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SPECTRAL SETS AND THE DRAZIN INVERSE WITH APPLICATIONS TO SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. The paper defines and studies the Drazin inverse for a closed linear operator A in a Banach space X in the case that 0 belongs to a spectral set of the spectrum of A. Results are applied to extend a result of Krein on a nonhomogeneous second order differential equation in a Banach space.

Keywords: Banach space, closed linear operators, Drazin inverse, spectral sets, second order differential equations

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1. Introduction and preliminaries

The main purpose of this paper is to introduce the Drazin inverse A^D of a closed linear operator A on a Banach space X when 0 belongs to a spectral set of the spectrum of A and apply it to represent solutions of differential equations in X.

Krein in his monograph [7] considered the nonhomogenous second order differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = B^2x(t) + f(t), \quad t \in [0,T], \tag{1.1}$$

where B is assumed to be invertible on X. We will demonstrate the usefulness of the introduced Drazin inverse by extending the above problem to a very general class of operators B whose spectrum has a spectral set containing 0.

By $C(X)$ we denote the space of all closed linear operators A with domain and range in X; $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the domain, nullspace and range of A, respectively. For $n \geqslant 1$, $\mathcal{D}(A^n)$ is the set of all $x \in X$ such that $x, Ax, \dots, A^{n-1}x$ are all in $\mathcal{D}(A)$; we also write $\mathcal{D}(A^\infty)$ for $\bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$. By $B(X)$ we denote the space of all bounded linear operators defined on all of X. An operator $A \in C(X)$ is *invertible* if there exists an operator $B \in B(X)$ such that $AB = I$ and $BAx = x$ for all $x \in \mathcal{D}(A)$; $A^{-1} = B$ is the *inverse* of A. In other words, $A \in C(X)$ is invertible if and only if $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = X$. If $A \in C(X)$, then $\varrho(A)$ denotes the *resolvent set* of A, that is, the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is invertible. The complement of $\varrho(A)$ in $\mathbb{C}$ is the *spectrum* $\sigma(A)$ of A. We also define the *extended spectrum* $\sigma_e(A)$ as the subset of the extended complex plane $\mathbb{C} \cup \{\infty\}$ equal to $\sigma(A)$ if $A \in B(X)$, and to $\sigma(A) \cup \{\infty\}$ otherwise. If $A \in B(X)$, we write $r(A)$ for the spectral radius of A. For $\lambda \in \varrho(A)$, $R(\lambda; A)$ denotes the *resolvent operator* $(\lambda I - A)^{-1}$ of A.

Let $A \in C(X)$ with $\sigma(A) \neq \mathbb{C}$. Then a subset σ of $\sigma_e(A)$ is called a *spectral set* of A if it is both open and closed in the relative topology of $\sigma_e(A)$ as a subset of $\mathbb{C} \cup \{\infty\}$. A singleton $\{\mu\}$ is a spectral set of A if and only if μ is an isolated singularity of the resolvent $R(\lambda; A)$ of A. We call μ a *pole of* A if μ is a pole of $R(\lambda; A)$.

If σ is a spectral set of A, then A admits a direct decomposition $A = A_1 \oplus A_2$ relative to a topological direct sum $X = X_1 \oplus X_2$ such that $\sigma(A_1) = \sigma$, $\sigma(A_2) = \sigma(A) \setminus \sigma$. The (bounded) projection P of X with $\mathcal{R}(P) = X_1$ and $\mathcal{N}(P) = X_2$ is the *spectral projection* of A corresponding to σ. If the spectral set σ is bounded, then $\mathcal{R}(P) \subset \mathcal{D}(A^n)$ for all n, and the restriction A_σ of A to $\mathcal{R}(P)$ is continuous ([10, Theorem V.9.2]).

2. Characterizations of the spectral sets and the Drazin inverse

In this section we characterize a spectral set of $A \in C(X)$ using the corresponding spectral projection, and rely on this characterization to define a Drazin inverse A^D of A.

Theorem 2.1. *Let $A \in C(X)$ be a noninvertible operator and let σ be a bounded set of $\mathbb{C}$ containing* 0. *Then σ is a spectral set of A if and only if there is a nonzero projection P such that*

(i) $\mathcal{R}(P) \subset \mathcal{D}(A)$,
(ii) $PAx = APx$ *for all* $x \in \mathcal{D}(A)$,
(iii) $\sigma(AP) = \sigma$,
(iv) $A - \mu I - \xi P$ *is invertible for all* $\mu \in \sigma$ *and for some (equivalently for all)* $\xi \in \mathbb{C}$ *such that* $|\xi| > 2r$ *where* $r = \sup_{\lambda \in \sigma} |\lambda|$.

