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POST-BUCKLING RANGE OF PLATES IN AXIAL COMPRESSION WITH UNCERTAIN INITIAL GEOMETRIC IMPERFECTIONS*

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Abstract. The method of reliable solutions alias the worst scenario method is applied to the problem of von Kármán equations with uncertain initial deflection. Assuming two-mode initial and total deflections and using Galerkin approximations, the analysis leads to a system of two nonlinear algebraic equations with one or two uncertain parameters-amplitudes of initial deflections. Numerical examples involve (i) minimization of lower buckling loads and (ii) maximization of the maximal mean reduced stress.

Keywords: elastic plates, Kármán equations, uncertain initial deflections, worst scenario

MSC 2000: 74G60, 74K20, 35J65, 65N30

Introduction

The post-buckling behaviour of thin elastic plates is an important topic in structural mechanics since plates are used extensively as load-carrying components up to and into the post-buckling range. Tests on plates in axial compression have shown that the waveforms of deflections adopted at the onset of buckling may undergo abrupt changes further into the post-buckling regimes [6]. This phenomenon has been justified by a mathematical model using the non-linear system of von Kármán equations involving initial geometric imperfections (see e.g. [7], [8], [2], [4]). It is sufficient to find approximate solutions of the latter system assuming two degrees of freedom for both the initial and total deflection forms and using the Ritz-Galerkin method (see [4], [6]).

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In practice, however, the initial geometric imperfections are uncertain and it is difficult to estimate the probability distribution of their amplitudes. Then the stochastic approach is not applicable and another method has to be adopted. One simple possibility is to apply the so called *method of reliable solutions* alias the *worst scenario* method. This approach was proposed by Ben-Haim and Elishakoff (see e.g. [1] or [3]) and independently by the author (see e.g. [5]), who established the method upon a more general mathematical background. The main features of the method are as follows: having a state problem (e.g. Kármán equations) and uncertain input data z_0 (e.g. initial imperfections), we assume that (i) the input data belong to a given set U_{ad} and (ii) the state problem has a solution $u(z_0)$ for any $z_0 \in U_{\mathrm{ad}}$. Then we choose a functional-criterion $\Phi(z_0, u(z_0))$ (e.g. a buckling load) and seek its extremal (minimal, i.e. most dangerous, worst) value over the set U_{ad}.

Starting in Section 1 from some old results of the author (see [4]), which were justified and extended by Supple in [6], we assume the initial imperfection in the form of one halfsinewave and define “two-mode solutions” of the Kármán system. We compute the equilibrium paths of relative loading vers. deflection amplitude on the basis of Galerkin approximations. Applying the worst scenario method in Section 2, we introduce two different functionals-criteria. We define the so called *lower buckling load* as the minimal relative loading which admits the possibility of an abrupt change of the waveform, i.e. a global loss of stability. We prove that the lower buckling load is a piecewise differentiable and increasing function of the initial deflection amplitude.

The second choice of the criterion is closely related to the so called mean reduced stress for a given loading parameter. By several numerical examples we have shown that this function of the initial deflection amplitude is decreasing.

In Section 3 we consider the initial imperfection as a combination of one and two halfwaves. We have justified a hypothesis that the above mentioned abrupt change of the waveform can be realized by a small geometric perturbation-imperfection of the antisymmetric form. The decisive part of maximal mean reduced stress was again computed, now as a function of the two amplitudes of the initial deflections. Several numerical examples in Section 4 show results of the worst scenario method, i.e. the maximization of the above-mentioned function over trapezoidal domains—the sets of admissible amplitudes of initial deflections. The search of maximum was accomplished by a simple direct method.

1. Buckling of a rectangular plate

Let us consider an elastic rectangular plate of a constant thickness h (Fig. 1). Let its middle plane occupy the rectangle $\Omega \equiv [-a/2, a/2] \times [-b/2, b/2]$. Assume that the material of the plate is homogeneous, isotropic and $a/b = 2$. Let the plate be simply supported and loaded by compressive uniformly distributed forces $\sigma \cdot h$ on the edges $x = \pm a/2$.

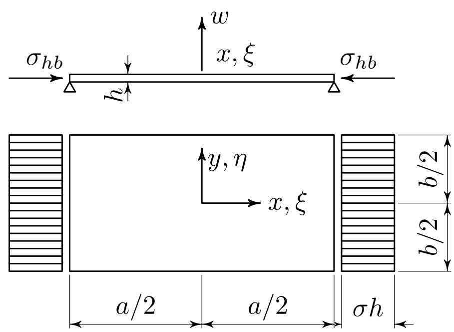

Figure 1. The coordinate system.

The mode of support is such that there are no out-of-plane deflections at the boundaries, the loaded edges remain straight and the longitudinal edges are not allowed to wave in the plane of the plate. The last condition applies to a single panel of a multi-panelled infinitely wide plate loaded in axial compression, the junctions of the panels being knife-edge supports.

It is further assumed that there is no restraint against lateral expansion of the plate in its plane.

If u, v, w denote the total displacements in the directions of axes x, y, z (cf. Fig. 1), these boundary conditions may be written as (cf. [6])

$$
\begin{aligned}
w &= 0 && \text{on } \partial\Omega, \\
w_{xx} + \nu w_{yy} &= 0 && \text{for } x = \pm a/2, \\
w_{yy} + \nu w_{xx} &= 0 && \text{for } y = \pm b/2, \\
u &= \text{const.} && \text{for } x = \pm a/2, \\
v &= \text{const.} && \text{for } y = \pm b/2.
\end{aligned} \tag{1.1}
$$

Here and in what follows the subscripts x and y denote partial derivatives with respect to x and y, respectively, ν is the Poisson constant.

