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Jiří Vala

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ON A SYSTEM OF EQUATIONS OF EVOLUTION WITH A NON-SYMMETRICAL PARABOLIC PART OCCURING IN THE ANALYSIS OF MOISTURE AND HEAT TRANSFER IN POROUS MEDIA*

Jiří Vala, Brno

Abstract. Most non-trivial existence and convergence results for systems of partial differential equations of evolution exclude or avoid the case of a non-symmetrical parabolic part. Therefore such systems, generated by the physical analysis of the processes of transfer of heat and moisture in porous media, cannot be analyzed easily using the standard results on the convergence of Rothe sequences (e.g. those of W. Jäger and J. Kačur). In this paper the general variational formulation of the corresponding system is presented and its existence and convergence properties are verified; its application to one model problem (preserving the symmetry in the elliptic, but not in the parabolic part) is demonstrated.

Keywords: PDE's of evolution, method of Rothe, porous media, moisture and heat transfer

MSC 2000: 35K05, 35K15

1. Introduction

In most European countries the so-called moisture behaviour of buildings and engineering constructions (more precisely: the moisture transfer conditioned by their thermal status) should be (by obligatory standards) evaluated using the methodology suggested by H. Glasser in [9]: the moisture transfer is regarded as a pure diffusion process connected with water condensation and evaporation. Unfortunately, moisture distributions predicted by [9] are in many cases far from those observed in practice and measured in laboratories. The strong effort to improve this approach is evident from many later papers, articles and books—[17] is probably the best known of them. An extensive critical analysis of Glasser's methodology, based on a large

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number of experiments with interactions of various types of materials with porous structure and moisture in all phases (liquid water, vapour and ice), has been carried out by K. Kiessl in [16]. The result of this study is a new (non-Glasser-type) model with two unknown fields: the temperature and the so-called moisture (or humidity) potential. Kiessl's model was originally derived for one spatial coordinate and time; its modifications, simplifications ([16] include many phenomenological both algebraic and differential relations, not user-friendly at all), improvements and generalizations to 2- and 3-dimensional problems can be found e.g. in [18], [20] and [24].

The Kiessl model takes into account many physical processes, neglected by the Glasser model. Nevertheless, it is based more on "ad hoc" tricks than on the proper physical formulation of the problem; even the definition of the moisture potential includes (for moisture sufficiently large) a strange phenomenological coefficient. All attempts to convert the system of Kiessl relations into a well-defined mathematical problem of evolution of two unknown fields, satisfying realistic a priori prescribed initial and boundary conditions, have led to rather complicated formulations with formal additional assumptions, having no clear physical interpretation. The solutions of some problems and algorithms for their numerical construction have been studied at the Faculty of Civil Engineering of the Technical University in Brno in the last years; the strong impulse for this research came from the practical problem concerning the compensation of damages caused by the climatic excesses in Moravia several years ago. Some existence and regularity results for the classical formulation can be found in [2]; the variational formulation in [3] removes some complications with non-continuous material properties (the constructions in civil engineering typically consist of several layers with quite different material properties—some of them are expected to have the primary insulation effect, which should be quantified), but some strongly non-linear terms (whose physical meaning is vague) do not allow to verify the convergence of sequences of approximate solutions to a weak solution of the problem on the whole finite time interval in some reasonable sense.

The above mentioned experience with the Glasser and Kiessl approaches made the authors of [4] start the development of an original model, based on correct physical thermodynamic principles (the cooperation with the Institute of Physics of Materials of the Academy of Sciences of the Czech Republic in Brno was very useful and substantial) and generating a clear variational formulation. The aim of this model is to remove the disadvantages of the Kiessl model and give a complete description of both time-dependent (very slow) processes—moisture transfer and heat transfer—for an arbitrary amount of water in various phases in the porous structure of material. The complete physical derivation of this model can be found in [4]; here we will only mention that it contains two time-variable unknown fields: like the Kiessl model the temperature $\tau(x,t)$ and unlike the Kiessl model (instead of the rather artificial moisture potential $\Phi(x,t)$) the hydrostatic pressure $p(x,t)$ (we will use the notation

$u(x,t) = (\tau(x,t), p(x,t))$ for the sake of brevity) where x are usually the Cartesian coordinates on a domain in $\mathbb{R}^N$ ($N \in \{1,2,3\}$ is the geometrical dimension of the problem) and t is time from a finite time interval $I = \{t \in \mathbb{R}\colon\ 0 \leqslant t \leqslant T\}$ of a given real length T.

All non-trivial mathematical models of moisture and heat transfer in porous media, including the model [4] (which initiated this study), contain systems of partial differential equations of evolution (usually with a non-symmetrical parabolic part, not covered by standard existence and convergence theorems) with a great number of various characteristics, whose values are known from literature (rarely) or have to be obtained from special measurements (in most cases). It cannot be the aim of this paper to analyze the classes of such characteristics and their physical meaning; thus we will present (for illustration) only the probably simplest special form of the system of two equations of "non-stationary transfer of heat and mass" (in practice, mass is understood as water in various phases) in a domain Ω in $\mathbb{R}^N$ for $N \in \{1,2,3\}$ from the textbook [20], p. 210:

$$\mathcal{A}\dot{\tau} = \nabla^2\tau + \mathcal{K}\mathcal{A}\dot{\Phi}, \quad \mathcal{A}\dot{\Phi} = \mathcal{L}\mathcal{P}\nabla^2\tau + \mathcal{L}\nabla^2\Phi.$$

In these equations (where dots denote time derivatives) all multiplicative factors are considered to be positive constants: $\mathcal{A}$ is the thermal diffusivity (as defined in [20], p. 203), $\mathcal{P}$ the Posnov number, $\mathcal{K}$ the Kossovich number and $\mathcal{L}$ the Lykov number (introduced in [20], pp. 132, 134, 138). Let us notice that a more perspicuous alternative form of this system is

$$\mathcal{A}\begin{pmatrix} \mathcal{K}\mathcal{L}\mathcal{P}+1 & \mathcal{L}\mathcal{P} \\ \mathcal{K}\mathcal{L} & \mathcal{L} \end{pmatrix}\begin{pmatrix} \dot{\tau} \\ \dot{\Phi} \end{pmatrix} = \begin{pmatrix} \mathcal{K}\mathcal{L}\mathcal{P}+1 & \mathcal{L}\mathcal{P} \\ \mathcal{K}\mathcal{L} & \mathcal{L} \end{pmatrix}\begin{pmatrix} \mathcal{K}\mathcal{L}\mathcal{P}+1 & \mathcal{K}\mathcal{L} \\ \mathcal{L}\mathcal{P} & \mathcal{L} \end{pmatrix}\begin{pmatrix} \nabla^2\tau \\ \nabla^2\Phi \end{pmatrix}.$$

The unknown time-dependent fields τ and Φ should be calculated from this system with appropriate initial and boundary conditions (discussed in [20], p. 213). The standard use of the Green-Ostrogradskiĭ theorem then shows that weak solvability of such a system (and similar generalized systems, too) can be studied with help of the arguments from our illustrative example; this example will demonstrate how our assumed formal properties (a), …, (k) of applied differential operators (non-symmetric in the parabolic part, but symmetric in the elliptic one) can be verified in practice. Of course, some additional assumptions on Ω must be accepted to ensure that the usual imbedding and trace theorems cannot be violated; the geometrical interpretation of such conditions has been discussed in great detail in [22], pp. 62, 220.

Through the whole paper we will apply the standard notation: all classes of special mappings applied here are introduced in [8] or [5], the notation of Lebesgue and Sobolev spaces is compatible with [22], the symbol * is reserved for adjoint spaces, the dot symbol (rarely) for time derivatives and $\mathbb{R}_0$ is sometimes used instead of $\mathbb{R}_+ \cup \{0\}$.

2. Basic assumptions and variational formulation

Following [26], we will formulate the abstract problem in a reflexive and separable Banach space V (u will be considered in general as an abstract function mapping every time from I into V, although V can be identified with some Sobolev or similar space of functions in most applications available). Using the method of discretization in time, we will then consider linear splines u^n instead of u, which enables us to decompose the problem of evolution into particular problems for discrete times. Finally, the limit passage for $n \to \infty$, making use of certain a priori estimates, will verify the existence of a variational solution. Unfortunately, the arguments from [26] cannot be applied directly to realistic problems with more unknown fields (unlike simple examples with one field in [26], pp. 490, 495) that are not generated by weak differentiation of certain potentials (no other case is studied in [12], [13], [14], [10] or [11]), which is generally not true for our model derived in [4].

In addition to a reflexive and separable Banach space V (in particular, for $\varrho \in \mathbb{R}_+$ the symbol V_ϱ is reserved for the set of all $v \in V$ such that $\|v\|_V \leqslant \varrho$ and the symbol V'_ϱ for the set of all $v \in V$ such that $\|v\|_V \geqslant \varrho$) let us consider another Banach space H and some mappings $A\colon V \to V^*$ and $B\colon H \to H^*$; the symbol $\langle \cdot, \cdot \rangle$ will be used for the duality between V and V^* and the symbol $(\cdot, \cdot)$ for the duality between H and H^*. Let these spaces and mappings possess the following properties:

(a) There exists a strongly continuous imbedding of V into H.

(b) A is weakly continuous.

(c) B is demicontinuous.

(d) The estimate

$$\sup_{v \in V'_\varrho} (\varphi(\|v\|_V)\|v\|_V)^{-1} \int_0^1 \langle A(w + \xi(v - w)), w \rangle \, \mathrm{d}\xi < \infty$$

is true for some radius $\varrho \in \mathbb{R}_+$ and arbitrary fixed $w \in V$; the function $\varphi(\|v\|_V)$ comes from (i).

