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STEADY PLANE FLOW OF VISCOELASTIC FLUID PAST AN OBSTACLE*

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Abstract. We consider the steady plane flow of certain classes of viscoelastic fluids in exterior domains with a non-zero velocity prescribed at infinity. We study existence as well as asymptotic behaviour of solutions near infinity and show that for sufficiently small data the solution decays near infinity as fast as the fundamental solution to the Oseen problem.

Keywords: viscoelastic fluid, Oseen problem, steady transport equation, weighted estimates

MSC 2000: 35B40, 76D99

0. Introduction

We study the steady flow of certain classes of viscoelastic fluids past an obstacle in two space dimensions. We consider a class of viscoelastic fluids for which the extra stress tensor $\mathbf{T}$ satisfies

$$\mathbf{T} + \lambda \frac{\mathcal{D}_a \mathbf{T}}{\mathcal{D}t} + \mathbf{B}(\mathbf{D}, \mathbf{T}) = 2\eta \mathbf{D}, \tag{0.1}$$

where

$$\frac{\mathcal{D}_a \mathbf{T}}{\mathcal{D}t} = (\mathbf{v} \cdot \nabla)\mathbf{T} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T} - a(\mathbf{D}\mathbf{T} + \mathbf{T}\mathbf{D}) \tag{0.2}$$

represents the objective derivative of a symmetric tensor in the stationary case, $\mathbf{v}(\mathbf{x})$ is the velocity of the fluid at the point $\mathbf{x}$, $\mathbf{D}$ is the symmetric part and $\mathbf{W}$ the skew

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part of the gradient of velocity, $\mathbf{B}(\mathbf{D}, \mathbf{T})$ will be in our case a bilinear tensor-valued function, λ characterizes the relaxation time, $\eta > 0$ is the viscosity and $a \in [-1; 1]$ is a given real parameter (see e.g. [1] for another possible choices of $\mathbf{B}$ and for characterizations of other models of viscoelastic fluids).

We will assume that the fluid occupies an exterior domain Ω in $\mathbb{R}^2$; without loss of generality we may take $\mathbb{R}^2 \setminus \Omega \subset B_1(\mathbf{0})$ and $\mathbf{0} \in \mathbb{R}^2 \setminus \Omega$.

Further, let $\mathbf{v} = \mathbf{0}$ at $\partial\Omega$ and $\mathbf{v} \to \mathbf{v}_\infty = |\mathbf{v}_\infty|\mathbf{e}_1$ as $|\mathbf{x}| \to \infty$, where e_1 is the unit vector in the direction of the x_1-axis. Throughout the paper the constant velocity at infinity $\mathbf{v}_\infty$ will be non-zero but small. We denote $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty$ and $|\mathbf{v}_\infty| = \beta$. Similarly as in [9] we put (0.1) and (0.2) into the general equations describing the steady flow of a fluid and using the idea of Renardy (see [11]) we end up with

$$
\begin{aligned}
&-\Delta \mathbf{u} + \mathcal{R}'\beta \frac{\partial \mathbf{u}}{\partial x_1} + \nabla \pi \\
&\quad = \mathbf{f} + \nabla \cdot \Big[\mathcal{W}'\mathbf{F}(\nabla \mathbf{u}, \mathbf{T}) - \mathcal{R}'\mathcal{W}'((\mathbf{u}\cdot\nabla)\mathbf{u}) \otimes \mathbf{u} - \mathcal{R}'\mathbf{u}\otimes\mathbf{u} - \mathcal{R}'\mathcal{W}'\beta^2 \frac{\partial \mathbf{u}}{\partial x_1} \otimes \mathbf{e}_1 \\
&\qquad - \mathcal{R}'\mathcal{W}'\beta\Big(\frac{\partial \mathbf{u}}{\partial x_1} \otimes \mathbf{u} + ((\mathbf{u}\cdot\nabla)\mathbf{u}) \otimes \mathbf{e}_1\Big) + \mathcal{W}'\mathbf{f} \otimes (\mathbf{u} + \beta\mathbf{e}_1) + \mathcal{W}' p (\nabla \mathbf{u})^T \Big],
\end{aligned} \tag{0.3}
$$

$$\nabla \cdot \mathbf{u} = 0, \tag{0.4}$$

$$
\begin{aligned}
\mathbf{u} &= -\beta\mathbf{e}_1 \quad \text{on } \partial\Omega, \\
\mathbf{u} &\to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty,
\end{aligned} \tag{0.5}
$$

$$\pi = p + \mathcal{W}'((\mathbf{u} + \beta\mathbf{e}_1)\cdot\nabla)p, \tag{0.6}$$

$$\mathbf{T} + \mathcal{W}'((\mathbf{u} + \beta\mathbf{e}_1)\cdot\nabla)\mathbf{T} + \mathcal{W}'\mathbf{G}(\nabla\mathbf{u}, \mathbf{T}) = 2\mathbf{D}(\mathbf{u}). \tag{0.7}$$

Here $p(\mathbf{x})$ denotes the pressure of the fluid at the point $\mathbf{x}$, $\mathbf{f}$ is the external force. The bilinear functions $\mathbf{F}$ and $\mathbf{G}$ are defined by

$$
\begin{aligned}
\mathbf{F}(\nabla\mathbf{u}, \mathbf{T}) &= -\mathbf{T}(\nabla\mathbf{u})^T - (\mathbf{TW} - \mathbf{WT}) + a(\mathbf{DT} + \mathbf{TD}) - \mathbf{B}(\mathbf{D}, \mathbf{T}), \\
\mathbf{G}(\nabla\mathbf{u}, \mathbf{T}) &= (\mathbf{TW} - \mathbf{WT}) - a(\mathbf{DT} + \mathbf{TD}) + \mathbf{B}(\mathbf{D}, \mathbf{T}),
\end{aligned}
$$

where $\mathcal{W}' = \lambda/L$, $\mathcal{R}' = \varrho L/\eta$, L is the diameter of the obstacle. Unlike the three-dimensional case, we cannot get the asymptotic structure of the solution without the assumption $|\mathbf{v}_\infty| = \beta$ sufficiently small. Thus it is not convenient to write the system in dimensionless form with $\mathcal{W}'$ replaced by the Weisenberg number $\mathcal{W} = \lambda\beta/L$, $\mathcal{R}'$ by the Reynolds number $\mathcal{R} = \varrho\beta L/\eta$, assume only these two numbers small and take $\beta = 1$. Since for β sufficiently small there are no further restrictions on the size of $\mathcal{R}'$ and $\mathcal{W}'$, in what follows we put $\mathcal{R}' = 1$, $\mathcal{W}' = 1$. Let us also note that in the existence part (i.e. in Section 2) we could, similarly as in [9], get the existence of a solution for $\mathcal{R}$ and $\mathcal{W}$ small, without any further restriction on the size of β. But, unlike [9], we would have to assume the external force $\mathbf{f}$ to be small.

We linearize the system (0.3)–(0.7) around $\mathbf{u} = \mathbf{0}$. Denoting $A(\mathbf{u}) = -\Delta\mathbf{u} + \beta^2\partial^2\mathbf{u}/\partial x_1^2$ we introduce an operator

$$\mathcal{M}\colon\ (\mathbf{w}, s) \mapsto (\mathbf{u}, \pi),$$

as the solution operator of the system

$$
\begin{aligned}
(0.8)\quad & A(\mathbf{u}) + \beta\frac{\partial\mathbf{u}}{\partial x_1} + \nabla\pi \\
&\quad = \mathbf{f} + \nabla\cdot\Big[\mathbf{F}(\nabla\mathbf{w}, \mathbf{T}) - ((\mathbf{w}\cdot\nabla)\mathbf{w})\otimes\mathbf{w} - \mathbf{w}\otimes\mathbf{w} \\
&\qquad - \beta\Big(\frac{\partial\mathbf{w}}{\partial x_1}\otimes\mathbf{w} + ((\mathbf{w}\cdot\nabla)\mathbf{w})\otimes\mathbf{e}_1\Big) + \mathbf{f}\otimes(\mathbf{w}+\beta\mathbf{e}_1) + p(\nabla\mathbf{w})^T\Big],
\end{aligned}
$$

$$(0.9)\qquad \nabla\cdot\mathbf{u} = 0,$$

$$(0.10)\qquad \mathbf{u}\to\mathbf{0}\quad\text{as }\ |\mathbf{x}|\to\infty,$$

$$\mathbf{u} = -\beta\mathbf{e}_1\quad\text{on }\ \partial\Omega,$$

$$(0.11)\qquad p + ((\mathbf{w}+\beta\mathbf{e}_1)\cdot\nabla)p = s,$$

$$(0.12)\qquad \mathbf{T} + ((\mathbf{w}+\beta\mathbf{e}_1)\cdot\nabla)\mathbf{T} + \mathbf{G}(\nabla\mathbf{w}, \mathbf{T}) = 2\mathbf{D}(\mathbf{w}).$$

We have decomposed the original problem into two kinds of standard problems; the Oseen-like problem (0.8)–(0.10) and the steady transport equations (0.11)–(0.12).

Similarly as in three dimensions (see [9]), our aim is to show that at least for small data the solution decays as fast as the fundamental solution to the Oseen problem. A similar result is in the two-dimensional case known e.g. for the incompressible (see [12]) or compressible (see [2]) Navier-Stokes equations. Denoting $s(\mathbf{x}) = |\mathbf{x}| - x_1$ we would like to prove that

$$
\begin{aligned}
(0.13)\qquad & |u_1(\mathbf{x})| \leqslant C|\mathbf{x}|^{-\frac{1}{2}}(1+s(\mathbf{x}))^{-\frac{1}{2}}, \\
& |u_2(\mathbf{x})| \leqslant C|\mathbf{x}|^{-1}
\end{aligned}
$$

for $|\mathbf{x}|$ sufficiently large.

While $(0.13)_1$ will be shown below, instead of $(0.13)_2$ we show only

$$|u_2(\mathbf{x})| \leqslant C|\mathbf{x}|^{-1}|\ln(2+|\mathbf{x}|)|^{-1}.$$

The situation is the same as in the case of the incompressible Navier-Stokes equations (see [12]). It is connected with the technique of estimates of certain convolutions and, at least using this technique, it seems not to be possible to get rid of the additional logarithmic term.

Similarly to the three-dimensional case, the L^p weighted estimates (Theorem 1.6) do not allow to consider instead of the modified Oseen problem (0.8)–(0.10) the (classical) Oseen problem, i.e. with $A(\mathbf{u}) = -\Delta\mathbf{u}$.

Throughout the paper we use standard notation for the Lebesgue spaces $L^p(\Omega)$ together with the norm $\|\cdot\|_p$, the Sobolev spaces $W^{k,p}(\Omega)$ with the norm $\|\cdot\|_{k,p}$, the homogeneous Sobolev spaces $D^{k,p}(\Omega)$ with the norm $|\cdot|_{k,p}$ and the spaces of continuously differentiable functions $C^k(\overline{\Omega})$ with the norm $\|\cdot\|_{C^k}$. Moreover, let $g \in L^1_{\mathrm{loc}}(\Omega)$ be a non-negative weight. Then $L^p(\Omega; g)$ denotes the weighted L^p space equipped with the norm

$$\|u\|_{p,(g)} = \|ug\|_p. \tag{1.1}$$

Similarly, the space $W^{k,p}(\Omega; g)$ denotes the space of weakly differentiable functions for which the norm

$$\|u\|_{k,p,(g)} = \|ug\|_{k,p}$$

is finite.

