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INTERPOLATION FORMULAS FOR FUNCTIONS OF EXPONENTIAL TYPE*

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Abstract. In the paper we present a derivative-free estimate of the remainder of an arbitrary interpolation rule on the class of entire functions which, moreover, belong to the space $L^2_{(-\infty,+\infty)}$. The theory is based on the use of the Paley-Wiener theorem. The essential advantage of this method is the fact that the estimate of the remainder is formed by a product of two terms. The first term depends on the rule only while the second depends on the interpolated function only. The obtained estimate of the remainder of Lagrange's rule shows the efficiency of the method of estimate. The first term of the estimate is a starting point for the construction of the optimal rule; only the optimal rule with prescribed nodes of the interpolatory rule is investigated. An example illustrates the developed theory.

Keywords: entire functions, Paley-Wiener theorem, numerical interpolation, optimal interpolatory rule with prescribed nodes, remainder estimate

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1. Preliminaries

Definition 1. By P_a we denote the space of functions f which have the following properties:

1) f is an entire function of exponential type with a fixed constant a, i.e. f is holomorphic in the whole complex plane and for every complex z the inequality

$$|f(z)| \leqslant C\mathrm{e}^{a|z|}, \quad C > 0 \tag{1}$$

holds (C depends on f).

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2) The restriction of the function f to the interval $(-\infty,+\infty)$ is an element of the space $L^2_{(-\infty,+\infty)}$ with the norm

$$\|f\|^2_{L^2_{(-\infty,+\infty)}} = \int_{-\infty}^{+\infty} |f(t)|^2 \,\mathrm{d}t < +\infty. \tag{2}$$

On P_a we define the scalar product in the following way:

$$(f,g) = \int_{-\infty}^{+\infty} f\overline{g} \,\mathrm{d}x \quad \text{for } f,g \in P_a.$$

R e m a r k 1. It is well known that the constant a from (1) can be calculated by the formula

$$a = \limsup_{|z|\to+\infty} \frac{\ln|f(z)|}{|z|}. \tag{3}$$

R e m a r k 2. Let F be an entire function of exponential type with a constant A. Let us assume the restriction of F to an interval $[\alpha,\beta]$ to be a real one, where $\alpha<\beta$ are reals. Then the function defined by

$$f(z) = F(\zeta), \tag{4}$$

$$z = \sqrt{\frac{2A}{\beta-\alpha}}\Big(\zeta - \frac{\alpha+\beta}{2}\Big), \quad \zeta = \sqrt{\frac{\beta-\alpha}{2A}}\, z + \frac{\alpha+\beta}{2} \tag{5}$$

is of exponential type with the constant a where $a = \sqrt{\frac{A(\beta-\alpha)}{2}}$.

P r o o f. For some $C'>0$ we easily obtain

$$|f(z)| = |F(\zeta)| \leqslant C'\mathrm{e}^{A|\zeta|} = C'\mathrm{e}^{A|\sqrt{\frac{\beta-\alpha}{2A}}\,z+\frac{\alpha+\beta}{2}|} \leqslant C'\mathrm{e}^{A|\frac{\alpha+\beta}{2}|}\mathrm{e}^{A\sqrt{\frac{\beta-\alpha}{2A}}\,|z|} = C\mathrm{e}^{a|z|},$$

where $C = C'\mathrm{e}^{A|\frac{\alpha+\beta}{2}|}$. $\square$

R e m a r k 3. In the sequel—in accordance with Remark 2—we suppose that f is a given function from the space P_a and the goal is to interpolate the restriction of the function f to the interval $[-a,+a]$.

2. Basic theorems

In this part we give a basic theorem from which the considerations concerning a general interpolation formula will be developed. Let us summarize the definition and basic properties of the Fourier-Plancherel transform in

Remark 4. By the symbol $\hat{f}$ we denote the Fourier-Plancherel transform of an arbitrary function f belonging to the class $L^2_{(-\infty,+\infty)}$. This transform and the inverse one are defined by the formulas

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}ts} - 1}{-\mathrm{i}s} f(s)\,\mathrm{d}s, \tag{6}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{\mathrm{i}ts} - 1}{\mathrm{i}s} \hat{f}(s)\,\mathrm{d}s.$$

If $f \in L^1_{(-\infty,+\infty)} \cap L^2_{(-\infty,+\infty)}$ then the Fourier-Plancherel transform $\hat{f}$ of the function f is the usual Fourier transform of the function f. For every $f \in L^2_{(-\infty,+\infty)}$ we have $\|\hat{f}\|_{L^2_{(-\infty,+\infty)}} = \|f\|_{L^2_{(-\infty,+\infty)}}$. Thus the mapping $f \to \hat{f}$ is the Hilbert isomorphism of the space $L^2_{(-\infty,+\infty)}$ onto $L^2_{(-\infty,+\infty)}$. The set $L^1_{(-\infty,+\infty)} \cap L^2_{(-\infty,+\infty)}$ is dense in $L^2_{(-\infty,+\infty)}$.

Definition 2. Let R be a continuous linear functional on the space P_a. Then we define a linear functional $\hat{R}$ on the space $L^2_{(-a,a)}$ by the relation

$$\hat{R}(\hat{f}) = R(f) \tag{7}$$

where $\hat{f}$ is the Fourier-Plancherel transform of the function f.