The von-Kármán large deflection equations in the presence of *initial geometric imperfections* w^0 may be written in terms of w, w^0 and Airy stress function F as

$$\begin{aligned}(1.2)\qquad \frac{1}{E}\Delta^2 F &= -\frac{1}{2}[w,w] + \frac{1}{2}[w^0,w^0],\\ \frac{D}{h}\Delta^2(w-w^0) &= [F,w],\end{aligned}$$

where

$$[u,v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}$$

is the Poisson bracket, E is the Young modulus and

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

denotes the bending rigidity of the plate. Recall that the stress tensor components are determined by formulae

$$\tau_{(x)} = F_{yy}, \quad \tau_{(y)} = F_{xx}, \quad \tau_{(xy)} = -F_{xy}.$$

Then we have also the boundary conditions

$$\begin{aligned}(1.3)\qquad \int_{-b/2}^{b/2} F_{yy}dy &= -\sigma b \qquad \text{for } x = \pm a/2 \ (\sigma = \text{const})\\ \int_{-a/2}^{a/2} F_{xx}dx &= 0 \qquad \text{for } y = \pm b/2,\\ F_{xy} &= 0 \qquad \text{on } \partial\Omega.\end{aligned}$$

An approximate solution of the system (1.2) with boundary conditions (1.1) and (1.3) is now obtained using the Ritz-Galerkin technique. We assume the following forms for w and w^0:

$$(1.4)\qquad w = (f_1 \cos\xi + f_2 \sin 2\xi)\cos\eta,$$

$$(1.5)\qquad w^0 = f_0 \cos\xi \cos\eta,$$

satisfying boundary conditions (1.1). Here f_1, f_2, f_0 are arbitrary constants and

$$\xi = \pi x/a, \qquad \eta = \pi y/b.$$

We introduce

$$k = \sigma/\sigma_0,$$

where σ_0 is the critical loading of an ideal plate, namely (see e.g. [7])

$$\sigma_0 = \frac{\pi^2 E h^2}{3(1-\nu^2)b^2}.$$

Let us denote $\zeta = f_1/h$, $\zeta_2 = f_2/h$, $z_0 = f_0/h$.

The expressions (1.4), (1.5) are substituted into the first of equations (1.2) and the resulting equation solved for F to give

$$\begin{aligned}(1.6)\qquad F = \frac{Eh^2}{32}\Big\{&(z_0^2-\zeta^2)\Big(4\cos 2\xi + \frac{1}{4}\cos 2\eta\Big) - \zeta\zeta_2\Big(\frac{32}{9}\sin 3\xi + 32\sin\xi\Big)\\ &- 32\zeta\zeta_2\cos 2\eta\Big(\frac{1}{625}\sin 3\xi + \frac{9}{289}\sin\xi\Big)\\ &+ \zeta_2^2(\cos 4\xi + \cos 2\eta)\Big\} - \frac{\sigma}{2}y^2.\end{aligned}$$

It is not difficult to verify that this expression satisfies all boundary conditions (1.3) and the corresponding u, v, calculated via the strain-stress relations, satisfy (1.1).

Next, the expressions for w and F are substituted into the second equation of (1.2) to obtain a residual R. The *Galerkin method* implies that

$$\int_\Omega R\cos\xi\cos\eta\,\mathrm{d}x\,\mathrm{d}y = 0, \qquad \int_\Omega R\sin 2\xi\cos\eta\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Performing the substitution and integrations these equations appear as (cf. [6], eq. (9))

$$(1.7)\qquad \frac{E\pi^2h^2}{8b^2}\Big[\frac{17}{8}(\zeta^2 - z_0^2)\zeta + 2\kappa\zeta\zeta_2^2 + \frac{25}{6(1-\nu^2)}(\zeta - z_0)\Big] - \sigma\zeta = 0,$$

$$(1.8)\qquad \zeta_2\Big(2\kappa\zeta^2 + 4\zeta_2^2 - z_0^2/2 + \frac{32}{3(1-\nu^2)} - \frac{\sigma}{E}\frac{32b^2}{\pi^2h^2}\Big) = 0,$$

where

$$\kappa = 4 + 1/4 + 1/625 + 81/289 \doteq 4.53188.$$

As the equation (1.8) yields that either

$$\zeta_2 = 0$$

or

$$(1.9)\qquad 2\kappa\zeta^2 + 4\zeta_2^2 - z_0^2/2 + \frac{32}{3(1-\nu^2)} - \frac{\sigma}{E}\frac{32b^2}{\pi^2h^2} = 0,$$

the solution of the system (1.7)–(1.8) branches in two parts:
1*st branch*:

$$\zeta_2 = 0,$$
$$\frac{17}{8}(\zeta^2 - z_0^2)\zeta + \frac{25}{6(1-\nu^2)}(\zeta - z_0) - \frac{\sigma}{E}\frac{8b^2}{\pi^2 h^2}\zeta = 0,$$

so that

$$k = k(\zeta, z_0) = \frac{25}{16}\frac{\zeta - z_0}{\zeta} + \frac{51}{64}(1-\nu^2)(\zeta^2 - z_0^2); \tag{1.10}$$

2*nd branch*: from (1.9) we obtain

$$\zeta_2^2 = (k-1)\frac{8}{3(1-\nu^2)} - \zeta^2\kappa/2 + z_0^2/8. \tag{1.11}$$