(e) The estimate

$$\sup_{v \in V'_\varrho} (\varphi(\|v\|_V)\|v\|_V)^{-1} (Bv, w) < \infty$$

is true for some radius $\varrho \in \mathbb{R}_+$ and arbitrary fixed $w \in V$; the function $\varphi(\|v\|_V)$ comes from (i).

(f) There exist $\psi_0 \in \mathbb{R}_+$ and an increasing continuous function $\psi\colon \mathbb{R}_0 \to \mathbb{R}_0$ such that $\psi(c) \geqslant \psi_0 c^2$ for any $c \in \mathbb{R}_0$ and

$$(Bv - Bw, v - w) \geqslant \psi(\|v - w\|_H)$$

for any $v, w \in V$.

(g) For the function ψ from (f) the estimate

$$\psi\left(\sum_{i=1}^{j} c_i\right) \leqslant \mu(j) \sum_{i=1}^{j} \psi(c_i)$$

is valid for every positive integer j, $c_i \in \mathbb{R}_+$ with $i \in \{1, \ldots, j\}$ and a certain increasing function μ mapping all positive integers into $\mathbb{R}_+$.

(h) The function μ from (g) has the limit behaviour

$$\lim_{j\to\infty} \frac{\mu(j)}{j} < \infty.$$

(i) There exist an increasing continuous function $\varphi\colon \mathbb{R}_0 \to \mathbb{R}$ and a $\nu \in \mathbb{R}_+$ such that

$$\int_0^1 \langle A(\xi v), v\rangle \,\mathrm{d}\xi + \nu(Bv, v) \geqslant \varphi(\|v\|_V)\|v\|_V$$

for any $v \in V$.

(j) There exists a $\gamma \in \mathbb{R}_+$ such that

$$0 \leqslant (Bv, v) \leqslant \gamma\psi(\|v\|_H)$$

for any $v \in V$.

(k) There exist $\omega, \kappa \in \mathbb{R}_+$ such that

$$\begin{aligned}\int_0^1 \langle A(w + \xi(v - w)), v - w\rangle \,\mathrm{d}\xi \geqslant \int_0^1 \langle A(\xi v), v\rangle \,\mathrm{d}\xi - \int_0^1 \langle A(\xi w), w\rangle \,\mathrm{d}\xi \\ - \kappa\sqrt{\varphi(\|v\|_V)\|v\|_V + \varphi(\|w\|_V)\|w\|_V + \omega}\,\sqrt{\psi(\|v - w\|_H)}\end{aligned}$$

for any $v, w \in V$.

Let us study the existence of $u\colon I \to V$ satisfying the equation of evolution

$$(1) \qquad (Bu(t) - Bu_0, v) + \int_0^t \langle Au(t'), v\rangle \,\mathrm{d}t' = 0$$

for all $v \in V$ and arbitrary $t \in I$ where the initial value $u(0) = u_0 \in V$ is prescribed. Let us choose an integer n and an $h_i \in \mathbb{R}_+$ for $i \in \{1, \ldots, n\}$ such that their sum is equal to T; later we will write only h instead of the largest and h_0 instead of the smallest h_i and apply the notation $\vartheta = h/h_0$. For $i \in \{1,\dots,n\}$ let us also consider the partial time intervals $I_i = \{t \in I\colon\ t_{i-1} < t \leqslant t_i\}$ where $t_0 = 0$ and $t_i = h_1 + \ldots + h_i$; for the sake of brevity let us define $J = \{t \in \mathbb{R}_0\colon\ t \leqslant 1\}$. Instead of $u(t)$ let us consider a linear spline

$$u^n(t) = u_{i-1} + \frac{t - t_{i-1}}{h_i}\,(u_i - u_{i-1})$$

for each I_i with $i \in \{1,\dots,n\}$ (evidently, $u_1,\dots,u_n$ as well as $h_1,\dots,h_n$ depend on the choice of n, but we will not emphasize it explicitly) which for an arbitrary $t = t_j$ with $j \in \{1,\dots,n\}$ simplifies (1) to the form

$$(2)\qquad (Bu_j - Bu_0, v) + \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} \left\langle A\left(u_{i-1} + \frac{t' - t_{i-1}}{h_i}\,(u_i - u_{i-1})\right), v\right\rangle \mathrm{d}t'$$

for any $v \in V$; formally we set $u^n(0) = u_0$.

In the next two sections we will verify the existence of u_i satisfying (2) with $i \in \{1,\dots,n\}$ in the first place; then we shall prove that some subsequence of $\{u^n\}_{n=1}^{\infty}$ has a limit u which coincides with a solution of (1). In the more exact form the first result will be presented in Theorem 1, the other in Theorem 2.

3. Solvability of a discrete scheme

Lemma 1. *For every integer n and $i \in \{1,\dots,n\}$ (2) can be converted into the discretized form*

$$(3)\qquad (Bu_i - Bu_{i-1}, v) + h_i \int_0^1 \langle A(u_{i-1} + \xi(u_i - u_{i-1})), v\rangle \,\mathrm{d}\xi = 0$$

with an arbitrary $v \in V$.

Proof. Let us formally rewrite (2) for $j = i$ and $j = i - 1$ where $i \in \{1,\dots,n\}$ and subtract the second equation from the first; in the special case $i = 1$ the second equation can be replaced by the identity $0 = 0$. We obtain

$$(Bu_i - Bu_{i-1}, v) + \int_{t_{i-1}}^{t_i} \left\langle A\left(u_{i-1} + \frac{t' - t_{i-1}}{h_i}\,(u_i - u_{i-1})\right), v\right\rangle \mathrm{d}t'$$

and using the simple linear transformation

$$\xi = \frac{t' - t_{i-1}}{h_i}$$

we arrive at (3). It is easy to see that this can be done in the opposite direction as well; the j-th equation (2) can be derived in this way as the sum of the first j equations (3). $\square$

Lemma 2. *For some $i \in \{1, \dots, n\}$ let T_i be the operator mapping each $w \in V$ into V^* defined by*

$$\langle T_i w, v\rangle = (Bw - Bu_{i-1}, v) + h_i \int_0^1 \langle A(u_{i-1} + \xi(w - u_{i-1})), v\rangle \, \mathrm{d}\xi$$

for all $v \in V$. Then for a fixed $u_{i-1} \in V$ the operator T_i is weakly continuous.

Proof. If the sequence $\{w^k\}_{k=1}^{\infty}$ from V has a weak limit w then also the sequence $\{\widetilde{w}^k(\xi)\}_{k=1}^{\infty}$ consisting of elements $\widetilde{w}^k(\xi) = \xi w^k + (1-\xi)u_{i-1}$ has a weak limit $\widetilde{w}(\xi) = \xi w + (1-\xi)u_{i-1}$ for each $\xi \in J$. Making use of the fact that every weakly convergent sequence is bounded (cf. [7], p. 193), the property (b) together with the Lebesgue dominated convergence theorem yields

$$\lim_{k\to\infty} \int_0^1 \langle A\widetilde{w}^k(\xi), v\rangle \, \mathrm{d}\xi = \int_0^1 \langle A\widetilde{w}(\xi), v\rangle \, \mathrm{d}\xi$$

for any $v \in V$. The property (c) implies

$$\lim_{k\to\infty} (Bw^k - Bu_{i-1}, v) = (Bw - Bu_{i-1}, v)$$

for any $v \in V$ provided $\{w^k\}_{k=1}^{\infty}$ has a strong limit w in H; but this follows directly from the property (a). As h_i is a positive constant, the weak continuity of T_i is now evident. □

Lemma 3. *For every $i \in \{1, \dots, n\}$, h_i small enough and a fixed u_{i-1} the operator T_i from Lemma 2 is coercive.*

Proof. Let us set $v = w$ in the definition of T_i in Lemma 2. We have

$$\langle T_i v, v\rangle = (Bv - Bu_{i-1}, v) + h_i \int_0^1 \langle A(u_{i-1} + \xi(v - u_{i-1})), v\rangle \, \mathrm{d}\xi$$

which, rewritten in another order, gives

$$\begin{aligned}\langle T_i v, v\rangle = \nu h_i (Bv, v) &+ (1 - \nu h_i)(Bv - Bu_{i-1}, v - u_{i-1}) \\ &+ h_i \int_0^1 \langle A(u_{i-1} + \xi(v - u_{i-1})), v - u_{i-1}\rangle \, \mathrm{d}\xi - \nu h_i (Bu_{i-1}, v) \\ &+ (1 - \nu h_i)(Bv - Bu_{i-1}, u_{i-1}) + h_i \int_0^1 \langle A(u_{i-1} + \xi(v - u_{i-1})), u_{i-1}\rangle \, \mathrm{d}\xi.\end{aligned}$$