Theorem 1. *Let $f \in P_a$, let R and $\hat{R}$ be the functionals from Definition* 2. *Then*

$$\|\hat{R}\|_{L^2_{(-a,a)}} = \|R\|_{L^2_{(-\infty,+\infty)}}. \tag{8}$$

Further, if we define the remainder R_n^x of the interpolation formula

$$L_n^x(f) = \sum_{k=0}^{n} t_k^{(n)}(x) f\big(x_k^{(n)}\big)$$

on the interval $[-a,+a]$ by the expression

$$R_n^x(f) = f(x) - \sum_{k=0}^{n} t_k^{(n)}(x) f\big(x_k^{(n)}\big) \tag{9}$$

where

$$\sum_{k=0}^{n}\left|t_k^{(n)}(x)\right| < +\infty \ \ \forall x \in [-a,+a]$$

then

$$\|R_n^x\|_{L^2_{(-\infty,+\infty)}} = \frac{1}{\sqrt{2\pi}}\left\|\mathrm{e}^{\mathrm{i}tx} - \sum_{k=0}^{n} t_k^{(n)}(x)\mathrm{e}^{\mathrm{i}tx_k^{(n)}}\right\|_{L^2_{(-a,a)}}. \tag{10}$$

P r o o f. 1) The assumption that the functional R is bounded means that

$$|R(f)| \leqslant \|R\|_{L^2_{(-\infty,+\infty)}}\|f\|_{L^2_{(-\infty,+\infty)}}, \quad f \in P_a.$$

In accordance with the Paley-Wiener theorem ([3], Theorem 19.3), for every $f \in P_a$ there exists a function $\tilde{f} \in L^2_{(-a,+a)}$ which has the following properties:

$$f(z) = \int_{-a}^{+a} \tilde{f}(t)\mathrm{e}^{\mathrm{i}tz}\,\mathrm{d}t \quad (z \text{ is complex}),$$
$$\tilde{f}(t) = 0 \quad \text{for} \quad t \notin [-a,a].$$

Let us set

$$\hat{\varphi}(t) = \sqrt{2\pi}\,\tilde{f}(t), \quad t \in (-\infty,+\infty).$$

Then

$$f(z) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{+a} \hat{\varphi}(t)\mathrm{e}^{\mathrm{i}tz}\,\mathrm{d}t \tag{11}$$

and by Plancherel's theorem ([3], Theorem 9.13) we have

$$\|f\|_{L^2_{(-\infty,+\infty)}} = \|\hat{\varphi}\|_{L^2_{(-\infty,+\infty)}} = \|\hat{\varphi}\|_{L^2_{(-a,a)}}.$$

Thus,

$$|\hat{R}(\hat{\varphi})| = |R(f)| \leqslant \|R\|_{L^2_{(-\infty,+\infty)}}\|f\|_{L^2_{(-\infty,+\infty)}} = \|R\|_{L^2_{(-\infty,+\infty)}}\|\hat{\varphi}\|_{L^2_{(-a,+a)}}$$

and then

$$\|\hat{R}\|_{L^2_{(-a,a)}} \leqslant \|R\|_{L^2_{(-\infty,+\infty)}}. \tag{12}$$

2) The converse inequality follows immediately from the relation

$$|R(f)| = |\hat{R}(\hat{\varphi})| \leqslant \|\hat{R}\|_{L^2_{(-a,a)}}\|\hat{\varphi}\|_{L^2_{(-a,a)}} = \|\hat{R}\|_{L^2_{(-a,a)}}\|f\|_{L^2_{(-\infty,+\infty)}},$$

which by virtue of (12) proves (8).

3) Let R_n^x be given by the formula (9). First we have to prove that R_n^x is bounded. Let $f \in P_a$. Then we have (see (11))

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} \hat{\varphi}(t) \mathrm{e}^{\mathrm{i}tx} \, \mathrm{d}t, \quad x \in (-a, a),$$

which gives by using the Cauchy inequality

$$\begin{aligned} |f(x)|^2 &\leqslant \frac{1}{2\pi} \int_{-a}^{+a} |\hat{\varphi}(t)|^2 \, \mathrm{d}t \int_{-a}^{+a} |\mathrm{e}^{\mathrm{i}tx}|^2 \, \mathrm{d}t \\ &= \frac{a}{\pi} \int_{-a}^{+a} |\hat{\varphi}(t)|^2 \, \mathrm{d}t = \frac{a}{\pi} \|\hat{\varphi}\|^2_{L^2_{(-a,a)}} = \frac{a}{\pi} \|f\|^2_{L^2_{(-\infty,+\infty)}}, \end{aligned}$$

and thus

$$|f(x)| \leqslant \sqrt{\frac{a}{\pi}} \|f\|_{L^2_{(-\infty,+\infty)}}.$$

Further,

$$\begin{aligned} |R_n^x(f)| &= \left| f(x) - \sum_{k=0}^{n} t_k^{(n)}(x) f\left(x_k^{(n)}\right) \right| \\ &\leqslant |f(x)| + \sum_{k=0}^{n} \left|t_k^{(n)}(x)\right| \left|f(x_k^{(n)})\right| \\ &\leqslant \sqrt{\frac{a}{\pi}} \|f\|_{L^2_{(-\infty,+\infty)}} + \sqrt{\frac{a}{\pi}} \|f\|_{L^2_{(-\infty,+\infty)}} \sum_{k=0}^{n} \left|t_k^{(n)}(x)\right| \\ &= \sqrt{\frac{a}{\pi}} \|f\|_{L^2_{(-\infty,+\infty)}} \left[1 + \sum_{k=0}^{n} \left|t_k^{(n)}(x)\right|\right]. \end{aligned}$$

This yields

$$\|R_n^x\|_{L^2_{(-\infty,+\infty)}} \leqslant \sqrt{\frac{a}{\pi}} \left[1 + \sum_{k=0}^{n} \left|t_k^{(n)}(x)\right|\right] < +\infty$$

for every positive integer n and $x \in [-a, +a]$. We see that R_n^x is bounded.

