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LINEAR CONFORM TRANSFORMATION: ERRORS IN BOTH COORDINATE SYSTEMS*

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Abstract. Linear conform transformation in the case of non-negligible errors in both coordinate systems is investigated. Estimation of transformation parameters and their statistical properties are described. Confidence ellipses of transformed nonidentical points and cross covariance matrices among them and identical points are determined. Some simulation for a verification of theoretical results are presented.

Keywords: linear conform transformation, regression model with constraints

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1. Introduction

Many problems lead to the necessity to transform coordinates of some points from one coordinate system to another. Parameters of the transformation can be determined on the basis of an n-tuple of points with coordinates given in both coordinate systems. Frequently the linear conform transformation has been used. A transformation of photogrammetry shots into cadastral maps may serve as an example.

The linear conform transformation (Helmert transformation) in its standard version has been derived under the assumption that non-negligible random errors occur at points of the coordinate system into which the transformation is performed; points of the inverse image coordinate system are assumed to be errorless. Statistical properties of such transformation are investigated, e.g., in [1], [2], [3].

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Nevertheless, the situations when the standard deviations of errors in coordinates of both systems are of the same order is frequent; in this case the standard procedure cannot be considered to be optimal. Several approaches were investigated for searching such situations, e.g. in [4], [6]. A simple example of some problems of this procedure is shown in Chapter 6.

The aim of the paper is to present a relatively simple and simultaneously approximately optimal solution of the latter situation.

2. Notation and auxiliary statements

A point whose coordinates are known in both coordinate systems is called an identical point.

Actual coordinates of the i-th identical point in System I are given by a vector $\boldsymbol{\eta}_{I,i} = (\varphi_{I,i}, \psi_{I,i})'$ ($'$ denotes transposition), the coordinates of the same point in System II are given by a vector $\boldsymbol{\eta}_{II,i} = (\varphi_{II,i}, \psi_{II,i})'$.

Let us suppose that n identical points are at our disposal for the Helmert transformation

$$\boldsymbol{\eta}_{II,i} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix} \boldsymbol{\eta}_{I,i}, \quad i = 1, \dots, n, \tag{2.1}$$

characterized by the parameters β_1, β_2, β_3, β_4.

Instead of the vectors of actual coordinates $\boldsymbol{\eta}_{I,i}$ (System I), $i = 1, \dots, n$, and $\boldsymbol{\eta}_{II,i}$ (System II), $i = 1, \dots, n$, only estimators of them are at our disposal.

These estimators have the form of random vectors $\mathbf{Y}_I \sim (\boldsymbol{\eta}_I, \boldsymbol{\Sigma}_{\mathbf{Y}_I)}$ and $\mathbf{Y}_{II} \sim (\boldsymbol{\eta}_{II}, \boldsymbol{\Sigma}_{\mathbf{Y}_{II}})$; ($\boldsymbol{\eta}_I = (\boldsymbol{\eta}'_{I,1}, \dots, \boldsymbol{\eta}'_{I,n})'$ is the mean value of the random vector $\mathbf{Y}_I$, i.e. $E(\mathbf{Y}_I) = \boldsymbol{\eta}_I = (\boldsymbol{\eta}'_{I,1}, \dots, \boldsymbol{\eta}'_{I,n})'$, and $\boldsymbol{\Sigma}_{\mathbf{Y}_I}$ is its covariance matrix $\operatorname{var}(\mathbf{Y}_I) = \boldsymbol{\Sigma}_{\mathbf{Y}_I}$; analogously $E(\mathbf{Y}_{II}) = \boldsymbol{\eta}_{II} = (\boldsymbol{\eta}'_{II,1}, \dots, \boldsymbol{\eta}'_{II,n})'$ and $\operatorname{var}(\mathbf{Y}_{II}) = \boldsymbol{\Sigma}_{Y_{II}}$).

The random vectors $\mathbf{Y}_I$ and $\mathbf{Y}_{II}$ are supposed to be stochastically independent and the matrices $\boldsymbol{\Sigma}_{\mathbf{Y}_I}$ and $\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}$ to be positive definite (i.e. regular).

The problem to determine the optimum estimators of the unknown transformation parameters and the transformed coordinates of the identical points simultaneously with the corrections of the coordinates within System II caused by the fact that they are not errorless, leads to an application of the following statement.

Lemma 2.1. *Consider the model of a direct incomplete measurement of the vector parameter*

$$\mathbf{Y} \sim (\boldsymbol{\eta}, \boldsymbol{\Sigma}), \quad \begin{pmatrix} \boldsymbol{\eta} \\ \boldsymbol{\beta} \end{pmatrix} \in \left\{ \begin{pmatrix} \boldsymbol{\eta} \\ \boldsymbol{\beta} \end{pmatrix} : \mathbf{b} + \mathbf{B}_1 \boldsymbol{\eta} + \mathbf{B}_2 \boldsymbol{\beta} = \mathbf{0} \right\}, \tag{2.2}$$

where $\mathbf{Y}$ *is a* k_1*-dimensional observation vector, the unknown vector parameter is formed by a* k_1*-dimensional directly observed parameter* $\boldsymbol{\eta}$ *and a* k_2*-dimensional parameter* $\boldsymbol{\beta}$*, which is not directly observed by the vector* $\mathbf{Y}$*,* $\boldsymbol{\Sigma}$ *is a positive definite matrix,* $\mathbf{b}$ *a known* q*-dimensional vector,* $\mathbf{B}_1$ *a given* $q \times k_1$ *matrix,* $\mathbf{B}_2$ *a given* $q \times k_2$ *matrix (it occurs in the vector condition only). If the ranks* $r(\mathbf{B}_1, \mathbf{B}_2)$ *and* $r(\mathbf{B}_2)$ *of the matrices* $(\mathbf{B}_1, \mathbf{B}_2)$ *and* $\mathbf{B}_2$*, respectively, satisfy the conditions*

$$r(\mathbf{B}_1, \mathbf{B}_2) = q < k_1 + k_2, \quad r(\mathbf{B}_2) = k_2 < q,$$

the best linear unbiased estimator of the vectors $\boldsymbol{\eta}$ *and* $\boldsymbol{\beta}$ *is*

$$\begin{pmatrix} \hat{\boldsymbol{\eta}}(\mathbf{Y}) \\ \hat{\boldsymbol{\beta}}(\mathbf{Y}) \end{pmatrix} = \begin{pmatrix} \mathbf{I} - \boldsymbol{\Sigma}\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1 \\ -\mathbf{Q}_{2,1}\mathbf{B}_1 \end{pmatrix} \mathbf{Y} + \begin{pmatrix} -\boldsymbol{\Sigma}\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{b} \\ -\mathbf{Q}_{2,1}\mathbf{b} \end{pmatrix},$$

where $\mathbf{I}$ *is the* $k_1 \times k_1$ *identical matrix,*

$$\begin{pmatrix} \mathbf{Q}_{1,1} & \mathbf{Q}_{1,2} \\ \mathbf{Q}_{2,1} & \mathbf{Q}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1\boldsymbol{\Sigma}\mathbf{B}_1' & \mathbf{B}_2 \\ \mathbf{B}_2' & \mathbf{0} \end{pmatrix}^{-1}$$

and the covariance matrix of this estimator is

$$\operatorname{var}\begin{pmatrix} \hat{\boldsymbol{\eta}}(\mathbf{Y}) \\ \hat{\boldsymbol{\beta}}(\mathbf{Y}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1\boldsymbol{\Sigma} & -\boldsymbol{\Sigma}\mathbf{B}_1'\mathbf{Q}_{1,2} \\ -\mathbf{Q}_{2,1}\mathbf{B}_1\boldsymbol{\Sigma} & -\mathbf{Q}_{2,2} \end{pmatrix}.$$