The estimate based on the properties (k), (i) and (f)

$$
\begin{aligned}
&\nu(Bv,v) + \int_0^1 \langle A(u_{i-1}+\xi(v-u_{i-1})), v-u_{i-1}\rangle \,\mathrm{d}\xi \\
&\quad \geqslant \nu(Bv,v) + \int_0^1 \langle A(\xi v), v\rangle \,\mathrm{d}\xi - \int_0^1 \langle A(\xi u_{i-1}), u_{i-1}\rangle \,\mathrm{d}\xi \\
&\qquad - \kappa\sqrt{\varphi(\|v\|_V)\|v\|_V + \varphi(\|u_{i-1}\|_V)\|u_{i-1}\|_V + \omega}\,\sqrt{\psi(\|v-u_{i-1}\|_H)} \\
&\quad \geqslant \varphi(\|v\|_V)\|v\|_V - \int_0^1 \langle A(\xi u_{i-1}), u_{i-1}\rangle \,\mathrm{d}\xi \\
&\qquad - \frac{1}{2}(\varphi(\|v\|_V)\|v\|_V + \varphi(\|u_{i-1}\|_V)\|u_{i-1}\|_V + \omega) - \frac{1}{2}\kappa^2\psi(\|v-u_{i-1}\|_H) \\
&\quad \geqslant \frac{1}{2}(\varphi(\|v\|_V)\|v\|_V - \varphi(\|u_{i-1}\|_V)\|u_{i-1}\|_V - \omega) - \int_0^1 \langle A(\xi u_{i-1}), u_{i-1}\rangle \,\mathrm{d}\xi \\
&\qquad - \frac{1}{2}\kappa^2(Bv - Bu_{i-1}, v-u_{i-1})
\end{aligned}
$$

enables us to conclude

$$
\begin{aligned}
\langle T_i v, v\rangle \geqslant{}& \frac{1}{2}h_i\varphi(\|v\|_V)\|v\|_V + \Big(1-\nu h_i - \frac{1}{2}\kappa^2 h_i\Big)(Bv-Bu_{i-1}, v-u_{i-1}) \\
&- \frac{1}{2}h_i\varphi(\|u_{i-1}\|_V)\|u_{i-1}\|_V - h_i\int_0^1 \langle A(\xi u_{i-1}), u_{i-1}\rangle \,\mathrm{d}\xi \\
&- (1-\nu h_i)(Bu_{i-1}, u_{i-1}) + (1-\nu h_i)(Bv, u_{i-1}) - \nu h_i(Bu_{i-1}, v) \\
&+ h_i\int_0^1 \langle A(u_{i-1}+\xi(v-u_{i-1}), u_{i-1}\rangle \,\mathrm{d}\xi - \frac{1}{2}\omega h_i.
\end{aligned}
$$

The second right-hand side additive term is always non-negative for h_i small enough due to the property (f); therefore it can be omitted. The third, fourth and fifth terms are independent of v, and therefore they cannot affect the coerciveness of T_i. The sixth, seventh and eighth terms are finite for $v \in V'_\varrho$ and ϱ large enough, which follows from the properties (d) and (e); the last term is a constant. Thus the first term is decisive and guarantees the coerciveness of V for h_i small enough. □

Theorem 1. *For every $i \in \{1\ldots,n\}$, h_i small enough and a fixed u_{i-1} there exists u_i satisfying* (3).

P r o o f. By [6], p. 46, every weakly continuous and coercive mapping of V into V^* is surjective (thanks to the reflexivity and separability of V). Thus Lemma 2 and Lemma 3 guarantee that for each $i \in \{1,\ldots,n\}$ at least one u_i can be calculated from (3) if u_{i-1} is given in advance. □

Lemma 4. *The sequence of piecewise linear abstract functions $\{u^n\}_{n=1}^{\infty}$ mapping I into V is equibounded.*

Proof. We shall construct more precise estimates than in the proof of Lemma 3. For arbitrary $i \in \{1, \dots, n\}$ let us choose $v = (u_i - u_{i-1})/h_i$ in (3). We obtain

$$\frac{1}{h_i}(Bu_i - Bu_{i-1}, u_i - u_{i-1}) + \int_0^1 \langle A(u_{i-1} + \xi(u_i - u_{i-1})), u_i - u_{i-1}\rangle \, \mathrm{d}\xi = 0. \tag{4}$$

Applying for arbitrary $j \in \{1, \dots, n\}$ the property (k) j-times (for $i \in \{1, \dots, j\}$) and summing up we obtain

$$\begin{aligned}
\sum_{i=1}^{j} \int_0^1 & \langle A(u_{i-1} + \xi(u_i - u_{i-1})), u_i - u_{i-1}\rangle \, \mathrm{d}\xi \\
&\geqslant \int_0^1 \langle A(\xi u_j), u_j\rangle \, \mathrm{d}\xi - \int_0^1 \langle A(\xi u_0), u_0\rangle \, \mathrm{d}\xi \\
&\quad - \kappa \sum_{i=1}^{j} \sqrt{\varphi(\|u_i\|_V)\|u_i\|_V + \varphi(\|u_{i-1}\|_V)\|u_{i-1}\|_V + \omega}\, \sqrt{\psi(\|u_i - u_{i-1}\|_H)} \\
&\geqslant \int_0^1 \langle A(\xi u_j), u_j\rangle \, \mathrm{d}\xi - \int_0^1 \langle A(\xi u_0), u_0\rangle \, \mathrm{d}\xi \\
&\quad - \kappa^2 h \sum_{i=0}^{j} \varphi(\|u_i\|_V)\|u_i\|_V - \frac{1}{2}\kappa^2 \omega j h - \frac{1}{2h}\sum_{i=1}^{j} \psi(\|u_i - u_{i-1}\|_H)
\end{aligned}$$

(clearly $jh \leqslant j\vartheta h_0 \leqslant jT$). Let us also recall simple consequences of the property (f):

$$\sum_{i=1}^{j} \frac{1}{h_i}(Bu_i - Bu_{i-1}, u_i - u_{i-1}) \geqslant \frac{1}{h}\sum_{i=1}^{j} \psi(\|u_i - u_{i-1}\|_H,$$

of the property (i):

$$\int_0^1 \langle A(\xi u_j), u_j\rangle \, \mathrm{d}\xi + \nu(Bu_j, u_j) \geqslant \varphi(\|u_j\|_V)\|u_j\|_V,$$

and of the property (j) with respect to the property (g):

$$\begin{aligned}
(Bu_j, u_j) &\leqslant \gamma\psi(\|u_j\|_H) \leqslant \gamma\mu(2)\psi(\|u_0\|_H) + \gamma\mu(2)\psi(\|u_j - u_0\|_H) \\
&\leqslant \gamma\mu(2)\psi(\|u_0\|_H) + \gamma\mu(2)\mu(j)\sum_{i=1}^{j} \psi(\|u_i - u_{i-1}\|_H)
\end{aligned}$$

(clearly $\mu(j) \leqslant \mu(n)$ here). Inserting all these estimates into the sum of (4) for $i \in \{1, \ldots, j\}$ we obtain

$$\Big(\frac{1}{2h} - \nu\gamma\mu(2)\mu(n)\Big) \sum_{i=1}^{j} \psi(\|u_i - u_{i-1}\|_H) + \varphi(\|u_j\|_V)\|u_j\|_V$$
$$\leqslant \int_0^1 \langle A(\xi u_0), u_0\rangle \,\mathrm{d}\xi + \frac{1}{2}\kappa^2\omega\vartheta T + \kappa^2 h \sum_{i=0}^{j} \varphi(\|u_i\|_V)\|u_i\|_V + \nu\gamma\mu(2)\psi(\|u_0\|_H).$$

Moreover, let us notice that

$$\frac{1}{2h} - \nu\gamma\mu(2)\mu(n) \geqslant \frac{n}{2\vartheta T} - \nu\gamma\mu(2)\mu(n) \geqslant \frac{n}{2\vartheta T}\Big(1 - 2\nu\vartheta T\gamma\mu(2)\frac{\mu(n)}{n}\Big)$$
$$\geqslant \frac{1}{2\vartheta h}\Big(1 - \nu\vartheta T\gamma\mu(2)\frac{\mu(n)}{n}\Big).$$

Since the property (h) is valid, the first left-hand side additive term must be positive for T sufficiently small; for greater T the interval I can be divided into a finite number of shorter time intervals and all arguments can be repeated (no special assumptions concerning the choice of u_0 are needed). Finally, for a certain positive constant ζ this gives

$$\varphi(\|u_j\|_V)\|u_j\|_V \leqslant \zeta + \kappa^2 h \sum_{i=1}^{j} \varphi(\|u_i\|_V)\|u_i\|_V. \tag{5}$$

Two cases are possible: in the first we evidently have $\varphi(\|u_j\|_V)\|u_j\|_V \leqslant 0$, in the other only $\varphi(\|u_j\|_V)\|u_j\|_V = |\varphi(\|u_j\|_V)|\, \|u_j\|_V > 0$ and

$$|\varphi(\|u_j\|_V)|\, \|u_j\|_V \leqslant \zeta + \kappa^2 h \sum_{i=1}^{j} |\varphi(\|u_i\|_V)|\, \|u_i\|_V,$$

which with help of the discrete version of the Gronwall lemma (see [12], p. 29, and [27], p. 370) gives

$$\varphi(\|u_j\|_V)\|u_j\|_V \leqslant \frac{\zeta}{1-\kappa^2 h}\exp\Big(\frac{\kappa^2(j-1)h}{1-\kappa^2 h}\Big) = \frac{\zeta}{1-\kappa^2 h}\exp\Big(\frac{\vartheta\kappa^2(j-1)h_0}{1-\kappa^2 h}\Big)$$
$$\leqslant \frac{\zeta}{1-\kappa^2 h}\exp\Big(\frac{\vartheta\kappa^2 T}{1-\kappa^2 h}\Big)$$

and for $2h \leqslant \kappa^{-2}$ finally

$$\varphi(\|u_j\|_V)\|u_j\|_V \leqslant \zeta_1 \tag{6}$$

where

$$\zeta_1 = 2\zeta \exp(2\vartheta\kappa^2 T);$$

this covers the first case, too. Now let us assume that for every $\varrho \in \mathbb{R}_+$ such a u_j can be found that $u_j \in V'_\varrho$. Consequently, this is possible also for $\varrho \geqslant \varrho_0$ where $\varrho_0 \in \mathbb{R}_+$ is chosen (using the property (i)) such that $\varphi(\varrho) \geqslant \varphi(\varrho_0) > 0$. Thus we have reached a contradiction

$$\varphi(\varrho_0)\varrho \leqslant 2\zeta \exp(2\vartheta\kappa^2 T).$$

Therefore there exists a $\varrho \in \mathbb{R}_+$ such that

$$\|u_j\|_V \leqslant \varrho \tag{7}$$

for every $j \in \{1, \ldots, n\}$, and for arbitrary $t \in I_j$ and a certain $\xi \in J$ we can conclude

$$\|u^n(t)\|_V = \|(1-\xi)u_{i-1} + \xi u_i\|_V \leqslant (1-\xi)\|u_{i-1}\|_V + \xi\|u_i\|_V \leqslant (1-\xi)\varrho + \xi\varrho = \varrho.$$

This does not depend on the choice of an integer n which expresses the required equiboundedness. □

Lemma 5. *The sequence* $\{u^n\}_{n=1}^\infty$ *from Lemma* 4 *is equicontinuous as a sequence of abstract functions mapping* I *into* H.