4) Let us define a functional $\hat{R}_n^x$ by the formula

$$\hat{R}_n^x(\hat{\theta}) = R_n^x(f),$$

where, as in part 3) of this proof,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} \hat{\theta}(t) \mathrm{e}^{\mathrm{i}tx} \, \mathrm{d}t.$$

Then

$$\begin{aligned}
\hat{R}_n^x(\hat{\theta}) = R_n^x(f) &= f(x) - \sum_{k=0}^{n} t_k^{(n)}(x) f(x_k^{(n)}) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} \hat{\theta}(t) \mathrm{e}^{\mathrm{i}tx} \, \mathrm{d}t - \sum_{k=0}^{n} t_k^{(n)}(x) \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} \hat{\theta}(t) \mathrm{e}^{\mathrm{i}tx_k^{(n)}} \, \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} \hat{\theta}(t) \left[\mathrm{e}^{\mathrm{i}tx} - \sum_{k=0}^{n} t_k^{(n)}(x) \mathrm{e}^{\mathrm{i}tx_k^{(n)}} \right] \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} \hat{\theta}(t) R_n^x(\mathrm{e}^{\mathrm{i}tx}) \, \mathrm{d}t,
\end{aligned}$$

where

$$R_n^x(\mathrm{e}^{\mathrm{i}tx}) = \mathrm{e}^{\mathrm{i}tx} - \sum_{k=0}^{n} t_k^{(n)}(x) \mathrm{e}^{\mathrm{i}tx_k^{(n)}}.$$

According to the Riesz theorem there exists a function $\hat{r}_n^x \in L^2_{(-a,a)}$ such that $\hat{R}_n^x(\hat{\theta}) = \int_{-a}^{+a} \hat{\theta}(t) \overline{\hat{r}_n^x(t)} \, \mathrm{d}t$ for every $\hat{\theta} \in L^2_{(-a,a)}$ and

$$\|\hat{R}_n^x\|_{L^2_{(-a,a)}} = \|\hat{r}_n^x\|_{L^2_{(-a,a)}}.$$

Thus

$$\overline{\hat{r}_n^x(t)} = \frac{1}{\sqrt{2\pi}} \left[\mathrm{e}^{\mathrm{i}tx} - \sum_{k=0}^{n} t_k^{(n)}(x) \mathrm{e}^{\mathrm{i}tx_k^{(n)}} \right]$$

and

$$\hat{r}_n^x(t) = \frac{1}{\sqrt{2\pi}} \left[\mathrm{e}^{-\mathrm{i}tx} - \sum_{k=0}^{n} t_k^{(n)}(x) \mathrm{e}^{-\mathrm{i}tx_k^{(n)}} \right]$$

is the representative of the functional $\hat{R}_n^x$. This implies that

$$\begin{aligned}
&\|\hat{R}_n^x\|_{L^2_{(-a,a)}} = \|\hat{r}_n^x\|_{L^2_{(-a,a)}} = \|\overline{\hat{r}_n^x}\|_{L^2_{(-a,a)}} \\
&= \frac{1}{\sqrt{2\pi}} \left\| \mathrm{e}^{\mathrm{i}tx} - \sum_{k=0}^{n} t_k^{(n)}(x) \mathrm{e}^{\mathrm{i}tx_k^{(n)}} \right\|_{L^2_{(-a,a)}} = \|R_n^x\|_{L^2_{(-\infty,+\infty)}}
\end{aligned}$$

by virtue of the relation $\hat{R}_n^x(\hat{\theta}) = R_n^x(f)$ for the corresponding $\hat{\theta}$, f. □

3. The estimate of the remainder of an interpolation formula

It is clear that for every function $f \in P_a$ the following inequality holds:

$$|R_n^x(f)| \leqslant \|R_n^x\|_{L^2_{(-\infty,+\infty)}} \|f\|_{L^2_{(-\infty,+\infty)}}. \tag{13}$$

This estimate has an important property that no derivatives of the function f appear in it and it consists of two terms; the first term depends on the rule only and the second on the function $f \in P_a$ only. The norm $\|f\|_{L^2_{(-\infty,+\infty)}}$ may be estimated more or less exactly.

Now we get back to a general interpolation rule of the form

$$f(x) = \sum_{k=0}^{n} t_k^{(n)}(x) f(x_k^{(n)}) + R_n^x(f)$$

where

$$\sum_{k=0}^{n} \left| t_k^{(n)}(x) \right| < +\infty \ \ \forall x \in [-a, +a] \tag{14}$$

in order to approximate an arbitrary function $f \in P_a$ over an interval $(-a, a)$, with $n \in \mathbb{N}$, where $x_k^{(n)}$, $k = 0, 1, \ldots, n$, are the nodes of the rule. The following theorem gives an explicit form for the expression (10).

Theorem 2. *Let $f \in P_a$, let R_n^x be the functional defined by* (9) *and by* (14). *Then*

$$\begin{aligned} \|R_n^x\|^2_{L^2_{(-\infty,+\infty)}} = \frac{1}{\pi} \Bigg\{ & a - 2 \sum_{k=0}^{n} t_k^{(n)}(x) \frac{\sin\big(a(x - x_k^{(n)})\big)}{x - x_k^{(n)}} \\ & + \sum_{k=0}^{n} \sum_{l=0}^{n} t_k^{(n)}(x) t_l^{(n)}(x) \frac{\sin\big(a(x_k^{(n)} - x_l^{(n)})\big)}{x_k^{(n)} - x_l^{(n)}} \Bigg\}, \end{aligned} \tag{15}$$

where

$$\frac{\sin\left(a(x_k^{(n)} - x_l^{(n)})\right)}{x_k^{(n)} - x_l^{(n)}} = a \quad \textit{for} \quad k = l.$$

P r o o f. According to the proof of Theorem 1 we may write

$$\|R_n^x\|^2_{L^2_{(-\infty,+\infty)}} = \frac{1}{2\pi}\|\mathrm{e}^{\mathrm{i}tx} - \sum_{k=0}^{n} t_k^{(n)}(x)\mathrm{e}^{\mathrm{i}tx_k^{(n)}}\|^2_{L^2_{(-a,a)}}$$

$$= \frac{1}{2\pi}\int_{-a}^{+a}\Big(\mathrm{e}^{\mathrm{i}tx} - \sum_{k=0}^{n} t_k^{(n)}(x)\mathrm{e}^{\mathrm{i}tx_k^{(n)}}\Big)\Big(\mathrm{e}^{-\mathrm{i}tx} - \sum_{l=0}^{n} t_l^{(n)}(x)\mathrm{e}^{-\mathrm{i}tx_l^{(n)}}\Big)\,\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-a}^{+a}\Big(1 - \mathrm{e}^{\mathrm{i}tx}\sum_{l=0}^{n} t_l^{(n)}(x)\mathrm{e}^{-\mathrm{i}tx_l^{(n)}} - \mathrm{e}^{-\mathrm{i}tx}\sum_{k=0}^{n} t_k^{(n)}(x)\mathrm{e}^{\mathrm{i}tx_k^{(n)}}$$
$$+ \sum_{k=0}^{n}\sum_{l=0}^{n} t_k^{(n)}(x)t_l^{(n)}(x)\mathrm{e}^{\mathrm{i}t(x_k^{(n)}-x_l^{(n)})}\Big)\,\mathrm{d}t$$