P r o o f. Cf. pages 138–143 in [9]. □

Lemma 2.2. *Under the assumptions of Lemma* 2.1 *we have*

$$\begin{aligned} \mathbf{Q}_{1,1} &= (\mathbf{B}_1\boldsymbol{\Sigma}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1} \\ &\quad - (\mathbf{B}_1\boldsymbol{\Sigma}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2[\mathbf{B}_2'(\mathbf{B}_1\boldsymbol{\Sigma}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2]^{-1} \\ &\quad \times \mathbf{B}_2'(\mathbf{B}_1\boldsymbol{\Sigma}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}, \\ \mathbf{Q}_{1,2} &= \mathbf{Q}_{2,1}' = (\mathbf{B}_1\boldsymbol{\Sigma}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2[\mathbf{B}_2'(\mathbf{B}_1\boldsymbol{\Sigma}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2]^{-1}, \\ \mathbf{Q}_{2,2} &= \mathbf{I} - [\mathbf{B}_2'(\mathbf{B}_1\boldsymbol{\Sigma}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2]^{-1}. \end{aligned}$$

P r o o f. It is based on the statement on the Pandora-Box matrix in [10], [5] and [7], p. 172 and on the assumption on the ranks of the matrices $\boldsymbol{\Sigma}$, $(\mathbf{B}_1, \mathbf{B}_2)$ and $\mathbf{B}_2$. □

3. Estimation of the transformation parameters β_i and the coordinates of the identical points

The above formulated transformation problem can be written in the form

$$
(3.1)\qquad \begin{pmatrix} \mathbf{Y}_I \\ \mathbf{Y}_{II} \end{pmatrix} \sim \left[\begin{pmatrix} \boldsymbol{\eta}_I \\ \boldsymbol{\eta}_{II} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{\mathbf{Y}_I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{\mathbf{Y}_{II}} \end{pmatrix}\right],
$$

$$
\begin{pmatrix} \boldsymbol{\eta}_I \\ \boldsymbol{\eta}_{II} \\ \boldsymbol{\beta} \end{pmatrix} \in \left\{ \begin{pmatrix} \boldsymbol{\eta}_I \\ \boldsymbol{\eta}_{II} \\ \boldsymbol{\beta} \end{pmatrix} ; \left(\mathbf{1} \otimes \mathbf{I};\ \ \boldsymbol{\eta}_I;\ \ \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \boldsymbol{\eta}_I \right) \boldsymbol{\beta} - \boldsymbol{\eta}_{II} = \mathbf{0} \right\}.
$$

Here $\mathbf{1} = (1, \dots, 1)'$ (n-dimensional) and $\otimes$ denotes the Kronecker multiplication of matrices.

The constraints are nonlinear and thus we need some good approximations $\boldsymbol{\eta}_{I,0}$, $\beta_{3,0}$ and $\beta_{4,0}$ of the vector $\boldsymbol{\eta}_I$ and the parameters β_3 and β_4, respectively, such that the vectors $\delta\boldsymbol{\eta}_I\delta\beta_3$ and $\delta\boldsymbol{\eta}_I\delta\beta_4$ can be neglected ($\delta\boldsymbol{\eta}_I = \boldsymbol{\eta}_I - \boldsymbol{\eta}_{I,0}$, $\delta\beta_3 = \beta_3 - \beta_{3,0}$, $\delta\beta_4 = \beta_4 - \beta_{4,0}$), in order that the constraints could be linearized.

The aim is to determine estimators of the vectors $\boldsymbol{\eta}_I$, $\boldsymbol{\eta}_{II}$ and $\boldsymbol{\beta}$ on the basis of the vectors $\mathbf{Y}_I$ and $\mathbf{Y}_{II}$. If we denote

$$
\begin{aligned}
\mathbf{b} &= \boldsymbol{\eta}_{I,0}\beta_{3,0} + \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \boldsymbol{\eta}_{I,0}\beta_{4,0}, \\
\mathbf{B}_1 &= \left(\beta_{3,0}\mathbf{I} + \beta_{4,0}\left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right], -\mathbf{I}\right) = (\mathbf{C}, -\mathbf{I}), \\
\mathbf{B}_2 &= \left(\mathbf{1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\eta}_{I,0}, \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \boldsymbol{\eta}_{I,0}\right),
\end{aligned}
$$

then the linearized version of the transformation problem (3.1) can be rewritten in the form

$$
(3.2)\qquad \begin{pmatrix} \mathbf{Y}_I - \boldsymbol{\eta}_{I,0} \\ \mathbf{Y}_{II} \end{pmatrix} \sim \left[\begin{pmatrix} \delta\boldsymbol{\eta}_I \\ \boldsymbol{\eta}_{II} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{\mathbf{Y}_I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{\mathbf{Y}_{II}} \end{pmatrix}\right],
$$

$$
\begin{pmatrix} \delta\boldsymbol{\eta}_I \\ \boldsymbol{\eta}_{II} \\ \beta_1 \\ \beta_2 \\ \delta\beta_3 \\ \delta\beta_4 \end{pmatrix} \in \left\{ \begin{pmatrix} \delta\boldsymbol{\eta}_I \\ \boldsymbol{\eta}_{II} \\ \beta_1 \\ \beta_2 \\ \delta\beta_3 \\ \delta\beta_4 \end{pmatrix} :\ \mathbf{b} + (\mathbf{C}, -\mathbf{I}) \begin{pmatrix} \delta\boldsymbol{\eta}_I \\ \boldsymbol{\eta}_{II} \end{pmatrix} + \mathbf{B}_2 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \delta\beta_3 \\ \delta\beta_4 \end{pmatrix} = \mathbf{0} \right\}.
$$

It is reasonable to choose the approximation $\boldsymbol{\eta}_{I,0}$, $\beta_{3,0}$ and $\beta_{4,0}$ in such a way that $\mathbf{b} = \mathbf{0}$.

A more detailed investigation what “a good approximation” of $\boldsymbol{\eta}_{I,0}$, $\beta_{3,0}$ and $\beta_{4,0}$ means is given in [11].

The next statement follows from Lemma 2.1.

Theorem 3.1. *The best linear unbiased estimators of*

$$\delta\boldsymbol{\eta}_I, \ \ \boldsymbol{\eta}_{II} \ \text{ and } (\beta_1, \beta_2, \delta\beta_3, \delta\beta_4)'$$

are

$$\widehat{\delta\boldsymbol{\eta}_I}(\mathbf{Y}_I, \mathbf{Y}_{II}) = (\mathbf{I} - \mathbf{T}_{\boldsymbol{\eta}_I,I}, \mathbf{T}_{\boldsymbol{\eta}_I,II}) \begin{pmatrix} \mathbf{Y}_I - \boldsymbol{\eta}_{I,0} \\ \mathbf{Y}_{II} \end{pmatrix} + \mathbf{t}_1, \tag{3.3}$$

$$\widehat{\boldsymbol{\eta}_{II}}(\mathbf{Y}_I, \mathbf{Y}_{II}) = (\mathbf{T}_{\boldsymbol{\eta}_{II},I}, \mathbf{I} - \mathbf{T}_{\boldsymbol{\eta}_{II},II}) \begin{pmatrix} \mathbf{Y}_I - \boldsymbol{\eta}_{I,0} \\ \mathbf{Y}_{II} \end{pmatrix} + \mathbf{t}_2 \tag{3.4}$$

and

$$\begin{pmatrix} \hat\beta_1(\mathbf{Y}_I, \mathbf{Y}_{II}) \\ \hat\beta_2(\mathbf{Y}_I, \mathbf{Y}_{II}) \\ \widehat{\delta\beta_3}(\mathbf{Y}_I, \mathbf{Y}_{II}) \\ \widehat{\delta\beta_4}(\mathbf{Y}_I, \mathbf{Y}_{II}) \end{pmatrix} = (\mathbf{T}_{\boldsymbol{\beta},I}, \mathbf{T}_{\boldsymbol{\beta},II}) \begin{pmatrix} \mathbf{Y}_I - \boldsymbol{\eta}_{I,0} \\ \mathbf{Y}_{II} \end{pmatrix} + \mathbf{t}_3, \tag{3.5}$$