Proof. By Lemma 4, for any $\xi \in J$, each $i \in \{1, \ldots, n\}$ and an arbitrary integer n we have

$$\|u_{i-1} + \xi(u_i - u_{i-1})\|_V \leqslant \max(\|u_i\|_V, \|u_{i-1}\|_V) \leqslant \varrho$$

where $\varrho \in \mathbb{R}_+$ comes from (7). Thanks to Lemma 4, to the related equiboundedness of $\{Au^n\}_{n=1}^\infty$ in V^* (forced by the property (b)) and to the property (f), this by (3) implies the existence of $\eta \in \mathbb{R}_+$ such that

$$\frac{1}{h_i}\psi(\|u_i - u_{i-1}\|_H) \leqslant \frac{1}{h_i}(Bu_i - Bu_{i-1}, u_i - u_{i-1}) \leqslant \eta.$$

Let us choose $t, t' \in I$ such that $t' < t$, $t \in I_k$ and $t' \in I_j$ where $j, k \in \{1, \ldots, n\}$ (clearly $j \leqslant k$). In the simplest case $k = j$ where $t = t_{j-1} + \xi h_j$ and $t' = t_{j-1} + \xi' h_j$ with $\xi, \xi' \in J$ we have

$$\begin{aligned}\|u^n(t) - u^n(t')\|_H^2 &= \|\xi u_j + (1-\xi)u_{j-1} - \xi' u_j - (1-\xi')u_{j-1}\|_H^2 \\ &= (\xi - \xi')^2\|u_j - u_{j-1}\|_H^2 \leqslant \frac{1}{\psi_0}(\xi - \xi')\psi(\|u_j - u_{j-1}\|_H) \\ &\leqslant \frac{\eta}{\psi_0}(\xi - \xi')h_j = \frac{1}{\psi_0}(t - t').\end{aligned}$$

Similarly, in the case $k = j + 1$ where $t = t_j + \xi h_j$ and $t' = t_j - \xi h_{j-1}$ with $\xi, \xi' \in J$ we have

$$\begin{aligned}\|u^n(t) - u^n(t')\|_H^2 &= \|\xi u_{j+1} + (1-\xi)u_j - \xi' u_{j-1} - (1-\xi')u_j\|_H^2 \\ &\leqslant 2(\xi^2\|u_{j+1} - u_j\|_H^2 + {\xi'}^2\|u_j - u_{j-1}\|_H^2) \\ &\leqslant \frac{2}{\psi_0}(\xi\psi(\|u_{j+1} - u_j\|_H) + \xi'\psi(\|u_j - u_{j-1}\|_H)) \\ &\leqslant \frac{2\eta}{\psi_0}(\xi h_{j+1} + \xi' h_j) = \frac{2\eta}{\psi_0}(t - t').\end{aligned}$$

It remains to deal with the general case $k > j + 1$. Following the proof of Lemma 4, let us rewrite the estimate (5) in a slightly improved form

$$\frac{\zeta_0}{h}\sum_{i=1}^{j}\psi(\|u_j - u_{j-1}\|_H) + \varphi(\|u_i\|_V)\|u_i\|_V \leqslant \zeta + \kappa^2 h\sum_{i=1}^{j}\varphi(\|u_i\|_V)\|u_i\|_V$$

where the existence of a $\zeta_0 \in \mathbb{R}_+$ (independent of n and j) follows from the preceding discussion based on the property (h). Using the discrete version of the Gronwall lemma again, we obtain finally the analogue of (6)

$$\frac{\zeta_0}{h}\sum_{i=1}^{j}\psi(\|u_i - u_{i-1}\|_H) + \varphi(\|u_j\|_V)\|u_j\|_V \leqslant \zeta_1.$$

If $\varphi(\|u_j\|_V) \geqslant 0$ the second left-hand side additive term can be omitted; in the opposite case we have

$$\frac{\zeta_0}{h}\sum_{i=1}^{j}\psi(\|u_i - u_{i-1}\|_H) \leqslant \zeta_1 - \varphi(\|u_j\|_V)\|u_j\|_V \leqslant \zeta_1 - \varphi(0)\|u_j\|_V \leqslant \zeta_2$$

with $\zeta_2 = \zeta_1 - \varphi(0)\varrho$ where $\varrho \in \mathbb{R}_+$ comes from (7). Thus, making use of the property (f), we can conclude

$$\sum_{i=1}^{j}\psi(\|u_i - u_{i-1}\|_H) \leqslant \zeta_* h \tag{8}$$

with the constant

$$\zeta_* = \frac{\max(\zeta_1, \zeta_2)}{\zeta_0}.$$

This yields the inequality

$$\begin{aligned}\|u^n(t)-u^n(t')\|_H^2 &\leqslant \Big(\sum_{i=1}^{k}\|u_i-u_{i-1}\|_H\Big)^2 \leqslant (k-j+1)\sum_{i=1}^{k}\|u_i-u_{i-1}\|_H^2\\ &\leqslant \frac{1}{\psi_0}(k-j+1)\sum_{i=1}^{k}\psi(\|u_i-u_{i-1}\|_H) \leqslant \frac{\zeta_* h(t-t'+2h)}{\psi_0 h_0}\\ &\leqslant \frac{\zeta_*}{\psi_0\vartheta}h(t-t'+2h).\end{aligned}$$

Taking into consideration that $h_0=\vartheta h\leqslant t-t'$ here, we obtain

$$\|u^n(t)-u^n(t')\|_H^2 \leqslant \frac{\zeta_*}{\psi_0}\frac{1+2\vartheta}{\vartheta}h(t-t'),$$

hence the estimate $h\leqslant T$ makes the required equicontinuity evident. □

Lemma 6. *There exists a* $u\colon I\to V$ *such that, up to a subsequence,* $u(t)$ *is the weak limit of* $\{u^n(t)\}_{n=1}^{\infty}$ *for every* $t\in I$ *and* u *is the strong limit of* $\{u^n\}_{n=1}^{\infty}$ *in* $C(I,H)$.

P r o o f. Lemma 4 and Lemma 5 guarantee the validity of all assumptions of the Arzelà-Ascoli theorem (see [19], p. 36 for the general formulation, and [12], p. 24, for its version directly applicable here), which yields the result. □

Theorem 2. *There exists an abstract function* $u\colon I\to V$ *satisfying* (1) *such that* $u\in C(I,H)$.

P r o o f. By Lemma 1 the sequence $\{u^n\}_{n=1}^{\infty}$ from Lemma 4, Lemma 5 and Lemma 6 evidently satisfies (2). By virtue of Lemma 4 and the property (b) the same arguments as in the proof of Lemma 2 yield

$$\lim_{n\to\infty}\int_0^t \langle Au^n(t')-Au(t'),v\rangle\,\mathrm{d}t'=0$$

and by the property (c) also

$$\lim_{n\to\infty}(Bu^n(t)-Bu(t),v)=0$$

for arbitrary $t\in I$ and every $v\in V$ (in both cases up to a subsequence); thus, the limit passage from (2) to (1) is possible. □

Lemma 7. *There exists $\Theta \in \mathbb{R}_+$ such that the estimate for the sequence $\{\dot{u}^n\}_{n=1}^{\infty}$ (consisting of the time derivatives of the sequence $\{u^n\}_{n=1}^{\infty}$) from Lemma 4,*

$$\int_I \|\dot{u}^n(t)\|_H^2 \,\mathrm{d}t \leqslant \Theta, \tag{9}$$

is valid independently of the choice of an integer n.

P r o o f. Let us apply the estimate (8) from the proof of Lemma 5. We obtain

$$\begin{aligned}\int_I \|\dot{u}^n(t)\|_H^2 \,\mathrm{d}t &= \sum_{i=1}^{n} \frac{1}{h_i}\|u_i - u_{i-1}\|_H^2 \leqslant \frac{\vartheta}{h}\sum_{i=1}^{n}\|u_i - u_{i-1}\|_H^2 \\ &\leqslant \frac{\vartheta}{\psi_0 h}\sum_{i=1}^{n}\psi(\|u_i - u_{i-1}\|_H) \leqslant \Theta\end{aligned}$$

with the constant

$$\Theta = \frac{\vartheta\zeta_*}{\psi_0}$$

(independent of n) for the inequality (9). $\square$

Theorem 3. *Let H be reflexive. Then every solution u of* (1) *in the sense of Theorem* 2 *belongs to $L^{\infty}(I, V) \cap W^{1,2}(I, H)$.*

P r o o f. Due to Lemma 7 it is possible to apply the convergence theorem for Rothe sequences (the reflexivity of H is substantial here) from [12], p. 25, which (under the assumption (9)) gives the desired result. $\square$

Now let us introduce a mapping $P\colon V \to V^*$ with the properties analogous to A:

(b′) P is weakly continuous.

(d′) For an arbitrary fixed $w \in V$ there are $\lambda_0, \lambda_1 \in \mathbb{R}_0$ such that

$$\langle Pv, w\rangle \geqslant -\lambda_0 - \lambda_1\|v\|_V$$

for any $v \in V$.

(i′) There exist $\varepsilon \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}_0$ such that

$$\langle Pv, v\rangle \geqslant \varepsilon\|v\|_V^2 - \lambda$$

for any $v \in V$.

(k′) In the original property (k) the square root of $\psi(\|v - w\|_H)$ is allowed to be substituted by the square root of the sum of $\psi(\|v - w\|_H) + \|v - w\|_V^2$ only (this makes (k) less strict).