$$= \frac{1}{2\pi}\Big(2a - \sum_{l=0}^{n} t_l^{(n)}(x)\int_{-a}^{+a}\mathrm{e}^{\mathrm{i}t(x-x_l^{(n)})}\,\mathrm{d}t - \sum_{k=0}^{n} t_k^{(n)}(x)\int_{-a}^{+a}\mathrm{e}^{\mathrm{i}t(x_k^{(n)}-x)}\,\mathrm{d}t$$
$$+ \sum_{k=0}^{n}\sum_{l=0}^{n} t_k^{(n)}(x)t_l^{(n)}(x)\int_{-a}^{+a}\mathrm{e}^{\mathrm{i}t(x_k^{(n)}-x_l^{(n)})}\,\mathrm{d}t\Big).$$

Because of

$$\int_{-a}^{+a}\mathrm{e}^{\mathrm{i}t(y-z)}\,\mathrm{d}t = 2\frac{\sin(a(z-y))}{z-y} \quad \text{for} \quad z \neq y,$$

we get

$$\|R_n^x\|^2_{L^2_{(-\infty,+\infty)}} = \frac{1}{\pi}\Big\{a - 2\sum_{k=0}^{n} t_k^{(n)}(x)\frac{\sin\big(a(x-x_k^{(n)})\big)}{x-x_k^{(n)}}$$
$$+ \sum_{k=0}^{n}\sum_{l=0}^{n} t_k^{(n)}(x)t_l^{(n)}(x)\frac{\sin\big(a(x_k^{(n)}-x_l^{(n)})\big)}{x_k^{(n)}-x_l^{(n)}}\Big\},$$

where $\frac{\sin a\alpha}{\alpha} = a$ for $\alpha = 0$. □

R e m a r k 5. The term in brackets in (15) can be calculated for every interpolation rule given on the interval $[-a,a]$ in the form of a table, or it is also possible to create a procedure with parameters $a, n, t_k^{(n)}(x), x_k^{(n)}$, $k = 0,1,\ldots,n$.

Now we formulate a theorem on convergence properties of the Lagrange interpolation rule.

Theorem 3. *Let $R_n^{L,x}$ be the error functional of the Lagrange interpolation formula on $[-a,a]$. Then*

$$\|R_n^{L,x}\|^2_{L^2_{(-\infty,+\infty)}} \leqslant 4\Big(\frac{\mathrm{e}}{\pi}\Big)^2\frac{a^5}{(2n+3)(n+1)^3}\,\mathrm{e}^{-2n\ln\frac{n+1}{2\mathrm{e}a^2}} \tag{16}$$

and thus

$$\|R_n^{L,x}\|^2_{L^2_{(-\infty,+\infty)}} \longrightarrow 0 \quad \textit{for} \quad n \to \infty.$$

P r o o f. We can write

$$\begin{aligned}
\|R_n^{L,x}\|^2_{L^2_{(-\infty,+\infty)}} &= \|\hat{r}_n^x\|^2_{L^2_{(-a,+a)}} \\
&= \frac{1}{2\pi}\int_{-a}^{+a} |R_n^{L,x}(\mathrm{e}^{\mathrm{i}tx})|^2 \,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-a}^{+a} |R_n^{L,x}(\cos(tx)) + \mathrm{i}R_n^{L,x}(\sin(tx))|^2 \,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-a}^{+a} \big[\big(R_n^{L,x}(\cos(tx))\big)^2 + \big(R_n^{L,x}(\sin(tx))\big)^2\big] \,\mathrm{d}t.
\end{aligned}$$

Let us denote $C_n^x = \frac{\omega_{n+1}(x)}{(n+1)!}$, $\omega_{n+1}(x) = (x - x_0^{(n)})(x - x_1^{(n)})\ldots(x - x_n^{(n)})$. It is well known that

$$R_n^{L,x}(f) = \frac{f^{n+1}(\xi_x)}{(n+1)!}\,\omega_{n+1}(x), \quad \xi_x \in (-a,+a),$$

for every function f having $(n+1)$ continuous derivatives on $(-a,+a)$. Using this fact, we can write

$$2\pi\|R_n^{L,x}\|^2_{L^2_{(-\infty,+\infty)}} = (C_n^x)^2 \int_{-a}^{+a} \Big\{ \big[(\cos(tx))_x^{(n+1)}(\xi_x)\big]^2 + \big[(\sin(tx))_x^{(n+1)}(\eta_x)^2\big] \Big\} \,\mathrm{d}t.$$

From the relations

$$\frac{\partial^n}{\partial x^n}(\cos(tx)) = \begin{cases} (-1)^{\frac{n}{2}} t^n \cos(tx) & \text{for } n \text{ even},\\ (-1)^{\frac{(n+1)}{2}} t^n \sin(tx) & \text{for } n \text{ odd},\end{cases}$$

$$\frac{\partial^n}{\partial x^n}(\sin(tx)) = \begin{cases} (-1)^{\frac{n}{2}} t^n \sin(tx) & \text{for } n \text{ even},\\ (-1)^{\frac{(n-1)}{2}} t^n \cos(tx) & \text{for } n \text{ odd},\end{cases}$$

we obtain

$$2\pi\|R_n^{L,x}\|^2_{L^2_{(-\infty,+\infty)}} = \begin{cases} (C_n^x)^2 \displaystyle\int_{-a}^{+a} t^{2n+2}[\sin^2(t\xi_x) + \cos^2(t\eta_x)] \,\mathrm{d}t & \text{for } n \text{ even},\\ (C_n^x)^2 \displaystyle\int_{-a}^{+a} t^{2n+2}[\cos^2(t\xi_x) + \sin^2(t\eta_x)] \,\mathrm{d}t & \text{for } n \text{ odd},\end{cases}$$

which can be rewritten with the aid of one formula only:

$$2\pi\|R_n^{L,x}\|^2_{L^2_{(-\infty,+\infty)}} = (C_n^x)^2 \int_{-a}^{+a} t^{2n+2}[\sin^2(t\alpha_x) + \cos^2(t\beta_x)] \,\mathrm{d}t.$$