Substituting this into (1.7), we arrive at

$$k = k(\zeta, z_0) = \frac{3(1-\nu^2)}{8(2\kappa - 1)}, \tag{1.12}$$
$$\Big[(\kappa^2 - 17/8)\zeta^2 + (17/8 - \kappa/4)z_0^2 + \frac{25}{6(1-\nu^2)}\Big(\frac{z_0}{\zeta} - 1\Big)\Big] + \frac{2\kappa}{2\kappa - 1}.$$

Let us compute the post-buckling equilibrium paths k vers. ζ and ζ_2 vers. ζ by (1.10), (1.12) and (1.11), respectively. Due to the symmetrical configuration of the problem we can restrict ourselves to positive values of the amplitudes z_0, ζ and ζ_2 only. Let $k_1(\zeta, z_0)$, $k_2(\zeta, z_0)$ be the graphs of the 1st and 2nd branch, respectively (see Fig. 2, where the solid lines represent the 1st branch and the dashed lines the 2nd branch, for $z_0 = 0.2, 0.4, 0.6, 0.8, 1.0$). Here and in the following $\nu = 0.3$ is substituted.

Note that the formula (1.12) can define a real graph $k_2(\zeta)$ if and only if $\zeta_2^2(\zeta)$ is non-negative. Substituting for k from (1.12) in (1.11), we obtain

$$\zeta_2^2 = \big(k(\zeta, z_0) - 1\big) \cdot 8/\big(3(1-\nu^2)\big) - \zeta^2\kappa/2 + z_0^2/8 \equiv g(\zeta, z_0). \tag{1.13}$$

The cubic equation

$$\zeta g(\zeta, z_0) = 0$$

has three real roots for $z_0 \leqslant \overline{z}_0 \doteq 0.405$. In fact, it reads

$$\mathcal{A}\zeta^3 + (z_0^2/8 - \mathcal{C})\zeta + \mathcal{B}z_0 = 0, \tag{1.14}$$

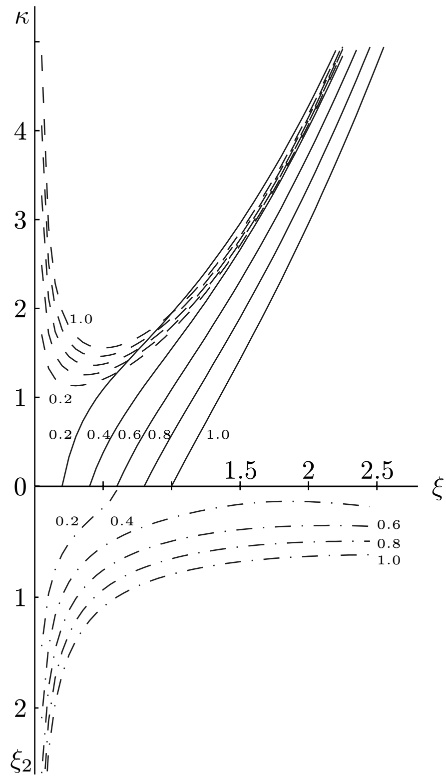

Figure 2. Equilibrium paths for initial deflection in one halfwave.

where

$$\mathcal{A} = \frac{\kappa - 17/4}{4\kappa - 2} \doteq 0.0174782,$$

$$\mathcal{B} = \frac{25}{6(1-\nu^2)(2\kappa - 1)} \doteq 0.567819,$$

$$\mathcal{C} = \frac{3}{2(1-\nu^2)(2\kappa - 1)} \doteq 0.204415.$$

The discriminant of the equation (1.14) is positive for $z_0 < \overline{z}_0$ and negative for $z_0 > \overline{z}_0$.

Let $z_0 \leqslant \overline{z}_0$ and let $z_2(z_0)$ and $z_3(z_0)$ be the middle and the maximal root, respectively, of the equation (1.14). Then ζ_2^2 is non-negative iff ζ belongs to the union of two intervals

$$(0, z_2(z_0)] \cup [z_3(z_0), +\infty).$$

Note that only positive z_0 and ζ are considered and $z_2(z_0) > 0$ for $z_0^2 < 8\mathcal{C}$. If $z_0 > \overline{z}_0$ then ζ_2^2 is positive for all $\zeta > 0$.

For the time being, assume that the function $k_2(\zeta, z_0)$ is defined for all positive ζ. Then the minimum of the extended function $k_2(\zeta, z_0)$ is attained at the point ζ_d, where

$$\partial k_2(\zeta_d, z_0)/\partial\zeta = 0.$$

We obtain

$$\zeta_d = \Big(\frac{25}{12(1-\nu^2)(\kappa^2 - 17/8)} z_0\Big)^{1/3} = C_0 z_0^{1/3}, \tag{1.15}$$

where

$$C_0 \doteq 0.499112.$$

The point

$$(\zeta_d, k_2(\zeta_d, z_0)) \quad \text{is real iff} \quad \zeta_2^2 \equiv g(\zeta_d, z_0) \geqslant 0.$$

Thus we arrive at the condition

$$\mathcal{A}\zeta_d^3 + (z_0^2/8 - \mathcal{C})\zeta_d + \mathcal{B}z_0 \geqslant 0 \tag{1.16}$$

and substituting from (1.15), we have

$$z_0^{1/3}(0.56999 z_0^{2/3} + 0.062389 z_0^2 - 0.102026) \geqslant 0.$$

Denoting $\chi := z_0^{2/3}$, we obtain

$$0.062389\chi^3 + 0.56999\chi \geqslant 0.102026,$$

which is satisfied iff

$$\chi \geqslant 0.178374.$$

Then (1.16) implies

$$z_0 \geqslant 0.178374^{3/2} \doteq 0.0753351 \equiv \widehat{z}_0.$$

As a consequence, the point of minimum is real iff $z_0 \geqslant \widehat{z}_0$.

