

$$\vartheta_h \geqslant \vartheta_0 \;\; \forall h \in (0, h_0), \tag{2.2}$$

which means that

(T7) the system of triangulations $\{\mathcal{T}_h\}_{h \in (0,h_0)}$ is regular.

We say that $T \in \mathcal{T}_h$ is a *boundary triangle*, if T has a side $S \subset \partial\Omega_h$. We denote by s_h the sets of all sides $S \subset \partial\Omega_h$ of boundary triangles $T \in \{\mathcal{T}\}_h$.

Further, let the following hold:

(T8) If $P \in \partial\Omega_h \cap \partial\Omega$, then either $P \in \sigma_h$ or there exists such a side $S \in s_h$ that $P \in S$ and $S \subset \partial\Omega \cap \partial\Omega_h$; if $P \in \partial\Omega \cap \partial T$ for some $T \in \mathcal{T}_h$, then $P \in \partial\Omega_h$.

In what follows we set $|T| =$ area of $T \in \mathcal{T}_h$ and $|S| =$ length of $S \in s_h$.

Let us assume that

(T9) the triangulations $\mathcal{T}_h$, $h \in (0, h_0)$, satisfy *locally an inverse assumption* at $\partial\Omega$: there exists $\nu > 0$ such that

$$h/|S| \leqslant \nu \;\; \forall S \in s_h, \;\; \forall h \in (0, h_0). \tag{2.3}$$

The assumptions (T9) and (2.2) give the existence of a constant $\sigma > 0$ such that

$$\sigma h^2 \leqslant |T| \leqslant h^2 \quad \text{for every boundary triangle } T \in \mathcal{T}_h \text{ and every } h \in (0, h_0). \tag{2.4}$$

Finally, because of our further considerations we suppose that

$$f \in W^{1,q}(\Omega^*) \text{ for some } q > 2 \text{ and } \varphi \in W^{1,p}(\partial\Omega) \text{ for some } p > 1. \tag{2.5}$$

An approximate solution to problem (1.4) will be sought in the space of triangular conforming piecewise linear elements $H_h \subset H^1(\Omega_h)$:

$$H_h = \{v \in C(\overline{\Omega}_h); v|_T \text{ is linear for each } T \in \mathcal{T}_h\}.$$

First we approximate the bilinear forms b, d, a and the linear functionals L^{Ω}, L^{Γ}, L by

$$
\begin{aligned}
\widetilde{b}_h(u,v) &= \int_{\Omega_h} \nabla u \cdot \nabla v \,\mathrm{d}x, \\
\widetilde{d}_h(u,v) &= \kappa \int_{\partial\Omega_h} |u|^{\alpha} uv \,\mathrm{d}S, \\
\widetilde{a}_h(u,v) &= \widetilde{b}_h(u,v) + \widetilde{d}_h(u,v), \\
\widetilde{L}_h^{\Omega}(v) &= \int_{\Omega_h} fv \,\mathrm{d}x, \quad \widetilde{L}_h^{\Gamma}(v) = \int_{\partial\Omega_h} \varphi_h v \,\mathrm{d}S, \\
\widetilde{L}_h(v) &= \widetilde{L}_h^{\Omega}(v) + \widetilde{L}_h^{\Gamma}(v), \\
& u, v \in H^1(\Omega_h),
\end{aligned}
\tag{2.6}
$$

where the function $\varphi_h \colon \partial\Omega_h \to \mathbb{R}$ is an approximation of φ defined later according to (3.6).

In practical computations, the integrals appearing in the definition (2.6) of the forms $\widetilde{d}_h(u,v)$ and $\widetilde{L}_h(v)$ are usually evaluated for $u, v \in H_h$ with the aid of quadrature formulae.

For $\psi \in C(\overline{\Omega}_h)$ we write

$$
\begin{aligned}
&\text{a)} \quad \int_{\Omega_h} \psi \,\mathrm{d}x = \sum_{T \in \mathcal{T}_h} \int_T \psi \,\mathrm{d}x, \\
&\text{b)} \quad \int_T \psi \,\mathrm{d}x \approx |T| \sum_{\mu=1}^{M} \omega_\mu \psi(x_{T,\mu}), \\
&\text{c)} \quad E_{\Omega}(\psi) = \int_{\Omega_h} \psi \,\mathrm{d}x - \sum_{T \in \mathcal{T}_h} |T| \sum_{\mu=1}^{M} \omega_\mu \psi(x_{T,\mu}),
\end{aligned}
\tag{2.7}
$$

where $\omega_\mu \in \mathbb{R}$ and $x_{T,\mu} \in T$. Similarly, for $\theta \in C(\partial\Omega_h)$ we evaluate integrals over $\partial\Omega_h$:

$$
\begin{aligned}
&\text{a)} \quad \int_{\partial\Omega_h} \theta \,\mathrm{d}S = \sum_{S \in s_h} \int_S \theta \,\mathrm{d}S, \\
&\text{b)} \quad \int_S \theta \,\mathrm{d}S \approx |S| \sum_{\mu=1}^{m} \beta_\mu \theta(x_{S,\mu}), \\
&\text{c)} \quad E_{\Gamma}(\theta) = \int_{\partial\Omega_h} \theta \,\mathrm{d}S - \sum_{S \in s_h} |S| \sum_{\mu=1}^{m} \beta_\mu \theta(x_{S,\mu})
\end{aligned}
\tag{2.8}
$$

with $\beta_\mu \in \mathbb{R}$ and $x_{S,\mu} \in S$.

We will assume that

(2.9) a) formula (2.7 b) is exact for all $\psi \in P_0(T)$, $T \in \mathcal{T}_h$ (i.e.$\Sigma\omega_\mu = 1$),

b) formula (2.8 b) is exact for all $\theta \in P_1(S)$, $S \in s_h$,

c) formula (2.8 b) is monotone, i.e., $\beta_\mu > 0$, $\mu = 1, \dots, m$.

Using the above formulae, we get the approximations d_h, L_h^Ω and L_h^Γ of $\widetilde{d}_h$, $\widetilde{L}_h^\Omega$ and $\widetilde{L}_h^\Gamma$:

(2.10)
$$\begin{aligned}
&\text{a)}\ d_h(u,v) = \kappa \sum_{S \in s_h} |S| \sum_{\mu=1}^{m} \beta_\mu (|u|^\alpha uv)(x_{S,\mu}),\\
&\text{b)}\ L_h^\Gamma(v) \quad = \sum_{S \in s_h} |S| \sum_{\mu=1}^{m} \beta_\mu (\varphi_h v)(x_{S,\mu}),\\
&\text{c)}\ L_h^\Omega(v) \quad = \sum_{T \in \mathcal{T}_h} |T| \sum_{\mu=1}^{M} \omega_\mu (fv)(x_{T,\mu}),\\
&\qquad u, v \in H_h.
\end{aligned}$$

For $u, v \in H_h$ the bilinear form $\widetilde{b}_h(u,v)$ is evaluated exactly because the elements used are linear. We set

$$\begin{aligned}
&\text{a)}\ a_h(u,v) = \widetilde{b}_h(u,v) + d_h(u,v), \quad u, v \in H_h,\\
&\text{b)}\quad L_h(v) = L_h^\Omega(v) + L_h^\Gamma(v), \qquad\quad v \in H_h.
\end{aligned}$$

For the approximation of the boundary forms $\widetilde{L}_h^\Gamma$ and $\widetilde{d}_h$ different integration formulae could be used, in general. For the sake of simplicity we will not distinguish them by notation.

Now we can formulate the *discrete problem*:

Definition 2. We call a function $u_h\colon \Omega_h \to \mathbb{R}$ an *approximate solution* of problem (1.4), if

(2.12)
$$\begin{aligned}
&\text{a)}\ u_h \in H_h,\\
&\text{b)}\ a_h(u_h, v_h) = L_h(v_h)\ \forall v_h \in H_h.
\end{aligned}$$

In the case of a nonpolygonal domain Ω we meet in the analysis of the finite element method some difficulties caused by the facts that $\Omega_h \neq \Omega$, $\partial\Omega_h \neq \partial\Omega$, $\bigcup_{T\in\mathcal{T}_h} T \neq \overline{\Omega}$ and $H_h \not\subset W^{1,2}(\Omega)$. Therefore, we introduce the concept of the so-called ideal triangulation, using the ideas from [30], [5], [11], [12], [13], [28], [29].

First we introduce the *ideal triangle*. Let $T \in \mathcal{T}_h$ be a boundary element, i.e., two vertices of T lie on $\partial\Omega$. We denote the vertices of this triangle by P_1, P_2, P_3 in such a way that $P_1, P_3 \in \partial\Omega$. By T^{id} we denote the "curved" element which we obtain from T by replacing the side $\overline{P_1P_3} \subset \partial\Omega_h$ by the arc $\widehat{P_1P_3} \subset \partial\Omega$. According to assumptions (T4) and (T8) the arc $\widehat{P_1P_3}$ has no common points with $\overline{P_1P_2}$ and $\overline{P_2P_3}$ except P_1 and P_3, and the boundary ∂T^{id} is a Jordan curve or $\overline{P_1P_3} = \widehat{P_1P_3}$. If T is not a boundary element, then we set $T^{\mathrm{id}} := T$ and the vertices can be numbered in any ordering. The element T^{id} obtained in this way will be called *the ideal element associated with the element* $T \in \mathcal{T}_h$.

Let now $\mathcal{T}_h$ be a triangulation of the domain Ω_h. We denote by $\mathcal{T}_h^{\mathrm{id}}$ the *ideal triangulation* of the domain Ω associated with $\mathcal{T}_h$, which is obtained from $\mathcal{T}_h$ by replacing the triangles $T \in \mathcal{T}_h$ by the ideal triangles T^{id} associated with them. It is obvious that $\bigcup_{T^{\mathrm{id}}\in\mathcal{T}_h^{\mathrm{id}}} T^{\mathrm{id}} = \overline{\Omega}$.

Let $\check{T}$ be the unit triangle with vertices $R_1 = (0,0)$, $R_2 = (1,0)$, $R_3 = (0,1)$ in the $\xi = (\xi_1, \xi_2)$-plane. Then the affine mapping

$$\xi \in \check{T},\ \ \xi = (\xi_1, \xi_2) \mapsto x = x^0(\xi) \equiv P_1 + (P_2 - P_1)\xi_1 + (P_3 - P_1)\xi_2 \in \mathbb{R}^2 \tag{3.1}$$

obviously maps the triangle $\check{T}$ one-to-one onto the triangle T.