(The numbers $\xi_x, \eta_x, \alpha_x, \beta_x$ are different in general and lie in the interval $(-a, +a)$.) Hence, for every real y, z we obtain

$$2\pi \|R_n^{L,x}\|^2_{L^2_{(-\infty,+\infty)}} \leqslant 2(C_n^x)^2 \int_{-a}^{+a} t^{2n+2}\,\mathrm{d}t = \frac{4}{2n+3}(C_n^x)^2 a^{2n+3}.$$

Now we can estimate

$$(C_n^x)^2 = \left(\frac{\omega_{n+1}(x)}{(n+1)!}\right)^2 = \left(\frac{(x-x_0^{(n)})(x-x_1^{(n)})\dots(x-x_n^{(n)})}{(n+1)!}\right)^2 \leqslant \left(\frac{(2a)^{n+1}}{(n+1)!}\right)^2.$$

This estimate can be rewritten using the Stirling formula

$$n! = (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} \mathrm{e}^{-n}(1+\omega_n),$$

where $0 < \omega_n \leqslant \mathrm{e}^{\frac{1}{4n}} - 1$:

$$\begin{aligned}\|R_n^{L,x}\|^2_{L^2_{(-\infty,+\infty)}} &\leqslant \frac{2}{\pi}\frac{a^{2n+3}}{(2n+3)}\frac{(2a)^{2n+2}}{((n+1)!)^2}\\ &= \frac{8a^5}{\pi(2n+3)}\frac{(2a^2)^{2n}}{((n+1)!)^2}\\ &\leqslant \left(\frac{\mathrm{e}}{\pi}\right)^2 \frac{4a^5}{(2n+3)}\frac{1}{(n+1)^3}\mathrm{e}^{-2n(\ln\frac{n+1}{2a^2}-1)}\end{aligned}$$

and, finally,

$$\|R_n^{L,x}\|^2_{L^2_{(-\infty,+\infty)}} \leqslant 4\left(\frac{\mathrm{e}}{\pi}\right)^2 \frac{a^5}{(2n+3)(n+1)^3}\,\mathrm{e}^{-2n(\ln\frac{n+1}{2\mathrm{e}a^2})},$$

which is (16). □

R e m a r k 6. The estimate (16) does not depend on $x \in [-a, +a]$ and, therefore, it is uniform on $[-a, +a]$.

R e m a r k 7. Let $a \in \mathbb{R}$ be arbitrary. Then there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ the inequality $\frac{n+1}{2a^2\mathrm{e}} > 1$ holds and then $\ln(\frac{n+1}{2a^2\mathrm{e}}) > 0$. The number $n_0 \in \mathbb{N}$ can be chosen as the whole part of $2a^2\mathrm{e} - 1$, i.e., $n_0 = [2a^2\mathrm{e} - 1]$.

In this section we shall construct the optimal interpolatory rule under the assumption that the nodes $x_i^{(n)} \in [-a, a]$, $i = 0, 1, \ldots, n$, are prescribed and such that $x_i^{(n)} \neq x_j^{(n)}$ for $i \neq j$, $i, j = 0, 1, \ldots, n$.

Definition 3. The interpolatory rule (9) with the property (14) is said to be optimal for given nodes $x_i^{(n)} \in [-a, a]$, $i = 0, 1, \ldots, n$, $x_i^{(n)} \neq x_j^{(n)}$, $i, j = 0, 1, \ldots, n$ and given $x \in [-a, +a]$, if the norm $\|R_n^x\|_{L^2_{(-\infty,+\infty)}}$ is minimal as a function of the coefficients $t_k^{(n)}(x)$, $k = 0, 1, \ldots, n$. The optimal coefficients $t_k^{(n)}(x)$, $k = 0, 1, \ldots, n$ will be denoted by ${}^{(\mathrm{opt})}t_i^{(n)}(x)$, $i = 0, 1, \ldots, n$.

Now we prove two lemmas which will be applied in the subsequent considerations.

Lemma 1. *The functions* $\frac{\sin(a(x-x_i^{(n)}))}{x-x_i^{(n)}}$, $i = 0, 1, \ldots, n$ *(defined to be equal to* a *at* $x = x_i^{(n)}$*) are linearly independent if and only if* $x_i^{(n)} \neq x_j^{(n)}$ *for* $i \neq j$, $i, j = 0, 1, \ldots, n$.

P r o o f. 1) Necessity is trivial.

2) To prove sufficiency, we show that the equation

$$\sum_{j=0}^{n} \alpha_j \frac{\sin\bigl(a(x - x_j^{(n)})\bigr)}{x - x_j^{(n)}} = 0, \quad \text{where} \quad \sum_{j=0}^{n} |\alpha_j| > 0 \tag{17}$$

has in $(-\infty, +\infty)$ isolated roots only. The equation (17) may be rewritten in the form

$$\sin(ax) \sum_{j=0}^{n} \alpha_j \cos(ax_j^{(n)}) \frac{1}{x - x_j^{(n)}} = \cos(ax) \sum_{j=0}^{n} \alpha_j \sin(ax_j^{(n)}) \frac{1}{x - x_j^{(n)}}.$$

Let us suppose that $x \neq x_i^{(n)}$, $i = 0, 1, \ldots, n$, and $\cos(ax) \neq 0$. Multiplying this equation by the polynomial $\prod_{i=0}^{n} (x - x_i^{(n)})$ and making some intermediate steps we get

$$\tan(ax) = \frac{\sum_{j=0}^{n} \alpha_j \sin(ax_j^{(n)}) \prod_{k=0,\, k\neq j}^{n} (x - x_k^{(n)})}{\sum_{j=0}^{n} \alpha_j \cos(ax_j^{(n)}) \prod_{k=0,\, k\neq j}^{n} (x - x_k^{(n)})},$$

where the right-hand side is a rational function of x. If $\cos(ax) = 0$, then $|\sin(ax)| = 1$. Thus for $x \neq x_k^{(n)}$, $k = 0, 1, \ldots, n$, we have obtained an equation which can have isolated roots only. $\square$