Definition 1.1. We call the couple $\{w, F\}$ a "two-mode solution" of the system (1.2), if w is the Galerkin approximation of the form (1.4) and F is the corresponding stress function (1.6).

Definition 1.2. The minimal relative loading k for which at least two two-mode solutions of the system (1.2) exist, will be called the *lower buckling load* and denoted $k_d(z_0)$.

Let us investigate the function $k_d(z_0)$ in what follows.

I. Let us consider the interval

$$z_0 \in [0, \widehat{z}_0).$$

Then the minimum of the function $k_2(\cdot, z_0)$ is attained for $\zeta = z_2(z_0)$.

In fact, differentiating the formula (1.12), we obtain

$$\partial k_2(\zeta, z_0)/\partial\zeta = \frac{3(1-\nu^2)}{8(2\kappa-1)\zeta^2}\Big[2(\kappa^2 - 17/8)\zeta^3 - \frac{25}{6(1-\nu^2)}z_0\Big]. \tag{1.17}$$

If $\zeta < \zeta_d(z_0)$, then

$$\zeta^3 < \zeta_d^3 = \frac{25z_0}{12(1-\nu^2)(\kappa^2 - 17/8)}$$

follows from (1.15), so that the expression (1.17) is negative. As a consequence, the extended function $k_2(\cdot, z_0)$ is decreasing for $\zeta \in (0, \zeta_d(z_0)]$. Since $z_2(z_0) < \zeta_d(z_0)$ for $z_0 < \widehat{z}_0$ (otherwise the point $(\zeta_d, k_2(\zeta_d, z_0))$ would be real, which contradicts $z_0 < \hat{z}_0$) and the function $k_2(\cdot, z_0)$ has a real meaning only for $\zeta \leqslant z_2(z_0)$, the minimum is attained at $\zeta = z_2(z_0)$.

R e m a r k 1.1. The branch of $k_2(\cdot, z_0)$ for $\zeta \geqslant z_3(z_0)$ will be neglected, because it corresponds to an unstable equilibrium (see [4], p. 182–183) and for the same loading k the potential energy of this unstable branch is higher than that of the stable branch (i.e., for $\zeta \leqslant z_2(z_0)$). $\square$

Using (1.12) we may write

$$k_d(z_0) = k_2(z_2(z_0), z_0).$$

Since

$$g(z_2(z_0), z_0) = 0,$$

the definition (1.13) yields

$$k_d(z_0) = k_2\big(z_2(z_0), z_0\big) = 1 + \tfrac{3}{8}(1-\nu^2)\big(-z_0^2/8 + \tfrac{1}{2}\kappa z_2^2(z_0)\big). \tag{1.18}$$

Since obviously

$$\lim_{z_0 \to 0+} z_2(z_0) = 0,$$

the formula (1.18) implies

$$\lim_{z_0 \to 0+} k_d(z_0) = 1.$$

Moreover, we have

$$\mathrm{d}k_d(z_0)/\,\mathrm{d}z_0 = \tfrac{3}{8}(1-\nu^2)(-z_0/4 + \kappa z_2(z_0)\,\mathrm{d}z_2(z_0)/\,\mathrm{d}z_0).$$

For $z_0 < \widehat{z_0}$ we may neglect the cubic term $\mathcal{A}\zeta^3$ in the equation (1.14) to get the estimate

$$z_2(z_0) > \frac{\mathcal{B}z_0}{\mathcal{C} - z_0^2/8} > \frac{\mathcal{B}}{\mathcal{C}}z_0.$$

Using the implicit function theorem for the equation (1.14), we obtain

$$\mathrm{d}z_2(z_0)/\,\mathrm{d}z_0 = (\mathcal{B} + \tfrac{1}{4}z_0 z_2(z_0))/(\mathcal{C} - \tfrac{1}{8}z_0^2 - 3\mathcal{A}z_2^2(z_0)) > \mathcal{B}/\mathcal{C}.$$

Then we have

$$\mathrm{d}k_d(z_0)/\,\mathrm{d}z_0 \geqslant \tfrac{3}{8}(1-\nu^2)z_0[-\tfrac{1}{4} + \kappa\mathcal{B}^2/\mathcal{C}^2] > 0, \tag{1.19}$$

since $\kappa\mathcal{B}^2/\mathcal{C}^2 - \frac{1}{4} \doteq 34.718 > 0$.

II. The interval $z_0 \geqslant \widehat{z}_0$.

Using (1.12) and (1.15), we obtain

$$\begin{aligned} k_d(z_0) &= k_2(\zeta_d(z_0), z_0) \\ &= \frac{3(1-\nu^2)}{8(2\kappa-1)}\Big[(\kappa^2 - 17/8)\zeta_d^2 + (17/8 - \kappa/4)z_0^2 + \frac{25}{6(1-\nu^2)} \\ &\quad \times(-1 + z_0/\zeta_d) + \frac{16\kappa}{3(1-\nu^2)}\Big] \\ &= \frac{1}{8(2\kappa-1)}\Big[3(1-\nu^2)(17/8-\kappa/4)z_0^2 + \frac{75}{4C_0}z_0^{2/3} + 16\kappa - 25/2\Big]. \end{aligned} \tag{1.20}$$

Moreover, we have

$$\mathrm{d}k_d(z_0)/\,\mathrm{d}z_0 = \frac{1}{8(2\kappa-1)}\Big[6(1-\nu^2)(17/8-\kappa/4)z_0 + \frac{25}{2C_0}z_0^{-1/3}\Big] > 0. \tag{1.21}$$

Theorem 1.1. *The function $z_0 \mapsto k_d(z_0)$ is increasing for all positive z_0.*