In the following lemmas and theorems we will study the analogue of (2),

$$(10)\qquad (Bu_j - Bu_0, v) + \sum_{i=1}^{j} h_i \Big\langle P\Big(\frac{u_i - u_{i-1}}{h_i}\Big), v\Big\rangle + \sum_{i=1}^{j} h_i \int_{t_{i-1}}^{t_i} \Big\langle A\Big(u_{i-1} + \frac{t' - t_{i-1}}{h_i}(u_i - u_{i-1})\Big), v\Big\rangle \,\mathrm{d}t' = 0$$

for any $v \in V$ (clearly the argument of P is equal to $\dot{u}^n$ everywhere).

Lemma 8. *For every integer n and $i \in \{1, \dots, n\}$ the equation* (10) *can be converted into the form similar to* (3),

$$(11)\qquad (Bu_i - Bu_{i-1}, v) + h_i \Big\langle P\Big(\frac{u_i - u_{i-1}}{h_i}\Big), v\Big\rangle + h_i \int_0^1 \langle A(u_{i-1} + \xi(u_i - u_{i-1})), v\rangle \,\mathrm{d}\xi = 0$$

with arbitrary $v \in V$. Moreover, the operator T_i' mapping each $w \in V$ into V^* and defined with help of the operator T_i from Lemma 3,

$$\langle T_i' w, v\rangle = \langle T_i w, v\rangle + h_i \Big\langle P\Big(\frac{u_i - w}{h_i}\Big), v\Big\rangle$$

for all $v \in V$, is weakly continuous and coercive.

P r o o f. Lemma 8 is nothing else than Lemma 1, Lemma 2 and Lemma 3 together reformulated for an additional mapping P. Therefore we will only sketch the difficulties in its proof. In Lemma 1 (relation between (10) and (11)) no complication occurs. In Lemma 2 the arguments based on the property (b′) must be applied also to P in the same way as those based on the property (b) to A. In Lemma 3 we have (instead of the estimate for $\langle T_i v, v\rangle$)

$$\begin{aligned}\langle T_i' v, v\rangle &\geqslant \langle T_i v, v\rangle + h_i \Big\langle P\Big(\frac{v - u_{i-1}}{h_i}\Big), v\Big\rangle - \frac{1}{2}\kappa^2 h_i \|v - u_{i-1}\|_V^2\\ &= \langle T_i v, v\rangle + h_i^2 \Big\langle P\Big(\frac{v - u_{i-1}}{h_i}\Big), \frac{v - u_{i-1}}{h_i}\Big\rangle + h_i \Big\langle P\Big(\frac{v - u_{i-1}}{h_i}\Big), u_{i-1}\Big\rangle\\ &\quad - \frac{1}{2}\kappa^2 h_i \|v - u_{i-1}\|_V^2\end{aligned}$$

where the last right-hand side additive term comes from the generalized property (k′). Taking into account the estimate based on the property (i′),

$$h_i^2 \Big\langle P\Big(\frac{v - u_{i-1}}{h_i}\Big), \frac{v - u_{i-1}}{h_i}\Big\rangle \geqslant \varepsilon \|v - u_{i-1}\|_V^2 - \lambda,$$

we obtain

$$\langle T_i' v, v\rangle \geqslant \langle T_i v, v\rangle + \Big(\varepsilon - \frac{1}{2}\kappa^2 h_i\Big)\|v-u_{i-1}\|_V^2 - \lambda + h_i\Big\langle P\Big(\frac{v-u_{i-1}}{h_i}\Big), u_{i-1}\Big\rangle,$$

hence the property (d′) (with λ_0 and λ_1 corresponding to $w = u_{i-1}$) for $h_i \leqslant \varepsilon/\kappa^2$ results in

$$\begin{aligned}\langle T_i' v, v\rangle &\geqslant \langle T_i v, v\rangle + \frac{\varepsilon}{2}\|v-u_{i-1}\|_V^2 - \lambda - \lambda_0 h_i - \lambda_1\|v-u_{i-1}\|_V \\ &\geqslant \langle T_i v, v\rangle + \frac{\varepsilon}{2}\|v-u_{i-1}\|_V^2 - \lambda - \lambda_0 h_i - \frac{\varepsilon}{2}\|v-u_{i-1}\|_V^2 - \frac{\lambda_1^2}{2\varepsilon} \\ &= \langle T_i v, v\rangle - \lambda - \lambda_0 h_i - \frac{\lambda_1^2}{2\varepsilon};\end{aligned}$$

thus the coerciveness of T_i' follows from the coerciveness of T_i. □

Theorem 4. *Theorem* 1 *holds with* (10) *instead of* (2), *too.*

P r o o f. It is the same as the proof of Theorem 1; Lemma 8 instead of Lemma 2 and Lemma 3 can be applied. □

Lemma 9. *The inequality* (9) *from Lemma* 7 *holds even with the norm of* V *instead of the norm of* H.

P r o o f. The proof of Lemma 4 can be repeated. The property (i′) gives

$$\begin{aligned}\sum_{i=1}^{j} h_i\Big\langle P\Big(\frac{u_i-u_{i-1}}{h_i}\Big), \frac{u_i-u_{i-1}}{h_i}\Big\rangle &\geqslant \varepsilon\sum_{i=1}^{j}\frac{1}{h_i}\|u_i-u_{i-1}\|_V^2 - \lambda\sum_{i=1}^{j} h_i \\ &\geqslant \frac{\varepsilon}{h}\sum_{i=1}^{j}\|u_i-u_{i-1}\|_V^2 - \lambda T.\end{aligned}$$

This occurs in the sum of (11) with $v = (u_i - u_{i-1})/h_i$. The modified right-hand side due to the property (k′) instead of (k) changes the factor ε to $\varepsilon - 1/2$ only; but if $\varepsilon < 1/2$ then κ in the corresponding estimates can be replaced by $\kappa/\sqrt{\varepsilon}$, which finally leads to the factor $\varepsilon/2$ instead of $\varepsilon - 1$. Similar arguments as in the proofs of Lemma 4 and Lemma 5 and the obvious equality

$$\int_I \|\dot u^n(t)\|_V^2 \,\mathrm{d}t = \sum_{i=1}^{n}\frac{1}{h_i}\|u_i-u_{i-1}\|_V^2$$

(the same with H instead of V can be found in the proof of Lemma 7) lead to our modified version of (9). □

Lemma 10. *There exists a $u\colon I \to V$ such that Lemma 6 is valid and (up to a subsequence) also $\dot u$ is a weak limit of $\{\dot u^n\}_{n=1}^{\infty}$ in $L^2(I,V)$.*

P r o o f. Lemma 9 similar to Lemma 7 is available, but unlike in the proof of Lemma 6 we cannot apply [12], p. 25, directly. Nevertheless, using the approach of [27], p. 371, we can see that the Eberlein-Shmul'yan theorem (see [27], p. 197) guarantees (thanks to the reflexivity of V, up to a subsequence) the existence of a weak limit u' of $\{\dot u^n\}_{n=1}^{\infty}$ in $L^2(I,V)$. This implies the weak convergence of $\{u^n(t)\}_{n=1}^{\infty}$ to $\int_0^t u'(t')\,\mathrm{d}t'$ in V, which can be identified with $u(t)$ for arbitrary $t\in I$. □

Theorem 5. *Theorem* 2 *holds with*

$$(Bu(t)-Bu_0,v)+\int_0^t\langle P\dot u(t'),v\rangle\,\mathrm{d}t'+\int_0^t\langle Au(t'),v\rangle\,\mathrm{d}t'=0 \tag{12}$$

instead of (1), *too. Moreover, let the mapping P have the following property (additional to* (i′)*):*
(i″) *If $w\in V_\varrho$ for some $\varrho\in\mathbb{R}_+$ then $\langle Pw,v\rangle=0$ for every $v\in V$.*
Then Theorem 2 *holds with* (1) *provided $\dot u(t)\in V_\varrho$ ($\varrho\in\mathbb{R}_+$ comes from the property* (i″)*) for arbitrary $t\in I$.*

P r o o f. The first part of Lemma 8 can be applied in the same way as Lemma 1 in the proof of Theorem 1. By [12], p. 24, the imbedding of $W^{1,2}(I,V)$ into $L^2(I,H)$ is compact (only the reflexivity of V and the property (a) are needed here). Thus by Lemma 10, up to a subsequence, $\{\dot u^n\}_{n=1}^{\infty}$ must have a strong limit in $L^2(I,H)$ which cannot be different from $\dot u$. Since (10) is an alternative form of (12) with u^n instead of u, the properties (b′) and (d′) complete the verification of the first assertion similarly to the proof of Theorem 2:

$$\lim_{n\to\infty}\int_0^t\langle P\dot u^n(t'),v\rangle\,\mathrm{d}t'=0$$

holds for arbitrary $t\in I$ and every $v\in V$ (up to a subsequence). The second assertion follows from the property (i″) applied directly to (12). □

Theorem 6. *Let H be reflexive. Then* (12) *can be differentiated with the result*

$$((Bu(t))^{\cdot},v+\langle P\dot u(t),v\rangle+\langle Au(t),v\rangle=0 \tag{13}$$

for every $v\in V$ and arbitrary $t\in I$.