Lemma 2. *Suppose that* $x_i^{(n)} \neq x_j^{(n)}$, $i \neq j$, $i, j, = 0, 1, \ldots, n$, $a > 0$. *Then the determinant* Δ_n *of the matrix*

$$\left\{ \frac{\sin\bigl(a(x_i^{(n)} - x_j^{(n)})\bigr)}{x_i^{(n)} - x_j^{(n)}} \right\}_{i,j=0}^{n}$$

is the Gram determinant of the linearly independent system of functions

$$\frac{\sin\bigl(a(x - x_i^{(n)})\bigr)}{\sqrt{\pi}(x - x_i^{(n)})}, \quad i = 0, 1, \ldots, n,$$

and, consequently, $\Delta_n > 0$.

If there exist indices i, j, $i \neq j$ *such that* $x_i^{(n)} = x_j^{(n)}$, *then obviously* $\Delta_n = 0$.

Proof. We know that the functions

$$\frac{\sin\bigl(a(x - x_i^{(n)})\bigr)}{\sqrt{\pi}(x - x_i^{(n)})}, \quad i = 0, 1, \ldots, n$$

where $x_i^{(n)} \neq x_j^{(n)}$, $i \neq j$, $i, j = 0, 1, \ldots, n$, are linearly independent in $L^2_{(-\infty,+\infty)}$. We have

$$\begin{aligned} I &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin\bigl(a(x - x_i^{(n)})\bigr) \sin\bigl(a(x - x_j^{(n)})\bigr)}{(x - x_i^{(n)})(x - x_j^{(n)})} \,\mathrm{d}x \\ &= \frac{1}{2\pi} \left\{ -\cos(2a\alpha) \int_{-\infty}^{+\infty} \frac{1}{\alpha^2 - y^2} \,\mathrm{d}y + \int_{-\infty}^{+\infty} \frac{\cos(2ay)}{\alpha^2 - y^2} \,\mathrm{d}y \right\}, \end{aligned}$$

where $\alpha = \frac{x_j^{(n)} - x_i^{(n)}}{2}$. Let us suppose that $x_j^{(n)} > x_i^{(n)}$, i.e., $\alpha > 0$ and put $y = \alpha x$. We can write the last expression in the form

$$I = \frac{1}{2\alpha\pi} \left\{ \int_{-\infty}^{+\infty} \frac{\cos(2a\alpha x)}{1 - x^2} \,\mathrm{d}x - \cos(2a\alpha) \int_{-\infty}^{+\infty} \frac{1}{1 - x^2} \,\mathrm{d}x \right\}.$$

In the sense of the principal value we have

$$\int_{-\infty}^{+\infty} \frac{1}{1 - x^2} \,\mathrm{d}x = \pi \cot \frac{\pi}{2} = 0$$

and

$$\int_{-\infty}^{+\infty} \frac{\cos(2a\alpha x)}{1 - x^2} \,\mathrm{d}x = 2 \int_{0}^{+\infty} \frac{\cos(2a\alpha x)}{1 - x^2} \,\mathrm{d}x = \pi \sin(2a\alpha).$$

In the sense of the principal value we finally find that

$$I = \frac{\sin(2a\alpha)}{2\alpha} = \frac{\sin\bigl(a(x_j^{(n)} - x_i^{(n)})\bigr)}{x_j^{(n)} - x_i^{(n)}}.$$

The case $x_j^{(n)} < x_i^{(n)}$ can be proved analogously. The case $x_j^{(n)} = x_i^{(n)}$ $(i = j = k)$ is easy to verify because of the relations

$$I = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin^2\bigl(a(x - x_k^{(n)})\bigr)}{(x - x_k^{(n)})^2} \,\mathrm{d}x = \frac{2}{\pi} \int_0^{+\infty} \frac{\sin^2(ay)}{y^2} \,\mathrm{d}y = a.$$

□

Theorem 4. *Let R_n^x be the functional of the remainder of the interpolatory rule (9) with property (14), where $x_i^{(n)} \in [-a, a]$, $i = 0, 1, \ldots, n$ are given mutually different nodes. Then for every $x \in [-a, +a]$ there exists exactly one system of weights ${}^{(\mathrm{opt})}t_k^{(n)}(x)$, $k = 0, 1, \ldots, n$, as a solution of the Gram system of linear equations*

$$\sum_{k=0}^{n} {}^{(\mathrm{opt})}t_k^{(n)}(x) \frac{\sin\bigl(a(x_k^{(n)} - x_i^{(n)})\bigr)}{x_k^{(n)} - x_i^{(n)}} = \frac{\sin\bigl(a(x - x_i^{(n)})\bigr)}{x - x_i^{(n)}}, \quad i = 0, 1, \ldots, n. \tag{18}$$

For the corresponding $\|{}^{(\mathrm{opt})}R_n^x\|^2_{L^2_{(-\infty,+\infty)}}$ the formulas

$$\|{}^{(\mathrm{opt})}R_n^x\|^2_{L^2_{(-\infty,+\infty)}} = \frac{1}{\pi}\biggl(a - \sum_{k=0}^{n} {}^{(\mathrm{opt})}t_k^{(n)}(x) \,\frac{\sin\bigl(a(x - x_i^{(n)})\bigr)}{x - x_i^{(n)}}\biggr), \tag{19}$$

$$\|{}^{(\mathrm{opt})}R_n^x\|^2_{L^2_{(-\infty,+\infty)}} = \frac{1}{\pi}\biggl(a - \sum_{k=0}^{n}\sum_{l=0}^{n} {}^{(\mathrm{opt})}t_k^{(n)}(x)\,{}^{(\mathrm{opt})}t_l^{(n)}(x) \,\frac{\sin\bigl(a(x_k^{(n)} - x_i^{(n)})\bigr)}{x_k^{(n)} - x_i^{(n)}}\biggr) \tag{20}$$

hold.