P r o o f. Observe that

$$\lim_{z_0 \to \hat{z}_0-} z_2(z_0) = \zeta_d(\hat{z}_0) \tag{1.22}$$

follows by comparing the equations (1.14) and (1.16). Using (1.15) and (1.22), we can express the difference

$$S := k_d(\hat{z}_0) - \lim_{z_0 \to \hat{z}_0-} k_d(z_0)$$

in terms of $\hat{z}_0$. Moreover, we can employ the equality in (1.16) and multiply it by $\frac{3}{8}(1-\nu^2)/\zeta_d(\hat{z}_0)$. In this way, we arrive at

$$S = \hat{z}_0^2\,\tfrac{3}{8}(1-\nu^2)(17/8-\kappa/4)/(2\kappa-1) \doteq 0.000238. \tag{1.23}$$

Numerically, we obtain

$$\lim_{z_0\to\hat{z}_0-} k_d(z_0) \doteq 1.03412,$$

whereas

$$k_d(\hat{z}_0) \doteq 1.03435$$

follows from (1.20).

The formulae (1.19), (1.21) and (1.23) imply that the function k_d is increasing in $(0,+\infty)$. $\square$

2. Worst-scenario method

Assume that the amplitude of the initial deflection (1.5) is *uncertain* and its probabilistic distribution function is not available. Let the only information we have be that the relative amplitude z_0 belongs to an interval

$$U_{\mathrm{ad}} = [z_{0\,\min}, z_{0\,\max}],$$

where

$$0 < z_{0\,\min} < z_{0\,\max}$$

are given bounds.

Then we may employ the "method of reliable solution", alias "worst scenario" (cf. [1], [5]), as follows. We choose a function-criterion $\Phi(z_0)$ and look for its "most dangerous" extremal value

$$\min_{z_0\in U_{\mathrm{ad}}} \Phi(z_0) \quad \text{or} \quad \max_{z_0\in U_{\mathrm{ad}}} \Phi(z_0),$$

in accordance with the physical meaning of the criterion Φ.

Example 2.1. Let us consider

$$\Phi_1(z_0) = k_d(z_0),$$

i.e., let the lower buckling load be the decisive criterion, and let us look for the minimum of Φ_1 over U_{ad}. Using Theorem 1.1, we immediately find

$$\min_{z_0 \in U_{\mathrm{ad}}} \Phi_1(z_0) = \Phi_1(z_{0\,\min}).$$

E x a m p l e 2.2. A less pessimistic choice is

$$(2.1) \qquad \Phi_2(z_0) = \max_{\{w,F\} \in \mathcal{K}(z_0)} \left(\max_{x \in [-a/2, a/2]} \int_{-b/2}^{b/2} \left| w_{xx} - w^0_{xx} + \nu(w_{yy} - w^0_{yy}) \right| \mathrm{d}y \right),$$

where $w = w(z_0)$ is the first component of a stable two-mode solution and $\mathcal{K}(z_0)$ is the set of all such solutions, corresponding to the parameter z_0 and to an a priori given fixed relative loading k.

The functional Φ_2 is linked with the so called "*mean reduced stress*" (see e.g. [2], chapt. III, § 15)

$$\sigma_s(x) = \frac{1}{b} \int_{-b/2}^{b/2} \left| \tau_{(x)} + \sigma^*_x / \sqrt{2} \right| \mathrm{d}y,$$

where

$$\sigma^*_x = \frac{Eh}{2(1-\nu^2)} (w_{xx} + \nu w_{yy})$$

is the stress caused by the pure bending at the surface of the plate. Indeed, we have the relation

$$(2.2) \qquad \max_{\{w,F\} \in \mathcal{K}(z_0)} \{ \max_{x \in [-\frac{a}{2}, \frac{a}{2}]} \sigma_s(x) \} = \sigma_0 \Big(k + \frac{3b}{2\sqrt{2}\pi^2 h} \Phi_2(z_0) \Big).$$

After some calculation, we derive

$$(2.3) \quad \Phi_2(z_0) = (2\pi h/b) \max_{\{w,F\} \in \mathcal{K}(z_0)} \Big[\max_{|\xi| \leqslant \pi/2} |(\tfrac{1}{4} + \nu)(\zeta - z_0) \cos \xi + (1+\nu)\zeta_2 \sin 2\xi| \Big].$$

If $k \geqslant k_d(z_0)$, which is the more interesting case, the set $\mathcal{K}(z_0)$ consists of two different two-mode solutions (see Fig. 2). The second maximum (over ξ) in (2.3) is attained at the point $\xi_m \in [0, \pi/2)$, which is determined by the equation

$$(2.4) \qquad \sin \xi_m = \big(-(1+4\nu)(\zeta - z_0) \\ + \big((1+4\nu)^2 (\zeta - z_0)^2 + 512(1+\nu)^2 \zeta_2^2 \big)^{1/2} \big) / \big(32(1+\nu)\zeta_2 \big).$$

For the first branch we have $\zeta_2 = 0$ and then $\xi_m = 0$ follows from (2.3), so that

$$\Phi_2(z_0) = (2\pi h/b)(\tfrac{1}{4} + \nu)\big(\zeta(z_0) - z_0\big)$$

holds for $k < k_d(z_0)$. $\square$

Computation of $\Phi_2(z_0)$—Algorithm I.

1° Compute $k_d(z_0)$.

2° Choose $k \geqslant k_d(z_0)$.