Let us now denote by S the segment $\overline{P_1P_3}$ of T and by $\widetilde{x}^0(r) = (\widetilde{x}_1^0(r), \widetilde{x}_2^0(r))$, $r \in \langle 0, |S|\rangle$ such a parametrization of S that r is the length of the part of S measured from P_1 to $\widetilde{x}^0(r)$. Analogously we denote by $\Sigma = \Sigma_S$ the arc $\widehat{P_1P_3}$ of T^{id}, by $|\Sigma|$ the length of Σ and by $\widetilde{x}(t) = (\widetilde{x}_1(t), \widetilde{x}_2(t))$, $t \in \langle 0, |\Sigma|\rangle$, such a parametrization of the arc Σ that t is the length of the part of Σ measured from P_1 to $\widetilde{x}(t)$.

It is possible to show that the mapping

$$\Phi_{T^{\mathrm{id}}}(\eta) = \frac{\widetilde{x}(|\Sigma|\eta) - x^0(0,\eta)}{1-\eta} = \frac{\widetilde{x}(|\Sigma|\eta) - P_1 - (P_3 - P_1)\eta}{1-\eta}, \quad \eta \in \langle 0, 1), \tag{3.2}$$

can be extended for $\eta = 1$ in such a way that it is of class C^2 on $\langle 0, 1\rangle$, the mapping

$$\xi \in \check{T},\ \ \xi = (\xi_1, \xi_2) \mapsto x = x(\xi) \equiv x^0(\xi) + (1 - \xi_1 - \xi_2)\Phi_{T^{\mathrm{id}}}(\xi_2) \in \mathbb{R}^2 \tag{3.3}$$

maps the triangle $\check{T}$ one-to-one onto the ideal triangle T^{id} and the mapping (3.3) as well as its inverse are of class C^2. (See [30], Theorem 1.)

R e m a r k s. a) We denote by ξ^0 and ξ the inverse mappings to x^0 and x, and by r and t the inverse mappings to $\widetilde{x}^0$ and $\widetilde{x}$.

b) We have $x(0,\xi_2) = \widetilde{x}(|\Sigma|\xi_2)$, $x^0(0,\xi_2) = \widetilde{x}^0(|S|\xi_2)$ for $\xi_2 \in \langle 0,1\rangle$.

c) If $T^{\rm id} \in \mathcal{T}^{\rm id} \cap \mathcal{T}_h$ (i.e., $\mathcal{T}^{\rm id} = T$), then obviously $\Phi_{T^{\rm id}}(\eta) \equiv 0$ in $\langle 0,1\rangle$ and $x = x^0$.

For $w \in L^p(\Omega_h)$, $p \geqslant 1$, we define *a function* $\hat{w}\colon \overline{\Omega} \to \mathbb{R}$ *associated with* w by

$$\hat{w}(x) = w(x^0(\xi(x)))\ \ \forall \mathcal{T}^{\rm id} \in \mathcal{T}_h^{\rm id},\ \ \forall x \in T^{\rm id} \tag{3.4}$$

and for a function $\gamma \in L^p(\partial\Omega_h)$, $p \geqslant 1$, *a function* $\hat{\gamma}\colon \partial\Omega \to \mathbb{R}$ *is associated with* γ by

$$\hat{\gamma}(x) = \gamma(x^0(\xi(x)))\ \ \forall S \in s_h,\ \ \forall x \in \Sigma_S. \tag{3.5}$$

On the other hand, we define *an approximation* $\gamma_h\colon \partial\Omega_h \to \mathbb{R}$ *of a function* $\gamma \in L^p(\partial\Omega)$ by

$$\gamma_h(x^0) = \gamma(x(\xi^0(x^0)))\ \ \forall S \in s_h,\ \ \forall x^0 \in S. \tag{3.6}$$

R e m a r k. It is obvious that $\hat{w}_h|_T = w_h|_T$ for every $T \in \mathcal{T}_h \cap \mathcal{T}_h^{\rm id}$.

For $v\colon M \to \mathbb{R}$, where $M = T^{\rm id}$ or $M = \Sigma = \widehat{P_1P_3}$, we define a function $\breve{v}\colon \breve{M} \to \mathbb{R}$, where $\breve{M} = \breve{T}$ or $\breve{M} = \breve{S} = \overline{R_1R_3}$, by

$$\breve{v}(\xi) = v(x(\xi)), \quad \xi \in \breve{M}, \tag{3.7}$$

and similarly for $v\colon M^0 \to \mathbb{R}$ with $M^0 = T$ or $M^0 = S = \overline{P_1P_3}$, we put

$$\breve{v}(\xi) = v(x^0(\xi)), \quad \xi \in \breve{M}. \tag{3.8}$$

It is easy to see that for $v\colon T \to \mathbb{R}$ or $v\colon S \to \mathbb{R}$ we have

$$(\hat{v})\breve{}(\xi) = \hat{v}(x(\xi)) = v(x^0(\xi(x(\xi)))) = v(x^0(\xi)) = \breve{v}(\xi). \tag{3.9}$$

Now let us introduce sets ω_h and τ_h and the natural extension of a function from H_h which we will use in several proofs. Let $T^{\rm id} \in \mathcal{T}_h^{\rm id} \setminus \mathcal{T}_h$ be an ideal triangle with vertices P_1, P_2, P_3 numbered as above. We denote again by Σ the arc $\widehat{P_1P_3} \subset \partial\Omega$ and by S the straight segment $\overline{P_1P_3}$. According to assumption (T8) we have

$$\Sigma \cap S = \{P_1, P_3\}$$

and the bounded open set $\mathcal{L}$ with the boundary formed by Σ and S is a simply connected domain.

Let us set

$$\omega_h = \Omega \setminus \overline{\Omega}_h, \quad \tau_h = \Omega_h \setminus \overline{\Omega}. \tag{3.10}$$

Due to assumption (T8), these sets are formed by a finite number of components of the type of the set $\mathcal{L}$ with the boundary $\Sigma \cup S$.

We say that a function $\overline{w}_h\colon \overline{\Omega}_h \cup \overline{\Omega} \to \mathbb{R}$ is *the natural extension of a function* $w_h \in H_h$ from $\overline{\Omega}_h$ to $\overline{\Omega}_h \cup \overline{\Omega}$, if for every $T \in \mathcal{T}_h$ and T^{id} associated with T we have

$$\overline{w}_h|_{T \cup T^{\mathrm{id}}} = p|_{T \cup T^{\mathrm{id}}},$$

where $p \in P_1(\mathbb{R}^2)$ is the polynomial of degree $\leqslant 1$ defined on $\mathbb{R}^2$ satisfying $p|_T = w_h|_T$.

Remark. In what follows c denotes a generic constant which can assume different values at different places and $c_1, c_2, \ldots$ are local generic constants that can have different values in different proofs.

Lemma 1. *Let $p \in \langle 1, \infty)$. Then there exist $h_0 > 0$ and $c = c(p) > 0$ such that for every $h \in (0, h_0)$ and $w \in L^p(\partial\Omega_h)$*

$$\begin{aligned} &\text{a)}\ \|w\|_{0,p,\partial\Omega_h} \leqslant c\|\hat{w}\|_{0,p,\partial\Omega}, \\ &\text{b)}\ \|\hat{w}\|_{0,p,\partial\Omega} \ \leqslant c\|w\|_{0,p,\partial\Omega_h}. \end{aligned} \tag{3.11}$$

Proof. We will proceed in several steps: I) Let $S \in s_h$, $\Sigma = \Sigma_S$ and $v \in L^1(S)$, $u \in L^1(\Sigma)$. Then by our choice of parametrizations $\widetilde{x}^0$ and $\widetilde{x}$ of S and Σ we have

$$\int_S v(x)\,\mathrm{d}S = |S| \int_0^1 \breve{v}(0, \xi_2)\,\mathrm{d}\xi_2 \tag{3.12}$$

and

$$\int_\Sigma u(x)\,\mathrm{d}S = |\Sigma| \int_0^1 \breve{u}(0, \xi_2)\,\mathrm{d}\xi_2. \tag{3.13}$$

This and (3.9) imply that

$$\left|\int_\Sigma \hat{v}(x)\,\mathrm{d}S - \int_S v(x)\,\mathrm{d}S\right| = \big||\Sigma| - |S|\big| \left|\int_0^1 \breve{v}(0, \xi_2)\,\mathrm{d}\xi_2\right|. \tag{3.14}$$

We can estimate the term $\big||\Sigma| - |S|\big|$ on the basis of the definition of the length of an arc, the smoothness of the boundary and the mean value theorem by

$$\big||\Sigma| - |S|\big| \leqslant c_1 h^2. \tag{3.15}$$

This, (3.14), (3.12) and assumption (T9) yield

$$\begin{aligned} \left|\int_\Sigma \hat{v}(x)\,\mathrm{d}S - \int_S v(x)\,\mathrm{d}S\right| &\leqslant c_1 h^2 \left|\int_0^1 \check{v}(0,\xi_2)\,\mathrm{d}\xi_2\right| \\ &\leqslant c_1 h^2 |S|^{-1} \left|\int_S v(x)\,\mathrm{d}S\right| \\ &\leqslant c_2 h \left|\int_S v(x)\,\mathrm{d}S\right|, \end{aligned} \tag{3.16}$$

i.e., for $v \in L^1(\partial\Omega_h)$ we have

$$\begin{aligned} \left|\int_{\partial\Omega} \hat{v}(x)\,\mathrm{d}S - \int_{\partial\Omega_h} v(x)\,\mathrm{d}S\right| &\leqslant \sum_{S\in s_h} \left|\int_\Sigma \hat{v}(x)\,\mathrm{d}S - \int_S v(x)\,\mathrm{d}S\right| \\ &\leqslant \sum_{S\in s_h} c_2 h \left|\int_S v(x)\,\mathrm{d}S\right| \\ &\leqslant c_2 h \int_{\partial\Omega_h} |v(x)|\,\mathrm{d}S. \end{aligned} \tag{3.17}$$

II) Let now $w \in L^p(\partial\Omega_h)$, $p \in \langle 1, \infty)$. If we put $v := |w|^p$ in (3.17) we get

$$\begin{aligned} \big|\|\hat{w}\|^p_{0,p,\partial\Omega} - \|w\|^p_{0,p,\partial\Omega_h}\big| &= \left|\int_{\partial\Omega} |\hat{w}|^p\,\mathrm{d}S - \int_{\partial\Omega_h} |w|^p\,\mathrm{d}S\right| \\ &\leqslant c_2 h \int_{\partial\Omega_h} |w|^p\,\mathrm{d}S = c_2 h \|w\|^p_{0,p,\partial\Omega_h} \end{aligned} \tag{3.18}$$

as $|\hat{w}|^p = \widehat{|w|^p}$.