P r o o f. Let $x \in [-a, +a]$ be given. Because $\|R_n^x\|^2_{L^2_{(-\infty,+\infty)}}$ is a nonnegative quadratic function of the variables $t_k^{(n)}(x)$, $k = 0, 1, \ldots, n$, necessary and sufficient conditions for an extreme point of $\|R_n^x\|^2_{L^2_{(-\infty,+\infty)}}$ as a function of $t_k^{(n)}(x)$ read

$$\frac{\partial \|R_n^x\|^2_{L^2_{(-\infty,+\infty)}}}{\partial t_k^{(n)}(x)} = 0, \quad k = 0, 1, \ldots, n.$$

According to (15) we have

$$\frac{\partial \|R_n^x\|^2_{L^2_{(-\infty,+\infty)}}}{\partial t_i^{(n)}(x)} = \frac{2}{\pi}\biggl\{-\frac{\sin\bigl(a(x - x_i^{(n)})\bigr)}{x - x_i^{(n)}} + \sum_{k=0}^{n} t_k^{(n)}(x) \frac{\sin\bigl(a(x_k^{(n)} - x_i^{(n)})\bigr)}{x_k^{(n)} - x_i^{(n)}}\biggr\},$$

which gives (18). The formulas (19), (20) follow from (15) by using (18). □

R e m a r k 8. From Theorem 4 the following results are obtained:

1) The functions

$$\varphi_i^{(n)}(x) = \frac{\sin\bigl(a(x - x_i^{(n)})\bigr)}{x - x_i^{(n)}}, \quad i = 0, 1, \ldots, n$$

are interpolated by the optimal rule exactly.

2) It is clear that ${}^{(\mathrm{opt})}t_k^{(n)}(x_j^{(n)}) = \delta_{j,k}$, $j, k = 0, 1, \ldots, n$. Thus, the optimal rule is an interpolatory one.

3) The corresponding ${}^{(\mathrm{opt})}t_k^{(n)}(x)$ can be calculated for given a, n, $x_i^{(n)}$, $i = 0, 1, \ldots, n$ and $x \in [-a, +a]$ once for ever.

In the next part we get the corresponding quadrature rule constructed by the optimal weights ${}^{(\mathrm{opt})}t_k^{(n)}(x)$, $k = 0, 1, \ldots, n$.

Corollary 1. *Let $f \in P_a$ and let $x_i^{(n)}$, $i = 0, 1, \ldots, n$, be given mutually different nodes of the interpolatory rule. Then there exists a quadrature rule of the form*

$$\int_{-a}^{+a} f(x)\,\mathrm{d}x = \sum_{k=0}^{n} {}^{(\mathrm{opt})}Q_k^{(n)} f(x_k^{(n)}) + R_n(f)$$

with coefficients ${}^{(\mathrm{opt})}Q_k^{(n)}$, $k = 0, 1, \ldots, n$, given by

$${}^{(\mathrm{opt})}Q_k^{(n)} = \int_{-a}^{+a} {}^{(\mathrm{opt})}t_k^{(n)}(x)\,\mathrm{d}x.$$

This formula is exact for the functions $\varphi_i^{(n)}$, $i = 0, 1, \ldots, n$.

P r o o f. We construct the interpolating operator of the Lagrange type

$$L_n^x(f) = \sum_{k=0}^{n} {}^{(\mathrm{opt})}t_k^{(n)}(x) f(x_k^{(n)}), \quad i = 0, 1, \ldots, n,$$

such that the identities $L_n^x(\varphi_i^{(n)}) = \varphi_i^{(n)}(x)$, $i = 0, 1, \ldots, n$ hold. That is

$$\sum_{k=0}^{n} {}^{(\mathrm{opt})}t_k^{(n)}(x) \frac{\sin\bigl(a(x_i^{(n)} - x_k^{(n)})\bigr)}{x_i^{(n)} - x_k^{(n)}} = \frac{\sin\bigl(a(x - x_i^{(n)})\bigr)}{x - x_i^{(n)}}, \quad i = 0, 1, \ldots, n.$$

Using the Cramer rule we obtain

$$\text{(21)} \qquad {}^{(\mathrm{opt})}t_k^{(n)}(x) = \frac{D_n^k(x)}{D_n}, \quad k = 0, 1, \ldots, n,$$

where $D_n^k(x)$ and $D_n > 0$ are the corresponding determinants. We have

$$^{(\mathrm{opt})}t_k^{(n)}(x_i^{(n)}) = \delta_{k,i}, \quad k, i = 0, 1, \dots, n.$$

From (21) it follows that

$$^{(\mathrm{opt})}Q_k^{(n)} = \int_{-a}^{+a} {}^{(\mathrm{opt})}t_k^{(n)}(x)\,\mathrm{d}x = \frac{1}{D_n}\int_{-a}^{+a} D_n^k(x)\,\mathrm{d}x, \quad k = 0, 1, \dots, n.$$

$\square$

5. The minimization of the estimate of the remainder $^{(\mathrm{opt})}R_n^x(f)$

In this part we will use the method used in the paper [2] in order to minimize the estimate of $|^{(\mathrm{opt})}R_n^x(f)|$.

Theorem 5. *Assume that $x_i^{(n)} \neq x_j^{(n)}$, $i \neq j$, $i, j = 0, 1, \dots, n$. Let us set*

$$\psi_i^{(n)}(x) = \frac{1}{\sqrt{\pi}}\,\varphi_i^{(n)}(x) = \frac{\sin\bigl(a(x - x_i^{(n)})\bigr)}{\sqrt{\pi}\,(x - x_i^{(n)})}, \quad i = 0, 1, \dots, n,$$

and let E_n be a projection operator from the space P_a to the subspace $S_n \subset P_a$, where

$$S_n = \operatorname{span}(\psi_0^{(n)}, \psi_1^{(n)}, \dots, \psi_n^{(n)}).$$

Then

$$|^{(\mathrm{opt})}R_n^x(f)|^2 \leqslant \|^{(\mathrm{opt})}R_n^x\|^2_{L^2_{(-\infty,+\infty)}}\bigl(\|f\|^2_{L^2_{(-\infty,+\infty)}} - \|E_n(f)\|^2_{L^2_{(-\infty,+\infty)}}\bigr)$$

for every function $f \in P_a$. Moreover,

$$E_n(f) = \sum_{i=0}^{n} e_i^{(n)}(f)\psi_i^{(n)}.$$

The coefficients $e_i^{(n)}$, $i = 0, 1, \dots, n$, can be calculated as the solution of the following system of normal equations with a positive determinant:

$$\sum_{k=0}^{n} e_k^{(n)}(f)(\psi_i^{(n)}, \psi_k^{(n)}) = (f, \psi_i^{(n)}), \quad i = 0, 1, \dots, n. \tag{22}$$