3° (First branch): compute the (unique) real root ζ' of the cubic equation (cf. (1.10))

$$x^3 + \Big(\frac{100 - 64k}{51(1-\nu^2)} - z_0^2\Big)x - \frac{100 z_0}{51(1-\nu^2)} = 0,$$

and set

$$f' = (\tfrac{1}{4} + \nu)(\zeta' - z_0).$$

4° (Second branch): compute the middle real root ζ'' of the cubic equation (cf. (1.12) and Fig. 2)

$$100.5346x^3 + [120.02 + 5.4165 z_0^2 - 129.02k]x + 25 z_0 = 0,$$

then set

$$\begin{aligned}
\zeta_1'' &= \zeta'' - z_0,\\
\zeta_2 &= \Big[(k-1)\frac{8}{3(1-\nu^2)} - \tfrac{1}{2}\kappa(\zeta'')^2 + z_0^2/8\Big]^{1/2},\\
\sin\xi_m &= \Big[-(1+4\nu)\zeta_1'' + \big((1+4\nu)^2(\zeta_1'')^2 + 512(1+\nu)^2\zeta_2^2\big)^{1/2}\Big]/(32(1+\nu)\zeta_2),\\
f'' &= \Big[\big(\nu + \tfrac{1}{4}\big)\zeta_1'' + 2(1+\nu)\zeta_2\sin\xi_m\Big](1-\sin^2\xi_m)^{1/2}.
\end{aligned} \tag{2.5}$$

5° $\Phi^*(z_0) = \frac{b}{2\pi h}\Phi_2(z_0) = \max\{f', f''\}$.

3. Numerical experiments

The computations justify the assertion of Theorem 1.1 that the function $z_0 \to k_d(z_0)$ is increasing—see Fig. 3 and some extracted values in Tab. 1.

z_0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$k_d(z_0)$	1	1.056	1.131	1.195	1.253	1.308	1.360	1.410	1.456	1.507

Table 1.

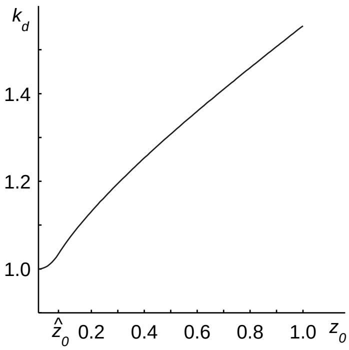

Figure 3. Dependence of lower buckling load on initial deflection amplitude.

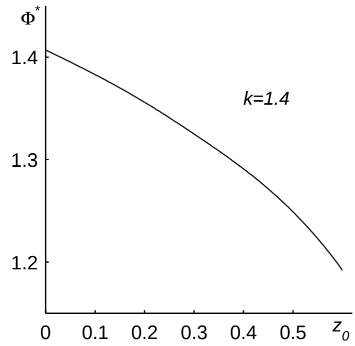

Figure. 4. Dependence of maximal mean reduced stress on initial deflection amplitude for $k = 1.4$.

3.1. Initial imperfection in one halfwave.

Algorithm I mentioned above has been accomplished for three different values of the relative loading, namely for $k = 1.4$, 1.25 and 1.1. We have considered

$$
\begin{aligned}
z_0 &\in [0, 0.6] \quad \text{if} \quad k = 1.4,\\
z_0 &\in [0, 0.4] \quad \text{if} \quad k = 1.25,\\
z_0 &\in [0, 0.14] \ \text{if} \quad k = 1.1,
\end{aligned}
$$

since $k \geqslant k_d(z_0)$ had to be fulfilled. The numerical results were calculated with the step 0.02. They show that the function $z_0 \to \Phi_2(z_0)$ is *decreasing* and concave—see Figs. 4–6. Although the function is differentiable, its derivative is not suitable for practical use due to its complexity.

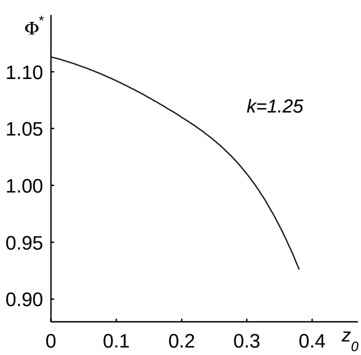

Figure 5. Dependence of maximal mean reduced stress on initial deflection amplitude for $k = 1.25$.

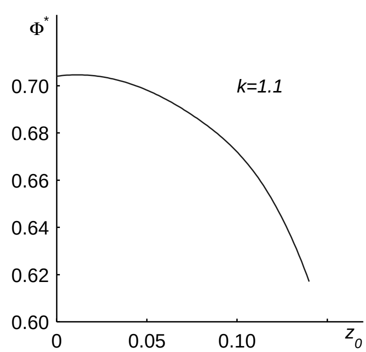

Figure 6. Dependence of maximal mean reduced stress on initial deflection amplitude for $k = 1.1$.

The method of "worst scenario" seeks the maximum of $\Phi_2(z_0)$ over the given interval U_{ad} and therefore

$$\max_{z_0 \in U_{\mathrm{ad}}} \Phi_2(z_0) = \Phi_2(z_{0\,\mathrm{min}}).$$

3.2. Initial imperfection form combined from one and two halfwaves. In the above study of the postcritical behaviour of rectangular plates under compression a possibility of a phenomenon called "snap-through" has been discovered. In fact, for $k \geqslant k_d(z_0)$ the deflection in the initial form of one halfwave (1st branch) can change abruptly into the form of a combination of one and two halfwaves (2nd branch). During this global loss of stability the plate loses part of its potential energy (see [4], [8]). On the other hand, the plate must overcome an energetic barrier by means of an amount of "perturbation energy". Assume that the influence of the latter can be realized by a small geometric perturbation in the form of initial imperfection of two halfwaves. Thus we are led to the assumption

$$w^0 = (f_{01} \cos \xi + f_{02} \sin 2\xi) \cos \eta, \tag{3.1}$$

where f_{02} denotes the amplitude of the geometric perturbation.