where

$$\begin{array}{llll} \mathbf{T}_{\eta_I,I} & = \boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{C}'\mathbf{Q}_{1,1}\mathbf{C}, & \mathbf{T}_{\eta_I,II} & = \boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{C}'\mathbf{Q}_{1,1}, \\ \mathbf{T}_{\eta_{II},I} & = \boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\mathbf{Q}_{1,1}\mathbf{C}, & \mathbf{T}_{\eta_{II},II} & = \boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\mathbf{Q}_{1,1}, \\ \mathbf{T}_{\beta,I} & = -\mathbf{Q}_{2,1}\mathbf{C}, & \mathbf{T}_{\beta,II} & = \mathbf{Q}_{2,1}, \end{array} \tag{3.6}$$

and

$$\mathbf{t}_1 = -\boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{C}'\mathbf{Q}_{1,1}\mathbf{b}, \quad \mathbf{t}_2 = \boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\mathbf{Q}_{1,1}\mathbf{b}, \quad \mathbf{t}_3 = -\mathbf{Q}_{2,1}b.$$

Corollary 3.2. *The covariance matrix of the estimator*

$$(\hat{\boldsymbol{\eta}}_I'(\mathbf{Y}_I, \mathbf{Y}_{II}), \hat{\boldsymbol{\eta}}_{II}'(\mathbf{Y}_I, \mathbf{Y}_{II}), \hat{\boldsymbol{\beta}}'(\mathbf{Y}_I, \mathbf{Y}_{II}))'$$

is

$$\operatorname{var}\begin{pmatrix} \hat{\boldsymbol{\eta}}_I \\ \hat{\boldsymbol{\eta}}_{II} \\ \hat\beta \end{pmatrix} = \begin{pmatrix} \operatorname{var}(\hat{\boldsymbol{\eta}}_I) & \operatorname{cov}(\hat{\boldsymbol{\eta}}_I, \hat{\boldsymbol{\eta}}_{II}) & \operatorname{cov}(\hat{\boldsymbol{\eta}}_I, \hat\beta) \\ \operatorname{cov}(\hat{\boldsymbol{\eta}}_{II}, \hat{\boldsymbol{\eta}}_I) & \operatorname{var}(\hat{\boldsymbol{\eta}}_{II}) & \operatorname{cov}(\hat{\boldsymbol{\eta}}_{II}, \hat\beta) \\ \operatorname{cov}(\hat\beta, \hat{\boldsymbol{\eta}}_I) & \operatorname{cov}(\hat\beta, \hat{\boldsymbol{\eta}}_{II}) & \operatorname{var}(\hat\beta) \end{pmatrix},$$

where

$$\begin{aligned}
\operatorname{var}(\hat{\boldsymbol{\eta}}_I) &= (\mathbf{I}-\mathbf{T}_{\eta_I,I})\boldsymbol{\Sigma}_{\mathbf{Y}_I}(\mathbf{I}-\mathbf{T}_{\eta_I,I})' + \mathbf{T}_{\eta_I,II}\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\mathbf{T}'_{\eta_I,II},\\
\operatorname{cov}(\hat{\boldsymbol{\eta}}_I,\hat{\boldsymbol{\eta}}_{II}) &= (\mathbf{I}-\mathbf{T}_{\eta_I,I})\boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{T}'_{\eta_{II},I} + \mathbf{T}_{\eta_I,II}\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}(\mathbf{I}-\mathbf{T}_{\eta_{II},II})',\\
\operatorname{cov}(\hat{\boldsymbol{\eta}}_I,\hat{\boldsymbol{\beta}}) &= (\mathbf{I}-\mathbf{T}_{\eta_I,I})\boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{T}'_{\beta,I} + \mathbf{T}_{\eta_I,II}\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\mathbf{T}'_{\beta,II},\\
\operatorname{var}(\hat{\boldsymbol{\eta}}_{II}) &= \mathbf{T}_{\eta_{II},I}\boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{T}'_{\eta_{II},I} + (\mathbf{I}-\mathbf{T}_{\eta_{II},II})\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}(\mathbf{I}-\mathbf{T}_{\eta_{II},II})',\\
\operatorname{cov}(\hat{\boldsymbol{\eta}}_{II},\hat{\boldsymbol{\beta}}) &= \mathbf{T}_{\eta_{II},I}\boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{T}'_{\beta,I} + (\mathbf{I}-\mathbf{T}_{\eta_{II},II})\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\mathbf{T}'_{\beta,II},\\
\operatorname{var}(\hat{\boldsymbol{\beta}}) &= \mathbf{T}_{\beta,I}\boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{T}'_{\beta,I} + \mathbf{T}_{\beta,II}\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\mathbf{T}'_{\beta,II}.
\end{aligned}$$

4. Accuracy characteristics of transformed points

Identical points characterized in System II before the transformation by the vector $\mathbf{Y}_{II}$ are characterized by the vector $\hat{\boldsymbol{\eta}}_{II}$ after the transformation. Its accuracy is now characterized by the matrix

$$\operatorname{var}(\hat{\boldsymbol{\eta}}_{II}) = \mathbf{T}_{\eta_{II},I}\boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{T}'_{\eta_{II},I} + (\mathbf{I}-\mathbf{T}_{\eta_{II},II})\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}(\mathbf{I}-\mathbf{T}_{\eta_{II},II})',$$

cf. Corollary 3.2.

Beside the identical points there exist also nonidentical points in System II and a similar situation can occur also in System I. By the nonidentical points we mean the points whose coordinates (or their measurements) are known only in one system (instead of the identical ones whose coordinates (or again their measurements) are known in both systems).

The point field of identical and nonidentical points of System I is thus characterized by the random vector

$$(4.1)\qquad \begin{pmatrix}\mathbf{Y}_I\\ \mathbf{X}_I\end{pmatrix} \sim \left[\begin{pmatrix}\boldsymbol{\eta}_I\\ \boldsymbol{\xi}_I\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Sigma}_{\mathbf{Y}_I} & \boldsymbol{\Sigma}_{\mathbf{Y}_I,\mathbf{X}_I}\\ \boldsymbol{\Sigma}_{\mathbf{X}_I,\mathbf{Y}_I} & \boldsymbol{\Sigma}_{\mathbf{X}_I}\end{pmatrix}\right]$$

and analogously in System II

$$(4.2)\qquad \begin{pmatrix}\mathbf{Y}_{II}\\ \mathbf{X}_{II}\end{pmatrix} \sim \left[\begin{pmatrix}\boldsymbol{\eta}_{II}\\ \boldsymbol{\xi}_{II}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Sigma}_{\mathbf{Y}_{II}} & \boldsymbol{\Sigma}_{\mathbf{Y}_{II},\mathbf{X}_{II}}\\ \boldsymbol{\Sigma}_{\mathbf{X}_{II},\mathbf{Y}_{II}} & \boldsymbol{\Sigma}_{\mathbf{X}_{II}}\end{pmatrix}\right].$$