We have decomposed the original problem into two kinds of linear problems: the (modified) Oseen problem and the steady transport equation. We will recall several known properties of these problems in exterior plane domains. Let us start with the former.

We denote by $(\boldsymbol{\mathcal{O}}^\mu, \mathbf{e})$ the fundamental solution to the modified Oseen problem (i.e. solution of

$$-\left[\Delta - \mu\frac{\partial^2}{\partial y_1^2} + 2\lambda\frac{\partial}{\partial y_1}\right]\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};2\lambda) - \frac{\partial}{\partial y_i}e_j(\mathbf{x}-\mathbf{y}) = \delta_{ij}\delta_{\mathbf{x}},$$
$$\frac{\partial\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};2\lambda)}{\partial y_i} = 0$$

in the sense of distributions). It can be verified that its asymptotic structure is the same as the asymptotic structure of the fundamental solution to the classical Oseen problem (see [3] or also [10]). Thus we have

$$\mathbf{e} = \nabla\mathcal{E}(\mathbf{x})$$

with $\mathcal{E}(\mathbf{x})$ the fundamental solution to the Laplace equation. The tensor $\boldsymbol{\mathcal{O}}^\mu$ satisfies the homogeneity property

$$\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\lambda) = \boldsymbol{\mathcal{O}}^\mu(\lambda\mathbf{x};1)$$

and therefore it is sufficient to study only the case $\lambda = 1$.

For small $|\mathbf{x}|$ we have

$$\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};1) = \boldsymbol{\mathcal{S}}(\mathbf{x}) + \boldsymbol{\mathcal{R}}(\mathbf{x}),$$

where $\boldsymbol{\mathcal{S}}$ denotes the singular part of $\boldsymbol{\mathcal{O}}^\mu$ (i.e. $D^\alpha \boldsymbol{\mathcal{S}}$ for $|\alpha| = 2$ is a singular integral kernel, see [3]) and

$$\left.\begin{aligned} \big|\boldsymbol{\mathcal{R}}(\mathbf{x})\big| &\leqslant C,\\ D^\alpha \boldsymbol{\mathcal{R}}(\mathbf{x}) &\leqslant \begin{cases} C \ln|\mathbf{x}|, & |\alpha| = 1,\\ C|\mathbf{x}|^{-|\alpha|+1}, & |\alpha| \geqslant 2 \end{cases} \end{aligned}\right\} \quad \text{for } |\mathbf{x}| \to 0.$$

Denote further $s(\mathbf{x}) = |\mathbf{x}| - x_1$. For $|\mathbf{x}| \to \infty$ we have[1]

$$\begin{aligned} \mathcal{O}^\mu_{11}(\mathbf{x}; 1) &\leqslant C|\mathbf{x}|^{-\frac{1}{2}}(1 + s(\mathbf{x}))^{-\frac{1}{2}},\\ \mathcal{O}^\mu_{ij}(\mathbf{x}; 1) &\leqslant C|\mathbf{x}|^{-1} \quad i + j \geqslant 3,\\ \frac{\partial \mathcal{O}^\mu_{11}}{\partial x_2}(\mathbf{x}; 1) &\leqslant C|\mathbf{x}|^{-1}(1 + s(\mathbf{x}))^{-1},\\ \frac{\partial \mathcal{O}^\mu_{ij}}{\partial x_k}(\mathbf{x}; 1) &\leqslant C|\mathbf{x}|^{-\frac{3}{2}}(1 + s(\mathbf{x}))^{-\frac{1}{2}}, \quad (i, j, k) \neq (1, 1, 2),\\ \nabla^2 \boldsymbol{\mathcal{O}}^\mu(\mathbf{x}; 1) &\leqslant C|\mathbf{x}|^{-\frac{3}{2}}(1 + s(\mathbf{x}))^{-\frac{3}{2}}. \end{aligned}$$

Next, let us consider the problem

$$\begin{gathered} -\Delta \mathbf{u} + \mu \frac{\partial^2 \mathbf{u}}{\partial x_1^2} + \beta \frac{\partial \mathbf{u}}{\partial x_1} + \nabla \pi = \mathbf{f} = \nabla \cdot \boldsymbol{\mathcal{G}},\\ \nabla \cdot \mathbf{u} = 0,\\ \mathbf{u} = \mathbf{u}_* \quad \text{on } \partial\Omega,\\ \mathbf{u} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty \end{gathered} \tag{1.2}$$

with Ω an exterior domain in $\mathbb{R}^2$, $0 \leqslant \mu < 1$.

Using a similar procedure as for the classical Oseen problem we can show (see [10], note that $\mu = \beta^2$ in our case; compare also with [4] or [7]).

Theorem 1.1. *Let $\mathbf{f} \in L^q(\Omega) \cap W^{k,p}(\Omega)$, $1 < q < \frac{6}{5}$, $1 < p < \infty$, $k \geqslant 0$, let Ω be a plane exterior domain of class C^{k+2}, $\mathbf{u}_* \in W^{k+2-\frac{1}{p},p}(\partial\Omega) \cap W^{2-\frac{1}{q},q}(\partial\Omega)$. Let $\mu(\beta) \leqslant C|\ln \beta|^{-1}$ for $\beta \to 0^+$. Then there exists a unique strong solution to the modified Oseen problem* (1.2) $(\mathbf{u}, \pi)$ *and it satisfies*

$$\begin{aligned} &\beta(\|u_2\|_{\frac{2q}{2-q}} + \|\nabla u_2\|_q) + \beta^{\frac{2}{3}}|\mathbf{u}|_{\frac{3q}{3-2q}} + \beta^{\frac{1}{3}}\|\nabla^2 \mathbf{u}\|_{\frac{3q}{3-q}}\\ &\qquad \leqslant C(\|\mathbf{f}\|_q + \beta^{2(1-\frac{1}{q})}|\ln \beta|^{-1}\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}),\\ &\beta^{2(1-\frac{1}{q})}(\|\nabla^2 \mathbf{u}\|_q + \|\nabla \pi\|_q + |\nabla^2 \mathbf{u}|_{k,p} + \|\nabla \pi\|_{k,p})\\ &\qquad \leqslant C\big(\|\mathbf{f}\|_q + \beta^{2(1-\frac{1}{q})}(\|\mathbf{f}\|_{k,p} + \|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)} + \|\mathbf{u}_*\|_{k+2-\frac{1}{p},p,(\partial\Omega)})\big). \end{aligned}$$

The constant C depends on k, q, p, Ω and, for $\beta \in (0; B]$, only on B.

[1] Although in some cases we can get better uniform behaviour, we do not use it and thus we will not write it out explicitly.

For obtaining weighted estimates we will use the integral representation of solutions to (1.2) constructed in Theorem 1.1.

Let us denote

$$
\begin{aligned}
(1.3)\qquad T_{ij}(\mathbf{u},\pi) &= \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \pi\delta_{ij} - \mu\delta_{1j}\frac{\partial u_i}{\partial x_1},\\
\mathcal{T}_{ij}(\mathbf{e}) &= \frac{\partial e_i}{\partial x_j} + \frac{\partial e_j}{\partial x_i} + \beta e_1\delta_{ij} + \mu\frac{\partial e_1}{\partial x_1}\delta_{ij} - \mu\delta_{1j}\frac{\partial e_i}{\partial x_1}.
\end{aligned}
$$

Theorem 1.2. *Let $\Omega \in C^2$ be a plane exterior domain, $\boldsymbol{\mathcal{G}} \in C_0^\infty(\overline{\Omega})$ and let $(\mathbf{u},\pi)$ be the unique solution to the Oseen problem (1.2). Let $\mathbf{T}$ be defined in $(1.3)_1$, let $(\boldsymbol{\mathcal{O}}^\mu,\mathbf{e})$ be the fundamental solution to the modified Oseen problem. Then*

$$
\begin{aligned}
(1.4)\qquad u_j(\mathbf{x}) = {}& \int_\Omega \frac{\partial}{\partial x_k}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\mathcal{G}_{ik}(\mathbf{y})\,\mathrm{d}\mathbf{y}\\
&+ \int_{\partial\Omega} [-\beta\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1k} + u_i(\mathbf{y})T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j},e_j)(\mathbf{x}-\mathbf{y};\beta)\\
&+ \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},\pi)(\mathbf{y}) + \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\mathcal{G}_{ik}(\mathbf{y})]n_k(\mathbf{y})\,\mathrm{d}S
\end{aligned}
$$

$$
\begin{aligned}
(1.5)\quad D^\alpha u_j(\mathbf{x}) = {}& \mathcal{A}_j^{(1),\alpha}(\boldsymbol{\mathcal{G}}) - \int_\Omega D^\alpha \frac{\partial \mathcal{R}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_k}\mathcal{G}_{ik}(\mathbf{y})\,\mathrm{d}\mathbf{y}\\
&+ \int_{\partial\Omega} [-\beta D^\alpha\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1k}\\
&+ u_i(\mathbf{y})D^\alpha T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j},e_j)(\mathbf{x}-\mathbf{y};\beta)\\
&+ D^\alpha\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},\pi)(\mathbf{y})\\
&+ D^\alpha\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\mathcal{G}_{ik}(\mathbf{y})]n_k(\mathbf{y})\,\mathrm{d}S
\end{aligned}
$$

if $|\alpha| = 1$,

$$
\begin{aligned}
(1.6)\qquad D^\alpha u_j(\mathbf{x}) = {}& \mathcal{A}_j^{(2),\alpha}(\nabla\cdot\boldsymbol{\mathcal{G}}) + \int_\Omega D^\alpha\mathcal{R}_{ij}(\mathbf{x}-\mathbf{y};\beta)\frac{\partial\mathcal{G}_{ik}(\mathbf{y})}{\partial y_k}\,\mathrm{d}\mathbf{y}\\
&+ \int_{\partial\Omega} [-\beta D^\alpha\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1k}\\
&+ u_i(\mathbf{y})D^\alpha T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j},e_j)(\mathbf{x}-\mathbf{y};\beta)\\
&+ D^\alpha\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},\pi)(\mathbf{y})]n_k(\mathbf{y})\,\mathrm{d}S
\end{aligned}
$$

if $|\alpha| = 2$, where $\mathcal{A}_j^{(i),\alpha}$ are operators which map $L^q(\Omega)$ into $L^q(\Omega)$, $L^q(\Omega;g)$ into $L^q(\Omega;g)$ for $1 < q < \infty$ and $g \geqslant 0$ are weights from the Muckenhoupt class A_q.

Let us recall that a weight g belongs to the Muckenhoupt class A_p, $1 < p < \infty$ if there exists a constant C such that for cubes Q in $\mathbb{R}^2$ we have

$$
\sup_Q\Big[\Big(\frac{1}{|Q|}\int_Q g^p(\mathbf{x})\,\mathrm{d}\mathbf{x}\Big)\Big(\frac{1}{|Q|}\int_Q g(\mathbf{x})^{-\frac{p}{p-1}}\,\mathrm{d}\mathbf{x}\Big)^{p-1}\Big] \leqslant C < \infty.
$$

Corollary 1.1. *The integral representation formulas in Theorem* 1.2 *hold for a.a.* $\mathbf{x} \in \Omega$ *if* $\mathbf{v} \in W^{2,q}_{\mathrm{loc}}(\overline{\Omega})$, $\pi \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega})$ *for some* $1 < q < \infty$ *and*

a) (1.4) *if* $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^r_{\mathrm{loc}}(\overline{\Omega})$, $1 < q < 3$, $1 < r < \infty$,

b) (1.5) *if* $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^r_{\mathrm{loc}}(\overline{\Omega})$, $1 < q, r < \infty$,

c) (1.6) *if* $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $1 < q < \infty$.