If (i)–(iv) *hold, P is the spectral projection of A corresponding to σ.*

P r o o f. Let σ be a spectral set of A containing 0. The spectral projection P of A corresponding to σ is bounded. Let $A = A_1 \oplus A_2$ be the decomposition of A relative to $X = \mathcal{R}(P) \oplus \mathcal{N}(P)$. Then conditions (i) and (ii) are clearly satisfied and $\sigma(AP) = \sigma(A_1 \oplus 0) = \sigma(A_1) \cup \{0\} = \sigma$ as $0 \in \sigma$. The operator A_2 is invertible since $0 \notin \sigma(A) \setminus \sigma = \sigma(A_2)$. Further, if $|\xi| > 2r$ and $\mu \in \sigma$ then

$$\begin{aligned} A - \mu I - \xi P &= A_1 \oplus A_2 - (\mu I_1 \oplus \mu I_2) - (\xi I_1 \oplus 0) \\ &= (A_1 - (\mu + \xi) I_1) \oplus (A_2 - \mu I_2) \end{aligned}$$

is invertible since $\mu \in \varrho(A_2)$ and $\mu + \xi \in \varrho(A_1)$ (as $|\mu + \xi| \geqslant |\xi| - |\mu| > 2r - r = r$).

Conversely, assume P is a projection satisfying (i)–(iv). Let $A = A_1 \oplus A_2$ be the decomposition of A relative to $X = \mathcal{R}(P) \oplus \mathcal{N}(P)$.

By condition (iv), there exists ξ with $|\xi| > 2r$ such that $A - \xi P = (A_1 - \xi I_1) \oplus A_2$ is invertible. Hence A_2 (and also $A_1 - \xi I_1$) is invertible. We have $\sigma = \sigma(AP) = \sigma(A_1 \oplus 0) = \sigma(A_1) \cup \{0\}$. But $A_1 \oplus A_2$ is noninvertible, which means that $0 \in \sigma(A_1)$ as A_2 is invertible. Thus $\sigma(A_1) = \sigma$.

Let $\mu \in \mathbb{C}$. Then

$$A - \mu I - \xi P = (A_1 - (\mu + \xi) I_1) \oplus (A_2 - \mu I_2)$$

is invertible by condition (iv), which implies that $A_2 - \mu I_2$ is invertible. Then $\sigma \subset \varrho(A_2)$, and $\sigma(A_1) \cap \sigma(A_2)$ is empty. Therefore σ is a spectral set of A, and P is the corresponding spectral projection. $\square$

Observe that the condition $0 \in \sigma$ is necessary for condition (iii) to hold and for the invertibility of A_2. We are in a position to introduce a generalization of the Drazin inverse.

Definition 2.2. Let $A \in C(X)$ be a noninvertible operator with a bounded spectral set σ containing 0 and the corresponding spectral projection P. We define the *Drazin inverse of A relative to σ* by

$$A^{D,\sigma} = (A - \xi P)^{-1}(I - P)$$

for some $\xi \in \mathbb{C}$ such that $|\xi| > 2r$ where $r = \sup\limits_{\lambda \in \sigma} |\lambda|$.

If $\sigma = \{0\}$, we write A^D in place of $A^{D,\sigma}$.

As we can observe from the proof of the previous theorem, condition $|\xi| > 2r$ is required to ensure that $A - \xi P$ is invertible. From the above definition, we can also immediately deduce that $P = I - AA^{D,\sigma}$. The next theorem gives a representation of the Drazin inverse and shows that the Drazin inverse depends on σ but not on the choice of ξ. A similar definition of the generalized Drazin inverse relative to a spectral set in a Banach algebra was given in [5].

Theorem 2.3. *Suppose that an operator $A \in C(X)$ possesses a Drazin inverse and that P is the spectral projection of A corresponding to σ. Then the Drazin inverse of A relative to σ is given by*

$$A^{D,\sigma} = 0 \oplus A_2^{-1}, \tag{2.1}$$

where $A = A_1 \oplus A_2$ is the decomposition of A with respect to the topological direct sum $X = \mathcal{R}(P) \oplus \mathcal{N}(P)$.