In the simplest case the resulting point field of System II is characterized by the vector

$$(4.3)\quad \begin{pmatrix} \hat{\boldsymbol{\eta}}_{II} \\ \mathbf{X}_{II} \\ \mathbf{1}\otimes\begin{pmatrix}\hat\beta_1\\ \hat\beta_2\end{pmatrix} + \left[\mathbf{I}\otimes\begin{pmatrix}\hat\beta_3 & \hat\beta_4\\ -\hat\beta_4 & \hat\beta_3\end{pmatrix}\right]\mathbf{X}_I \end{pmatrix}$$

$$\sim \left[\begin{pmatrix} \eta_{II} \\ \boldsymbol{\xi}_{II} \\ \mathbf{1}\otimes\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \left[\mathbf{I}\otimes\begin{pmatrix}\beta_3 & \beta_4\\ -\beta_4 & \beta_3\end{pmatrix}\right]\boldsymbol{\xi}_I + \mathbf{R}_{X_I} \end{pmatrix}, \begin{pmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} & \mathbf{C}_{1,3} \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} & \mathbf{C}_{2,3} \\ \mathbf{C}_{3,1} & \mathbf{C}_{3,2} & \mathbf{C}_{3,3} \end{pmatrix}\right],$$

where

$$\mathbf{R}_{X_I} = \left\{\mathbf{I}\otimes\left[\begin{pmatrix} (\mathbf{e}_3^{(4)})' & (\mathbf{e}_4^{(4)})' \\ -(\mathbf{e}_4^{(4)})' & (\mathbf{e}_3^{(4)})' \end{pmatrix}(\mathbf{I}\otimes(\mathbf{T}_{\beta,I}\boldsymbol{\Sigma}_{Y_I,X_I}))\right]\right\}\operatorname{vec}(\mathbf{I}),$$
$$\mathbf{e}_3^{(4)} = (0,0,1,0)' \text{ and } \mathbf{e}_4^{(4)} = (0,0,0,1)'.$$

Remark 4.1. The final estimators (3.3), (3.4) and (3.5) are unbiased only within the linear model (3.2). But the "reality" is described by the nonlinear model (3.1) which consequently causes the existence of some bias of these estimators, i.e. that for example $E(\hat{\boldsymbol{\eta}}_{II}) \neq \boldsymbol{\eta}_{II}$ but $E(\hat{\boldsymbol{\eta}}_{II}) = \boldsymbol{\eta}_{II} + \text{bias}_{\eta_{II}}$. Specifying the conditions under which these biases are neglectable was the aim of the paper [11]. In the present paper we suppose that these conditions are satisfied and this is why the biases are not mentioned in the characteristics of the resulting point field (4.3). The same idea will be used in the case of the field (5.2).

The matrices $\mathbf{C}_{1,1} = \operatorname{var}(\hat{\boldsymbol{\eta}}_{II})$, $\mathbf{C}_{2,2} = \boldsymbol{\Sigma}_{\mathbf{X}_{II}}$ are already determined; the others have to be determined for solving standard problems in the resulting point field (e.g. to determine the standard deviation in the distance between a point whose coordinates are given by a subvector $\hat{\boldsymbol{\eta}}_{II,i}$ of the vector $\hat{\boldsymbol{\eta}}_{II}$ and a point with coordinates given by a subvector of the vector $\mathbf{X}_{II}$).

The following two statements will be used for solving the problem.

Lemma 4.1. *Let the random vectors $\mathbf{Y}_I$, $\mathbf{X}_I$, $\mathbf{Y}_{II}$ and $\mathbf{X}_{II}$ fulfil (4.1) and (4.2). Let the vector functions $\mathbf{f}(\mathbf{Y}_I,\mathbf{Y}_{II})$ and $\mathbf{g}(\mathbf{Y}_I,\mathbf{X}_I,\mathbf{Y}_{II})$ possess continuous second derivatives and let the diagonal elements of all considered covariance matrices be sufficiently small (in detail cf. [8]). Then*

$$\operatorname{cov}[\mathbf{f}(\mathbf{Y}_I,\mathbf{Y}_{II}),\mathbf{g}(\mathbf{Y}_I,\mathbf{X}_I,\mathbf{Y}_{II}) = (\partial\mathbf{f}/\partial\boldsymbol{\eta}_I')\boldsymbol{\Sigma}_{\mathbf{Y}_I}\partial\mathbf{g}'/\partial\boldsymbol{\eta}_I + (\partial\mathbf{f}/\partial\boldsymbol{\eta}_I')\boldsymbol{\Sigma}_{\mathbf{Y}_I,\mathbf{X}_I}\partial\mathbf{g}'/\partial\boldsymbol{\xi}_I + (\partial\mathbf{f}/\partial\boldsymbol{\eta}_{II}')\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\partial\mathbf{g}'/\partial\boldsymbol{\eta}_{II},$$

where

$$\partial \mathbf{f}/\partial \boldsymbol{\eta}'_I = (\partial \mathbf{f}(\mathbf{Y}_I, \mathbf{Y}_{II})/\partial \mathbf{Y}'_I)|_{\mathbf{Y}_I=\boldsymbol{\eta}_I, \mathbf{Y}_{II}=\boldsymbol{\eta}_{II}},$$
$$\{\partial \mathbf{f}/\partial \boldsymbol{\eta}'_I\}_{i,j} = \partial f_i/\partial \eta_{I,j}, \quad \partial \mathbf{g}'/\partial \boldsymbol{\xi}_I = (\partial \mathbf{g}/\partial \boldsymbol{\xi}'_I)', \ \textit{etc.}$$

P r o o f. It is based on procedures given in [8] and on neglecting the second and higher derivatives in the Taylor series of the functions $\mathbf{f}(\cdot,\cdot\cdot)$ and $\mathbf{g}(\cdot,\cdot\cdot,\cdots)$. Further, by virtue of the assumptions, $\operatorname{cov}(\mathbf{Y}_I, \mathbf{Y}_{II}) = \mathbf{0}$ and $\operatorname{cov}(\mathbf{Y}_{II}, \mathbf{X}_I) = \mathbf{0}$. $\square$

Theorem 4.2. *The matrices* $\mathbf{C}_{i,j}$, $i,j = 1,2,3$, *from* (4.3) *are given by the relations*

$$\mathbf{C}_{1,1} = \operatorname{var}(\hat{\boldsymbol{\eta}}_{II}) = \mathbf{T}_{\eta_{II},I}\boldsymbol{\Sigma}_{Y_I}\mathbf{T}'_{\eta_{II},I} + (\mathbf{I}-\mathbf{T}_{\eta_{II},II})\boldsymbol{\Sigma}_{Y_{II}}(\mathbf{I}-\mathbf{T}_{\eta_{II},II})',$$

$$\mathbf{C}_{1,2} = \operatorname{cov}(\hat{\boldsymbol{\eta}}_{II}, \mathbf{X}_{II}) = (\mathbf{I}-\mathbf{T}_{\eta_{II},II})\boldsymbol{\Sigma}_{Y_{II},X_{II}} = \mathbf{C}'_{2,1},$$

$$\begin{aligned}\mathbf{C}_{1,3} = {} & \mathbf{T}_{\eta_{II},I}\boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{T}'_{\beta,I}\begin{pmatrix}\mathbf{1}'\otimes\mathbf{I}\\ \boldsymbol{\xi}'_I\\ \boldsymbol{\xi}'_I\left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}'\right]\end{pmatrix}\\ & + \mathbf{T}_{\eta_{II},I}\boldsymbol{\Sigma}_{\mathbf{Y}_I,\mathbf{X}_I}\left[\mathbf{I}\otimes\begin{pmatrix}\beta_3 & \beta_4\\ -\beta_4 & \beta_3\end{pmatrix}'\right]\\ & + (\mathbf{I}-\mathbf{T}_{\eta_{II},II})\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\mathbf{T}'_{\beta,II}\begin{pmatrix}\mathbf{1}'\otimes\mathbf{I}\\ \boldsymbol{\xi}'_I\\ \boldsymbol{\xi}'_I\left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}'\right]\end{pmatrix} = \mathbf{C}'_{3,1},\end{aligned}$$