Similarly, for the pressure we have

Theorem 1.3. *Let* $\Omega \in C^2$ *be a plane exterior domain,* $\boldsymbol{\mathcal{G}} \in C_0^\infty(\overline{\Omega})$ *and let* $(\mathbf{u}, \pi)$ *be the unique solution to the Oseen problem* (1.2). *Let* $\mathcal{T}_{il}(\mathbf{e})$ *be defined in* $(1.3)_2$ *and* $T_{ij}(\mathbf{u}, \pi)$ *in* $(1.3)_1$. *Then*[2]

$$\begin{aligned}\pi(\mathbf{x}) = {} & \text{v.p.} \int_\Omega \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_k} \mathcal{G}_{ik}(\mathbf{y})\,\mathrm{d}\mathbf{y} + c_{ik}\mathcal{G}_{ik}(\mathbf{x}) \\ & + \int_{\partial\Omega} [-\beta e_i(\mathbf{x}-\mathbf{y}) u_i(\mathbf{y})\delta_{1l} + u_i(\mathbf{y})\mathcal{T}_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) \\ & + e_i(\mathbf{x}-\mathbf{y}) T_{il}(\mathbf{u},\pi)(\mathbf{y}) + e_i(\mathbf{x}-\mathbf{y})\mathcal{G}_{il}(\mathbf{y})] n_l(\mathbf{y})\,\mathrm{d}_{\mathbf{y}}S,\end{aligned} \tag{1.7}$$

$$\begin{aligned}\frac{\partial \pi(\mathbf{x})}{\partial x_j} = {} & \text{v.p.} \int_\Omega \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_j} \frac{\partial}{\partial y_k} \mathcal{G}_{ik}(\mathbf{y})\,\mathrm{d}\mathbf{y} + c_{ij} \frac{\partial \mathcal{G}_{ik}(\mathbf{x})}{\partial x_k} \\ & + \int_{\partial\Omega} \Big[-\beta \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_j} u_i(\mathbf{y})\delta_{1l} + u_i(\mathbf{y}) \frac{\partial}{\partial x_j} \mathcal{T}_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) \\ & + \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_j} T_{il}(\mathbf{u},\pi)(\mathbf{y})\Big] n_l(\mathbf{y})\,\mathrm{d}_{\mathbf{y}}S.\end{aligned} \tag{1.8}$$

Corollary 1.2. *The integral representation in Theorem* 1.3 *holds a.e. in* Ω *if* $\mathbf{v} \in W^{2,q}_{\mathrm{loc}}(\overline{\Omega})$, $\pi \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega})$ *for some* $1 < q < \infty$ *and*

a) (1.7) *if* $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^r_{\mathrm{loc}}(\overline{\Omega})$, $1 < q, r < \infty$,

b) (1.8) *if* $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $1 < q < \infty$.

In order to capture the asymptotic structure of solutions to the original problem (especially the existence of the wake region), we will use the anisotropic weights

$$\begin{aligned}\eta^A_B(\mathbf{x}) &= (1+|\mathbf{x}|)^A(1+s(\mathbf{x}))^B, \\ \nu^A_B(\mathbf{x}) &= |\mathbf{x}|^A(1+s(\mathbf{x}))^B, \\ \mu^{A,\omega}_B(\mathbf{x}) &= \eta^{A-\omega}_B(\mathbf{x})\nu^\omega_0(\mathbf{x}), \\ \eta^A_B(\mathbf{x};\beta) &= (1+|\beta\mathbf{x}|)^A(1+s(\beta\mathbf{x}))^B, \\ \nu^A_B(\mathbf{x};\beta) &= |\mathbf{x}|^A(1+s(\beta\mathbf{x}))^B, \\ \mu^{A,\omega}_B(\mathbf{x};\beta) &= \eta^{A-\omega}_B(\mathbf{x};\beta)\nu^\omega_0(\mathbf{x};\beta).\end{aligned}$$

[2] v.p. in front of an integral means that the integral is taken in the principal value sense.

with real exponents A, B. In some cases, disturbing logarithmic terms appear. Thus we denote

$$\overline{\eta}_F^E(\mathbf{x};\beta) = \eta_F^E(\mathbf{x};\beta) P(\ln^{-1}|\beta\mathbf{x}|),$$

where $P(\cdot)$ denotes a polynomial of zero, first or second degree, depending on the fact whether a logarithmic term appears or not, see the theorems below. Analogously for $\overline{\nu}_F^E(\mathbf{x};\beta)$.

We will use the following estimates proved in [5]; $B^* = \min(B, \frac{1}{2})$ and ∂_k denotes the k-th component of the gradient, i.e. the derivative with respect to x_k.

Theorem 1.4. *Let $A+B > 1/2$, $(A+B)^* > 1/2$, and $i,j = 1,2$. Then for $f \in L^\infty(\mathbb{R};\eta_B^A(\cdot;\beta))$ we have $\partial_2\mathcal{O}_{11}^\mu(\cdot;\beta) * f \in L^\infty(\mathbb{R};\overline{\eta}_F^E(\cdot;\beta))$, where*[3]

$$E = \begin{cases} A - \frac{1}{2} & \text{for} \quad -\frac{1}{2} < A \leqslant \frac{3}{2},\ \ A \leqslant B+1,\ \ B \geqslant 0, \\ 1 & \text{for} \quad A + B^* \geqslant 2, \\ A + B - \frac{1}{2} & \text{for} \quad B < 0,\ \ A + B \leqslant 1, \\ \frac{1}{2}(A+B) & \text{for} \quad B \leqslant A - 1,\ \ 1 \leqslant A + B \leqslant 2, \end{cases}$$

$$E + F = \begin{cases} A + B^* & \text{for} \quad -\frac{1}{2} < A \leqslant \frac{3}{2},\ \ B \geqslant 1, \\ A + B - \frac{1}{2} & \text{for} \quad A + B \leqslant \frac{5}{2},\ \ B \leqslant 1, \\ 2 & \text{for} \quad A + B \geqslant \frac{5}{2},\ \ A \geqslant \frac{3}{2}, \end{cases}$$

with logarithmic factors

$$\ln_+(\beta|\mathbf{x}|) \quad \text{for} \quad \begin{cases} A + B^* = 2, \\ A = B + 1,\ \ 0 \leqslant B \leqslant \frac{1}{2}, \\ A + B = 1,\ \ B \leqslant 0, \end{cases}$$

$$\ln_+(\beta s(\mathbf{x})) \quad \text{for} \quad A + B = 1,\ \ 0 < B \leqslant 1.$$

Moreover, we have

$$\|\partial_2\mathcal{O}_{12}^\mu(\cdot;\beta) * f\|_{\infty,(\overline{\eta}_F^E(\cdot;\beta)),\mathbb{R}^2} \leqslant C\beta^{-1}\|f\|_{\infty,(\eta_B^A(\cdot;\beta)),\mathbb{R}^2}.$$

Let in addition A, B satisfy the following conditions:

$$\tfrac{1}{2} < A < 2, \quad B \geqslant -\tfrac{1}{2}, \quad A \leqslant B + 2.$$

[3] The proof in [5] is done for the (classical) Oseen problem, i.e. $\mu = 0$; nevertheless, as follows from [3], the results are applicable also to the case $0 \leqslant \mu < 1$, i.e. to the modified Oseen problem.

Then $f \in L^\infty(\mathbb{R}^2; \nu_B^A(\cdot;\beta))$ *we have* $\partial_2 \mathcal{O}_{11}^\mu * f \in L^\infty(\mathbb{R}^2; \overline{\nu}_F^E(\cdot;\beta))$ *and*

$$\|\partial_2 \mathcal{O}_{11}^\mu(\cdot;\beta) * f\|_{\infty,(\overline{\nu}_F^E(\cdot;\beta)),\mathbb{R}^2} \leqslant C\beta^{-1+A-E}\|f\|_{\infty,(\nu_B^A(\cdot;\beta)),\mathbb{R}^2}.$$

Theorem 1.5. *Let* $A+B^* > 0$ *and* $R = |\partial_k \mathcal{O}_{ij}|$, $i,j,k = 1,2$, *except of* $|\partial_2 \mathcal{O}_{11}|$. *Then for* $f \in L^\infty(\mathbb{R}^2; \eta_B^A(\cdot;\beta))$ *we have* $R * f \in L^\infty(\mathbb{R}^2; \overline{\eta}_F^E(\cdot;\beta))$, *where*

$$E = \begin{cases} \frac{3}{2} & \text{for} \quad A + B^* \geqslant 2, \\ A & \text{for} \quad A \leqslant \frac{3}{2}, \ \ B \geqslant 0, \ \ A \leqslant B+1, \\ \frac{1}{2}(A+B+1) & \text{for} \quad 1 \leqslant A+B \leqslant 2, \ \ A \geqslant B+1, \\ A+B & \text{for} \quad B \leqslant 0, \ \ A+B \leqslant 1, \end{cases}$$

$$E + F = \begin{cases} 2 & \text{for} \quad A + B^* \geqslant 2, \\ A + B^* & \text{for} \quad A + B^* \leqslant 2, \end{cases}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|) \ \ \text{for} \ \ A + B^* \leqslant 2.$$

Moreover, we have

$$\|\,|\partial_k \mathcal{O}_{ij}(\cdot;\beta)| * f\|_{\infty,(\overline{\eta}_F^E(\cdot;\beta)),\mathbb{R}^2} \leqslant C\beta^{-1}\|f\|_{\infty,(\eta_B^A(\cdot;\beta)),\mathbb{R}^2}.$$

Let in addition A, B *satisfy the following conditions:*

$$0 \leqslant A < 2, \quad B \geqslant -1.$$

Then for $f \in L^\infty(\mathbb{R}^2; \nu_B^A(\cdot;\beta))$ *we have* $R * f \in L^p(\mathbb{R}^2; \overline{\nu}_F^E(\cdot;\beta))$ *and*

$$\|\,|\partial_k \mathcal{O}_{ij}(\cdot;\beta)| * f\|_{\infty,(\overline{\nu}_F^E(\cdot;\beta)),\mathbb{R}^2} \leqslant C\beta^{-1+A-E}\|f\|_{\infty,(\nu_B^A(\cdot;\beta)),\mathbb{R}^2}.$$

Theorem 1.6. *Let* T *be an integral operator defined by the volume terms in* (1.5) *or* (1.6) *and let* $1 < p < \infty$. *Then* T *is a well defined continuous operator*

a) $$L^p(\mathbb{R}^2; \eta_B^{A+1/2}(\cdot;\beta)) \longmapsto L^p(\mathbb{R}^2; \eta_B^{A+1/2-\varepsilon}(\cdot;\beta))$$

for $\max\{-1/(2p), -\varepsilon(p-1)/p\} < B < 1/2 - 1/(2p)$, $-1/2 - 1/p - \varepsilon/p < A + B < 3/2 - 2/p$, $A - B < 1/2 - 1/p + \varepsilon + \varepsilon(p-1)/p$, $-1 - 1/(2p) - \varepsilon/p < A < 1 - 3/(2p) + \varepsilon$, $0 < \varepsilon \leqslant 1/2$;

b) $$L^p(\mathbb{R}^2 \setminus \mathcal{B}, \mu_B^{A+1/2,k\omega}(\cdot;\beta)) \longmapsto L^p(\mathbb{R}^2 \setminus \mathcal{B}, \mu_B^{A+1/2-\varepsilon,\omega}(\cdot;\beta))$$

for $\max\{-1/(2p), -\varepsilon(p-1)/p\} < B < 1/2 - 1/(2p)$, $-1/2 - 1/p - \varepsilon/p < A + B < 3/2 - 2/p$, $-1 - 1/(2p) - \varepsilon/p < A < 1 - 3/(2p) + \varepsilon$, $A - B < 1/2 - 1/p + \varepsilon + \varepsilon(p-1)/p$, $0 < \varepsilon \leqslant 1/2$, $\omega < \min(1/(2k) + A/k,\ A + 1/2 - \varepsilon)$, $\mathcal{B} \subset \mathbb{R}^2$*—an arbitrary domain,* $\mathbf{0} \in \mathcal{B}$, $k \in \mathbb{N}_0$.