P r o o f. The Eberlein-Shmul'yan theorem (the reflexivity of H is needed here, cf. the proof of Lemma 10) guarantees the existence of a strong limit of $\{\dot u^n\}_{n=1}^{\infty}$ (up to a subsequence); this cannot be other than $\dot{u}$. Using the property (b) we obtain

$$\lim_{n\to\infty}\int_0^t (B\dot{u}^n(t') - B\dot{u}^n(t'), v)\,\mathrm{d}t' = 0$$

for every $v \in V$ and arbitrary $t \in I$. Taking into account similar limits from the proofs of Theorem 2 and Theorem 5 we can finally conclude that the differentiation of (12) is well-defined; its result is (13). □

6. Illustrative example

In Introduction we started with some physical considerations, but up to now we have studied an abstract variational problem with certain mappings A and B possessing formal properties (a), ..., (k) or their slight modifications. Some of these properties are rather complicated; the reader could be afraid that no reasonable equations of technical practice are covered by the presented theory. To weaken this doubt, we will demonstrate what such properties mean for special mappings applied to functions defined on a domain Ω in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. The symbol m will be reserved for the Lebesgue measure in $\mathbb{R}^2$, the symbol s for the Hausdorff measure on $\partial\Omega$. Let us introduce V as a suitable subspace of the Sobolev space $W^{1,2}(\Omega,\mathbb{R}^2)$. Moreover, let us set $H = L^2(\Omega,\mathbb{R}^2)$ and $X = L^2(\partial\Omega,\mathbb{R}^2)$. Evidently, such special V, H and X are separable Hilbert spaces. For the sake of brevity, we will not emphasize explicitly that the functions from V, H and X depend on the Cartesian coordinates $x = (x_1, x_2)$ (e.g., v instead of $v(x)$ and $\mathrm{d}m$ instead of $\mathrm{d}m(x)$ will be used). Let $b_i\colon H \to H$ with $i \in \{1,2\}$ be differentiable mappings whose derivatives $b_{i,j}$ with respect to the j-th variable for $j \in \{1,2\}$ satisfy the following assumptions (i and j here and later also k and l are applied as Einstein summation indices from the set $\{1,2\}$):

(B1) The mapping B has the form

$$(Bv, w) = \int_\Omega b_i(v_1, v_2) w_i \,\mathrm{d}m$$

for all $v = (v_1, v_2)$ and $w = (w_1, w_2)$ from H.

(B2) There exist positive constants $b_\diamond$ and b_* such that

$$b_\diamond w_i w_i \leqslant b_{i,j}(v_1, v_2) w_j w_i \leqslant b_* w_i w_i$$

on Ω for all $v = (v_1, v_2)$ and $w = (w_1, w_2)$ from H.

Let us remark that the symmetry of the Jacobi matrix $b_{1,2}(v_1, v_2) = b_{2,1}(v_1, v_2)$ is not needed; in the simplest special case (of linear B), $b_{1,1}$, $b_{1,2}$, $b_{2,1}$ and $b_{2,2}$ generate a positive definite matrix of constants from $\mathbb{R}^{2\times 2}$.

Let us begin with checking the properties concerning the spaces V and H (X is not needed yet) or the mapping B only (not the mapping A):

(a) The following result is a direct consequence of the Sobolev imbedding theorem (see [7], pp. 136, 217): there exists a compact imbedding of V into H.

(b) For any $\hat{\xi}$, $\tilde{\xi} \in \mathbb{R}$ and arbitrary $v, w \in H$ we have

$$\begin{aligned}(B(v+\hat{\xi}w) &- B(v+\tilde{\xi}w), w)\\ &= \int_\Omega (b_i(v_1+\hat{\xi}w_1, v_2+\hat{\xi}w_2) - b_i(v_1+\tilde{\xi}w_1, v_2+\tilde{\xi}w_2))w_i \,\mathrm{d}m\\ &= \int_\Omega \int_{\tilde{\xi}}^{\hat{\xi}} b_{i,j}(v_1+\xi w_1, v_2+\xi w_2)\,\mathrm{d}\xi\,(\hat{\xi}-\tilde{\xi})w_j w_i \,\mathrm{d}m.\end{aligned}$$

Thus the upper bound from the assumption (B2) gives

$$(B(v+\hat{\xi}w) - B(v+\tilde{\xi}w), w) \leqslant b_*|\hat{\xi}-\tilde{\xi}|\,\|w\|_H^2;$$

this yields the radial continuity of B. By [8], p. 66, in all reflexive and separable Banach spaces demicontinuity and radial continuity of monotone operators coincide. Since the property (f) (which will be verified independently) forces the monotony of B, the demicontinuity of B is guaranteed.

(f) From the assumption (B1) we obtain

$$\begin{aligned}(Bv &- Bw, v-w)\\ &= \int_\Omega (b_i(v_1,v_2) - b_i(w_1,w_2))(v_i-w_i)\,\mathrm{d}m\\ &= \int_\Omega \int_0^1 b_{i,j}(w_1+\xi(v_1-w_1), w_2+\xi(v_2-w_2))\,\mathrm{d}\xi(v_j-w_j)(v_i-w_i)\,\mathrm{d}m\end{aligned} \tag{14}$$

for all $v=(v_1,v_2)$ and $w=(w_1,w_2)$ from H. Thus the lower bound from the assumption (B2) gives

$$(Bv-Bw, v-w) \geqslant b_\diamond \|v-w\|_H^2;$$

we can clearly choose $\psi(c) = b_\diamond c^2$ for any $c \in \mathbb{R}$.

(g) For every positive integer j and $c_1,\dots,c_j \in \mathbb{R}_+$ we have

$$\psi(c_1+\ldots+c_j) = b_\diamond(c_1+\ldots+c_j)^2 \leqslant j b_\diamond(c_1^2+\ldots+c_j^2).$$

Consequently, we can put $\mu(j) = j$.

(h) Then we have also

$$\lim_{j\to\infty} \frac{\mu(j)}{j} = 1 < \infty.$$

(j) Let us recall (14). Both bounds from the assumption (B2) and the special choice of w as a zero point of H imply

$$0 \leqslant b_\diamond \|v\|_H^2 \leqslant (Bv, v) \leqslant b_* \|v\|_H^2 = \frac{b_*}{b_\diamond} \psi(\|v\|_H).$$

To present more useful arguments for the verification of all the remaining properties (in which the mapping A occurs), we shall construct this mapping as a sum of 3 mappings A_1, A_2 and A_3 in the sense

$$\langle Av, w\rangle = \langle A_1 v, w\rangle + \langle A_2 v, w\rangle + \langle A_3 v, w\rangle$$

for each $v, w \in V$. We will suppose $A_1\colon\ V \to V$, $A_2\colon\ X \to X$ and $A_3\colon\ H \to H$. The mapping A is well-defined; this follows from the Sobolev imbedding theorem (cf. the verificaton of the property (a)) and from the trace theorem (see [23], p. 32): there exists a compact imbedding of V into X.

Let λ_{ijkl} with $i, j, k, l \in \{1, 2\}$ be functions from $L^\infty(\Omega)$. Let g_i with $i \in \{1, 2\}$ be mappings from X to X and f_i analogous mappings from H to H. Let us assume:

(A1) The mapping A_1 has the form

$$\langle A_1 v, w\rangle = \int_\Omega \lambda_{ijkl}\, v_{i/j} w_{k/l}\, \mathrm{d}m$$

for all $v = (v_1, v_2)$ and $w = (w_1, w_2)$ from V where $v_{i/j}$ and $w_{k/l}$ is the simplified notation for the partial derivatives $\partial v_i/\partial x_j$ and $\partial w_k/\partial x_l$.

(A2) The operator A_2 has the form

$$\langle A_2 v, w\rangle = \int_{\partial\Omega} g_i(v_1, v_2) w_i\, \mathrm{d}s$$

for all $v = (v_1, v_2)$ and $w = (w_1, w_2)$ from X.

(A3) The operator A_3 has the form

$$\langle A_3 v, w\rangle = \int_\Omega f_i(v_1, v_2) w_i\, \mathrm{d}m$$

for all $v = (v_1, v_2)$ and $w = (w_1, w_2)$ from H.

(A4) For every $i, j, k, l \in \{1, 2\}$ and for a certain $\lambda_\diamond \in \mathbb{R}_+$,

$$\lambda_{ijkl}\, a_{ij} a_{kl} \geqslant \lambda_\diamond a_{ij} a_{ij}$$

holds on Ω for each matrix $a \in \mathbb{R}^{2\times 2}$.

(A5) The following growth condition is valid: there exist positive constants $\bar{g}$ and g_* such that

$$\max_{j\in\{1,2\}} g_j^2(v_1,v_2) \leqslant \bar{g}^2 + g_*^2 v_i v_i$$

on $\partial\Omega$ for all $v = (v_1, v_2)$ from X.

(A6) The following Lipschitz continuity condition is true: there exists a positive constant $\hat{f} \in \mathbb{R}_0$ such that

$$\max_{j\in\{1,2\}} (f_j(v_1,v_2) - f_j(w_1,w_2))^2 \leqslant \hat{f}^2 (v_i - w_i)(v_i - w_i)$$

on Ω for all $v = (v_1, v_2)$ and $w = (w_1, w_2)$ from H.

(A7) The symmetry $\lambda_{ijkl} = \lambda_{klij}$ is preserved on Ω for every $i, j, k, l \in \{1, 2\}$.

(A8) There exists a mapping $G\colon X \to X$ such that $g_i(v_1, v_2) = G_{,i}(v_1, v_2)$ for all $v = (v_1, v_2)$ from X and every $i \in \{1, 2\}$; $G_{,i}$ here means the derivative of G with respect to the i-th variable.

Now we are ready to complete the verification of those properties where A occurs:

(b) The weak continuity of A_1 follows from its linearity, forced by (A1). The Sobolev imbedding and the trace theorems (due to the continuity of f_1, f_2, g_1 and g_2) guarantee the weak continuity of A_2 and A_3 (by (A2) and (A3)), hence A is weakly continuous, too.