$$\mathbf{C}_{2,2} = \operatorname{var}(\mathbf{X}_{II}) = \boldsymbol{\Sigma}_{X_{II}},$$

$$\mathbf{C}_{2,3} = \boldsymbol{\Sigma}_{\mathbf{X}_{II},\mathbf{Y}_{II}}\mathbf{T}'_{\beta,II}\begin{pmatrix}\mathbf{1}'\otimes\mathbf{I}\\ \boldsymbol{\xi}'_I\\ \boldsymbol{\xi}'_I\left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}'\right]\end{pmatrix} = \mathbf{C}'_{3,2},$$

$$\begin{aligned}\mathbf{C}_{3,3} = {} & \left(\mathbf{1}\otimes\mathbf{I}, \boldsymbol{\xi}_I, \left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]\boldsymbol{\xi}_I\right)\mathbf{T}_{\beta,I}\boldsymbol{\Sigma}_{\mathbf{Y}_I}\mathbf{T}'_{\beta,I}\begin{pmatrix}\mathbf{1}'\otimes\mathbf{I}\\ \boldsymbol{\xi}'_I\\ \boldsymbol{\xi}'_I\left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}'\right]\end{pmatrix}\\ & + \left(\mathbf{1}\otimes\mathbf{I}, \boldsymbol{\xi}_I, \left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]\boldsymbol{\xi}_I\right)\\ & \quad \times \mathbf{T}_{\beta,II}\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}\mathbf{T}'_{\beta,II}\begin{pmatrix}\mathbf{1}'\otimes\mathbf{I}\\ \boldsymbol{\xi}'_I\\ \boldsymbol{\xi}'_I\left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}'\right]\end{pmatrix}\end{aligned}$$

$$+\left(\mathbf{1}\otimes\mathbf{I},\boldsymbol{\xi}_I,\left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]\boldsymbol{\xi}_I\right)\mathbf{T}_{\beta,I}\boldsymbol{\Sigma}_{\mathbf{Y}_I,\mathbf{X}_I}\left[\mathbf{I}\otimes\begin{pmatrix}\beta_3 & \beta_4\\ -\beta_4 & \beta_3\end{pmatrix}'\right]$$

$$+\left[\mathbf{I}\otimes\begin{pmatrix}\beta_3 & \beta_4\\ -\beta_4 & \beta_3\end{pmatrix}\right]\boldsymbol{\Sigma}_{\mathbf{X}_I,\mathbf{Y}_I}\mathbf{T}'_{\beta,1}\begin{pmatrix}\mathbf{1}'\otimes\mathbf{I}\\ \boldsymbol{\xi}'_I\\ \boldsymbol{\xi}'_I\left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}'\right]\end{pmatrix}$$

$$+\left[\mathbf{I}\otimes\begin{pmatrix}\beta_3 & \beta_4\\ -\beta_4 & \beta_3\end{pmatrix}\right]\boldsymbol{\Sigma}_{\mathbf{X}_I}\left[\mathbf{I}\otimes\begin{pmatrix}\beta_3 & \beta_4\\ -\beta_4 & \beta_3\end{pmatrix}'\right].$$

P r o o f. The form of $\mathbf{C}_{1,1}$ follows from the relation

$$\mathbf{f}(\mathbf{Y}_I,\mathbf{Y}_{II})=\mathbf{T}_{\eta_{II},I}(\mathbf{Y}_I-\boldsymbol{\eta}_{I,0})+(\mathbf{I}-\mathbf{T}_{\eta_{II},II})\mathbf{Y}_{II}+\mathbf{t}_2.$$

Since

$$\mathbf{C}_{1,2}=\operatorname{cov}(\hat{\boldsymbol{\eta}}_{II},\mathbf{X}_{II})=\operatorname{cov}\left[(\mathbf{T}_{\eta_{II},I},\mathbf{I}-\mathbf{T}_{\eta_{II},II})\begin{pmatrix}\mathbf{Y}_I-\boldsymbol{\eta}_{I,0}\\ \mathbf{Y}_{II}\end{pmatrix},\mathbf{X}_{II}\right],$$

obviously $\mathbf{C}_{1,2}=(\mathbf{I}-\mathbf{T}_{\eta_{II},II})\boldsymbol{\Sigma}_{Y_{II},X_{II}}$.

Let

$$\mathbf{g}(\mathbf{Y}_I,\mathbf{X}_I,\mathbf{Y}_{II})=\mathbf{1}\otimes\begin{pmatrix}\hat{\beta}_1(\mathbf{Y}_I,\mathbf{Y}_{II})\\ \hat{\beta}_2(\mathbf{Y}_I,\mathbf{Y}_{II})\end{pmatrix}$$

$$+\left[\mathbf{I}\otimes\begin{pmatrix}\hat{\beta}_3(\mathbf{Y}_I,\mathbf{Y}_{II}) & \hat{\beta}_4(\mathbf{Y}_I,\mathbf{Y}_{II})\\ -\hat{\beta}_4(\mathbf{Y}_I,\mathbf{Y}_{II}) & \hat{\beta}_3(\mathbf{Y}_I,\mathbf{Y}_{II})\end{pmatrix}\right]\mathbf{X}_I.$$

Then

$$\partial\mathbf{g}/\partial\boldsymbol{\eta}'_I=\left(\mathbf{1}\otimes\mathbf{I},\boldsymbol{\xi}_I,\left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]\boldsymbol{\xi}_I\right)\mathbf{T}_{\beta,I},$$

$$\partial\mathbf{g}/\partial\boldsymbol{\eta}'_{II}=\left(\mathbf{1}\otimes\mathbf{I},\boldsymbol{\xi}_I,\left[\mathbf{I}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]\boldsymbol{\xi}_I\right)\mathbf{T}_{\beta,II}$$

and

$$\partial\mathbf{g}/\partial\boldsymbol{\xi}'_I=\mathbf{I}\otimes\begin{pmatrix}\beta_3 & \beta_4\\ -\beta_4 & \beta_3\end{pmatrix}.$$

Further,

$$\partial \mathbf{f}/\partial \boldsymbol{\eta}_I' = \mathbf{T}_{\eta_{II},I}, \quad \partial \mathbf{f}/\partial \boldsymbol{\eta}_{II}' = \mathbf{I} - \mathbf{T}_{\eta_{II},II}.$$

The matrix $\mathbf{C}_{1,3}$ can be expressed in the form

$$\begin{aligned}\mathbf{C}_{1,3} = \; & \operatorname{cov}(\mathbf{f}(\mathbf{Y}_I, \mathbf{Y}_{II}), \mathbf{g}(\mathbf{Y}_I, \mathbf{X}_I, \mathbf{Y}_{II})) \\ & = \partial \mathbf{f}/\partial \boldsymbol{\eta}_I' \operatorname{var}(\mathbf{Y}_I) \partial \mathbf{g}'/\partial \boldsymbol{\eta}_I + \partial \mathbf{f}/\partial \boldsymbol{\eta}_{II}' \operatorname{var}(\mathbf{Y}_{II}) \partial \mathbf{g}'/\partial \boldsymbol{\eta}_{II} \\ & \quad + \partial \mathbf{f}/\partial \boldsymbol{\eta}_I' \boldsymbol{\Sigma}_{Y_I,X_I} \partial \mathbf{g}'/\partial \boldsymbol{\xi}_I.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{C}_{1,3} = \mathbf{T}_{\eta_{II},I} \boldsymbol{\Sigma}_{\mathbf{Y}_I} \mathbf{T}_{\beta,I}' & \begin{pmatrix} \mathbf{1}' \otimes \mathbf{I} \\ \boldsymbol{\xi}_I' \\ \boldsymbol{\xi}_I' \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}' \right] \end{pmatrix} \\ & + \mathbf{T}_{\eta_{II},I} \boldsymbol{\Sigma}_{\mathbf{Y}_I,\mathbf{X}_I} \left[\mathbf{I} \otimes \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix}' \right] \\ & + (\mathbf{I} - \mathbf{T}_{\eta_{II},II}) \boldsymbol{\Sigma}_{\mathbf{Y}_{II}} \mathbf{T}_{\beta,II}' \begin{pmatrix} \mathbf{1}' \otimes \mathbf{I} \\ \boldsymbol{\xi}_I' \\ \boldsymbol{\xi}_I' \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}' \right] \end{pmatrix}.\end{aligned}$$