Assume that the total deflection has again the form (1.4) and use the same Galerkin approach as in Section 1. Let us denote

$$\begin{aligned} z_0 &= f_{01}/h, \quad t_0 = f_{02}/h, \\ z &= f_1/h, \qquad t = f_2/h. \end{aligned}$$

Due to the symmetry it is sufficient to consider positive amplitudes f_{01}, f_{02} only. Then we obtain the system of two equations

$$[\tfrac{17}{4}(z^2 - z_0^2) + t^2 - t_0^2]z + (4\kappa - 1)(zt - z_0 t_0)t + \frac{25}{3(1-\nu^2)}(z - z_0) = 16b^2\sigma z/(\pi^2 h^2 E), \tag{3.2}$$

$$[8(t^2 - t_0^2) + z^2 - z_0^2]t + (4\kappa - 1)(zt - z_0 t_0)z + \frac{64}{3(1-\nu^2)}(t - t_0) = 64b^2\sigma t/(\pi^2 h^2 E). \tag{3.3}$$

By elimination of the loading parameter σ we arrive at the following cubic equation in terms of the variable z:

$$a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0, \tag{3.4}$$

where

$$
\begin{aligned}
a_3 &= -(\kappa - 17/4)/16t \doteq -0.0176175t, \\
a_2 &= (\kappa - 1/4)/16z_0t_0 \doteq 0.2676175z_0t_0, \\
a_1 &= (\kappa - 1/2)/4t^3 - (z_0^2/4 - t_0^2/16 - 3/(16(1-\nu^2)))t + t_0/(3(1-\nu^2)) \\
&\doteq 1.00797t^3 - (z_0^2/4 - t_0^2/16 - 0.206044)t + t_0/2.73, \\
a_0 &= -25z_0t/(48(1-\nu^2)) - (\kappa - 1/4)/4z_0t_0t^2 \\
&\doteq -0.572344z_0t - 1.07047z_0t_0t^2.
\end{aligned}
$$

Let us introduce a new variable y by the formula

$$z = y - a_2/(3a_3) \doteq y + 5.06348z_0t_0/t. \tag{3.5}$$

Thus we obtain a reduced cubic equation

$$y^3 + p(t)y + q(t) = 0, \tag{3.6}$$

where

$$
\begin{aligned}
p(t) \doteq\ & -57.2141t^2 + 14.19043z_0^2 - 3.54761t_0^2 - 11.6954 \\
& - 20.7918t_0/t - 76.9164(z_0t_0/t)^2, \\
q(t) \doteq\ & 32.4872z_0 - 228.94z_0t_0t + [(71.8526z_0^2 - 17.9633t_0^2 - 59.2196)/t \\
& - 105.279t_0/t^2]z_0t_0 - 259.643(z_0t_0/t)^3.
\end{aligned}
$$

Choosing t we substitute it in the equation (3.6), solve for y (taking the middle root) and calculate $z(t)$ according to (3.5). Then the relative loading $k = k(z,t) = \sigma/\sigma_0$ is obtained from (3.3). The numerical results are plotted for $z_0 = 0.4$ and $t_0 = 0.1$ in Fig. 7 together with the previous results (from Fig. 2) for $z_0 = 0.4$ and $t_0 = 0$. Obviously, the above-mentioned "snap-through" phenomenon can be realized by means of a small antisymmetric geometric perturbation of two halfwaves, which enables the plate to overcome the energetic barrier.

Next, let us consider again the functional (2.1), i.e., a decisive part of the maximal "mean reduced stress"—see (2.2). Instead of the formula (2.3), however, we derive a simpler formula

$$\Phi_2(z_0,t_0) = (2\pi h/b) \max_{|\xi|\leqslant\pi/2} |(1/4+\nu)(z-z_0)\cos\xi + (1+\nu)(t-t_0)\sin 2\xi|, \tag{3.7}$$

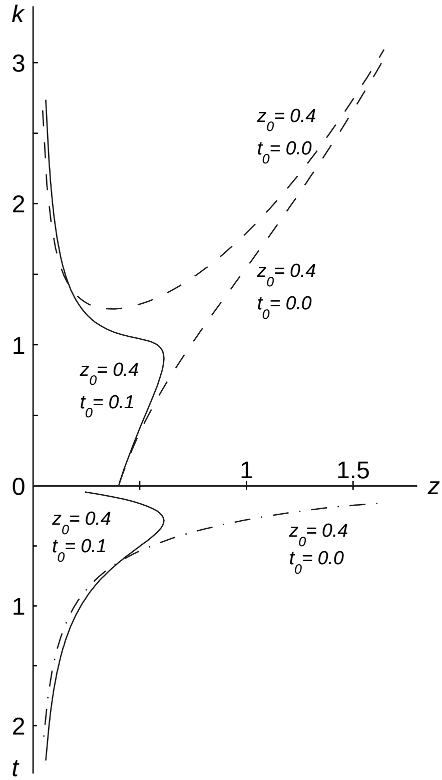

Figure 7. Equilibrium paths for combined initial deflections.

since the set $\mathcal{K}(z_0, t_0)$ of stable two-mode solutions, corresponding to given parameters k, z_0, t_0, consists of a unique pair $\{w, F\}$. The maximum in the expression (3.7) is attained at a point $\xi_m \in [0, \pi/2]$, which is determined by the equation

$$\begin{aligned}\sin \xi_m = \big[&-(1+4\nu)(z-z_0) + \big((1+4\nu)^2(z-z_0)^2 \\ &+ 512(1+\nu)^2(t-t_0)^2\big)^{1/2}\big] / \big(32(1+\nu)(t-t_0)\big). \end{aligned} \tag{3.8}$$

Finally, we obtain

$$\begin{aligned}\Phi^*(z_0, t_0) &\equiv b(2\pi h)^{-1}\Phi_2(z_0, t_0) \\ &= [(\tfrac{1}{4}+\nu)(z-z_0) + 2(1+\nu)(t-t_0)\sin \xi_m](1-\sin^2 \xi_m)^{1/2}. \end{aligned} \tag{3.9}$$

Computation of $\Phi^*(z_0, t_0)$.