(d) It is easy to see that

$$\lambda_{ijkl}\, v_{i/j} w_{k/l} \leqslant \lambda_* \|v\|_V \|w\|_V$$

holds for any $v, w \in V$ with a certain $\lambda_* \in \mathbb{R}_+$ independent of v, w. Let K_0 and K_1 be positive constants; for any $c \in \mathbb{R}_0$ let us set $\varphi(c) = K_1 c - K_0$. (We believe that such φ will be suitable for the property (i), too.) From the assumption (A1) we have

$$\begin{aligned}\int_0^1 \langle A_1(w + \xi(v - w)), w\rangle \,\mathrm{d}\xi &= \frac{1}{2}\int_\Omega \lambda_{ijkl}\,(v_{i/j} + w_{i/j}) w_{k/l}\,\mathrm{d}m\\ &\leqslant \frac{\lambda_*}{2}(\|v\|_V + \|w\|_V)\|w\|_V.\end{aligned}$$

The assumptions (A2) and (A5) give

$$
\begin{aligned}
&\int_0^1 \langle A_2(w+\xi(v-w)), w\rangle \,\mathrm{d}\xi \\
&= \int_{\partial\Omega}\int_0^1 g_i(w_1+\xi(v_1-w_1), w_2+\xi(v_2-w_2))\,\mathrm{d}\xi w_i\,\mathrm{d}s \\
&\leqslant \max_{j\in\{1,2\}}\biggl(\int_{\partial\Omega}\int_0^1 g_j^2(w_1+\xi(v_1-w_1), w_2+\xi(v_2-w_2))\,\mathrm{d}\xi\,\mathrm{d}s\biggr)^{1/2} \\
&\quad\times\sum_{i=1}^2\biggl(\int_{\partial\Omega} w_i^2\,\mathrm{d}s\biggr)^{1/2} \\
&\leqslant 2^{1/2}\max_{j\in\{1,2\}}\biggl(\int_{\partial\Omega}\int_0^1 g_j^2(w_1+\xi(v_1-w_1), w_2+\xi(v_2-w_2))\,\mathrm{d}\xi\,\mathrm{d}s\biggr)^{1/2}\|w\|_X \\
&\leqslant 2^{1/2}\biggl(\bar g^2 s(\partial\Omega)+g_*^2\int_{\partial\Omega}\int_0^1 (w_i+\xi(v_i-w_i))(w_i+\xi(v_i-w_i))\,\mathrm{d}\xi\,\mathrm{d}s\biggr)^{1/2}\|w\|_X \\
&= 2^{1/2}\biggl(\bar g^2 s(\partial\Omega)+g_*^2\int_{\partial\Omega} g\int_0^1 ((1-\xi)^2 w_iw_i+\xi^2 v_iv_i+2\xi(1-\xi)v_iw_i)\,\mathrm{d}\xi\,\mathrm{d}s\biggr)^{1/2} \\
&\quad\times\|w\|_X \\
&\leqslant 2^{1/2}\biggl(\bar g^2 s(\partial\Omega)+g_*^2\biggl(\frac{1}{3}\|w\|_X^2+\frac{1}{3}\|v\|_X^2+2\cdot\frac{1}{6}\cdot\frac{1}{2}(\|w\|_X^2+\|v\|_X^2)\biggr)\biggr)^{1/2}\|w\|_X \\
&= (2\bar g^2 s(\partial\Omega)+g_*^2\|w\|_X^2+g_*^2\|v\|_X^2)^{1/2}\|w\|_X \\
&\leqslant (\bar g(2s(\partial\Omega))^{1/2}+g_*\|w\|_X+g_*\|v\|_X)\|w\|_X;
\end{aligned}
$$

moreover (from the trace theorem), an estimate $\|v\|_X \leqslant K\|v\|_V$ with a positive constant K independent of v is available. The assumption (A3) implies

$$\int_0^1 \langle A_3(w+\xi(v-w)), w\rangle\,\mathrm{d}\xi = \int_{\partial\Omega}\int_0^1 f_i(w_1+\xi(v_1-w_1), w_2+\xi(v_2-w_2))\,\mathrm{d}\xi\, w_i\,\mathrm{d}m;$$

this seems to be similar to the previous case with A_2. Indeed, choosing w in the assumption (A6) as (o,o) where o is the zero-valued function defined on Ω, for arbitrary $j\in\{1,2\}$ we obtain

$$(f_j(v_1,v_2)-f_j(o,o))^2 \leqslant \hat f^2 v_iv_i$$

and consequently

$$f_j^2(v_1,v_2)-\frac{1}{2}f_j^2(v_1,v_2)-2f_j^2(o,o)+f_j^2(o,o)\leqslant \hat f^2 v_iv_i,$$

which yields

$$f_j^2(v_1,v_2)\leqslant 2f_j^2(o,o)+2\hat f^2 v_iv_i.$$

Thus we can rewrite the growth condition from the assumption (A5) with f_j instead of g_j on Ω instead of $\partial\Omega$ formally with the constants $f_* = \sqrt{2}\hat{f}$ instead of g_* and $\overline{f} = \sqrt{2}\max(f_1(o,o), f_2(o,o))$ instead of $\bar{g}$. Then $m(\Omega)$ instead of $s(\partial\Omega)$ and all norms in H instead of X (every $K \geqslant 1$ is acceptable here) occur in the final estimate. For the sum of A_1, A_2 and A_3 we can conclude

$$\int_0^1 \langle A(v+\xi(v-w)), w\rangle \,\mathrm{d}\xi \leqslant \overline{K}_1\|v\|_V + \overline{K}_0$$

where $\overline{K}_0$ and $\overline{K}_1$ are positive constants independent of v, w. Under the assumption $v \in V'_\varrho$ this inequality yields

$$\begin{aligned}(\varphi(\|v\|_V)\|v\|_V)^{-1}\int_0^1 \langle A(w+\xi(v-w)), w\rangle \,\mathrm{d}\xi &\leqslant \frac{\overline{K}_1\|v\|_V + \overline{K}_0}{(K_1\|v\|_V - K_0)\|v\|_V} \\ &\leqslant \frac{\overline{K}_1 + \overline{K}_0/\varrho}{K_1\varrho - K_0},\end{aligned}$$

but this must be finite for any ϱ large enough.

(e) The preceding argumentation, concerning the verification of the property (d), can be easily adapted to this case: the estimate

$$(Bv, w) = \int_\Omega \int_0^1 b_{i,j}(v_1, v_2)\,\mathrm{d}\xi\, v_i w_j \,\mathrm{d}m \leqslant b_*\|v\|_H\|w\|_H \leqslant b_*\|v\|_V\|w\|_H$$

leads for any $v \in V'_\varrho$ directly to

$$(\varphi(\|v\|_V)\|v\|_V)^{-1}\int_0^1 (Bv, w)\,\mathrm{d}\xi \leqslant \frac{b_*\|w\|_H}{K_1\|v\|_V - K_0} \leqslant \frac{b_*\|w\|_H}{K_1\varrho - K_0}.$$

(i) The assumptions (A1) and (A4) guarantee

$$\int_0^1 \langle A_1(\xi v), v\rangle \,\mathrm{d}\xi = \frac{1}{2}\int_\Omega \lambda_{ijkl}\, v_{i/j} v_{k/l} \,\mathrm{d}m \geqslant \frac{\lambda_\diamond}{2}\|v\|_V^2$$

and together with the property (j) they yield

$$\int_0^1 \langle A_1(\xi v), v\rangle \,\mathrm{d}\xi + (Bv, v) \geqslant \frac{\lambda_\diamond}{2}\|v\|_V^2 + \nu b_\diamond\|v\|_H^2.$$

Following the verification of the property (d) we obtain

$$
\begin{aligned}
\int_0^1 \langle A_2(\xi v), v\rangle \,\mathrm{d}\xi = & \int_{\partial\Omega}\int_0^1 g_i(\xi v_1, \xi v_2)\,\mathrm{d}\xi\, v_i\,\mathrm{d}s \\
\geqslant & -\max_{j\in\{1,2\}}\left(\int_{\partial\Omega}\int_0^1 g_j^2(\xi v_1,\xi v_2)\,\mathrm{d}\xi\,\mathrm{d}s\right)^{1/2}\sum_{i=1}^{2}\left(\int_{\partial\Omega} v_i^2\,\mathrm{d}s\right)^{1/2} \\
\geqslant & -2^{1/2}\max_{j\in\{1,2\}}\left(\int_{\partial\Omega}\int_0^1 g_j^2(\xi v_1,\xi v_2)\,\mathrm{d}\xi\,\mathrm{d}s\right)^{1/2}\|v\|_X \\
\geqslant & -2^{1/2}\left(\bar{g}^2 s(\partial\Omega) + g_*^2\int_{\partial\Omega}\int_0^1 \xi^2\,\mathrm{d}\xi\, v_i v_i\,\mathrm{d}s\right)^{1/2}\|v\|_X \\
= & -\left(2\bar{g}^2 s(\partial\Omega) + \frac{2g_*^2}{3}\|v\|_X^2\right)^{1/2}\|v\|_X \\
\geqslant & -(\bar{g}(2s(\partial\Omega))^{1/2} + (2/3)^{1/2} g_*\|v\|_X)\|v\|_X \\
\geqslant & -K\bar{g}(2s(\partial\Omega))^{1/2}\|v\|_V - (2/3)^{1/2} g_*\|v\|_X^2 .
\end{aligned}
$$

Now we need a more precise estimate for $\|v\|_X$ in the last additive term: from [22], p. 222, we know that

$$\|v\|_X \leqslant \delta\|v\|_V + \frac{\overline{K}}{\delta}\|v\|_H$$

for every $v \in V$, some constant $\overline{K} \in \mathbb{R}_+$, and any $\delta \in \mathbb{R}_+$. This inequality implies

$$-\|v\|_X^2 \geqslant -2\delta^2\|v\|_V^2 - \frac{2\overline{K}^2}{\delta^2}\|v\|_H^2 .$$

The same result can be obtained for A_3 with $s(\partial\Omega)$ replaced by $m(\Omega)$, g replaced by f and X replaced by H; the last step can be omitted because all norms are in H only. For ν large and δ small enough all these estimates together give

$$\int_0^1 \langle A(\xi v), v\rangle\,\mathrm{d}\xi + (Bv, v) \geqslant (K_1\|v\|_V - K_0)\|v\|_V + b_\diamond\|v\|_H^2$$

where K_0 and K_1 are some positive constants independent of v; this by virtue of the special choice of φ and ψ (introduced in the verification of the properties (f) and (d) above) completes this verification.