Moreover, we have for A, B *specified in* a) *and* b), *respectively,*

$$\|Tf\|_{p,(\eta_B^{A+1/2-\varepsilon}(\cdot;\beta)),\mathbb{R}^2} \leqslant C\|f\|_{p,(\eta_B^{A+1/2}(\cdot;\beta)),\mathbb{R}^2}$$

and

$$\|Tf\|_{p,(\mu_B^{A+1/2-\varepsilon,\omega}(\cdot;\beta)),\mathbb{R}^2\setminus\mathcal{B}} \leqslant C\beta^{(k-1)\omega}\|f\|_{p,(\mu_B^{A+1/2,k\omega}(\cdot;\beta)),\mathbb{R}^2\setminus\mathcal{B}}.$$

Theorem 1.7. *Let*

$$Tf(\mathbf{x}) = \frac{\partial}{\partial x_i}\int_{\mathbb{R}^N} \frac{\partial \mathcal{E}(\mathbf{x}-\mathbf{y})}{\partial x_j} f(\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad i,j = 1,\ldots,N,$$

$f \in C_0^\infty(\mathbb{R}^N)$, *where* $\mathcal{E}$ *denotes the fundamental solution to the Laplace equation. Let* $1 < p < \infty$ *and let* g *stand for one of the weights* η_B^A, ν_B^A *or* $\mu_B^{A,\omega}$. *Let* A, B, ω *be such that* g *is an* A_p *weight in* $\mathbb{R}^N$. *Then* T *maps* $C_0^\infty(\mathbb{R}^N)$ *into* $L^p(\mathbb{R}^N; g)$ *and we have*

$$\|Tf\|_{p,(g),\mathbb{R}^N} \leqslant C\|f\|_{p,(g),\mathbb{R}^N}.$$

T *can be therefore continuously extended onto* $L^p(\mathbb{R}^N; g)$.[4] *Especially, for* $k\omega < A$ ($\omega < A$ *if* $k = 0$), $\mathbf{0} \in \mathcal{B} \subset \mathbb{R}^N$, $k \in \mathbb{N}_0$ *we have*

$$\|Tf\|_{p,(\mu_B^{A,\omega}(\cdot;\beta)),\mathbb{R}^N\setminus\mathcal{B}} \leqslant C\beta^{(k-1)\omega}\|f\|_{p,(\mu_B^{A,k\omega}(\cdot;\beta)),\mathbb{R}^N\setminus\mathcal{B}}.$$

In two space dimensions we have

Theorem 1.8.

(i) *Let* $-1/2 < Bp < 1/2(p-1)$, $-2 < (A+B)p < 2(p-1)$. *Then the weight* η_B^A *is an* A_p *weight in* $\mathbb{R}^2$ *for* $p \in (1;\infty)$.

(ii) *Let* $-1/2 < Bp < 1/2(p-1)$, $-2 < (A+B)p < 2(p-1)$, $-2 < Ap < 2(p-1)$ *and* $0 \leqslant \omega \leqslant A$. *Then the weights* ν_B^A *and* $\mu_B^{A,\omega}$ *are* A_p *weights in* $\mathbb{R}^2$ *for* $p \in (1;\infty)$.

Further we consider the steady transport equation

$$z + \mathbf{w}\cdot\nabla z + az = f \tag{1.9}$$

[4] More precisely, to the closure of $C^\infty(\mathbb{R}^N)$ in the norm $\|\cdot\|_{p,(g)}$. But, as shown e.g. in [13], this space coincides with $L^p(\mathbb{R}^N; g)$.

in Ω. Although the equation (1.9) is scalar, everything below holds also for the vector case. For the proofs see [6].

Theorem 1.9. *Let $\Omega \in C^{0,1}$ be an exterior domain, $a \in C^{k-1}(\Omega)$, $\mathbf{w} \in C^k(\Omega)$, $\mathbf{w} \cdot \mathbf{n} = 0$ on $\partial\Omega$, $\nabla^k a \in L^q(\Omega)$, $kq > 2$, $f \in W^{k,q}(\Omega)$. Then there exists $\alpha > 0$ such that if*

$$\|a\|_{C^{k-1}} + \|\nabla \mathbf{w}\|_{C^{k-1}} + \|\nabla^k a\|_q < \alpha,$$

then there exists a unique solution $z \in W^{k,q}(\Omega)$ to (1.9) satisfying the estimate

$$\|z\|_{k,q} \leqslant C(\alpha)\|f\|_{k,q}.$$

Another situation is considered in

Theorem 1.10. *Let $\Omega \in C^{0,1}$ be an exterior domain, $a \in C(\Omega)$, $\mathbf{w} \in C^1(\Omega)$, $\mathbf{w} \cdot \mathbf{n} = 0$ on $\partial\Omega$, $\nabla a \in L^2(\Omega)$, $f \in L^p(\Omega) \cap D^{1,q}(\Omega)$, $1 < q < 2$, $1 < p < \infty$. Then there exists $\alpha > 0$ such that if*

$$\|a\|_{C^0} + \|\nabla \mathbf{w}\|_{C^0} + \|\nabla a\|_2 < \alpha,$$

then there exists a unique solution $z \in D^{1,q}(\Omega) \cap L^p(\Omega)$ to (1.9) satisfying the estimates

$$|z|_{1,q} \leqslant C(\alpha)|f|_{1,q},$$
$$\|z\|_p \leqslant C(\alpha)\|f\|_p.$$

Let the weight g be such that $W^{k,q}(\Omega; g) \subset W^{k,q}(\Omega)$. Then we have

Theorem 1.11. *Let k, q, Ω, a, $\mathbf{w}$ and f satisfy the assumptions of Theorem 1.9 (ii). Moreover, let $g \in C^k(\Omega)$ be a positive weight and let*

$$\|\mathbf{w} \cdot \nabla \ln g\|_{C^{k-1}} + |\mathbf{w} \cdot \nabla \ln g|_{k,q}$$

be sufficiently small. If $\mathbf{f} \in W^{k,q}(\Omega; g)$, then z, the solution to (1.9), belongs to $W^{k,q}(\Omega; g)$ and

$$\|z\|_{k,q,(g)} \leqslant C\|f\|_{k,q,(g)}.$$

In this head we only shortly sketch the construction of a solution to the system (0.3)–(0.7). It is essentially based on the following version of the Banach fixed point theorem:

Lemma 2.1. *Let X, Y be Banach spaces such that X is reflexive and $X \hookrightarrow Y$. Let H be a non-empty, closed, convex and bounded subset of X and let $\mathcal{M}\colon H \mapsto H$ be a mapping such that*

$$\|\mathcal{M}(u) - \mathcal{M}(v)\|_Y \leqslant \kappa \|u - v\|_Y \quad \forall u, v \in H,$$

$0 \leqslant \kappa < 1$. *Then $\mathcal{M}$ has a unique fixed point in H.*

The proof of existence of a solution is straightforward but slightly technical. Combining Theorem 1.1 with Theorems 1.9 and 1.10 gives

Theorem 2.1. *Let $\Omega \in C^{k+1}$ be an exterior domain in $\mathbb{R}^2$. Let $\mathbf{f} \in W^{2,q}(\Omega) \cap W^{k,p}(\Omega)$, $k \geqslant 2$, $p \in (2;\infty)$, $1 < q < \frac{6}{5}$. Let β_0 and $\|\mathbf{f}\|_{2,q} + \|\mathbf{f}\|_{k,p}$ be sufficiently small. Then for any $0 < \beta \leqslant \beta_0$ there exists a solution $(\mathbf{u}, \pi)$ to the problem* (0.3)–(0.7) *such that*[5]

$$\mathbf{u} \in L^{\frac{3q}{3-2q}}(\Omega) \cap D^{1,\frac{3q}{3-q}}(\Omega),$$
$$\nabla^2 \mathbf{u}, \nabla p, \nabla \pi \in W^{1,q}(\Omega) \cap W^{k-1,p}(\Omega).$$

Remark 2.1. The construction of a solution can be viewed as a successive approximation procedure; namely

$$(\mathbf{u}_{n+1}, p_{n+1}) = \mathcal{M}(\mathbf{u}_n, p_n), \quad n \geqslant 0.$$

This is a direct consequence of the fact that Lemma 2.1 is a version of the Banach fixed point theorem. It can be shown that there exist $\varepsilon \in (0;1)$, $\alpha \in (\frac{2}{3};1)$ depending on β_0 and $\|\mathbf{f}\|_{2,q} + \|\mathbf{f}\|_{k,p}$ such that

$$\beta(\|(u_2)_n\|_{\frac{2q}{2-q}} + \|\nabla (u_2)_n\|_q) + \beta^{\frac{2}{3}} \|\mathbf{u}_n\|_{\frac{3q}{3-2q}} + \beta^{\frac{1}{3}} \|\nabla \mathbf{u}_n\|_{\frac{3q}{3-q}} \leqslant \varepsilon \beta^{2(1+1/q)+1}, \tag{2.1}$$

$$\|\nabla^2 \mathbf{u}_n\|_q + \|\nabla \pi_n\|_q + \|\nabla^2 \mathbf{u}_n\|_{k-1,p} + \|\nabla \pi_n\|_{k-1,p} \leqslant \varepsilon \beta^\alpha \tag{2.2}$$

for all $n \in \mathbb{N}_0$, $0 < \beta \leqslant \beta_0$. The details can be found in [10].

[5] Let us recall (see (0.6)) that π plays the role of the effective pressure; the real pressure is p.