III) Now we can prove estimates (3.11 a) and (3.11 b).

a) From (3.18) we have

$$\|w\|^p_{0,p,\partial\Omega_h} \leqslant \|\hat{w}\|^p_{0,p,\partial\Omega} + c_2 h \|w\|^p_{0,p,\partial\Omega_h}, \tag{3.19}$$

which gives for $h < h_0 < \frac{1}{c_2}$ the inequalities

$$\|w\|^p_{0,p,\partial\Omega_h} \leqslant \frac{1}{1 - c_2 h} \|\hat{w}\|^p_{0,p,\partial\Omega} < \frac{1}{1 - c_2 h_0} \|\hat{w}\|^p_{0,p,\partial\Omega}, \tag{3.20}$$

i.e.,

$$\|w\|_{0,p,\partial\Omega_h} \leqslant c \|\hat{w}\|_{0,p,\partial\Omega}.$$

b) Similarly we have

$$\|\hat{w}\|^p_{0,p,\partial\Omega} \leqslant \|w\|^p_{0,p,\partial\Omega_h}(1+c_2h) < \|w\|^p_{0,p,\partial\Omega_h}(1+c_2h_0), \tag{3.21}$$

i.e.,

$$\|\hat{w}\|_{0,p,\partial\Omega} \leqslant c\|w\|_{0,p,\partial\Omega_h}.$$

$\square$

Lemma 2. *Let $p \in \langle 1,\infty)$. Then there exists a constant $c = c(p) > 0$ such that*

$$\begin{aligned} &\text{a)}\ \ |v|_{1,p,\partial\Omega_h} \leqslant c|\hat{v}|_{1,p,\partial\Omega}, \\ &\text{b)}\ \ |\hat{v}|_{1,p,\partial\Omega} \ \leqslant c|v|_{1,p,\partial\Omega_h} \end{aligned} \tag{3.22}$$

for every $h \in (0,h_0)$ and $v \in W^{1,p}(\partial\Omega_h)$.

P r o o f. For $S = \overline{P_1P_3} \in s_h$, $\Sigma = \Sigma_S$ and $v \in W^{1,p}(\partial\Omega_h)$ we define

$$v_\Sigma(t) = \hat{v}(\widetilde{x}(t)) = \hat{v}\Bigl(x\Bigl(0,\frac{t}{|\Sigma|}\Bigr)\Bigr) = \breve{v}\Bigl(0,\frac{t}{|\Sigma|}\Bigr), \qquad t \in \langle 0,|\Sigma|\rangle, \tag{3.23}$$

$$v_S(r) = v(\widetilde{x}^0(r)) = v\Bigl(x^0\Bigl(0,\frac{r}{|S|}\Bigr)\Bigr) = \breve{v}\Bigl(0,\frac{r}{|S|}\Bigr), \quad r \in \langle 0,|S|\rangle. \tag{3.24}$$

a) By the definition of the seminorm $|.|_{1,p,S}$ and by the choice of $\widetilde{x}^0$ and $\widetilde{x}$ we have

$$\begin{aligned} |v|^p_{1,p,S} &= \int_S |v'_S(r(x))|^p\,\mathrm{d}S = \int_S \Bigl|\frac{\partial\breve{v}}{\partial\xi_2}\Bigl(0,\frac{r}{|S|}\Bigr)\frac{1}{|S|}\Bigr|^p\,\mathrm{d}S \\ &= \int_0^{|S|} \Bigl|\frac{\partial\breve{v}}{\partial\xi_2}\Bigl(0,\frac{r}{|S|}\Bigr)\frac{1}{|S|}\Bigr|^p\,\mathrm{d}r = \int_0^{|\Sigma|} \Bigl|\frac{\partial\breve{v}}{\partial\xi_2}\Bigl(0,\frac{t}{|\Sigma|}\Bigr)\frac{1}{|S|}\Bigr|^p\frac{|S|}{|\Sigma|}\,\mathrm{d}t \\ &= \Bigl(\frac{|S|}{|\Sigma|}\Bigr)^{1-p}\int_0^{|\Sigma|} \Bigl|\frac{\partial\breve{v}}{\partial\xi_2}\Bigl(0,\frac{t}{|\Sigma|}\Bigr)\frac{1}{|\Sigma|}\Bigr|^p\,\mathrm{d}t = \Bigl(\frac{|S|}{|\Sigma|}\Bigr)^{1-p}|\hat{v}|^p_{1,p,\Sigma}. \end{aligned} \tag{3.25}$$

Now we use (3.15) and the local inverse assumption (T9) to estimate the fraction $|\Sigma|/|S|$:

$$\frac{|\Sigma|}{|S|} \leqslant \frac{|S| + \bigl||\Sigma|-|S|\bigr|}{|S|} \leqslant 1 + \frac{c_1h^2}{|S|} \leqslant 1 + c_2h < 1 + c_2h_0. \tag{3.26}$$

This means that

$$|v|^p_{1,p,S} \leqslant \bigl(1+c_2h_0\bigr)^{p-1}|\hat{v}|^p_{1,p,\Sigma}, \tag{3.27}$$

i.e.,

$$|v|^p_{1,p,\partial\Omega_h} = \sum_{S\in s_h}|v|^p_{1,p,S} \leqslant \bigl(1+c_2h_0\bigr)^{p-1}|\hat{v}|^p_{1,p,\partial\Omega}. \tag{3.28}$$

b) The second estimate can be obtained immediately from (3.25) as $|S| \leqslant |\Sigma|$ and $p \geqslant 1$. $\square$

Corollary 1. *Let $p \in \langle 1, \infty)$. Then there exist $h_0 > 0$ and $c = c(p) > 0$ such that for every $h \in (0, h_0)$ and $w \in W^{1,p}(\partial\Omega_h)$ we have*

$$\text{(3.29)} \qquad \begin{aligned} &\text{a)}\ \ \|\hat{w}\|_{1,p,\partial\Omega} \leqslant c\|w\|_{1,p,\partial\Omega_h},\\ &\text{b)}\ \ \|w\|_{1,p,\partial\Omega_h} \leqslant c\|\hat{w}\|_{1,p,\partial\Omega}. \end{aligned}$$

Let us now deal with the norms over Ω and Ω_h.

Lemma 3. *There exists $c > 1$ such that for every $h \in (0, h_0)$ and $v \in H_h$ we have*

$$\text{(3.30)} \qquad |\hat{v}|_{1,2,\Omega} \leqslant c|v|_{1,2,\Omega_h}.$$

P r o o f. Estimate (3.30) is a special case of Lemma 2.7 in [24]. $\square$

Lemma 4. *There exist constants $c > 0$ and $h_0 > 0$ such that for every $h \in (0, h_0)$ and $v \in H_h$ we have*

$$\text{(3.31)} \qquad \begin{aligned} &\text{a)}\ \ \|\hat{v}\|_{1,2,\Omega} \leqslant c\|v\|_{1,2,\Omega_h},\\ &\text{b)}\ \ \|v\|_{1,2,\Omega_h} \leqslant c\|\hat{v}\|_{1,2,\Omega}. \end{aligned}$$

P r o o f. Let $v \in H_h$ and let $\overline{v}_h \colon \overline{\Omega} \cup \overline{\Omega}_h \to \mathbb{R}$ be its natural extension. By Lemma 3.3.12 in [12] and the results obtained in the proof of Lemma 5.1.6 in [13], $\|\overline{v}\|^2_{1,2,\omega_h \cup \tau_h} \leqslant ch\|v\|^2_{1,2,\Omega_h}$ and hence

$$\begin{aligned} \|\hat{v}\|_{1,2,\Omega} &\leqslant \|\overline{v}\|_{1,2,\Omega} + \|\hat{v} - \overline{v}\|_{1,2,\Omega}\\ &\leqslant (\|v\|^2_{1,2,\Omega_h} + \|\overline{v}\|^2_{1,2,\omega_h \cup \tau_h})^{1/2} + \|\hat{v} - \overline{v}\|_{1,2,\Omega}\\ &\leqslant \|v\|_{1,2,\Omega_h}(1 + c_1 h)^{1/2} + c_2 h\|v\|_{1,2,\Omega_h}, \end{aligned}$$

where $c_1, c_2 > 0$ are constants. It means that for any $h_0 > 0$ there exists such a $c > 0$ that (3.31 a) holds for every $h \in (0, h_0)$.

In virtue of the definition of $\overline{v}$ we have

$$\|\overline{v}\|^2_{1,2,\Omega} = \|v\|^2_{1,2,\Omega_h} + \|\overline{v}\|^2_{1,2,\omega_h} - \|\overline{v}\|^2_{1,2,\tau_h} \geqslant \|v\|^2_{1,2,\Omega_h} - \|\overline{v}\|^2_{1,2,\omega_h \cup \tau_h}.$$

Similarly as above, we get the existence of a constant $c_3 > 0$ such that

$$\|\overline{v}\|^2_{1,2,\Omega} \geqslant \|v\|^2_{1,2,\Omega_h} - c_3 h\|v\|^2_{1,2,\Omega_h} = (1 - c_3 h)\|v\|^2_{1,2,\Omega_h} > (1 - c_3 h)^2\|v\|^2_{1,2,\Omega_h}$$

as $0 < 1 - c_3 h < 1$. Then we have

$$\begin{aligned}\|\hat{v}\|_{1,2,\Omega} &\geqslant \|\overline{v}\|_{1,2,\Omega} - \|\overline{v} - \hat{v}\|_{1,2,\Omega}\\ &\geqslant (1 - c_3 h)\|v\|_{1,2,\Omega_h} - c_4 h\|v\|_{1,2,\Omega_h}\\ &\geqslant (1 - (c_3 + c_4)h)\|v\|_{1,2,\Omega_h}\end{aligned}$$

with a constant c_4 independent of h and v.