For determining the matrix

$$\mathbf{C}_{2,3} = \operatorname{cov}\left(\mathbf{X}_{II}, \mathbf{1} \otimes \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} + \left[\mathbf{I} \otimes \begin{pmatrix} \hat{\beta}_3 & \hat{\beta}_4 \\ -\hat{\beta}_4 & \hat{\beta}_3 \end{pmatrix} \right] \mathbf{X}_I \right),$$

the relationship

$$\mathbf{C}_{2,3} = \boldsymbol{\Sigma}_{\mathbf{X}_{II},\mathbf{Y}_{II}} \partial \mathbf{g}'/\partial \boldsymbol{\eta}_{II} = \boldsymbol{\Sigma}_{\mathbf{X}_{II},\mathbf{Y}_{II}} \mathbf{T}_{\beta,II}' \begin{pmatrix} \mathbf{1}' \otimes \mathbf{I} \\ \boldsymbol{\xi}_I' \\ \boldsymbol{\xi}_I' \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}' \right] \end{pmatrix}$$

has to be applied. □

The last step is to determine the matrix $\mathbf{C}_{3,3}$. In the analogous way we obtain

$$\begin{aligned}
\mathbf{C}_{3,3} &= \frac{\partial \mathbf{g}}{\partial \boldsymbol{\eta}_I'} \boldsymbol{\Sigma}_{\mathbf{Y}_I} \frac{\partial \mathbf{g}'}{\partial \boldsymbol{\eta}_I} + \frac{\partial \mathbf{g}}{\partial \boldsymbol{\eta}_{II}'} \boldsymbol{\Sigma}_{\mathbf{Y}_{II}} \frac{\partial \mathbf{g}'}{\partial \boldsymbol{\eta}_{II}} + \frac{\partial \mathbf{g}}{\partial \boldsymbol{\eta}_I'} \boldsymbol{\Sigma}_{\mathbf{Y}_I,\mathbf{X}_I} \frac{\partial \mathbf{g}'}{\partial \boldsymbol{\xi}_I} \\
&\quad + \frac{\partial \mathbf{g}}{\partial \boldsymbol{\xi}_I'} \boldsymbol{\Sigma}_{\mathbf{X}_I,\mathbf{Y}I} \frac{\partial \mathbf{g}'}{\partial \boldsymbol{\eta}_I} + \frac{\partial \mathbf{g}}{\partial \boldsymbol{\xi}_I'} \boldsymbol{\Sigma}_{\mathbf{X}_I} \frac{\partial \mathbf{g}'}{\partial \boldsymbol{\xi}_I} \\
&= \left(\mathbf{1} \otimes \mathbf{I}, \boldsymbol{\xi}_I, \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \boldsymbol{\xi}_I \right) \\
&\quad \times \mathbf{T}_{\beta,I} \boldsymbol{\Sigma}_{\mathbf{Y}_I} \mathbf{T}_{\beta,I}' \begin{pmatrix} \mathbf{1}' \otimes \mathbf{I} \\ \boldsymbol{\xi}_I' \\ \boldsymbol{\xi}_I' \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}' \right] \end{pmatrix} \\
&\quad + \left(\mathbf{1} \otimes \mathbf{I}, \boldsymbol{\xi}_I, \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \boldsymbol{\xi}_I \right) \\
&\quad \times \mathbf{T}_{\beta,II} \boldsymbol{\Sigma}_{\mathbf{Y}_{II}} \mathbf{T}_{\beta,II}' \begin{pmatrix} \mathbf{1}' \otimes \mathbf{I} \\ \boldsymbol{\xi}_I' \\ \boldsymbol{\xi}_I' \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}' \right] \end{pmatrix} \\
&\quad + \left(\mathbf{1} \otimes \mathbf{I}, \boldsymbol{\xi}_I, \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \boldsymbol{\xi}_I \right) \\
&\quad \times \mathbf{T}_{\beta,I} \boldsymbol{\Sigma}_{\mathbf{Y}_I,\mathbf{X}_I} \left[\mathbf{I} \otimes \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix}' \right] \\
&\quad + \left[\mathbf{I} \otimes \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix} \right] \boldsymbol{\Sigma}_{\mathbf{X}_I,\mathbf{Y}_I} \mathbf{T}_{\beta,1}' \begin{pmatrix} \mathbf{1}' \otimes \mathbf{I} \\ \boldsymbol{\xi}_I' \\ \boldsymbol{\xi}_I' \left[\mathbf{I} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}' \right] \end{pmatrix} \\
&\quad + \left[\mathbf{I} \otimes \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix} \right] \boldsymbol{\Sigma}_{\mathbf{X}_I} \left[\mathbf{I} \otimes \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix}' \right].
\end{aligned}$$

$\square$

5. Corrections of coordinates of the nonidentical points

The resulting point field and its characteristics of accuracy given by the matrices $\mathbf{C}_{i,j}$, $i,j = 1,\dots,3$, do not respect the substitution of the vector $\mathbf{Y}_{II}$ by the vector $\hat{\boldsymbol{\eta}}_{II}$ in the relations for the vector $\mathbf{X}_{II}$. To say it more precisely: the relative position of the ith identical point, i.e. $\mathbf{Y}_{II,i}$, and the jth nonidentical point, i.e. $\mathbf{X}_{II,j}$, is given by the difference $\mathbf{X}_{II,j} - \mathbf{Y}_{II,i}$. After the transformation this difference is substituted by the difference $\mathbf{X}_{II,j} - \hat{\boldsymbol{\eta}}_{II,i}$; the nearer is the jth nonidentical point to the ith identical point, the more we feel a necessity to change the coordinates $\mathbf{X}_{II,j}$ into the corrected coordinates $\tilde{\mathbf{X}}_{II,j}$, which reflects the change of $\mathbf{Y}_{II}$ into $\hat{\boldsymbol{\eta}}_{II}$.

A commonly accepted procedure is the following:

The domain, where the points given by the vector $\mathbf{X}_{II}$ are located, is divided into several disjoint domains in such a way that their boundaries are formed by polygons of properly chosen identical points. For the sake of simplicity, let the polygon be given by points $\mathbf{Y}_{II,1}, \dots, \mathbf{Y}_{II,p}$ which are relocated into points $\hat{\boldsymbol{\eta}}_{II,1}, \dots, \hat{\boldsymbol{\eta}}_{II,p}$. Let the nonidentical points of the domain considered be $\mathbf{X}_{II,1}, \dots, \mathbf{X}_{II,r}$.

The corrected positions of these nonidentical points, i.e. $\tilde{\mathbf{X}}_{II,1}, \dots, \tilde{\mathbf{X}}_{II,r}$, are given by the formulae

$$\tilde{\mathbf{X}}_{II,j} = \mathbf{X}_{II,j} + \sum_{i=1}^{p} \frac{f(s_{i,j})(\hat{\boldsymbol{\eta}}_{II,i} - \mathbf{Y}_{II,i})}{\sum\limits_{i=1}^{p} f(s_{i,j})}, \quad j = 1, \dots, r, \tag{5.1}$$

where $s_{i,j}$ is the horizontal distance between the ith identical point and the jth nonidentical point and the function $f(\cdot)$ is chosen in such a way that

$$\lim_{s_{i,j} \to 0} \frac{f(s_{k,j})}{\sum\limits_{i=1}^{p} f(s_{i,j})} = \begin{cases} 1 & k = i, \\ 0 & k \neq i. \end{cases}$$

This requirement can be satisfied e.g. by the functions $f_1(s) = 1/s$, $f_2(s) = 1/s^2, \dots$, etc.