Proof. By [10, Theorem V.9.2], $X = \mathcal{R}(P) \oplus \mathcal{N}(P)$ and $A = A_1 \oplus A_2$ with A_1 bounded, $\sigma(A_1) = \sigma$, and A_2 closed invertible. Then for any $\xi \in \mathbb{C}$ such that $|\xi| > 2r$, where $r = \sup_{\lambda\in\sigma} |\lambda|$,

$$A^{D,\sigma} = (A - \xi P)^{-1}(I - P) = \big((A_1 - \xi I)^{-1} \oplus A_2^{-1}\big)(0 \oplus I) = 0 \oplus A_2^{-1}$$

and the result follows. □

3. Applications to second order differential equations

In this section we consider an abstract differential equationon on a Banach space that was studied by Krein [7, Chapter 3]. Let B be the infinitesimal generator of a strongly continuous group of bounded linear operators $T(t)$. Following [7, Definition 3.1.1], we say that a function $x\colon [0,T] \to X$ is a solution of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = B^2x(t) + f(t), \quad t \in [0,T], \tag{3.1}$$

if x takes values in $\mathcal{D}(B^2)$, is twice continuously differentiable and satisfies (3.1) on the interval $[0,T]$. Observe that if x is twice continuously differentiable on $[0,T]$, so is Px for any $P \in B(X)$.

Let f be a continuous function on $[0,T]$. We define a *primitive* of f by

$$F(t) = \int_0^t f(s)\,\mathrm{d}s \quad \text{for} \quad t \in [0,T].$$

Observe that $\sup_{t\in[0,T]} \|F^{(n)}\| \leqslant MT^n$ for each $n \in \mathbb{N}$ where $F^{(n)}$ denotes the nth primitive of f and $M = \sup_{t\in[0,T]} \|f(t)\|$. We begin with a theorem on the local solution of a nonhomogeneous second order differential equation.

Theorem 3.1. *Let B be the infinitesimal generator of a strongly continuous group of bounded linear operators $T(t)$. If f is continuously differentiable on $[0,T]$ and there exists a spectral set σ of B such that $0 \in \sigma \subset D_r$ where $D_r = \{\lambda\colon\ |\lambda| < r\}$ for some $r > 0$, then the unique solution of Equation 3.1 with initial conditions $x(0) = u_0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 x(t) = v_0$ can be expressed explicitly as*

$$x(t) = \sum_{j=1}^{\infty} B^{2(j-1)} P F^{(2j)}(t) \tag{3.2}$$
$$+ \frac{1}{2}\big(T(t) + T(-t)\big)(I-P)u_0 + \frac{1}{2} B^{D,\sigma}\big(T(t) - T(-t)\big)(I-P)v_0$$
$$+ \int_0^t B^{D,\sigma}\big(T(t-s) - T(s-t)\big)(I-P)f(s)\,\mathrm{d}s$$

for each $t \in [0, r^{-1}]$ provided that $u_0 \in \mathcal{D}(B^2)$ satisfies

$$\sum_{j=1}^{\infty} B^{2(j-1)} P F^{(2j)}(0) = P u_0 \tag{3.3}$$

and $v_0 \in \mathcal{D}(B)$ satisfies

$$\sum_{j=1}^{\infty} B^{2(j-1)} P F^{(2j-1)}(0) = P v_0, \tag{3.4}$$

where P is the spectral projection of B corresponding to σ and $F^{(j)}$ is the jth primitive of f.

P r o o f. Since σ is a spectral set, let $B = B_1 \oplus B_2$ be the decomposition corresponding to the topological direct sum $X = X_1 \oplus X_2$ where $X_1 = \mathcal{R}(P)$ and $X_2 = \mathcal{N}(P)$. The operator B_1 is bounded with $r(B_1) < r$ and B_2 is a closed invertible operator with $\mathcal{D}(B_2) = X_2 \cap \mathcal{D}(B)$. We observe that (3.1) with $T = r^{-1}$ has a unique solution if and only if the following two differential equations

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = B_1^2 x_1(t) + f_1(t), \quad t \in [0, r^{-1}], \tag{3.5}$$
$$x_1(0) = P u_0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 x_1(t) = P v_0$$

and

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = B_2^2 x_2(t) + f_2(t), \quad t \in [0, r^{-1}], \tag{3.6}$$
$$x_2(0) = (I-P) u_0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 x_2(t) = (I-P) v_0$$

have unique solutions on the subspaces X_1 and X_2 respectively. Since B_2 is an invertible generator of a C_0-group in X_2, the existence and uniqueness of the solution of (3.6) follow from [7, Theorem 3.1.5]. The existence and uniqueness of the solution of (3.5) can be seen from writing (3.5) as a Cauchy problem in the product space $X_1 \times X_1$ and from observing that B_1 is a bounded operator. It remains to express the solution explicitly in terms of B and its Drazin inverse.