Hence, if h_0 is chosen such that $1 - (c_3 + c_4)h_0 > 0$, (3.31 b) holds with

$$c = (1 - (c_3 + c_4)h_0)^{-1}.$$

$\square$

4. Convergence of the method

In virtue of (2.5), we have $f \in C(\overline{\Omega^*})$ and $\varphi \in C(\partial\Omega)$. Hence, these functions are bounded by $\|f\|_{0,\infty,\Omega^*}$ and $\|\varphi\|_{0,\infty,\partial\Omega}$, respectively.

Lemma 5. *There exists $c > 0$ such that*

$$|L^\Omega_h(v)| \leqslant c\|f\|_{0,\infty,\Omega^*}\|v\|_{0,2,\Omega_h}, \quad v \in H_h, \;\; h \in (0, h_0). \tag{4.1}$$

Proof. Let us set $K_\omega = \sum\limits_{\mu=1}^{M} |\omega_\mu|$. Then we find that

$$\begin{aligned}|L^\Omega_h(v)| &\leqslant \sum_{T\in\mathcal{T}_h} |T| \sum_{\mu=1}^{M} |\omega_\mu (fv)(x_{T,\mu})|\\ &\leqslant \|f\|_{0,\infty,\Omega^*} K_\omega \sum_{T\in\mathcal{T}_h} |T|\,\|\breve{v}\|_{0,\infty,\breve{T}}.\end{aligned} \tag{4.2}$$

As $v|_T \in P_1(T)$, we have $\breve{v} \in P_1(\breve{T})$. Since $P_1(\breve{T})$ is a finite dimensional space, there exists a constant $c_1 > 0$ independent of $\breve{w} \in P_1(\breve{T})$ such that

$$\begin{aligned}\|\breve{w}\|_{0,\infty,\breve{T}} &\leqslant c_1\|\breve{w}\|_{0,2,\breve{T}} = c_1\left(\int_{\breve{T}} |\breve{w}|^2\,\mathrm{d}x\right)^{1/2}\\ &= c_1\left(|T|^{-1}\int_T |w|^2\,\mathrm{d}x\right)^{1/2} = c_1|T|^{-1/2}\|w\|_{0,2,T}.\end{aligned} \tag{4.3}$$

Using the Hölder inequality, we get

$$|L_h^{\Omega}(v)| \leqslant c_1 K_\omega \|f\|_{0,\infty,\Omega^*} \sum_{T\in\mathcal{T}_h} |T|^{1/2}\|v\|_{0,2,T} \qquad (4.4)$$
$$\leqslant c_1 K_\omega \|f\|_{0,\infty,\Omega^*}[\mathrm{meas}(\Omega^*)]^{1/2}\|v\|_{0,2,\Omega_h}.$$

□

In what follows we denote by $\breve{S}$ the side $\overline{R_1R_3}$ of the reference triangle $\breve{T}$.

Lemma 6. *There exists a constant $c > 0$ such that for every $h \in (0, h_0)$, $v \in H_h$ the following estimate holds:*

$$|L_h^{\Gamma}(v)| \leqslant c\|\varphi\|_{0,\infty,\partial\Omega}\|v\|_{1,2,\Omega_h}. \qquad (4.5)$$

Proof. By the definition of L_h^{Γ} and φ_h we have similarly as in (4.2)

$$|L_h^{\Gamma}(v)| \leqslant \|\varphi\|_{0,\infty,\partial\Omega} K_\beta \sum_{S\in s_h} |S|\,\|\breve{v}\|_{0,\infty,\breve{S}}, \qquad (4.6)$$

where $K_\beta = \sum\limits_{\mu=1}^{m} |\beta_\mu|$.

Further, we use again the fact that $\breve{v}|_{\breve{S}} \in P_1(\breve{S})$, $\dim P_1(\breve{S}) < \infty$, which implies the existence of $c_2 > 0$ such that for every $\breve{w} \in P_1(\breve{S})$

$$\|\breve{w}\|_{0,\infty,\breve{S}} \leqslant c_2\|\breve{w}\|_{0,2,\breve{S}} = c_2|S|^{-1/2}\|w\|_{0,2,S}. \qquad (4.7)$$

By (4.7) and the definition of φ_h,

$$|L_h^{\Gamma}(v)| \leqslant c_2 K_\beta \|\varphi\|_{0,\infty,\partial\Omega} \sum_{S\in s_h} |S|^{1/2}\|v\|_{0,2,S} \qquad (4.8)$$
$$\leqslant c_2 K_\beta \|\varphi\|_{0,\infty,\partial\Omega}\, \mathrm{meas}(\partial\Omega_h)]^{1/2}\|v\|_{0,2,\partial\Omega_h}.$$

Let us now denote by c_3 the constant from Lemma 1 and by c_4 the constant from the imbedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$. Then, due to the inequality $\mathrm{meas}(\partial\Omega_h) \leqslant \mathrm{meas}(\partial\Omega)$,

$$|L_h^{\Gamma}(v)| \leqslant c_2 K_\beta \|\varphi\|_{0,\infty,\partial\Omega}[\mathrm{meas}(\partial\Omega)]^{1/2} c_3\|\hat{v}\|_{0,2,\partial\Omega} \qquad (4.9)$$
$$\leqslant c_2 c_3 K_\beta \|\varphi\|_{0,\infty,\partial\Omega}[\mathrm{meas}(\partial\Omega)]^{1/2} c_4\|\hat{v}\|_{1,2,\Omega}.$$

Now it is sufficient to use Lemma 4 a). □

R e m a r k. By (2.5), the functions f and φ satisfy the assumptions of Lemmas 5 and 6.

From Lemmas 5 and 6 we immediately get the following assertion:

Corollary 2. *There exists a constant* $c > 0$ *such that*

$$|L_h(v)| \leqslant c\|v\|_{1,2,\Omega_h} \tag{4.10}$$

for every $h \in (0, h_0)$, $v \in H_h$.

R e m a r k. The following lemma is proved in [8]. Although the problem with a polygonal domain Ω is studied in that paper, the fact that $\Omega_h = \Omega$ is not used in the proof. The same will be true whenever we refer to any proof from [8].

Lemma 7. *Let* $p \in (1, \infty\rangle$ *and* (2.9 b) *hold. Then there exists a constant* $c > 0$ *such that*

$$|E_\Gamma(Qv)| \leqslant ch|Q|_{1,p,\partial\Omega_h}\|v\|_{1,2,\Omega_h} \tag{4.11}$$

for every $h \in (0, h_0)$, $v \in H_h$, $Q \in W^{1,p}(\partial\Omega_h)$, *where* E_Γ *is defined by* (2.8 c).

P r o o f. See Lemma 3.44 in [8]. □

Lemma 8. *Let* (2.5) *and* (2.9 b) *hold. Let* h_0 *be as in Lemma* 1. *Then there exists a constant* $c > 0$ *such that*

$$|L^\Gamma(\hat{v}_h) - L^\Gamma_h(v_h)| \leqslant ch\|v_h\|_{1,2,\Omega_h}, \quad v_h \in H_h,\ h \in (0, h_0). \tag{4.12}$$

P r o o f. By the triangle inequality we have

$$|L^\Gamma(\hat{v}_h) - L^\Gamma_h(v_h)| \leqslant |L^\Gamma(\hat{v}_h) - \widetilde{L}^\Gamma_h(v_h)| + |\widetilde{L}^\Gamma_h(v_h) - L^\Gamma_h(v_h)|. \tag{4.13}$$

First we estimate the second term on the right-hand side. Let us denote the constants from Lemmas 7 and 2 by c_1 and c_2, respectively. From the definition of the functions φ_h it follows that $\varphi_h \in W^{1,p}(\partial\Omega_h)$. Hence, by Lemma 7,

$$|\widetilde{L}^\Gamma_h(v_h) - L^\Gamma_h(v_h)| = |E_\Gamma(\varphi_h v_h)| \leqslant c_1 h|\varphi_h|_{1,p,\partial\Omega_h}\|v_h\|_{1,2,\Omega_h}.$$

Since $\varphi = \hat{\varphi}_h$, Lemma 2 implies that

$$|\varphi_h|_{1,p,\partial\Omega_h} \leqslant c_2|\varphi|_{1,p,\partial\Omega} \leqslant c_2\|\varphi\|_{1,p,\partial\Omega}$$

and thus

$$|\widetilde{L}^\Gamma_h(v_h) - L^\Gamma_h(v_h)| \leqslant c_1 c_2 h\|\varphi\|_{1,p,\partial\Omega}\|v_h\|_{1,2,\Omega_h}. \tag{4.14}$$

Let us now deal with the first term on the right-hand side in (4.13). With the aid of (3.17) and the Hölder inequality we get

$$\begin{aligned}|L^{\Gamma}(\hat{v}_h) - \widetilde{L}_h^{\Gamma}(v_h)| &= \left|\int_{\partial\Omega} \varphi\hat{v}_h \,\mathrm{d}S - \int_{\partial\Omega_h} \varphi_h v_h \,\mathrm{d}S\right| \\ &= \left|\int_{\partial\Omega} \widehat{\varphi_h v_h} \,\mathrm{d}S - \int_{\partial\Omega_h} \varphi_h v_h \,\mathrm{d}S\right| \\ &\leqslant c_3 h \int_{\partial\Omega_h} |\varphi_h v_h| \,\mathrm{d}S \leqslant c_3 h \|\varphi_h\|_{0,2,\partial\Omega_h} \|v_h\|_{0,2,\partial\Omega_h},\end{aligned}$$

where c_3 is the constant from (3.17).