3. Weighted estimates

This head is devoted to the study of weighted estimates of solutions to the original problem constructed in Theorem 2.1. Due to the construction of a solution (see Remark 2.1) it is sufficient to verify that the operator $\mathcal{M}$ maps sufficiently small balls in some weighted spaces into themselves. Then also the solution (i.e. the limit of the sequence) belongs to the same ball. Let us denote

$$\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta) = \mu_0^{1,\omega}(\mathbf{x};\beta)|\ln(2+|\beta\mathbf{x}|)|^{-1}.$$

We will search the solution in the space

$$\begin{aligned} V_\beta = \big\{(\mathbf{u},\pi);\ u_1 \in L^\infty(\Omega;\mu_{1/2}^{1/2,\omega}(\cdot;\beta)),\ \ u_2 \in L^\infty(\Omega;\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\\ \nabla\mathbf{u},\nabla^2\mathbf{u} \in L^r(\Omega;\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta)),\ \ \pi,\nabla\pi \in L^r(\Omega;\mu_0^{1-3/r,\omega}(\cdot;\beta))\big\} \end{aligned}$$

with the norm

$$\begin{aligned} \|(\mathbf{u},\pi)\|_{V_\beta} = \|u_1\|_{\infty,(\mu_{1/2}^{1/2,\omega}(\cdot;\beta))} + \|u_2\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta))}\\ + \|\nabla\mathbf{u},\nabla^2\mathbf{u}\|_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta))} + \|\pi,\nabla\pi\|_{r,\left(\mu_0^{1-3/r,\omega}(\cdot;\beta)\right)}, \end{aligned}$$

where $r \in (1;\infty)$ is a sufficiently large power and $\omega > 0$ will be made more precise later. Note that the asymptotic structure of $\mathbf{u}$ is almost the same as the asymptotic structure of $\boldsymbol{\mathcal{O}}^\mu$.[6]

Let us assume that $(\mathbf{w},s)$ is such that

$$\|(\mathbf{w},s)\|_{V_\beta} \leqslant \delta_0 = \varepsilon\beta^{1-\omega}$$

with ε, β sufficiently small. Moreover, let $(\mathbf{w},s)$ satisfy (2.1) and (2.2). Our aim is to show that $(\mathbf{u},\pi)$ defined by

$$(\mathbf{u},\pi) = \mathcal{M}(\mathbf{w},s)$$

remains in the same ball in V_β for β and $\mathbf{f}$ sufficiently small. Recall that from the existence part we already know that $(\mathbf{u},\pi)$ satisfies (2.1) and (2.2). Applying Theorem 1.11 we get

[6] This is no more true for higher gradients of the velocity; nevertheless, under more restrictive assumptions on the right-hand side $\mathbf{f}$ we can get better asymptotic structure also here, see Remark 3.2.

Lemma 3.1. *Let β and $\|\mathbf{w}\|_{C^1}$, $\|\nabla^2\mathbf{w}\|_r$ $(2 < r < \infty)$ be sufficiently small. Then for any $0 \leqslant \omega \leqslant a$, $0 \leqslant b$ and p, $\mathbf{T}$ solutions to* (0.11) *and* (0.12), *respectively, we have*

$$\begin{aligned}\|p\|_{r,(\mu_b^{a,\omega}(\cdot;\beta))} &\leqslant C\|s\|_{r,(\mu_b^{a,\omega}(\cdot;\beta))},\\ \|p\|_{1,r,(\mu_b^{a,\omega}(\cdot;\beta))} &\leqslant C\|s\|_{1,r,(\mu_b^{a,\omega}(\cdot;\beta))},\\ \|\mathbf{T}\|_{r,(\mu_b^{a,\omega}(\cdot;\beta))} &\leqslant C\|\nabla\mathbf{w}\|_{r,(\mu_b^{a,\omega}(\cdot;\beta))},\\ \|\mathbf{T}\|_{1,r,(\mu_b^{a,\omega}(\cdot;\beta))} &\leqslant C\|\nabla\mathbf{w}\|_{1,r,(\mu_b^{a,\omega}(\cdot;\beta))}\end{aligned}$$

with the constant C independent of β.

Proof. It follows from Theorem 1.11 since for $\mathbf{0} \in \mathbb{R}^2 \setminus \Omega$, $|\nabla \ln \mu_b^{a,\omega}(\mathbf{x};\beta)|$ is independent of β. $\square$

Remark 3.1. There exist $C_i = C_i(\Omega, a, b, r)$, $i = 1, 2$, independent of β such that for any $a, b \geqslant 0$, $\beta \leqslant 1$ and any $g \in W^{1,r}(\Omega; \mu_b^{a,\omega})$ we have

$$\begin{aligned}(3.1)\qquad C_1\|g\|_{1,r,(\mu_b^{a,\omega}(\cdot;\beta))} &\leqslant [\|g\|_{r,(\mu_b^{a,\omega}(\cdot;\beta))} + \|\nabla g\|_{r,(\mu_b^{a,\omega}(\cdot;\beta))}]\\ &\leqslant C_2\|g\|_{1,r,(\mu_b^{a,\omega}(\cdot;\beta))}.\end{aligned}$$

This follows easily due to the fact that there exists a constant C independent of β such that

$$\|g\|_{r,(\nabla\mu_b^{a,\omega}(\cdot;\beta))} \leqslant C\|g\|_{r,(\mu_b^{a,\omega}(\cdot;\beta))}$$

for $\mathbf{0} \in \mathbb{R}^2 \setminus \Omega$. Moreover, for $r > 2$ we have

$$\|g\mu_b^{a,\omega}(\mathbf{x};\beta)\|_\infty \leqslant C\|g\mu_b^{a,\omega}(\mathbf{x};\beta)\|_{1,r}$$

and therefore by (3.1) also

$$\|g\mu_b^{a,\omega}(\mathbf{x};\beta)\|_\infty \leqslant C(\|g\|_{r,(\mu_b^{a,\omega}(\cdot;\beta))} + \|\nabla g\|_{r,(\mu_b^{a,\omega}(\cdot;\beta))}).$$

Let

$$\boldsymbol{\mathcal{N}}(\mathbf{f}, \mathbf{w}, \mathbf{T}(\mathbf{w}), p(\mathbf{w}, s)) = \nabla \cdot \boldsymbol{\mathcal{G}}(\mathbf{f}, \mathbf{w}, \mathbf{T}(\mathbf{w}), p(\mathbf{w}, s)),$$

$\boldsymbol{\mathcal{N}}$ denoting the right-hand side of (0.8). Let $\mathbf{f} = \nabla \cdot \mathbf{h}$. From Theorem 1.2 and Corollary 1.1 we have

$$\begin{aligned}(3.2)\quad u_j(\mathbf{x}) = \int_\Omega \frac{\partial \mathcal{O}_{ij}^\mu(\mathbf{x}-\mathbf{y};\beta)}{\partial x_k}\Big[h_{ik} + F_{ik}(\nabla\mathbf{w}, \mathbf{T}) + p\frac{\partial w_k}{\partial y_i} - w_l w_k \frac{\partial w_i}{\partial y_l}\\ - w_i w_k - \beta\Big(w_k\frac{\partial w_i}{\partial y_1} + w_l\frac{\partial w_i}{\partial y_l}\delta_{1k}\Big) + f_i(w_k + \beta\delta_{1k})\Big](\mathbf{y})\,\mathrm{d}\mathbf{y}\\ + \int_{\partial\Omega} [-\beta\mathcal{O}_{ij}^\mu(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1k} + u_i(\mathbf{y})T_{ik}(\boldsymbol{\mathcal{O}}_{\cdot j}^\mu, e_j)(\mathbf{x}-\mathbf{y};\beta)\\ + \mathcal{O}_{ij}^\mu(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},\pi)(\mathbf{y}) + \mathcal{O}_{ij}^\mu(\mathbf{x}-\mathbf{y};\beta)\mathcal{G}_{ik}(\mathbf{y})]n_k(\mathbf{y})\,\mathrm{d}_\mathbf{y}S\end{aligned}$$

for $j = 1, 2$.

Due to different asymptotic properties of $\partial\mathcal{O}^\mu_{11}/\partial x_2$ and the other components of $\nabla\boldsymbol{\mathcal{O}}^\mu$ we have to study separately u_1 and u_2. We denote by u_j^V the part of u_j corresponding to the volume integrals, by $u_j^{S,i}$, $i = 1,2,3,4$, the parts corresponding to the surface integrals. Applying Theorems 1.4 and 1.5 we obtain[7]

$$\begin{aligned}\|u_1^V\|_{\infty,(\mu^{1/2,\omega}_{1/2}(\cdot;\beta))} &= \|u_1^V\mu^{1/2,\omega}_{1/2}(\cdot;\beta)\|_\infty \\ &\leqslant C\beta^{-1+(k-1)\omega}\big[\,\|\mathcal{G}_{12}\|_{\infty,(\mu^{1,k\omega}_{1/2}(\cdot;\beta))} + \|\mathcal{G}_{11},\mathcal{G}_{21},\mathcal{G}_{22}\|_{\infty,(\mu^{1/2+\delta,k\omega}_{1/2}(\cdot;\beta))}\big],\end{aligned}$$

$\omega \geqslant 0$, $(k-1)\omega \leqslant 1$, $k \in \mathbb{N}_0$, $\delta > 0$ can be taken arbitrarily small. We will assume that $\mathbf{h}$ and $\mathbf{f}$ are sufficiently smooth with sufficiently fast decay at infinity and we will collect the precise assumptions in the main theorem. We estimate each term separately. Applying Lemma 3.1 and Remark 3.1 we obtain for $r \geqslant 5$

$$\begin{aligned}&\|\mathbf{F}(\nabla\mathbf{w},\mathbf{T}\|_{\infty,(\mu^{1,2\omega}_{1/2}(\cdot;\beta))} + \|p\nabla\mathbf{w}\|_{\infty,(\mu^{1,2\omega}_{1/2}(\cdot;\beta))} \\ &\qquad\leqslant C(\|\mathbf{T}\|_{1,r,(\mu^{2/5,\omega}_{1/4}(\cdot;\beta))} \\ &\qquad\quad + \|p\|_{1,r,(\mu^{2/5,\omega}_{1/4}(\cdot;\beta))}\|\nabla\mathbf{w}\|_{1,r,(\mu^{3/5,\omega}_{1/4}(\cdot;\beta))} \leqslant \frac{\varepsilon}{60}\beta^{2-2\omega}, \\ &\|\,|\mathbf{w}|^2\nabla\mathbf{w}\|_{\infty,(\mu^{1,2\omega}_{1/2}(\cdot;\beta))} \leqslant \|\mathbf{w}\|^2_{\infty,(\mu^{1/2,\omega}_{1/4}(\cdot;\beta))}\|\nabla\mathbf{w}\|_\infty \leqslant \frac{\varepsilon}{60}\beta^{2-2\omega}\end{aligned}$$

(see (2.1) and (2.2)). Next,

$$\|\,|\mathbf{w}|^2\|_{\infty,(\mu^{1,2\omega}_{1/2}(\cdot;\beta))} \leqslant \|\mathbf{w}\|^2_{\infty,(\mu^{1/2,\omega}_{1/4}(\cdot;\beta))} \leqslant \frac{\varepsilon}{60}\beta^{2-2\omega}.$$

For $r \geqslant 4$ it is also easy to establish

$$\beta\|\mathbf{w}\nabla\mathbf{w}|_{\infty,(\mu^{1,2\omega}_{1/2}(\cdot;\beta))} \leqslant \beta\|\mathbf{w}\|_{\infty,(\mu^{1/2,\omega}_{1/2}(\cdot;\beta))}\|\nabla\mathbf{w}\|_{\omega,(\mu^{1/2,\omega}_{0}(\cdot;\beta))} \leqslant \frac{\varepsilon}{60}\beta^{2-2\omega}$$

and finally,

$$\begin{aligned}&\|\mathbf{f}\,\mathbf{w}\|_{\infty,(\mu^{1,\omega}_{1/2}(\cdot;\beta))} + \beta\|\mathbf{f}\|_{\infty,(\eta^1_{1/2}(\cdot;\beta))} \\ &\qquad\leqslant \|\mathbf{w}\|_{\infty,(\mu^{1/2,\omega}_{1/2}(\cdot;\beta))}\|\mathbf{f}\|_{\infty,(\eta_0^{1/2}(\cdot))} + \beta\|\mathbf{f}\|_{\infty,(\eta^1_{1/2}(\cdot))} \leqslant \frac{\varepsilon}{60}\beta^2.\end{aligned}$$

Summarizing the calculations above we have

$$\|u_1^V\|_{\infty,(\mu^{1/2,\omega}_{1/2}(\cdot;\beta))} \leqslant \frac{\varepsilon}{10}\beta^{1-\omega}$$

for β and ε sufficiently small.