Choosing $k > 1$, we find t from the "implicit" equation (3.3), where the function $z(t)$ is substituted for z, i.e., the middle root of the cubic equation (3.4). In this way we obtain a transcendental equation

$$(3.10)\quad 1-k-t_0/t+3(1-\nu^2)\big(\kappa z^2(t)/2+t^2-t_0^2-z_0^2/8-(\kappa-1/4)z_0t_0z/(2t)\big)/8=0,$$

which has to be solved for t by an iterative method. Let us choose the first approximation t_1 as the amplitude

$$t_1=\zeta_2$$

of two halfwaves from the previous case with $t_0\equiv 0$ (see Algorithm I, (2.5)). Then t_2 will be determined as follows: denote by $g(t)$ the left-hand side of (3.10). Choose a small $\delta>0$ (e.g. $\delta=5.10^{-3}$) and set

$$t_2=t_1+\delta \quad \text{or} \quad t_2=t_1-\delta,$$

which realizes

$$(3.11)\qquad \min\{|g(t_1+\delta)|,\ |g(t_1-\delta)|\}.$$

The following approximations will be determined by the secant method:

$$(3.12)\qquad t_{i+1}=t_i-g(t_i)(t_i-t_{i-1})/(g(t_i)-g(t_{i-1})),\quad i=2,3,\dots$$

Having a root t of the equation (3.10), we calculate $\sin\xi_m$ according to (3.8) where $z=z(t)$, and $\Phi^*(z_0,t_0)$ from (3.9).

4. Worst scenario method for uncertain two-mode imperfections

In production, the amplitudes of two halfwaves do not exceed those of one halfwave. Therefore we assume the set of uncertain amplitudes of imperfections as follows:

$$U_{\rm ad}=\{(z_0,t_0)\colon\ z_{0\,\min}\leqslant z_0\leqslant z_{0\,\max},\ t_0\leqslant z_0\},$$

where $0<z_{0\,\min}<z_{0\,\max}$ are given bounds.

We accomplished the algorithm described above to get $\Phi^*(z_0,t_0)$. The maximum over the set $U_{\rm ad}$ has been found by a simple direct method, since the derivatives $\partial\Phi^*/\partial z_0$, $\partial\Phi^*/\partial t_0$ are too complicated for practical use.

Example 4.1. We consider $k=1.4$, $z_{0\,\min}=0.2$, $z_{0\,\max}=0.6$. The maximal value 1.3751 of Φ^* was found at the point

$$z_0=0.2,\quad t_0=0.13$$

(see Fig. 8). Some values of $\Phi^*(0.2,t_0)$ for $t_0\in[0,0.2]$ are displayed in Tab. 2, those of $\Phi^*(z_0,0.13)$ for $z_0\in[0.2,0.60]$ are in Tab. 3.

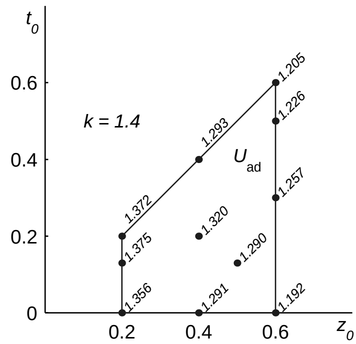

Figure 8. Decisive part of the maximal mean reduced stress over the set of admissible two-mode initial deflections for $k = 1.4$.

t_0	0	0.04	0.08	0.12	0.13	0.14	0.16	0.20
$\Phi^*(0.2, t_0)$	1.356	1.367	1.373	1.3750	1.37512	1.3750	1.374	1.372

Table 2.

z_0	0.20	0.24	0.28	0.34	0.40	0.46	0.50	0.54	0.58
$\Phi^*(z_0, 0.13)$	1.375	1.365	1.354	1.338	1.320	1.302	1.290	1.276	1.262

Table 3.

E x a m p l e 4.2. Let $k = 1.1$, $z_{0\,\min} = 0.07$, $z_{0\,\max} = 0.14$. Then

$$\max_{U_{\mathrm{ad}}} \Phi^* = \Phi^*(0.14, 0.14) \doteq 0.939.$$

Tables 4, 5, 6 display some values of Φ^* for

(i) $z_0 \in [0.07, 0.14]$, $t_0 = 0$,

(ii) $z_0 = 0.07$ and $t_0 \in [0, 0.07]$,

(iii) $z_0 = 0.14$ and $t_0 \in [0, 0.14]$.

z_0	0.07	0.08	0.10	0.12	0.14
$\Phi^*(z_0, 0)$	0.691	0.685	0.672	0.651	0.617

Table 4.

t_0	0	0.02	0.04	0.06	0.07
$\Phi^*(0.07, t_0)$	0.691	0.774	0.826	0.865	0.880

Table 5.

t_0	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14
$\Phi^*(0.14, t_0)$	0.617	0.744	0.805	0.847	0.878	0.903	0.923	0.939

Table 6.

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