Further, on the basis of Lemma 1, the continuous imbeddings $W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$, $W^{1,p}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ and Lemma 7, there exist positive constants c_4, c_5, c_6 independent of $h \in (0, h_0)$ and $v_h \in H_h$ such that

$$\begin{aligned}|L^{\Gamma}(\hat{v}_h) - \widetilde{L}_h^{\Gamma}(v_h)| &\leqslant c_4 h \|\hat{v}_h\|_{0,2,\partial\Omega} \|\varphi\|_{0,2,\partial\Omega} \\ &\leqslant c_5 h \|\hat{v}_h\|_{1,2,\Omega} \|\varphi\|_{1,p,\partial\Omega} \\ &\leqslant c_6 h \|v_h\|_{1,2,\Omega_h} \|\varphi\|_{1,p,\partial\Omega}.\end{aligned} \tag{4.15}$$

Combination of (4.13), (4.14) and (4.15) already gives estimate (4.12). $\square$

Lemma 9. *Let* (2.5) *and* (2.9 a) *hold. Let h_0 be as in Lemma* 4. *Then there exists such a constant $c > 0$ that*

$$|L^{\Omega}(\hat{v}_h) - L_h^{\Omega}(v_h)| \leqslant ch\|v_h\|_{1,2,\Omega_h} \tag{4.16}$$

for every $h \in (0, h_0)$ and $v_h \in H_h$.

P r o o f. Similarly as in (4.13) we first estimate

$$|L^{\Omega}(\hat{v}_h) - L_h^{\Omega}(v_h)| \leqslant |L^{\Omega}(\hat{v}_h) - \widetilde{L}_h^{\Omega}(v_h)| + |\widetilde{L}_h^{\Omega}(v_h) - L_h^{\Omega}(v_h)|. \tag{4.17}$$

It is possible to show in the same way as in the proof of Lemma 3.23 in [8] that there exists a constant $c_1 > 0$ independent of $h \in (0, h_0)$ and $v_h \in H_h$ such that

$$|\widetilde{L}_h^{\Omega}(v_h) - L_h^{\Omega}(v_h)| \leqslant c_1 h (\operatorname{meas}(\Omega^*))^{\frac{q-2}{2q}} \|f\|_{1,q,\Omega^*} \|v_h\|_{1,2,\Omega_h}, \tag{4.18}$$

as $\Omega_h \subset \Omega^*$ for every $h \in (0, h_0)$.

This means that it remains to estimate the first term on the right-hand side in (4.17). Because $\overline{v}_h = v_h$ in $\Omega_h \cap \Omega$, we have

$$
\begin{aligned}
(4.19)\quad |L^{\Omega}(\hat{v}_h) - \widetilde{L}_h^{\Omega}(v_h)| &= \left|\int_{\Omega} f\hat{v}_h \,\mathrm{d}x - \int_{\Omega_h} f v_h \,\mathrm{d}x\right| \\
&\leqslant \left|\int_{\Omega\cap\Omega_h} f(\hat{v}_h - \overline{v}_h)\,\mathrm{d}x\right| + \left|\int_{\omega_h} f\hat{v}_h \,\mathrm{d}x\right| + \left|\int_{\tau_h} f v_h \,\mathrm{d}x\right| \\
&\leqslant \|f\|_{0,2,\Omega\cap\Omega_h} \|\hat{v}_h - \overline{v}_h\|_{0,2,\Omega\cap\Omega_h} \\
&\quad + \|f\|_{0,\infty,\omega_h} \int_{\omega_h} |\hat{v}_h| \,\mathrm{d}x + \|f\|_{0,\infty,\tau_h} \int_{\tau_h} |v_h| \,\mathrm{d}x \\
&\leqslant \|f\|_{0,2,\Omega^*} \|\hat{v}_h - \overline{v}_h\|_{1,2,\Omega} \\
&\quad + \|f\|_{0,\infty,\Omega^*} \left(\int_{\omega_h} |\hat{v}_h| \,\mathrm{d}x + \int_{\tau_h} |v_h| \,\mathrm{d}x\right).
\end{aligned}
$$

By Lemma 2.3 in [24], there exists a constant $c_2 > 0$ independent of h and v_h such that

$$
(4.20)\qquad \|\hat{v}_h - \overline{v}_h\|_{1,2,\Omega} \leqslant c_2 h \|v_h\|_{1,2,\Omega_h}.
$$

Further, by [12], there exists $c_3 > 0$ independent of $h \in (0, h_0)$ such that

$$
(4.21)\qquad \operatorname{meas}(\omega_h \cup \tau_h) \leqslant c_3 h^2.
$$

This, the Hölder inequality and Lemma 4 give us the estimates

$$
(4.22)\quad \int_{\omega_h} |\hat{v}_h| \,\mathrm{d}x \leqslant \operatorname{meas}(\omega_h))^{1/2} \|\hat{v}_h\|_{0,2,\omega_h} \leqslant c_3^{1/2} h \|\hat{v}_h\|_{1,2,\Omega} \leqslant c_3^{1/2} c_4 h \|v_h\|_{1,2,\Omega_h}
$$

and

$$
(4.23)\quad \int_{\tau_h} |v_h| \,\mathrm{d}x \leqslant (\operatorname{meas}(\tau_h))^{1/2} \|v_h\|_{0,2,\tau_h} \leqslant c_3^{1/2} h \|v_h\|_{1,2,\Omega_h}.
$$

(By c_4 we denote the constant from (3.31 a).)

Finally, the continuous imbeddings $W^{1,q}(\Omega^*) \hookrightarrow L^2(\Omega^*)$ and $W^{1,q}(\Omega^*) \hookrightarrow L^\infty(\Omega^*)$ imply that there exists $c_5 > 0$ independent of f such that

$$
(4.24)\qquad \|f\|_{0,2,\Omega^*} \leqslant c_5 \|f\|_{1,q,\Omega^*}, \quad \|f\|_{0,\infty,\Omega^*} \leqslant c_5 \|f\|_{1,q,\Omega^*}.
$$

If we now summarize (4.19)–(4.24), we get

$$
(4.25)\qquad |L^{\Omega}(\hat{v}_h) - \widetilde{L}_h^{\Omega}(v_h)| \leqslant ch \|f\|_{1,q,\Omega^*} \|v_h\|_{1,2,\Omega_h}.
$$

This and (4.18) already yield estimate (4.16). $\square$

As a consequence of the last two lemmas we get

Corollary 3. *Let h_0 be as in Lemmas* 1 *and* 3. *Then under assumptions* (2.9 a–b) *and* 2.5 *there exists a constant $c > 0$ such that*

$$|L(\hat{v}_h) - L_h(v_h)| \leqslant ch\|v_h\|_{1,2,\Omega_h}, \quad v_h \in H_h, \ \ h \in (0, h_0). \tag{4.26}$$

Lemma 10. *Let h_0 be as in Lemmas* 1 *and* 3. *Then, under assumptions* (T7), (T9) *and* (2.9 b), *for every $r \in \langle 1, \infty)$ there exists a constant $c = c(r) > 0$ such that*

$$|d_h(u_h, v_h) - d(\hat{u}_h, \hat{v}_h)| \leqslant ch^{1/2-\alpha/r}\|u_h\|^{\alpha+1}_{1,2,\Omega_h}\|v_h\|_{1,2,\Omega_h} \tag{4.27}$$

for all $h \in (0, h_0)$ and all $u_h, v_h \in H_h$.

P r o o f. First we show the existence of a constant $c_1 > 0$ such that

$$\begin{gathered}|\widetilde{d}_h(u_h, v_h) - d(\hat{u}_h, \hat{v}_h)| \leqslant c_1 h\|u_h\|^{\alpha+1}_{1,2,\Omega_h}\|v_h\|_{1,2,\Omega_h}, \\ u_h, v_h \in H_h, \ \ h \in (0, h_0).\end{gathered} \tag{4.28}$$

By the definitions of $\widetilde{d}_h$, d and estimates (3.17),

$$\begin{aligned}|\widetilde{d}_h(u_h, v_h) - d(\hat{u}_h, \hat{v}_h)| &= \kappa\left|\int_{\partial\Omega_h} |u_h|^\alpha u_h v_h \,\mathrm{d}S - \int_{\partial\Omega} |\hat{u}_h|^\alpha \hat{u}_h \hat{v}_h \,\mathrm{d}S\right| \\ &\leqslant \kappa c_2 h \int_{\partial\Omega_h} |u_h|^{\alpha+1}|v_h| \,\mathrm{d}S,\end{aligned}$$

where c_2 denotes the constant from (3.17). Further, using the Hölder inequality we get

$$\begin{aligned}\int_{\partial\Omega_h} |u_h|^{\alpha+1}|v_h| \,\mathrm{d}S &\leqslant \left(\int_{\partial\Omega_h} |u_h|^{\alpha+2} \,\mathrm{d}S\right)^{\frac{\alpha+1}{\alpha+2}} \left(\int_{\partial\Omega_h} |v_h|^{\alpha+2} \,\mathrm{d}S\right)^{1/(\alpha+2)} \\ &= \|u_h\|^{\alpha+1}_{0,\alpha+2,\partial\Omega_h}\|v_h\|_{0,\alpha+2,\partial\Omega_h}.\end{aligned}$$

To obtain estimate (4.28), it is now enough to use Lemma 1, the continuity of the imbedding $H^1(\Omega) \hookrightarrow L^{\alpha+2}(\Omega)$ and Lemma 4.

In the next step we show that there exists a constant $c_3 = c_3(r) > 0$ independent of $h \in (0, h_0)$ and $u_h, v_h \in H_h$ such that

$$|d_h(u_h, v_h) - \widetilde{d}_h(u_h, v_h)| \leqslant c_3 h^{1/2-\alpha/r}\|u_h\|^{\alpha+1}_{1,2,\Omega_h}\|v_h\|_{1,2,\Omega_h}. \tag{4.29}$$

In what follows, we will proceed similarly as in the proof of Theorem 3.51 in [8].