[7] Note that the other components of $\nabla\boldsymbol{\mathcal{O}}^\mu$ which appear in (3.2) for $j = 1$ behave better and thus the most restrictive condition comes from the term $\partial\mathcal{O}^\mu_{11}/\partial x_2$.

Next we estimate the L^∞-weighted norm of u_2^V. As in the integral representation the term $\partial\mathcal{O}_{11}^\mu/\partial y_2$ does not appear, we have

$$\|u_2^V\|_{\infty,(\mu_0^{1,\omega}(\cdot;\beta)P|\ln(2+2|\beta\mathbf{x}|)|^{-1})} \leqslant C\beta^{(k-1)\omega-1}\|\boldsymbol{\mathcal{G}}\|_{\infty,(\mu_0^{1,k\omega}(\cdot;\beta))}, \tag{3.3}$$

where P is a polynomial of the first or second order. The most delicate term in $\boldsymbol{\mathcal{G}}$ will be w_1w_1. We therefore write (3.3) in a different way, namely

$$\begin{aligned}\|u_2^V\|_{\infty,(\mu_0^{1,\omega}(\cdot;\beta)P|\ln(2+2|\beta\mathbf{x}|)|^{-1})} &\leqslant C\beta^{\omega-1}\|w_1w_1\|_{\infty,(\mu_0^{1,2\omega}(\cdot;\beta))}\\ &\quad + C\beta^{(k-1)\omega-1}\|\boldsymbol{\mathcal{G}}'\|_{\infty,(\mu_0^{1+\delta,k\omega}(\cdot;\beta))},\end{aligned}$$

where $\delta > 0$, $\mathcal{G}'_{ij} = \mathcal{G}_{ij} + w_1w_1\delta_{1i}\delta_{1j}$. Thus we get (see [10]; compare also with Tab. 3 and Tab. 4 in [5]) that the power of the polynomial P, which is determined by the term $\partial\mathcal{O}_{12}^\mu/\partial y_1 * (w_1w_1)$, is equal to 1.

Moreover,

$$\|w_1w_1\|_{\infty,(\mu_0^{1,2\omega}(\cdot;\beta))} \leqslant \|w_1\|^2_{\infty,(\mu_0^{1/2,\omega}(\cdot;\beta))} \leqslant C\varepsilon^2\beta^{2-2\omega}.$$

So we have

$$\|u_2^V\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta))} \leqslant C\varepsilon^2\beta^{1-\omega} + C\beta^{(k-1)\omega-1}\|\boldsymbol{\mathcal{G}}'\|_{\infty,(\mu_0^{1+\delta,k\omega}(\cdot;\beta))}.$$

We can now estimate all the other terms in the weighted L^∞-spaces, analogously as for u_1^V. It can be easily checked that the estimates were not "optimal". We only have to restrict a little bit more the values of r, namely to $r > 5$. We get

$$\|u_2^V\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta))} \leqslant \frac{\varepsilon}{10}\beta^{1-\omega}.$$

Next, we continue with the surface integrals. We distinguish three situations:

a) $|\mathbf{x}| \leqslant 1$

b) $1 \leqslant |\mathbf{x}| \leqslant 1/\beta$, $(\beta < 1)$

c) $\beta|\mathbf{x}| > 1$.

In the case a) we will not use the integral representation; we rather use the estimates (2.1) and (2.2). Thus

$$\|\mathbf{u}\|_{\infty,\Omega_1} \leqslant C\|\mathbf{u}\|_{2,q,\Omega_1} \leqslant C(\|\mathbf{u}\|_{\frac{2q}{2-q},\Omega_1} + \|\nabla^2\mathbf{u}\|_{q,\Omega_1}).$$

However, due to the Friedrichs inequality we have

$$\|\mathbf{u}\|_{\frac{2q}{2-q},\Omega_1} \leqslant C(\beta + \|\nabla^2\mathbf{u}\|_q)$$

and then, using (2.1) and (2.2), we get for $\omega > 1-\alpha$

$$\|\mathbf{u}\|_{\infty,(\mu_1^{1,\omega}(\cdot;\beta)),\Omega_1} \leqslant \frac{\varepsilon}{20}\beta^{1-\omega}.$$

In the other two situations we can use the integral representation (3.2). We easily observe that the boundary terms decay sufficiently fast at infinity. Thus we are only left with the checking that these terms are sufficiently small. This is evidently true for the first term due to the presence of the number β in the boundary integral. The second integral can be estimated easily; in the case b) using the fact that $\beta|\mathbf{x}| \leqslant 1$, $|\mathbf{x}| > 1$, we obtain

$$\begin{aligned}
&|u_1^{S,2}(\mathbf{x})\mu_{1/2}^{1/2,\omega}(\mathbf{x};\beta)| + |u_2^{S,2}(\mathbf{x})\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta)| \\
&\quad\leqslant C\beta|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{1/2-\omega}(1+s(\beta\mathbf{x}))^{1/2} \\
&\qquad\times\Big[|\nabla\mathcal{O}_{11}^{\mu}(\mathbf{x};\beta)| + \Big|\nabla^2\mathcal{O}_{11}^{\mu}\Big(\frac{\mathbf{x}}{2};\beta\Big)\Big| + |\mathbf{e}(\mathbf{x})| + \Big|\nabla\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)\Big|\Big] \\
&\qquad + C\beta|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{1-\omega}|\ln(2+|\beta\mathbf{x}|)|^{-1} \\
&\qquad\times\Big[|\nabla\mathcal{O}_{12}^{\mu}(\mathbf{x};\beta)| + \Big|\nabla^2\mathcal{O}_{12}^{\mu}\Big(\frac{\mathbf{x}}{2};\beta\Big)\Big| + |\mathbf{e}(\mathbf{x})| + \Big|\nabla\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)\Big|\Big] \\
&\quad\leqslant C\beta\Big(\frac{1}{|\mathbf{x}|^{1-\omega}} + \frac{1}{|\mathbf{x}|^{2-\omega}}\Big) \leqslant C\beta.
\end{aligned}$$

Therefore

$$\|u_1^{S,2}\|_{\infty,(\mu_{1/2}^{1/2,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} + \|u_2^{S,2}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} \leqslant \frac{\varepsilon}{40}\beta^{1-\omega}.$$

In the case c) we use $\beta|\mathbf{x}| > 1$ and thus

$$\begin{aligned}
&|u_1^{S,2}(\mathbf{x})\mu_{1/2}^{1/2,\omega}(\mathbf{x};\beta)| + |u_2^{S,2}(\mathbf{x})\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta)| \\
&\leqslant C\beta^{2-\omega}\Big(\frac{1}{(1+|\beta\mathbf{x}|)^{1/2}(1+s(\beta\mathbf{x}))^{1/2}} + \frac{(1+s(\beta\mathbf{x}))^{1/2}}{(1+|\beta\mathbf{x}|)^{1/2}} + |\ln(2+|\beta\mathbf{x}|)|^{-1}\Big),
\end{aligned}$$

i.e.

$$\|u_1^{S,2}\|_{\infty,(\mu_{1/2}^{1/2,\omega}(\cdot;\beta)),\Omega^{1/\beta}} + \|u_2^{S,2}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega^{1/\beta}} \leqslant \frac{\varepsilon}{40}\beta^{1-\omega}.$$

We must proceed very carefully in the estimate of the third term. We put $\mathbf{u} = {}^{I}\mathbf{u} + {}^{II}\mathbf{u}$ where ${}^{I}\mathbf{u}$ solves the Oseen problem with zero right-hand side and a non-zero boundary condition while ${}^{II}\mathbf{u}$ solves the Oseen problem with zero boundary conditions and a non-zero right-hand side. Then

$$|u_j^{S,3}(\mathbf{x})| \leqslant \Big|\int_{\partial\Omega}\mathcal{O}_{ij}^{\mu}(\mathbf{x}-\mathbf{y};\beta)\Big[T_{ik}({}^{I}\mathbf{u},{}^{I}\pi) + T_{ik}({}^{II}\mathbf{u},{}^{II}\pi)\Big](\mathbf{y})n_k(\mathbf{y})\,\mathrm{d}_{\mathbf{y}}S\Big|.$$

From [4][8] we have for $|\beta\mathbf{x}| \leqslant 1$

$$\left|\int_{\partial\Omega} T_{ik}({}^I\mathbf{u},{}^I\pi)(\mathbf{y})n_k(\mathbf{y})\,\mathrm{d}_{\mathbf{y}}S\right| \leqslant C|\ln\beta|^{-1}\|\mathbf{u}_*\|_{2-1/q,q,(\partial\Omega)} \leqslant C\beta|\ln\beta|^{-1}$$

and so

$$\begin{aligned}
&|{}^I u_1^{S,3}(\mathbf{x})\mu_{1/2}^{1/2,\omega}(\mathbf{x};\beta)| + |{}^I u_2^{S,3}(\mathbf{x})\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta)| \\
&\quad \leqslant C|\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1/2-\omega}[(1+|\beta\mathbf{x}|)^{1/2}\ln(2+|\beta\mathbf{x})|)^{-1} \\
&\qquad + (1+s(\beta\mathbf{x}))^{1/2}]\left|\int_{\partial\Omega} T_{ik}({}^I\mathbf{u},{}^I\pi)(\mathbf{y})n_k(\mathbf{y})\,\mathrm{d}_{\mathbf{y}}S\right| \\
&\qquad + C\left|\nabla\boldsymbol{\mathcal{O}}^\mu\left(\frac{\mathbf{x}}{2};\beta\right)\right||\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1/2-\omega}[(1+|\beta\mathbf{x}|)^{1/2}|\ln(2+|\beta\mathbf{x}|)|^{-1} \\
&\qquad + (1+s(\beta\mathbf{x}))^{1/2}]\int_{\partial\Omega}(|\nabla\mathbf{u}|+|p|)\,\mathrm{d}S \\
&\quad \leqslant C(\beta^{1-\omega}|\ln\beta|^{-1}+\beta).
\end{aligned}$$