The entries of the matrix and the elements of the right-hand side, respectively, of the system (22) *are given by the formulas*

$$(\psi_i^{(n)},\psi_k^{(n)})=\frac{\sin\bigl(a(x_i^{(n)}-x_k^{(n)})\bigr)}{x_i^{(n)}-x_k^{(n)}},\quad i,k=0,1,\dots,n,\tag{23}$$

and

$$(f,\psi_i^{(n)})=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}f(x)\,\frac{\sin\bigl(a(x-x_i^{(n)})\bigr)}{x-x_i^{(n)}}\,\mathrm{d}x,\quad i=0,1,\dots,n,\tag{24}$$

respectively.

P r o o f. The optimal interpolatory formula with prescribed nodes is exact for the functions $\psi_i^{(n)}$, $i=0,1,\dots,n$. Then the equality ${}^{(\mathrm{opt})}R_n^x(\psi_i^{(n)})=0$, $i=0,1,\dots,n$, follows. If $g=\sum\limits_{i=0}^{n}\alpha_i^{(n)}\psi_i^{(n)}$, then ${}^{(\mathrm{opt})}R_n^x(f+g)={}^{(\mathrm{opt})}R_n^x(f)+{}^{(\mathrm{opt})}R_n^x(g)={}^{(\mathrm{opt})}R_n^x(f)$ for every $f\in P_a$. From Lemma 1 we find that $\psi_i^{(n)}$, $i=0,1,\dots,n$, are linearly independent and there exist numbers $e_i^{(n)}(f)$, $i=0,1,\dots,n$, such that

$$E_n(f)=\sum_{k=0}^{n}e_k^{(n)}(f)\psi_k^{(n)}$$

is an orthogonal projector. Now we have

$$\begin{aligned}|{}^{(\mathrm{opt})}R_n^x(f)|^2&=|{}^{(\mathrm{opt})}R_n^x(f-E_n(f))|^2\\&\leqslant\|{}^{(\mathrm{opt})}R_n^x\|^2_{L^2_{(-\infty,+\infty)}}\,\|f-E_n(f)\|^2_{L^2_{(-\infty,+\infty)}}\\&=\|{}^{(\mathrm{opt})}R_n^x\|^2_{L^2_{(-\infty,+\infty)}}\bigl(\|f\|^2_{L^2_{(-\infty,+\infty)}}-\|E_n(f)\|^2_{L^2_{(-\infty,+\infty)}}\bigr).\end{aligned}$$

Now we can compute the coefficients $e_k^{(n)}(f)$, $k=0,1,\dots,n$. We have

$$\bigl(E_n(f),\psi_j^{(n)}\bigr)=(f,E_n\psi_j^{(n)})=(f,\psi_j^{(n)}),$$

where

$$\bigl(E_n(f),\psi_j^{(n)}\bigr)=\sum_{k=0}^{n}e_k^{(n)}(f)(\psi_k^{(n)},\psi_j^{(n)})$$

and

$$(f,\psi_i^{(n)})=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}f(x)\,\frac{\sin\bigl(a(x-x_i^{(n)})\bigr)}{x-x_i^{(n)}}\,\mathrm{d}x,\quad i=0,1,\dots,n.$$

These integrals are convergent because f and $\psi_i^{(n)}$, $i=0,1,\dots,n$, are elements of the space $L^2_{(-\infty,+\infty)}$.

The matrix of (22) as the Gram matrix of the linear independent system of functions $\psi_i^{(n)}$, $i=0,1,\dots,n$, gives the unique solution of our problem. □

6. Numerical example

We shall interpolate the function

$$\left(\frac{\sin\zeta}{\zeta}\right)^2$$

by the classical Lagrange rule on the interval $[-1,+1]$.

It is known that (cf. [1])

$$\int_{-\infty}^{+\infty}\left(\frac{\sin\zeta}{\zeta}\right)^4 \mathrm{d}\zeta = \frac{2\pi}{3}.$$

The function $F(\zeta)=(\frac{\sin\zeta}{\zeta})^2$ is an entire function of exponential type with constant 2. The function $f(z)=F(\zeta)$, $z=\sqrt{2}\zeta$, $z\in[-\sqrt{2},\sqrt{2}]$, is of exponential type with constant $\sqrt{2}$. Thus, according to Theorem 3 and (13), we get the inequality

$$|R_n^{L,x}(f)| \leqslant 4\mathrm{e}\sqrt{\frac{2\sqrt{2}}{3\pi}}\frac{1}{(n+1)\sqrt{(n+1)(2n+3)}}\,\mathrm{e}^{-n\ln\frac{n+1}{4\mathrm{e}}}.$$

Tab. 1 gives the numerically computed estimates of the remainder of the Lagrange rule.

Number of nodes	Estimate of $\lvert R_n^{L,x}(f)\rvert$
8	0.230
16	$1.127_{10}-\ \ 5$
32	$1.143_{10}-18$
64	$1.983_{10}-53$

Table 1.

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References

[1] *I. S. Gradshtejn, I. M. Ryzhik*: Tables of Integrals, Sums, Series and Products. Fizmatgiz, Moskva, 1963. (In Russian.)

[2] *G. Hämmerlin*: Über ableitungsfreie Schranken für Quadraturfehler II. Ergänzungen und Möglichkeiten zur Verbesserung. Numer. Math. *7* (1965), 232–237.

[3] *W. Rudin*: Real and Complex Analysis. McGraw-Hill, New York, 1966.

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