If we put $Q := |u_h|^\alpha u_h$ and $p = 2$ in Lemma 7, we get

$$
\begin{aligned}
(4.30)\qquad |d_h(u_h, v_h) - \widetilde{d}_h(u_h, v_h)| &= \kappa |E_\Gamma(|u_h|^\alpha u_h v_h)| \\
&\leqslant \kappa c_4 h \big| |u_h|^\alpha u_h \big|_{1,2,\partial\Omega_h} \|v_h\|_{1,2,\Omega_h},
\end{aligned}
$$

where c_4 denotes the constant from Lemma 7. Further, similarly as in the proof of Theorem 3.51 from [8],

$$\big| |u_h|^\alpha u_h \big|_{1,2,\partial\Omega_h} \leqslant (\alpha+1) \|u_h\|^\alpha_{0,\infty,\partial\Omega_h} |u_h|_{1,2,\partial\Omega_h}.$$

By [8], Lemma 3.18 and Corollary 3.16, there exist constants $c_5 = c_5(r)$ ($r \in \langle 1, \infty)$) and $c_6 > 0$ independent of $h \in (0, h_0)$ and $w_h \in H_h$ such that

$$\|w_h\|_{0,\infty,\partial\Omega_h} \leqslant c_5 h^{-1/r} \|w_h\|_{0,r,\partial\Omega_h}$$

and

$$|w_h|_{1,2,\partial\Omega_h} \leqslant c_6 h^{-1/2} |w_h|_{1,2,\Omega_h}.$$

This gives

$$(4.31)\qquad \big| |u_h|^\alpha u_h \big|_{1,2,\partial\Omega_h} \leqslant (\alpha+1) c_5^\alpha(r) c_6 h^{-1/2-\alpha/r} \|u_h\|^\alpha_{0,r,\partial\Omega_h} |u_h|_{1,2,\Omega_h}.$$

Now (3.11 a), the continuity of the imbedding $H^1(\Omega) \hookrightarrow L^r(\partial\Omega)$ and (3.31 a) imply the existence of a $c_7 = c_7(r) > 0$ such that

$$(4.32)\qquad \|u_h\|^\alpha_{0,r,\partial\Omega_h} \leqslant c_7^\alpha \|u_h\|^\alpha_{1,2,\Omega_h}.$$

Combining (4.30)–(4.32), we get estimate (4.29).

The assertion of the theorem follows immediately from (4.28) and (4.29). $\square$

Lemma 11. *Let the assumptions of Lemma 10 be satisfied. Let us denote by $\widetilde{c}$ the constant from Lemma 4. Then there exist constants $c > 0$ and $h_0 > 0$ such that*

$$(4.33)\qquad a_h(u_h, u_h) \geqslant c$$

for every $h \in (0, h_0)$ and every $u_h \in H_h$ with $\|u_h\|_{1,2,\Omega_h} = \widetilde{c}$.

Proof. Let us denote by c_1 and c_2 the constants from (3.30) and (4.27), respectively. Then for $u_h \in H_h$, $\|u_h\|_{1,2,\Omega_h} = \widetilde{c}$ we can write

$$\begin{aligned} a_h(u_h,u_h) &= |u_h|^2_{1,2,\Omega_h} + d_h(u_h,u_h) \\ &\geqslant \frac{1}{c_1^2}|\hat{u}_h|^2_{1,2,\Omega} + d(\hat{u}_h,\hat{u}_h) + \big(d_h(u_h,u_h) - d(\hat{u}_h,\hat{u}_h)\big) \\ &\geqslant \frac{1}{c_1^2}a(\hat{u}_h,\hat{u}_h) - |d_h(u_h,u_h) - d(\hat{u}_h,\hat{u}_h)| \\ &\geqslant \frac{1}{c_1^2}a(\hat{u}_h,\hat{u}_h) - c_3 h^{1/2-\alpha/r}\|u_h\|^{\alpha+2}_{1,2,\Omega_h} \\ &= \frac{1}{c_1^2}a(\hat{u}_h,\hat{u}_h) - c_2\widetilde{c}^{\alpha+2}h^{1/2-\alpha/r}. \end{aligned}$$

As $\|u_h\|_{1,2,\Omega_h} = \widetilde{c}$, we have by Lemma 4

$$\|\hat{u}_h\|_{1,2,\Omega} \geqslant \frac{1}{\widetilde{c}}\|u_h\|_{1,2,\Omega_h} = 1.$$

It was shown in the proof of Theorem 1.14 in [8] that there exists such a constant $c_3 > 0$ that

$$a(w,w) \geqslant c_3\|w\|^2_{1,2,\Omega}$$

for every $w \in H^1(\Omega)$ with $\|w\|_{1,2,\Omega} \geqslant 1$. This and Lemma 4 yield the estimate

$$a(\hat{u}_h,\hat{u}_h) \geqslant c_3\|\hat{u}_h\|^2_{1,2,\Omega} \geqslant c_3\frac{1}{\widetilde{c}^2}\|u_h\|^2_{1,2,\Omega_h} = c_3$$

and hence

$$a_h(u_h,u_h) \geqslant \frac{c_3}{c_1^2} - c_2\widetilde{c}^{\alpha+2}h^{1/2-\alpha/r}.$$

Now, if we choose $r > 2\alpha$, we see that there exists $h_0 > 0$ such that for every $h \in (0,h_0)$ the estimate (4.33) holds. □

Lemma 12. *Let the assumptions of Lemma* 10 *be satisfied and let h_0, $\widetilde{c}$ be as in Lemma* 11. *Then there exists $c > 0$ such that*

$$a_h(u_h,u_h) \geqslant c\|u_h\|^2_{1,2,\Omega_h} \tag{4.34}$$

for every $h \in (0,h_0)$ and $u_h \in H_h$ with $\|u_h\|_{1,2,\Omega_h} \geqslant \widetilde{c}$.

Proof. Let $u_h \in H_h$, $\|u_h\|_{1,2,\Omega_h} \geqslant \widetilde{c}$ and let us put

$$w_h = \frac{u_h}{\|u_h\|_{1,2,\Omega_h}}\widetilde{c}.$$

As $\|w_h\|_{1,2,\Omega_h} = \widetilde{c}$ we have $a_h(w_h, w_h) \geqslant c_1$, where c_1 is the constant from Lemma 11. If we multiply this inequality by the term $\widetilde{c}^{-2}\|u_h\|^2_{1,2,\Omega_h}$, which is greater or equal to 1, according to assumption (2.9 c) we find that

$$\begin{aligned}
\frac{c_1}{\widetilde{c}^2}\|u_h\|^2_{1,2,\Omega_h} &\leqslant a_h(w_h, w_h)\|u_h\|^2_{1,2,\Omega_h}\frac{1}{\widetilde{c}^2} \\
&= \left(\int_{\Omega_h} |\nabla w_h|^2 \,\mathrm{d}x + \kappa \sum_{S\in s_h} |S| \sum_{\mu=1}^{m} \beta_\mu |w_h|^{\alpha+2}(x_{S,\mu})\right)\frac{\|u_h\|^2_{1,2,\Omega_h}}{\widetilde{c}^2} \\
&= \widetilde{b}_h(u_h, u_h) + \left(\kappa \sum_{S\in s_h} |S| \sum_{\mu=1}^{m} \beta_\mu |u_h|^{\alpha+2}(x_{S,\mu})\right)\left(\frac{\|u_h\|_{1,2,\Omega_h}}{\widetilde{c}}\right)^{-\alpha} \\
&\leqslant \widetilde{b}_h(u_h, u_h) + d_h(u_h, u_h) = a_h(u_h, u_h)
\end{aligned}$$

and so estimate (4.34) holds with the constant $c = c_1\widetilde{c}^{-2}$. $\square$

Theorem 4.1. *Let assumptions* (T1)–(T9), (2.5) *and* (2.9) *be satisfied and let* h_0 *be as in Lemma* 11. *Then for every* $h \in (0, h_0)$ *the discrete problem* (2.12) *has a unique solution* $u_h \in H_h$. *Moreover, there exists a constant* $c > 0$ *such that*

$$\|u_h\|_{1,2,\Omega_h} \leqslant c \tag{4.35}$$

for every $h \in (0, h_0)$.

Proof. The existence and uniqueness of the approximate solutions can be established in the same way as in Theorem 4.13 from [8], if we replace the reference to Lemma 4.8 in [8] by our Lemma 12. The boundedness of the approximate solutions can be derived in a similar way as well:

Let $\widetilde{c}$ be again the constant from Lemma 11. If $\|u_h\|_{1,2,\Omega_h} < \widetilde{c}$ nothing is to be proved. Therefore we will assume that $\|u_h\|_{1,2,\Omega_h} \geqslant \widetilde{c}$. Since u_h is the approximate solution, in view of Corollary 2 and Lemma 12,

$$c_1\|u_h\|^2_{1,2,\Omega_h} \leqslant a_h(u_h, u_h) = L_h(u_h) \leqslant c_2\|u_h\|_{1,2,\Omega_h},$$

where c_1 and c_2 are the constants from estimates (4.34) and (4.10), respectively. It means that for every $h \in (0, h_0)$

$$\|u_h\|^2_{1,2,\Omega_h} \leqslant \max\left(\widetilde{c}, \frac{c_2}{c_1}\right).$$

$\square$

Lemma 13. *Let $\{w_n\} \subset H^1(\Omega)$, $\{v_n\} \subset H^1(\Omega)$ and let $w, v \in H^1(\Omega)$ be such functions that $w_h \rightharpoonup w$, $v_h \rightharpoonup v$ weakly in $H^1(\Omega)$ as $n \to \infty$. Then*

$$|d(w_n, v_n) - d(w, v)| \to 0 \quad \text{as } n \to \infty. \tag{4.36}$$

Proof. We have

$$\begin{aligned} |d(w_n, v_n) - d(w, v)| &= \kappa \left| \int_{\partial\Omega} (|w_n|^\alpha w_n v_n - |w|^\alpha w v) \, \mathrm{d}S \right| \\ &\leqslant \kappa \int_{\partial\Omega} \big(\big| |w_n|^\alpha w_n - |w|^\alpha w \big| |v_n| \big) \, \mathrm{d}S \\ &\quad + \int_{\partial\Omega} (|w|^{\alpha+1} |v_n - v|) \, \mathrm{d}S = I_1 + I_2. \end{aligned} \tag{4.37}$$

Similarly as in the proof of Lemma 1.20 from [8] we get an estimate of I_1:

$$0 \leqslant I_1 \leqslant (\alpha + 1) \int_{\partial\Omega} |w_n - w| (|w_n|^\alpha + |w|^\alpha) |v_n| \, \mathrm{d}S.$$