Here we have used the fact that (see Theorem 1.1, recall that ${}^I\mathbf{f} = \mathbf{0}$)

$$\int_{\partial\Omega}(|\nabla^I\mathbf{u}|+|{}^I\pi|)\,\mathrm{d}S \leqslant C(\|\nabla^{2I}\mathbf{u}\|_{q,\Omega}+\|\nabla^I\pi\|_{q,\Omega}).$$

Thus

$$\|{}^I u_1^{S,3}\|_{\infty,(\mu_{1/2}^{1/2,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} + \|{}^I u_2^{S,3}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} \leqslant \frac{\varepsilon}{80}\beta^{1-\omega}.$$

Analogously we have

$$\begin{aligned}
&\|{}^{II}u_1^{S,3}\|_{\infty,(\mu_{1/2}^{1/2,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} + \|{}^{II}u_2^{S,3}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} \\
&\qquad \leqslant C\beta^{-\omega-2(1-1/q)}\|\nabla\cdot\boldsymbol{\mathcal{G}}\|_q.
\end{aligned}$$

However, estimating the right-hand side term by term, we get

$$\|\nabla\cdot\boldsymbol{\mathcal{G}}\|_q \leqslant C[\varepsilon^2\beta^{1+2(1-\frac{1}{q})}+\varepsilon^2\beta^{2\alpha}+\varepsilon^3\beta^{\frac{4}{3}+2(1-1/q)}+c(\mathbf{f})],$$

and thus

$$\|u_1^{S,3}\|_{\infty,(\mu_{1/2}^{1/2,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} + \|u_2^{S,3}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} \leqslant \frac{\varepsilon}{40}\beta^{1-\omega}.$$

In a completely analogous way we can proceed for $|\mathbf{x}| > \frac{1}{\beta}$ and thus

$$\|u_1^{S,3}\|_{\infty,(\mu_{1/2}^{1/2,\omega}(\cdot;\beta)),\Omega^{1/\beta}} + \|u_2^{S,3}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega^{1/\beta}} \leqslant \frac{\varepsilon}{40}\beta^{1-\omega}.$$

[8] The proof in [4] is done for the classical Oseen problem; in the case of the modified version, the proof is essentially the same, see [10].

For the fourth term we get (in both cases b) and c))

$$\|u_1^{S,4}\|_{\infty,(\mu_{1/2}^{1/2,\omega}(\cdot;\beta)),\Omega^1} + \|u_2^{S,4}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega^1} \leqslant C\beta^{-\omega}\int_{\partial\Omega} |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|\,\mathrm{d}S.$$

But

$$\int_{\partial\Omega} |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|\,\mathrm{d}S \leqslant C\int_{\Omega_1} (|\boldsymbol{\mathcal{G}}| + |\nabla\cdot\boldsymbol{\mathcal{G}}|)\,\mathrm{d}\mathbf{x}$$

and estimating the right-hand side term by term we get

$$\|u_1^{S,4}\|_{\infty,(\mu_{1/2}^{1/2,\omega}(\cdot;\beta)),\Omega} + \|u_2^{S,4}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega} \leqslant \frac{\varepsilon}{5}\beta^{1-\omega}.$$

Next we continue with the first and second gradients of the velocity. We will simultaneously treat the pressure and its gradients as these terms can be estimated analogously as the gradients of the velocity. We denote again by the upper index V the volume parts of the integral representation and by the upper index S the surface parts.

Applying Theorem 1.6 we get $(0<\delta<\frac{1}{2r},\ k=0,1,2,3)$

$$\begin{aligned}\|\nabla\mathbf{u}^V\|_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta))} &\leqslant C\beta^{(k-1)\omega}\|\boldsymbol{\mathcal{G}}\|_{r,(\mu_{1/2-1/r}^{1-2/r+\delta,k\omega}(\cdot;\beta))},\\ \|\nabla^2\mathbf{u}^V\|_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot\beta))} &\leqslant C\beta^{(k-1)\omega}\|\nabla\cdot\boldsymbol{\mathcal{G}}\|_{r,(\mu_{1/2-1/r}^{1-2/r+\delta,k\omega}(\cdot;\beta))}.\end{aligned}\tag{3.4}$$

Similarly, applying Theorems 1.7 and 1.8 we get

$$\begin{aligned}\|\pi^V\|_{r,(\mu_0^{1-3/r,\omega}(\cdot;\beta))} &\leqslant C\beta^{(k-1)\omega}\|\boldsymbol{\mathcal{G}}\|_{r,(\mu_0^{1-3/r,k\omega}(\cdot;\beta))},\\ \|\nabla\pi^V\|_{r,(\mu_0^{1-3/r,\omega}(\cdot;\beta))} &\leqslant C\beta^{(k-1)\omega}\|\nabla\cdot\boldsymbol{\mathcal{G}}\|_{r,(\mu_0^{1-3/r,k\omega}(\cdot;\beta))}.\end{aligned}\tag{3.5}$$

However, the right-hand side of (3.5) can be estimated by the right hand side of (3.4) and thus it is sufficient to consider only the latter case. We thus estimate $\boldsymbol{\mathcal{G}}$ and $\nabla\cdot\boldsymbol{\mathcal{G}}$ term by term in the norm indicated in (3.4). Let $\delta>0$ be sufficiently small. First, evidently

$$\|\mathbf{h},\mathbf{f},\nabla\mathbf{f}\|_{r,(\mu_{1/2-1/r}^{1-2/r+\delta,0}(\cdot;\beta))} \leqslant C\|\mathbf{h},\mathbf{f},\nabla\mathbf{f}\|_{\infty,(\eta_{1/2}^1(\cdot))}.$$

Next,

$$\|\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_{1/2-1/r}^{1-2/r+\delta,2\omega}(\cdot;\beta))} \leqslant \|\mathbf{T}\|_{\infty,(\mu_{1/4-1/(2r)}^{1/2-1/r,\omega}(\cdot;\beta))}\|\nabla\mathbf{w}\|_{r,(\mu_{1/4-1/(2r)}^{1/2-1/r+\delta,\omega}(\cdot;\beta))}$$

and due to Remark 3.1 and Lemma 3.1 we have for $r>5$

$$\|\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_{1/2-1/r}^{1-2/r+\delta,2\omega}(\cdot;\beta))} \leqslant C\|\nabla\mathbf{w}\|^2_{1,r(\mu_{1/4-1/(2r)}^{1/2-1/r+\delta,\omega}(\cdot;\beta))} \leqslant \frac{\varepsilon}{120}\beta^{1-\omega}.$$

In a completely analogous way we obtain

$$\|\nabla(\mathbf{T}\nabla\mathbf{w})\|_{r,(\mu^{1-2/r+\delta,2\omega}_{1/2-1/r}(\cdot;\beta))} + \|p\nabla\mathbf{w}\|_{r,(\mu^{1-2/r+\delta,2\omega}_{1/2-1/r}(\cdot;\beta))} + \|\nabla(p\nabla\mathbf{w})\|_{r,(\mu^{1-2/r+\delta,2\omega}_{1/2-1/r}(\cdot;\beta))} \leqslant \frac{\varepsilon}{120}\beta^{1-\omega}.$$

In order to estimate the trilinear term it is enough to use

$$\||\mathbf{w}|^2\nabla^l\mathbf{w}\|_r \leqslant \|\nabla^l\mathbf{w}\|_r\|\mathbf{w}\|^2_\infty \quad (l=1,2),$$
$$\||\nabla\mathbf{w}|^2\mathbf{w}\|_r \leqslant \|\nabla\mathbf{w}\|_r\|\nabla\mathbf{w}\|_{1,r}\|\mathbf{w}\|_\infty$$

and we get

$$\||\mathbf{w}|^2\nabla\mathbf{w}\|_{r,(\mu^{1-2/r+\delta,3\omega}_{1/2-1/r}(\cdot;\beta))} + \|\nabla(|\mathbf{w}|^2\nabla\mathbf{w})\|_{r,(\mu^{1-2/r+\delta,3\omega}_{1/2-1/r}(\cdot;\beta))} \leqslant \frac{\varepsilon}{120}\beta^{1-\omega}.$$

For estimating the convective term we use the L^∞ estimate $\left(\delta r < \frac{1}{2}\right)$

$$\begin{aligned}
\||\mathbf{w}|^2\|^r_{r,(\mu^{1-2/r+\delta,2\omega}_{1/2-1/r}(\cdot;\beta))} &= \int_\Omega |\mathbf{w}|^{2r}|\mathbf{x}|^{2\omega r}(1+|\beta\mathbf{x}|)^{r-2r\omega-2+\delta r}(1+s(\beta\mathbf{x})^{r/2-1}\,\mathrm{d}\mathbf{x}\\
&\leqslant \|\mathbf{w}\|^{2r}_{\infty,(\mu^{1/2,\omega}_{1/2}(\cdot;\beta))}\int_\Omega (1+|\beta\mathbf{x}|)^{-2+\delta r}(1+s(\beta\mathbf{x}))^{-r/2-1}\,\mathrm{d}\mathbf{x}\\
&\leqslant C\beta^{-2}\|\mathbf{w}\|^{2r}_{\omega,(\mu^{1/2,\omega}_{1/2}(\cdot;\beta))}
\end{aligned}$$

concluding

$$\beta\|\mathbf{w}\otimes\mathbf{w}\|_{r,(\mu^{1-2/r+\delta,2\omega}_{1/2-1/r}(\cdot;\beta))} \leqslant \frac{\varepsilon}{120}\beta^{1-\omega}$$

for $r > 5$.

The other bilinear terms are estimated using the standard inequalities

$$\beta\|\mathbf{w}\nabla\mathbf{w}\|_{r,(\mu^{1-2/r+\delta,2\omega}_{1/2-1/r}(\cdot;\beta))} + \beta\|\mathbf{w}\cdot\nabla\mathbf{w}\|_{r,(\mu^{1-2/r+\delta,2\omega}_{1/2-1/r}(\cdot;\beta))} + \beta\|\nabla(\mathbf{w}\cdot\nabla)\mathbf{w}\|_{r,(\mu^{1-2/r,2\omega}_{1/2-1/r}(\cdot;\beta))} \leqslant \frac{\varepsilon}{120}\beta^{1-\omega}.$$

The last term which contains $\mathbf{f}\mathbf{w}$ and $\nabla(\mathbf{f}\mathbf{w})$ can be estimated analogously and thus we get

$$\|\nabla\mathbf{u}^V\|_{r,(\mu^{1-2/r,\omega}_{1/2-1/r}(\cdot;\beta))} + \|\nabla^2\mathbf{u}^V\|_{r,(\mu^{1-2/r,\omega}_{1/2-1/r}(\cdot;\beta))} + \|\pi^V\|_{r,(\mu^{1-3/r,\omega}_{0}(\cdot;\beta))} + \|\nabla\pi^V\|_{r,(\mu^{1-3/r,\omega}_{0}(\cdot;\beta))} \leqslant \frac{\varepsilon}{5}\beta^{1-\omega}$$

for β sufficiently small. We continue with the boundary terms. As above, we distinguish three cases, i.e. $|\mathbf{x}| \leqslant 1$, $1 < |\mathbf{x}| \leqslant \frac{1}{\beta}$ and $|\mathbf{x}| > \frac{1}{\beta}$.