(k) The mappings A_1 and A_2 are negligible here; this is evident from an equality based on the properties (A1) and (A7):

$$\begin{aligned}
&\int_0^1 \langle A_1(w+\xi(v-w)), v-w\rangle \,\mathrm{d}\xi - \int_0^1 \langle A_1(\xi v), v\rangle \,\mathrm{d}\xi + \int_0^1 \langle A_1(\xi w), w\rangle \,\mathrm{d}\xi \\
&= \frac{1}{2}\int_\Omega \lambda_{ijkl}\,(v_{i/j}+w_{i/j})(v_{k/l}-w_{k/l}) - \frac{1}{2}\int_\Omega \lambda_{ijkl}\, v_{i/j}v_{k/l} + \frac{1}{2}\int_\Omega \lambda_{ijkl}\, w_{i/j}w_{k/l} \\
&\quad + \frac{1}{2}\int_\Omega (\lambda_{ijkl}-\lambda_{klij}) v_{i/j}w_{k/l} = 0,
\end{aligned}$$

and from another one based on the properties (A2) and (A8):

$$\begin{aligned}
&\int_0^1 \langle A_2(w+\xi(v-w)), v-w\rangle \,\mathrm{d}\xi - \int_0^1 \langle A_2(\xi v), v\rangle \,\mathrm{d}\xi + \int_0^1 \langle A_2(\xi w), w\rangle \,\mathrm{d}\xi \\
&= \int_{\partial\Omega}\int_0^1 G_{,i}(w_1+\xi(v_1-w_1), w_2+\xi(v_2-w_2))\,\mathrm{d}\xi\,(v_i-w_i)\,\mathrm{d}s \\
&\quad - \int_{\partial\Omega}\int_0^1 G_{,i}(\xi v_1, \xi v_2)\,\mathrm{d}\xi\, v_i\,\mathrm{d}s + \int_{\partial\Omega}\int_0^1 G_{,i}(\xi w_1, \xi w_2)\,\mathrm{d}\xi\, v_i\,\mathrm{d}s \\
&= \int_{\partial\Omega} (G(v_1,v_2)-G(w_1,w_2))\,\mathrm{d}s - \int_{\partial\Omega} G(v_1,v_2)\,\mathrm{d}s + \int_{\partial\Omega} G(w_1,w_2)\,\mathrm{d}s = 0.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\int_0^1 \langle A(w+\xi(v-w)), v-w\rangle \,\mathrm{d}\xi - \int_0^1 \langle A(\xi v), v\rangle \,\mathrm{d}\xi + \int_0^1 \langle A(\xi w), w\rangle \,\mathrm{d}\xi \\
&= \int_0^1 \langle A_3(w+\xi(v-w)), v-w\rangle \,\mathrm{d}\xi - \int_0^1 \langle A_3(\xi v), v\rangle \,\mathrm{d}\xi + \int_0^1 \langle A_3(\xi w), w\rangle \,\mathrm{d}\xi \\
&= \int_\Omega\int_0^1 f_i(w_1+\xi(v_1-w_1), w_2+\xi(v_2-w_2))\,\mathrm{d}\xi\,(v_i-w_i)\,\mathrm{d}m \\
&\quad - \int_\Omega\int_0^1 f_i(\xi v_1,\xi v_2)\,\mathrm{d}\xi\, v_i\,\mathrm{d}m + \int_\Omega\int_0^1 f_i(\xi w_1,\xi w_2)\,\mathrm{d}\xi\, w_i\,\mathrm{d}m \\
&= \int_\Omega\int_0^1 (f_i(w_1+\xi(v_1-w_1), w_2+\xi(v_2-w_2)) - f_i(\xi v_1,\xi v_2)\,\mathrm{d}\xi\,(v_i-w_i)\,\mathrm{d}m \\
&\quad - \int_\Omega\int_0^1 (f_i(\xi v_1,\xi v_2) - f_i(\xi w_1,\xi w_2))\,\mathrm{d}\xi\, w_i\,\mathrm{d}m \\
&\geqslant -\max_{j\in\{1,2\}}\Big(\int_\Omega\int_0^1 \big(f_j(w_1+\xi(v_1-w_1), w_2+\xi(v_2-w_2)) \\
&\quad - f_j(\xi v_1,\xi v_2)^2\,\mathrm{d}\xi\,\mathrm{d}m\Big)^{1/2} \sum_{i=1}^2 \Big(\int_\Omega (v_i-w_i)^2\,\mathrm{d}m\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&- \max_{j\in\{1,2\}} \Big(\int_\Omega \int_0^1 (f_j(\xi v_1, \xi v_2) - f_j(\xi w_1, \xi w_2))^2 \,\mathrm{d}\xi \,\mathrm{d}m\Big)^{1/2} \sum_{i=1}^{2} \Big(\int_\Omega w_i^2 \,\mathrm{d}m\Big)^{1/2} \\
\geqslant\ & - 2^{1/2} \hat{f} \Big(\int_\Omega \int_0^1 (1-\xi)^2 \,\mathrm{d}x\, w_i w_i \,\mathrm{d}m\Big)^{1/2} \|v-w\|_H \\
&- 2^{1/2} \hat{f} \Big(\int_\Omega \int_0^1 \xi^2 \,\mathrm{d}\xi\, (v_i - w_i)(v_i - w_i) \,\mathrm{d}m\Big)^{1/2} \|w\|_H \\
\geqslant\ & - 2(2/3)^{1/2} \hat{f} \|w\|_H \|v-w\|_H \\
\geqslant\ & - 2(2/(3b_\diamond))^{1/2} \hat{f} \|w\|_V \sqrt{\psi(\|v-w\|_H)},
\end{aligned}
$$

which is even stronger than needed: the estimate

$$
\begin{aligned}
\|w\|_V^2 &= \|w\|_V^2 - 2\varepsilon\|w\|_V + 2\varepsilon\|w\|_V \leqslant \|w\|_V^2 - 2\varepsilon\|w\|_V + \|w\|_V^2 + \varepsilon^2 \\
&= 2\|w\|_V(\|w\|_V - \varepsilon) + \varepsilon^2
\end{aligned}
$$

for $\varepsilon = K_0/K_1$ converts the result easily into the required form.

We can sum up: Theorems 1, 2 and 3 hold. Theorems 4, 5 and 6 cannot be applied directly—no mapping P has been given. In more difficult problems the property (k′) may be more realistic then (k); to handle such cases, some "penalty term" using the mapping P (with reasonable interpretation in physics, if possible) should be included.

7. Conclusions and remarks to applications

In this paper we have demonstrated that for a rather large class of problems of evolution (including at least some problems of moisture and heat transfer in porous media, as the one from the introduction) certain reasonable existence and convergence results can be derived using the properties of the Rothe sequences (the method of discretization in time). Nevertheless, one can see that the problem is not closed: e.g. the property (k) can be violated in many situations of practical interest and the construction of a mapping P, required by the generalized property (k′), need not have a transparent physical interpretation. An other disadvantage, especially from the point of view of practical computations, is that the formulation in spaces of abstract functions avoids the discretization in $\mathbb{R}^N$ completely; this must be done using the finite element or similar techniques, but to verify the convergence properties of the corresponding sequences of functions from finite-dimensional subspaces of V may be not easy as a consequence of the nonlinearity of A and B (and P, if needed).

We have not discussed the case of weakly continuous operators B because this seems to be a very strong and physically non-realistic assumption in applications: e.g. from [1], pp. 63, 103, and [27], p. 360, we know that in the Lebesgue space $H = L^2(\Omega, \mathbb{R}^N)$, where Ω is an open set in $\mathbb{R}^N$, every weakly continuous mapping is linear. This is not true for weakly continuous operators A (and P, if necessary) in the Sobolev spaces: e.g. if V is some subspace of $W^{1,2}(\Omega, \mathbb{R}^N)$ containing $W_0^{1,2}(\Omega, \mathbb{R}^N)$ (to include prescribed boundary conditions of Dirichlet type) then many nonlinear weakly continuous mappings (as in examples from [6], pp. 52, 53) exist and the weak continuity can be tested effectively using the theorem on Nemytskiĭ operators (cf. [7], p. 75, [5], p. 288, and [25], p. 36). Other than Sobolev spaces can be also applied: e.g. in [2] the regularity questions are analyzed in the Morrey-Campanato spaces (cf. [23], p. 35).

For numerical modelling of problems of heat and moisture transfer based on [4] (especially for the numerical construction of Rothe sequences in a sufficiently general case) no standard software seems to be available. To verify theoretical results by numerical experiments, several special PC programs have been written by the authors of [4] in the Fortran and Pascal code. The complete development of an original software package would be very expensive (consuming both much time and much money) and would require periodical update in future due to the hardware and software progress (probably in every new Windows based operation system). To avoid most complications of this type, experiments with the PDE toolbox of the MATLAB software package (new updates of this toolbox have a modified commercial name FEMLAB) and with its compatibility to the user-defined functions in C++ code via the so-called MEX-files (dynamically linked subroutines that the MATLAB interpreter can automatically load and execute—cf. [21], page 2-2) have been made successfully with interesting numerical outputs. This should be a promising way even in such cases when professional development of algorithms and the corresponding programs in commercially oriented software incorporations (as ABAQUS, ANSYS, etc.) cannot be expected.

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Author's address: *Jiří Vala*, Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Technical University in Brno, 662 37 Brno, Žižkova 17, Czech Republic, e-mail: `Vala.J@fce.vutbr.cz`.