Due to the compact imbedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\partial\Omega)$ for $q \geqslant 1$, we have $w_n \to w$, $v_n \to v$ strongly in $L^q(\partial\Omega)$. But it means that for any $q \geqslant 1$ the norms $\{\|w_n\|_{0,q,\partial\Omega}\}_{n=1}^\infty$, $\{\|v_n\|_{0,q,\partial\Omega}\}_{n=1}^\infty$ are bounded by some constant $c(q) > 0$. Let us now choose $p_1, p_2, p_3 \geqslant 1$ in such a way that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. The double use of the Hölder inequality gives

$$\begin{aligned} 0 \leqslant I_1 &\leqslant (\alpha + 1) \|w_n - w\|_{0,p_1,\partial\Omega} (\|w_n\|^\alpha_{0,\alpha p_2,\partial\Omega} + \|w\|^\alpha_{0,\alpha p_2,\partial\Omega}) \|v_n\|_{0,p_3,\partial\Omega} \\ &\leqslant 2(\alpha + 1)(c(\alpha p_2))^\alpha c(p_3) \|w_n - w\|_{0,p_1,\partial\Omega} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.38}$$

The term I_2 can be estimated with the use of the Hölder inequality as well. We have

$$\begin{aligned} 0 \leqslant I_2 &\leqslant \kappa \|w\|^{\alpha+1}_{0,2(\alpha+1),\partial\Omega} \|v_n - v\|_{0,2,\partial\Omega} \\ &\leqslant \kappa (2c(\alpha + 1))^{\alpha+1} \|v_n - v\|_{0,2,\partial\Omega} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.39}$$

Finally, (4.37), (4.38) and (4.39) already give the required result. □

Lemma 14. *Let $w_n \rightharpoonup w$ weakly in $H^1(\Omega)$, $v_n \to v$ strongly in $H^1(\Omega)$. Then*

$$b(w_n, v_n) \to b(w, v) \quad \text{as } n \to \infty. \tag{4.40}$$

Proof. We have

$$|b(w_n, v_n) - b(w, v)| \leqslant |b(w_n, v_n - v)| + |b(w_n - w, v)|. \tag{4.41}$$

The boundedness of the norms $\|w_n\|_{1,2,\Omega}$ and the strong convergence $v_n \to v$ imply that

$$|b(w_n, v_n - v)| \leqslant |w_n|_{1,2,\Omega}\, |v_n - v|_{1,2,\Omega} \leqslant c^* \|v_n - v\|_{1,2,\Omega} \to 0.$$

As $b(., v)$ is a continuous linear functional on $H^1(\Omega)$, the second term in (4.41) tends to zero according to the definition of the weak convergence $w_n \rightharpoonup w$. □

Lemma 15. *There exists a constant $c > 0$ such that*

$$|\widetilde{b}_h(w_h, v_h) - b(\hat{w}_h, \hat{v}_h)| \leqslant ch\|w_h\|_{1,2,\Omega_h}\, \|v_h\|_{1,2,\Omega_h} \tag{4.42}$$

for every $h \in (0, h_0)$ and $w_h, v_h \in H_h$.

Proof. We have

$$\begin{aligned}
|\widetilde{b}_h(w_h, v_h) - b(\hat{w}_h, \hat{v}_h)| &= |\widetilde{b}_h(\overline{w}_h, \overline{v}_h) - b(\hat{w}_h, \hat{v}_h)| \\
&\leqslant |\widetilde{b}_h(\overline{w}_h, \overline{v}_h) - b(\overline{w}_h, \overline{v}_h)| + |b(\overline{w}_h, \overline{v}_h) - b(\hat{w}_h, \hat{v}_h)| \\
&\leqslant \int_{\omega_h \cup \tau_h} |\nabla \overline{w}_h \cdot \nabla \overline{v}_h| \,\mathrm{d}x \\
&\quad + |b(\overline{w}_h, \overline{v}_h - \hat{v}_h)| + |b(\overline{w}_h - \hat{w}_h, \hat{v}_h)|.
\end{aligned} \tag{4.43}$$

By [12], Lemma 3.3.12,

$$\begin{aligned}
\int_{\omega_h \cup \tau_h} |\nabla \overline{w}_h \cdot \nabla \overline{v}_h| \,\mathrm{d}x &\leqslant |\overline{w}_h|_{1,2,\omega_h \cup \tau_h} \cdot |\overline{v}_h|_{1,2,\omega_h \cup \tau_h} \\
&\leqslant c_1 h\|w_h\|_{1,2,\Omega_h} \cdot \|v_h\|_{1,2,\Omega_h}.
\end{aligned} \tag{4.44}$$

As was shown in the proof of Lemma 5.1.2 in [13], there exists such a constant $c_2 > 0$ independent of w_h, h that

$$\|\hat{w}_h - \overline{w}_h\|_{1,2,\Omega} \leqslant c_2 h\|w_h\|_{1,2,\Omega_h}.$$

This, the definition of the bilinear form $b(.,.)$ and Lemma 4 (with constant c_3) give us

$$\begin{aligned}
|b(\overline{w}_h, \overline{v}_h - \hat{v}_h)| &\leqslant (\|\hat{w}_h\|_{1,2,\Omega} + \|\hat{w}_h - \overline{w}_h\|_{1,2,\Omega})\|\overline{v}_h - \hat{v}_h\|_{1,2,\Omega} \\
&\leqslant c_3 h(c_3 + c_2 h)\|w_h\|_{1,2,\Omega_h}\, \|v_h\|_{1,2,\Omega_h}.
\end{aligned} \tag{4.45}$$

In the same way we get

$$|b(\overline{w}_h - \hat{w}_h, \hat{v}_h)| \leqslant \|\overline{w}_h - \hat{w}_h\|_{1,2,\Omega}\|\hat{v}_h\|_{1,2,\Omega} \leqslant c_2 c_3 h\|w_h\|_{1,2,\Omega_h}\|v_h\|_{1,2,\Omega_h}. \tag{4.46}$$

Estimate (4.42) now immediately follows from (4.43)–(4.46). □

Theorem 4.2. *Let the assumptions of Theorem* 4.1 *be satisfied and let* h_0 *be as in Lemma* 11. *Let* u *be the solution of problem* (1.4) *and* u_h $(h \in (0, h_0))$ *the solution of problem* (2.12). *Then*

$$\lim_{h\to 0} \|\hat{u}_h - u\|_{1,2,\Omega} = 0. \tag{4.47}$$

P r o o f. I) We get from (4.35) and Lemma 4 a) that there exists a constant $c_1 > 0$ such that

$$\|\hat{u}_h\|_{1,2,\Omega} \leqslant c_1 \text{ for every } h \in (0, h_0). \tag{4.48}$$

Since the space $H^1(\Omega)$ is reflexive and the imbedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, there exist a sequence $\{h_n\}_{n=1}^{\infty}$ and a function $u \in H^1(\Omega)$ such that

$$\begin{gathered} h_n \in (0, h_0), \ \ h_n \to 0 \text{ as } n \to \infty, \\ \hat{u}_{h_n} \rightharpoonup u \text{ weakly in } H^1(\Omega) \text{ as } n \to \infty \text{ and} \\ \hat{u}_{h_n} \to u \text{ strongly in } L^2(\Omega) \text{ as } n \to \infty. \end{gathered} \tag{4.49}$$

As a consequence of (4.49) we also get an estimate

$$\|\hat{u}_{h_n}\|_{0,2,\partial\Omega} \leqslant c_2 \ \ \forall n = 0, 1, \ldots. \tag{4.50}$$

We will show that

$$\hat{u}_{h_n} \to u \text{ strongly in } H^1(\Omega) \text{ as } n \to \infty$$

for any sequence $\{u_{h_n}\}$ with $\{h_n\} \subset (0, h_0)$, $h_n \to 0$, and that the limit function u is a weak solution of the problem.

II) Let $v \in C^\infty(\overline{\Omega})$ be an arbitrary fixed function and $v_c \in H^2(\Omega^*)$ its Calderon extension to Ω^* (cf., e.g., [23]). Let v_h be the Lagrange interpolation of the function v_c in the space H_h. By Theorem 2 from [30], there exists a constant $c_3 > 0$ such that

$$\|\hat{v}_h - v\|_{1,2,\Omega} \leqslant c_3 h \|v_c\|_{2,2,\Omega^*} \tag{4.51}$$

for every $h \in (0, h_0)$ and $v \in C^\infty(\overline{\Omega})$. As an easy consequence of (4.51) and Lemma 4 we get

$$\|\hat{v}_h\|_{1,2,\Omega} \leqslant \|v\|_{1,2,\Omega} + c_3 h \|v_c\|_{2,2,\Omega^*} \leqslant c_4 \|v_c\|_{2,2,\Omega^*} \tag{4.52}$$

and

$$\|v_h\|_{1,2,\Omega_h} \leqslant c_5 \|v_c\|_{2,2,\Omega^*}. \tag{4.53}$$

Moreover,

$$\hat{v}_h \to v \text{ strongly in } H^1(\Omega) \text{ for } h \to 0. \tag{4.54}$$

By virtue of the compactness of the imbedding of $H^1(\Omega)$ into $L^2(\partial\Omega)$, (4.52) and (4.54) we have

$$\begin{aligned} &\text{a) } \|\hat{v}_h\|_{0,2,\partial\Omega} \leqslant c_6, \\ &\text{b) } \hat{v}_h \to v \text{ strongly in } L^2(\Omega) \text{ for } h \to 0. \end{aligned} \tag{4.55}$$

III) Let us now choose $r > 2\alpha$ and let us put $\sigma := 1/2 - \alpha/r > 0$. In what follows we will omit for simplicity the index n at h and suppose that $h = h_n \to 0$ as $n \to \infty$.