Let us start with the first case. Using the Sobolev imbedding theorem we have ($q < 2$)

$$\begin{aligned}
\|\nabla\mathbf{u}, \nabla^2\mathbf{u}\|_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta)),\Omega_1} &\leqslant C(\|\nabla\mathbf{u}, \nabla^2\mathbf{u}\|_{r,\Omega_1} \\
&\leqslant C(\|\nabla^2\mathbf{u}\|_{1,p,\Omega_1} + \|\nabla^2\mathbf{u}\|_{q,\Omega}) \leqslant \frac{\varepsilon}{20}\beta^{1-\omega}, \\
\|\pi, \nabla\pi\|_{r,(\mu_0^{1-3/r,\omega}(\cdot;\beta)),\Omega_1} &\leqslant C(\|\nabla\pi\|_{1,p,\Omega_1} + \|\nabla\pi\|_{q,\Omega}) \leqslant \frac{\varepsilon}{20}\beta^{1-\omega}.
\end{aligned}$$

Let us continue with the case $1 < |\mathbf{x}| \leqslant \frac{1}{\beta}$, $\beta < 1$. Using the integral representation (see Theorems 1.2 and 1.3) we easily check that it is sufficient to consider only $\nabla\mathbf{u}^S$ and π^S; $\nabla^2\mathbf{u}^S$ and $\nabla\pi^S$ are estimated similarly and even easier as the higher gradients of $\boldsymbol{\mathcal{O}}^\mu$ and $\mathcal{E}$ decay faster.

First, let $1 < |\mathbf{x}| \leqslant \frac{1}{\beta}$. We again skip the term $\nabla\mathbf{u}^{S,1}$ and concentrate on the other terms. We have

$$\begin{aligned}
&|\nabla\mathbf{u}^{S,2}(\mathbf{x})\mu_{1/2-1/r}^{1-2/r,\omega}(\mathbf{x};\beta)| \\
&\quad\leqslant C\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-2/r}(1+s(\beta\mathbf{x}))^{1/2-1/r} \\
&\qquad\times\Big(|\nabla^2\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)| + |\nabla\mathbf{e}(\mathbf{x})| + \Big|\nabla^3\boldsymbol{\mathcal{O}}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)\Big| + \Big|\nabla^2\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)\Big|\Big),
\end{aligned}$$

i.e.

$$\begin{aligned}
\|\nabla\mathbf{u}^{S,2}\|^r_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} &\leqslant C\beta^{(3-\omega)r}\int_{\Omega^1_{1/\beta}}(|\beta\mathbf{x}|^{(\omega-2)r} + \beta^r|\beta\mathbf{x}|^{(\omega-3)r}) \\
&\qquad\times(1+|\beta\mathbf{x}|)^{(1-\omega-2/r)r}(1+s(\beta\mathbf{x}))^{r/2-1}\,\mathrm{d}\mathbf{x} \\
&\leqslant C\beta^{(3-\omega)r-2}\int_\beta^1 y^{\omega r-2r+1}\,\mathrm{d}y \leqslant \Big(\frac{\varepsilon}{40}\beta^{1-\omega}\Big)^r.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\|\nabla\mathbf{u}^{S,3}\|^r_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} + \|\nabla\mathbf{u}^{S,4}\|^r_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} \\
&\qquad\leqslant C\int_{\partial\Omega}(|\mathbf{u}| + |\pi| + |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|)\,\mathrm{d}S \leqslant \Big(\frac{\varepsilon}{40}\beta^{1-\omega}\Big)^r.
\end{aligned}$$

For the pressure we proceed similarly. Again, the gradient of the pressure is estimated easier than the pressure itself. We skip the term $\pi^{S,1}$ and consider only the other three terms. We have

$$|\pi^{S,2}(\mathbf{x})\mu_0^{1-\frac{3}{r},\omega}(\mathbf{x};\beta)| \leqslant C\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-\frac{3}{r}}\Big(\frac{1}{|\mathbf{x}|^2} + \frac{1}{|\mathbf{x}|^4}\Big)$$

and then

$$\|\pi^{S,2}\|^r_{r,(\mu_0^{1-3/r,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} \leqslant C\beta \int_1^{1/\beta} (y^{(\omega-2)r} + y^{(\omega-4)r}) y \, \mathrm{d}y \leqslant \Big(\frac{\varepsilon}{40}\beta^{1-\omega}\Big)^r.$$

The last two terms can be estimated as above:

$$\begin{aligned}\|\pi^{S,3}\|_{r,(\mu_0^{1-3/r,\omega}(\cdot;\beta)),\Omega^1{}_{1/\beta}} &+ \|\pi^{S,4}\|_{r,(\mu_0^{1-3/r,\omega}(\cdot;\beta)),\Omega^1_{1/\beta}} \\ &\leqslant C \int_{\partial\Omega} (|\nabla \mathbf{u}| + |\pi| + |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|) \, \mathrm{d}S \leqslant \frac{\varepsilon}{40}\beta^{1-\omega}.\end{aligned}$$

Finally, let $|\mathbf{x}| > \frac{1}{\beta}$. Here the most restrictive terms are the first, third and fourth. We consider only the first term

$$\begin{aligned}&|\nabla \mathbf{u}^{S,1}(\mathbf{x})\mu_{1/2-1/r}^{1-2/r,\omega}(\mathbf{x};\beta)| \\ &\quad\leqslant C\beta^2|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-2/r}(1+s(\beta\mathbf{x}))^{1/2-1/r} \\ &\qquad\times\Big[\frac{\beta}{(1+|\beta\mathbf{x}|)(1+s(\beta\mathbf{x}))} + \frac{\beta^2}{(1+|\beta\mathbf{x}|)^{3/2}(1+s(\beta\mathbf{x}))^{3/2}}\Big]\end{aligned}$$

and so

$$\begin{aligned}&\|\nabla \mathbf{u}^{S,1}\|^r_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta)),\Omega^{1/\beta}} \\ &\quad\leqslant C\beta^{(3-\omega)r}\int_{\Omega^{1/\beta}} (1+|\beta\mathbf{x}|)^{-2-\omega r}|\beta\mathbf{x}|^{\omega r}(1+s(\beta\mathbf{x}))^{-1-r/2}\,\mathrm{d}\mathbf{x} \\ &\quad\leqslant C\beta^{(3-\omega)r-2}\int_{\Omega^1}(1+\mathbf{y})^{-2}(1+s(\mathbf{y}))^{-1-r/2}\,\mathrm{d}\mathbf{y} \leqslant \Big(\frac{\varepsilon}{40}\beta^{(1-\omega)}\Big)^r.\end{aligned}$$

The other terms decay similarly or even faster and thus cause no problems.[9]

For the pressure, the most restrictive terms are the first, the third and the fourth. We demonstrate the estimate on the first term:

$$|\pi^{S,1}(\mathbf{x})\mu_0^{1-3/r,\omega}(\mathbf{x};\beta)| \leqslant C\beta^2|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-3/r}\Big[\frac{1}{|\mathbf{x}|}+\frac{1}{|\mathbf{x}|^2}\Big]$$

and therefore

$$\|\pi^{S,1}\|^r_{r,(\mu_0^{1-3/r,\omega}(\cdot;\beta)),\Omega^{1/\beta}} \leqslant C\beta^{(3-\omega)r-2}\int_{\Omega^1}|\mathbf{y}|^{-3}\,\mathrm{d}\mathbf{y} \leqslant C\beta^{(3-\omega)r-2}.$$

[9] Note that we could even get a better result, i.e. an estimate of the surface terms in $L^r(\Omega;\mu_{1-1/r}^{1-2/r,\omega}(\cdot;\beta))$; the bound comes therefore from the volume terms, see also Remark 3.2.

The other terms can be estimated similarly. Thus we get

$$\|\nabla \mathbf{u}^S\|_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta)),\Omega} + \|\nabla^2 \mathbf{u}^S\|_{r,(\mu_{1/2-1/r}^{1-2/r,\omega}(\cdot;\beta)),\Omega} \leqslant \frac{\varepsilon}{5}\beta^{1-\omega},$$
$$\|\pi^S\|_{r,(\mu_0^{1-3/r,\omega}(\cdot;\beta)),\Omega} + \|\nabla \pi^S\|_{r,(\mu_0^{1-3/r,\omega}(\cdot;\beta)),\Omega} \leqslant \frac{\varepsilon}{5}\beta^{1-\omega}.$$

Combining this with the estimates above we have

$$\|(\mathbf{u},\pi)\|_{V_\beta} \leqslant \varepsilon\beta^{1-\omega} = \delta_0$$

and the operator $\mathcal{M}$ maps sufficiently small balls into themselves. We have proved

Theorem 3.1. *Let $\mathbf{f} = \nabla\cdot\mathbf{h}$ and let $\mathbf{h} \in L^1_{\rm loc}(\Omega)$, $\mathbf{f} \in W^{2,q}(\Omega)\cap W^{k,p}(\Omega)$, $q \in (1;\frac{6}{5})$, $k \geqslant 2$, $p \in (2;\infty)$ with norms sufficiently small. Let $\Omega \in C^{k+1}$ be an exterior domain in $\mathbb{R}^2$. Moreover, let*

$$\mathbf{h},\mathbf{f},\nabla\mathbf{f} \in L^\infty(\Omega;\eta_{1/2}^{1+\delta}(\cdot)) \tag{3.6}$$

for some $\delta > 0$. Let $\beta = |\mathbf{v}_\infty|$ and $\|\mathbf{h},\mathbf{f},\nabla\mathbf{f}\|_{\infty,\eta_{1/2}^{1+\delta}(\cdot)}$ be sufficiently small.

Then $(\mathbf{v} = \mathbf{u}+\mathbf{v}_\infty, p)$, the solution to the problem (0.3)–(0.7) constructed in Theorem 2.1, has the asymptotic properties

$$\begin{aligned} u_1 = v_1 - \beta &\in L^\infty(\Omega;\eta_{1/2}^{1/2}(\cdot)),\\ v_2 &\in L^\infty(\Omega;\eta_0^1(\cdot)|\ln(2+\cdot)|^{-1}),\\ \nabla\mathbf{v},\nabla^2\mathbf{v} &\in L^r(\Omega;\eta_{1/2-1/r}^{1-2/r}(\cdot)),\\ p,\nabla p &\in L^r(\Omega;\eta_0^{1-3/r}(\cdot)), \end{aligned} \tag{3.7}$$

where $r \in (5;\infty)$.

Remark 3.2. The assumptions on $\mathbf{f}$ were in some sense "minimal" in order to ensure the fastest possible decay of the velocity itself. Assuming $\mathbf{h},\mathbf{f},\nabla\mathbf{f} \in L^\infty(\Omega;\eta_{1/2}^{3/2}(\cdot))$ we would get $\nabla\mathbf{v}$, $\nabla^2\mathbf{v} \in L^r(\Omega;\eta_{1-1/r}^{1-2/r}(\cdot))$. Thus, using the Sobolev imbedding theorem we have that $\nabla\mathbf{v} \in L^\infty(\Omega;\eta_{1-1/r}^{1-2/r}(\cdot))$ and $\pi \in L^\infty(\Omega;\eta_0^{1-3/r}(\cdot))$. Passing with $r\to\infty$ we get almost the same asymptotic structure for $\nabla\mathbf{v}$ and p as for $\nabla\boldsymbol{\mathcal{O}}^\mu$ and $\nabla\mathcal{E}$, respectively. The details can be found in [10].

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