If the differences $\hat{\boldsymbol{\eta}}_{II,i} - \mathbf{Y}_{II,i}$ cannot be neglected, then the above mentioned corrections should be realized.

An analogous consideration is connected with the vectors $\mathbf{Y}_I$, $\mathbf{X}_I$ and $\hat{\boldsymbol{\eta}}_I$ if the difference $\hat{\boldsymbol{\eta}}_I - \mathbf{Y}_I$ cannot be neglected.

The above mentioned corrections of the vectors $\mathbf{X}_I$ and $\mathbf{X}_{II}$ lead to the resulting point field

$$\begin{pmatrix} \hat{\boldsymbol{\eta}}_{II} \\ \tilde{\mathbf{X}}_{II} \\ \mathbf{1} \otimes \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} + \left[\mathbf{I} \otimes \begin{pmatrix} \hat{\beta}_3 & \hat{\beta}_4 \\ -\hat{\beta}_4 & \hat{\beta}_3 \end{pmatrix} \right] \tilde{\mathbf{X}}_I \end{pmatrix} \tag{5.2}$$

$$\sim \left[\begin{pmatrix} \boldsymbol{\eta}_{II} \\ \boldsymbol{\xi}_{II} \\ \mathbf{1} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \left[\mathbf{I} \otimes \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix} \right] \boldsymbol{\xi}_I + \mathbf{R}_{\tilde{X}_I} \end{pmatrix}, \begin{pmatrix} \mathbf{C}_{1,1}, & \mathbf{D}_{1,2} & \mathbf{D}_{1,3} \\ \mathbf{D}_{2,1} & \mathbf{D}_{2,2} & \mathbf{D}_{2,3} \\ \mathbf{D}_{3,1} & \mathbf{D}_{3,2} & \mathbf{D}_{3,3} \end{pmatrix} \right],$$

where

$$\mathbf{R}_{\tilde{X}_I} = \left\{ \mathbf{I} \otimes \left[\begin{pmatrix} (\mathbf{e}_3^{(4)})' & (\mathbf{e}_4^{(4)})' \\ -(\mathbf{e}_4^{(4)})' & (\mathbf{e}_3^{(4)})' \end{pmatrix} (\mathbf{I} \otimes \mathbf{W}) \right] \right\} \mathrm{vec}(\mathbf{I}),$$

$$\mathbf{W} = \mathbf{T}_{\beta,I}\boldsymbol{\Sigma}_{Y_I,X_I} - \mathbf{T}_{\beta,I}\boldsymbol{\Sigma}_{Y_I}\mathbf{T}'_{\eta_I,I}\mathbf{U}'_{Y_I,X_I} + \mathbf{T}_{\beta,II}\boldsymbol{\Sigma}_{Y_{II}}\mathbf{T}'_{\eta_I,II}\mathbf{U}'_{Y_I,X_I},$$

and where the matrices $\mathbf{D}_{i,j}$ must be determined. This is simple; it is sufficient to realize that (5.1) leads to a linear transformation

$$\tilde{\mathbf{X}}_{II} = \mathbf{X}_{II} + \mathbf{U}_{Y_{II},X_{II}}(\hat{\boldsymbol{\eta}}_{II} - \mathbf{Y}_{II})$$

and

$$\tilde{\mathbf{X}}_I = \mathbf{X}_I + \mathbf{U}_{Y_I,X_I}(\hat{\boldsymbol{\eta}}_I - \mathbf{Y}_I).$$

In a way analogous to that given in the preceding section we can obtain expressions for $\mathbf{D}_{i,j}$, e.g.

$$\begin{aligned}
\mathbf{D}_{2,2} = {} & \mathrm{var}\{\mathbf{U}_{Y_{II},X_{II}}[\mathbf{T}_{\eta_{II},I}(\mathbf{Y}_I - \boldsymbol{\eta}_{I,0}) + (\mathbf{I} - \mathbf{T}_{\eta_{II},II})\mathbf{Y}_{II}] \\
& - \mathbf{U}_{Y_{II},X_{II}}\mathbf{Y}_{II} + \mathbf{X}_{II}\} \\
= {} & \mathrm{var}\left\{ [\mathbf{U}_{Y_{II},X_{II}}\mathbf{T}_{\eta_{II},I}, \mathbf{U}_{Y_{II},X_{II}}(\mathbf{I} - \mathbf{T}_{\eta_{II},II}) - \mathbf{U}_{Y_{II},X_{II}}, \mathbf{I}] \begin{pmatrix} \mathbf{Y}_I \\ \mathbf{Y}_{II} \\ \mathbf{X}_{II} \end{pmatrix} \right\} \\
= {} & (\mathbf{U}_{Y_{II},X_{II}}\mathbf{T}_{\eta_{II},I}, -\mathbf{U}_{Y_{II},X_{II}}\mathbf{T}_{\eta_{II},II}, \mathbf{I}) \\
& \times \begin{pmatrix} \boldsymbol{\Sigma}_{\mathbf{Y}_I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_{\mathbf{Y}_{II}} & \boldsymbol{\Sigma}_{\mathbf{Y}_{II},\mathbf{X}_{II}} \\ \mathbf{0} & \boldsymbol{\Sigma}_{\mathbf{X}_{II},\mathbf{Y}_{II}} & \boldsymbol{\Sigma}_{\mathbf{X}_{II}} \end{pmatrix} \begin{pmatrix} \mathbf{T}'_{\eta_{II},I}\mathbf{U}'_{Y_{II},X_{II}} \\ -\mathbf{T}'_{\eta_{II},II}\mathbf{U}'_{Y_{II},X_{II}} \\ \mathbf{I} \end{pmatrix}.
\end{aligned}$$

Since these formulae are space consuming and the technique of their derivation is clear, they are omitted here.

6. Numerical example

In this section a verification of the theoretical results is given.

As the model of the identical points let's use the grid of the square whose sides are 300 meters long. In System I we have located this square at the following points:

$$\boldsymbol{\eta}_{I,1} = (100, 100)', \ \ \boldsymbol{\eta}_{I,2} = (400, 100)', \ \ \boldsymbol{\eta}_{I,3} = (400, 400)', \ \ \boldsymbol{\eta}_{I,4} = (100, 400)'$$

and in System II, for easy verification of the results, we transformed its grid to the points $\boldsymbol{\eta}_{II,i}$ satisfying the relation

$$\boldsymbol{\eta}_{II,i} = \begin{pmatrix} 200 \\ 200 \end{pmatrix} + \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \boldsymbol{\eta}_{I,i}, \quad i = 1, ..., 4.$$

These relations correspond to the Helmert transformation with parameters

$$\beta = (\beta_1, \beta_2, \beta_3, \beta_4) = (200, 200, \cos\tfrac{\pi}{3}, \sin\tfrac{\pi}{3}).$$

As the model of the nonidentical points let us use the following one: $\boldsymbol{\xi}_I = (250, 200)$ in System I and

$$\boldsymbol{\xi}_{II} = \begin{pmatrix} 200 \\ 200 \end{pmatrix} + \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} 330 \\ 140 \end{pmatrix}$$

in System II.