It is obvious that

$$\begin{aligned} a_h(u_h, v_h) &= b(\hat{u}_h, \hat{v}_h) + \big(\widetilde{b}_h(u_h, v_h) - b(\hat{u}_h, \hat{v}_h)\big) \\ &\quad + d(\hat{u}_h, \hat{v}_h) + \big(d_h(u_h, v_h) - d(\hat{u}_h, \hat{v}_h)\big) \\ &= L(\hat{v}_h) + \big(L_h(v_h) - L(\hat{v}_h)\big). \end{aligned} \tag{4.56}$$

On the basis of Lemma 10, (4.35) and (4.53) we have

$$|d_h(u_h, v_h) - d(\hat{u}_h, \hat{v}_h)| \leqslant ch^{1/2-\alpha/r} \|u_h\|^{\alpha+1}_{1,2,\Omega_h} \|v_h\|_{1,2,\Omega_h} \to 0. \tag{4.57}$$

It follows from Corollary 3 and (4.53) that

$$|L_h(v_h) - L(\hat{v}_h)| \leqslant ch\|v_h\|_{1,2,\Omega_h} \to 0. \tag{4.58}$$

Finally, by Lemma 15 and (4.35), (4.53), we have

$$|\widetilde{b}_h(u_h, v_h) - b(\hat{u}_h, \hat{v}_h)| \to 0 \text{ for } h \to 0. \tag{4.59}$$

IV) It follows from the continuity of the functional L and from (4.54) that

$$L(\hat{v}_h) \to L(v). \tag{4.60}$$

In virtue of Lemma 13,

$$d(\hat{u}_h, \hat{v}_h) \to d(u, v) \tag{4.61}$$

and due to Lemma 14,

$$b(\hat{u}_h, \hat{v}_h) \to b(u, v). \tag{4.62}$$

Now, summarizing (4.57)–(4.62), we find out that the limit function $u \in H^1(\Omega)$ satisfies the identity

$$a(u,v) = L(v) \;\; \forall v \in C^\infty(\overline{\Omega}).$$

Since $a(u,.)$ and $L(.)$ are continuous linear functionals on $H^1(\Omega)$ and $C^\infty(\Omega)$ is dense in $H^1(\Omega)$, the function u is the weak solution of the continuous problem.

V) We prove now the strong convergence $\hat{u}_h \to u$ in $H^1(\Omega)$. Obviously,

$$\begin{aligned}(4.63) \qquad |\hat{u}_h - u|^2_{1,2,\Omega} &= b(\hat{u}_h - u, \hat{u}_h - u) \\ &= \big(b(\hat{u}_h, \hat{u}_h) - \widetilde{b}_h(u_h, u_h)\big) + L_h(u_h) - d_h(u_h, u_h) \\ &\quad - [2b(\hat{u}_h, u) - b(u,u)].\end{aligned}$$

According to Lemma 15,

$$b(\hat{u}_h, \hat{u}_h) - \widetilde{b}_h(u_h, u_h) \to 0.$$

In virtue of Corollary 3, the continuity of the linear functional L on $H^1(\Omega)$ and the definition of the weak convergence we have

$$|L_h(u_h) - L(u)| \leqslant |L_h(u_h) - L(\hat{u}_h)| + |L(\hat{u}_h - u)| \to 0,$$

and thus

$$L_h(u_h) \to L(u).$$

The combination of Lemma 10, Lemma 13 and (4.35) implies that

$$|d_h(u_h,u_h) - d(u,u)| \leqslant |d_h(u_h,u_h) - d(\hat{u}_h,\hat{u}_h)| + |d(\hat{u}_h,\hat{u}_h) - d(u,u)| \to 0,$$

i.e.,

$$d_h(u_h,u_h) \to d(u,u).$$

Finally, on the basis of the definition of the weak convergence and the continuity of the bilinear form $b(.,.)$ the last term in (4.63) converges to $b(u,u)$. But this means that $|\hat{u}_h - u|^2_{1,2,\Omega} \to L(u) - d(u,u) - b(u,u) = 0$, as u is the weak solution. The convergence $\|\hat{u}_h - u\|_{0,2,\Omega} \to 0$ follows immediately from (4.49).

In such a way we have proved the strong convergence $\hat{u}_{h_n} \to u$ in $H^1(\Omega)$ as $n \to \infty$.

VI) Finally, since u is the only weak solution of the continuous problem and the weakly convergent sequence $\{\hat{u}_{h_n}\}$ was chosen arbitrarily, the whole system $\{\hat{u}_h\}_{h \in (0,h_0)}$ converges strongly to u in $H^1(\Omega)$ as $h \to 0$, which is what we wanted to prove. □

Concluding Remarks. The above analysis represents the extension of results from [8] concerning the convergence of approximate solutions of the elliptic problem with nonlinear Newton boundary conditions to the exact solution for the case of a nonpolygonal domain with a curved boundary. We have shown here that the approximation of the boundary by a piecewise linear curve does not effect the convergence which has been proved above without any assumption on the regularity of a weak solution.

Several problems still remain unsolved. It is, e.g., the influence of the approximation of the boundary on error estimates (established in [9] for the problem in a polygonal domain). In the papers [6], [8], [9], [10] concerned with the boundary value problem equipped with nonlinear Newton boundary conditions, the analysis was carried out for piecewise linear conforming finite element approximations. An extension to higher order finite elements will be the subject of a future work. (See, e.g., [26]). Another interesting topic is the use of nonconforming finite elements for the numerical solution of our problem, which represents again one of the finite element variational crimes.

References

[1] *R. Bialecki, A. J. Nowak*: Boundary value problems in heat conduction with nonlinear material and nonlinear boundary conditions. Appl. Math. Modelling *5* (1981), 417–421.

[2] *S. S. Chow*: Finite element error estimates for nonlinear elliptic equations of monotone type. Numer. Math. *54* (1988), 373–393.

[3] *P. G. Ciarlet*: The Finite Element Method for Elliptic Problems. North Holland, Amsterdam, 1978.

[4] *P. G. Ciarlet, P. A. Raviart*: The combined effect of curved boundaries and numerical integration in isoparametric finite element method. In: The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations (A. K. Aziz, ed.). Academic Press, New York, 1972, pp. 409–474.

[5] *M. Feistauer*: On the finite element approximation of a cascade flow problem. Numer. Math. *50* (1987), 655–684.

[6] *M. Feistauer, H. Kalis and M. Rokyta*: Mathematical modelling of an electrolysis process. Comment. Math. Univ. Carolin. *30* (1989), 465–477.

[7] *M. Feistauer, M. Křížek and V. Sobotíková*: An analysis of finite element variational crimes for a nonlinear elliptic problem of a nonmonotone type. East-West J. Numer. Math. *1* (1993), 267–285.

[8] *M. Feistauer, K. Najzar*: Finite element approximation of a problem with a nonlinear Newton boundary condition. Numer. Math. *78* (1998), 403–425.

[9] *M. Feistauer, K. Najzar and V. Sobotíková*: Error estimates for the finite element solution of elliptic problems with nonlinear Newton boundary conditions. Numer. Funct. Anal. Optim. *20* (1999), 835–851.

[10] *M. Feistauer, K. Najzar, V. Sobotíková and P. Sváček*: Numerical analysis of problems with nonlinear Newton boundary conditions. In: Numerical Mathematics and Advanced Applications, Proc. of the Conf. ENUMATH99 (P. Neittaanmäki, T. Tiihonen and P. Tarvainen, eds.). World Scientific, Singapore, 2000, pp. 486–493.

[11] *M. Feistauer, V. Sobotíková*: Finite element approximation of nonlinear elliptic problems with discontinuous coefficients. RAIRO Modél. Math. Anal. Numér. *24* (1990), 457–500.
[12] *M. Feistauer, A. Ženíšek*: Finite element solution of nonlinear elliptic problems. Numer. Math. *50* (1987), 451–475.
[13] *M. Feistauer, A. Ženíšek*: Compactness method in the finite element theory of nonlinear elliptic problems. Numer. Math. *52* (1988), 147–163.
[14] *J. Franců*: Monotone operators. A survey directed to applications to differential equations. Appl. Math. *35* (1990), 257–301.
[15] *M. Ganesh, I. G. Graham and J. Sivaloganathan*: A pseudospectral three-dimensional boundary integral method applied to a nonlinear model problem from finite elasticity. SIAM J. Numer. Anal. *31* (1994), 1378–1414.
[16] *M. Ganesh, O. Steinbach*: Boundary element methods for potential problems with nonlinear boundary conditions. Applied Mathematics Report AMR98/17, School of Mathematics, The University of New South Wales, Sydney (1998).
[17] *M. Ganesh, O. Steinbach*: Nonlinear boundary integral equations for harmonic problems. Applied Mathematics Report AMR98/20, School of Mathematics, The University of New South Wales, Sydney (1998).
[18] *J. Gwinner*: A discretization theory for monotone semicoercive problems and finite element convergence for p-harmonic Signorini problems. Z. Angew. Math. Mech. *74* (1994), 417–427.
[19] *M. Křížek, L. Liu and P. Neittaanmäki*: Finite element analysis of a nonlinear elliptic problem with a pure radiation condition. In: Applied Nonlinear Analysis. Kluwer, Amsterdam, 1999, pp. 271–280.
[20] *A. Kufner, O. John and S. Fučík*: Function Spaces. Academia, Praha, 1977.
[21] *L. Liu, M. Křížek*: Finite element analysis of a radiation heat transfer problem. J. Comput. Math. *16* (1998), 327–336.
[22] *R. Moreau, J. W. Ewans*: An analysis of the hydrodynamics of aluminium reduction cells. J. Electrochem. Soc. *31* (1984), 2251–2259.
[23] *J. Nečas*: Les méthodes directes en théorie des équations elliptiques. Academia, Prague, 1967.
[24] *V. Sobotíková*: Finite elements on curved domains. East-West J. Numer. Math. *4* (1996), 137–149.
[25] *G. Strang*: Variational crimes in the finite element method. In: The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations (A. K. Aziz, ed.). Academic Press, New York, 1972, pp. 689–710.
[26] *P. Sváček*: Higher order finite element method for a problem with nonlinear boundary condition. In: Proc. of the 13th Summer School "Software and Algorithms of Numerical Mathematics". University of West Bohemia in Pilsen, 1999, pp. 301–308.
[27] *A. Ženíšek*: Nonhomogeneous boundary conditions and curved triangular finite elements. Appl. Math. *26* (1981), 121–141.
[28] *A. Ženíšek*: The finite element method for nonlinear elliptic equations with discontinuous coefficients. Numer. Math. *58* (1990), 51–77.
[29] *A. Ženíšek*: Nonlinear Elliptic and Evolution Problems and Their Finite Element Approximations. Academic Press, London, 1990.
[30] *M. Zlámal*: Curved elements in the finite element method, I. SIAM J. Numer. Anal. *10* (1973), 229–240.

Authors' addresses: *M. Feistauer, K. Najzar*, Institute of Numerical Mathematics, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: `{feist, knaj}@karlin.mff.cuni.cz`; *V. Sobotíková*, Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University, Technická 2, 166 27 Praha 6, Czech Republic, e-mail: `sobotik@math.feld.cvut.cz`.