On the subspace X_1, consider the infinite series $\sum_{j=1}^{\infty} B_1^{2(j-1)} F_1^{(2j)}(t)$ for each $t \in [0, r^{-1}]$. The series converges since

$$\sum_{j=1}^{\infty} \left\| B_1^{2(j-1)} F_1^{(2j)}(t) \right\| \leqslant \sum_{j=0}^{\infty} \left\| B_1^{j} F_1^{(j+2)}(t) \right\|$$

and

$$\begin{aligned} \limsup_{j\to\infty} \left\| B_1^{j} F_1^{(j+2)}(t) \right\|^{1/j} &\leqslant \limsup_{j\to\infty} \| B_1^{j} \|^{1/j} \limsup_{j\to\infty} \left\| F_1^{(j+2)}(t) \right\|^{1/j} \\ &< r r^{-1} = 1. \end{aligned}$$

By direct verification, we can see that the series satisfies (3.5) and hence it is the unique solution of (3.5).

Finally, on X_2, the closed operator B_2 is the infinitesimal generator of a strongly continuous group $T_2(t)$ which is equal to $T(t)(I-P)$. By [7, Theorem 3.1.5] and the relation $B^{D,\sigma} = 0 \oplus B_2^{-1}$, the unique solution of (3.6) is the last three terms of (3.2). The proof is complete. □

In the preceding theorem, we were able to express explicitly only the local solution of (3.1). The domain of the solution depends on r, the radius of the disc that contains the spectral set σ. However, if 0 is an isolated spectral point of B then the global solution of (3.1) can be expressed explicitly in terms of B and its Drazin inverse.

Theorem 3.2. *Let B be the infinitesimal generator of a strongly continuous group of bounded linear operators $T(t)$. If f is continuously differentiable on $[0, T]$ for any $T > 0$ and 0 is an isolated spectral point of B then the unique solution of Equation 3.1 with initial conditions $x(0) = u_0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 x(t) = v_0$ can be expressed explicitly as*

$$\begin{aligned} x(t) = {} & \sum_{j=1}^{\infty} B^{2(j-1)} P F^{(2j)}(t) \\ & + \frac{1}{2}\big(T(t) + T(-t)\big)(I-P)u_0 + \frac{1}{2} B^D \big(T(t) - T(-t)\big)(I-P)v_0 \\ & + \int_0^t B^D \big(T(t-s) - T(s-t)\big)(I-P) f(s)\,\mathrm{d}s, \end{aligned} \tag{3.7}$$

provided that $u_0 \in \mathcal{D}(B^2)$ *satisfies*

$$\sum_{j=1}^{\infty} B^{2(j-1)} P F^{(2j)}(0) = P u_0 \tag{3.8}$$

and $v_0 \in \mathcal{D}(B)$ *satisfies*

$$\sum_{j=1}^{\infty} B^{2(j-1)} P F^{(2j-1)}(0) = P v_0, \tag{3.9}$$

where P *is the spectral projection corresponding to* 0 *and* $F^{(j)}$ *is the* j*th primitive of* f.

When B is a nilpotent operator, the infinite series in the solution is truncated at the nilpotency index.

Theorem 3.3. *Let* B *be the infinitesimal generator of a strongly continuous group of bounded linear operators* $T(t)$. *If* f *is continuously differentiable on* $[0,T]$ *and* 0 *is a pole of* B *of order* $2k$ *for* $k \in \mathbb{N}$, *then the solution to Equation 3.1 with initial conditions* $x(0) = u_0$ *and* $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 x(t) = v_0$ *is*

$$\begin{aligned} x(t) = {} & \sum_{j=1}^{k} B^{2(j-1)} P F^{(2j)}(t) \\ & + \frac{1}{2}\big(T(t) + T(-t)\big)(I-P)u_0 + \frac{1}{2} B^D \big(T(t) - T(-t)\big)(I-P)v_0 \\ & + \int_0^t B^D \big(T(t-s) - T(s-t)\big)(I-P) f(s) \, \mathrm{d}s, \end{aligned} \tag{3.10}$$

provided that $u_0 \in \mathcal{D}(B^2)$ *satisfies*

$$\sum_{j=1}^{k} B^{2(j-1)} P F^{(2j)}(0) = P u_0 \tag{3.11}$$

and $v_0 \in \mathcal{D}(B)$ *satisfies*

$$\sum_{j=1}^{k} B^{2(j-1)} P F^{(2j-1)}(0) = P v_0, \tag{3.12}$$

where P *is the spectral projection corresponding to* 0 *and* $F^{(j)}$ *is the* j*th primitive of* f. *If* f *is continuous differentiable and* 0 *is a pole of* B *of order* $2k+1$ *then the term* $B^{2k} P F^{(2k+2)}(t)$ *is added to the solution.*

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