So the actual values of all parameters which we are interested in are at our disposal. Now we have to realize the simulation of the vectors $\mathbf{Y}_I$, $\mathbf{Y}_{II}$, $\mathbf{X}_I$ and $\mathbf{X}_{II}$ and of their covariance matrices $\boldsymbol{\Sigma}_{\mathbf{Y}_I}$, $\boldsymbol{\Sigma}_{\mathbf{Y}_{II}}$, $\boldsymbol{\Sigma}_{\mathbf{X}_I}$, $\boldsymbol{\Sigma}_{\mathbf{X}_{II}}$, $\boldsymbol{\Sigma}_{\mathbf{Y}_I,\mathbf{X}_I}$ and $\boldsymbol{\Sigma}_{\mathbf{Y}_{II},\mathbf{X}_{II}}$. The technique is as follows:

At first let us choose two ancillary points in both the coordinate systems, e.g. $\mathbf{P}_1 = (0, 0)$ and $\mathbf{P}_2 = (500, 0)$ whose coordinates we will consider to be errorless. Now we can simulate the measurements $\hat{\mathbf{s}}_I$ and $\hat{\mathbf{s}}_{II}$ of at least 10 (= dimension of $(\mathbf{Y}_I', \mathbf{X}_I')'$) various distances between all the mentioned points (identical, nonidentical and ancillary ones) $\mathbf{s}_I$ and $\mathbf{s}_{II}$, respectively, in both systems. The actual values of $\mathbf{s}_I$ and $\mathbf{s}_{II}$, are known. They are obviously nonlinear functions of the coordinates $(\boldsymbol{\eta}_I, \boldsymbol{\xi}_I)$ and $(\boldsymbol{\eta}_{II}, \boldsymbol{\xi}_{II})$, respectively. So if we know their sufficiently precise approximations $(\boldsymbol{\eta}_{I,0}, \boldsymbol{\xi}_{I,0})$ and $(\boldsymbol{\eta}_{II,0}, \boldsymbol{\xi}_{II,0})$, we can use the linearization

$$\mathbf{s}_I(\boldsymbol{\eta}_I, \boldsymbol{\xi}_I) \doteq \mathbf{s}_I(\boldsymbol{\eta}_{I,0}, \boldsymbol{\xi}_{I,0}) + \mathbf{F}_I \begin{pmatrix} \delta\boldsymbol{\eta}_I \\ \delta\boldsymbol{\xi}_I \end{pmatrix}, \quad \mathbf{F}_I = \frac{\partial \mathbf{s}_I}{\partial(\boldsymbol{\eta}_I', \boldsymbol{\xi}_I')} (\boldsymbol{\eta}_{I,0}, \boldsymbol{\xi}_{I,0}).$$

Analogously for $\mathbf{s}_{II}$.

Now we can gain the measurements $\hat{\mathbf{s}}_I$ and $\hat{\mathbf{s}}_{II}$ by generating normally distributed errors with zero mean values and with dispersions σ_I^2 and σ_{II}^2, on the linearized actual values $\mathbf{s}_I$ and $\mathbf{s}_{II}$, respectively, i.e.

$$\hat{\mathbf{s}}_I \sim \mathbf{N}\left(\mathbf{s}_{I,0} + \mathbf{F}_I \begin{pmatrix} \delta\boldsymbol{\eta}_I \\ \delta\boldsymbol{\xi}_I \end{pmatrix}, \sigma_I^2\mathbf{I}\right), \quad \hat{\mathbf{s}}_{II} \sim \mathbf{N}\left(\mathbf{s}_{II,0} + \mathbf{F}_{II} \begin{pmatrix} \delta\boldsymbol{\eta}_{II} \\ \delta\boldsymbol{\xi}_{II} \end{pmatrix}, \sigma_{II}^2\mathbf{I}\right),$$

where $\mathbf{s}_{I,0} = \mathbf{s}_I(\boldsymbol{\eta}_{I,0}, \boldsymbol{\xi}_{I,0})$ and $\mathbf{s}_{II,0} = \mathbf{s}_{II}(\boldsymbol{\eta}_{II,0}, \boldsymbol{\xi}_{II,0})$.

Now as the simulations of $\mathbf{Y}_I$, $\mathbf{X}_I$ and of their covariance matrices we can take

$$\text{(6.1)} \qquad \begin{pmatrix} \mathbf{Y}_I \\ \mathbf{X}_I \end{pmatrix} = (\mathbf{F}'_I\mathbf{F}_I)^{-1}\mathbf{F}'_I(\hat{\mathbf{s}}_I - \mathbf{s}_{I,0}) + \begin{pmatrix} \boldsymbol{\eta}_{I,0} \\ \boldsymbol{\xi}_{I,0} \end{pmatrix},$$

$$\operatorname{cov}\begin{pmatrix} \mathbf{Y}_I \\ \mathbf{X}_I \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{\mathbf{Y}_I} & \boldsymbol{\Sigma}_{\mathbf{Y}_I,\mathbf{X}_I} \\ \boldsymbol{\Sigma}_{\mathbf{X}_I,\mathbf{Y}_I} & \boldsymbol{\Sigma}_{\mathbf{X}_I} \end{pmatrix} = \sigma_I^2(\mathbf{F}'_I\mathbf{F}_I)^{-1}.$$

Analogously for $\mathbf{Y}_{II}$ and $\mathbf{X}_{II}$.

These are our data. Now let us try to determine some optimum estimators of the transformation parameters $\boldsymbol{\beta}$ and the coordinates of the identical points in both systems, i.e. $\boldsymbol{\eta}_I$ and $\boldsymbol{\eta}_{II}$.

Standard solution of this problem which is based on the assumption of errorless coordinates in System I, i.e. on the assumption that $\mathbf{Y}_I = \boldsymbol{\eta}_I$, leads to an easy linear regression model

$$\text{(6.2)} \qquad \mathbf{Y}_{II} \sim \mathbf{N}(\mathbf{X}\boldsymbol{\beta},\ \boldsymbol{\Sigma}_{\mathbf{Y}_{II}}),$$

where

$$\text{(6.3)} \qquad \mathbf{X} = \begin{pmatrix} 1 & 0 & x_1 & y_1 \\ 0 & 1 & y_1 & -x_1 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & x_n & y_n \\ 0 & 1 & y_n & -x_n \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}.$$

It is well known that the best linear unbiased estimators of the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\eta}_{II}$ $(= \mathbf{X}\boldsymbol{\beta})$ within this model are

$$\text{(6.4)} \qquad \hat{\hat{\boldsymbol{\beta}}} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}_{\mathbf{Y}_{II}}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}_{\mathbf{Y}_{II}}\mathbf{Y}_{II}, \quad \operatorname{var}(\hat{\hat{\boldsymbol{\beta}}}) = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}_{\mathbf{Y}_{II}}\mathbf{X})^{-1},$$

$$\text{(6.5)} \qquad \hat{\hat{\boldsymbol{\eta}}}_{II} = \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}, \quad \operatorname{var}(\hat{\hat{\boldsymbol{\eta}}}_{II}) = \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}_{\mathbf{Y}_{II}}\mathbf{X})^{-1}\mathbf{X}'.$$

The resulting point field can be characterized by the vector

$$\text{(6.6)} \qquad \begin{pmatrix} \hat{\hat{\boldsymbol{\eta}}}_{II} \\ \tilde{\mathbf{X}}_{II} \\ \mathbf{1}\otimes\begin{pmatrix} \hat{\hat{\beta}}_1 \\ \hat{\hat{\beta}}_2 \end{pmatrix} + \left[\mathbf{I}\otimes\begin{pmatrix} \hat{\hat{\beta}}_3 & \hat{\hat{\beta}}_4 \\ -\hat{\hat{\beta}}_4 & \hat{\hat{\beta}}_3 \end{pmatrix}\right]\boldsymbol{\xi}_I \end{pmatrix}$$

$$\sim \left[\begin{pmatrix} \boldsymbol{\eta}_{II} \\ \boldsymbol{\xi}_{II} \\ \mathbf{1}\otimes\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \left[\mathbf{I}\otimes\begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix}\right]\boldsymbol{\xi}_I \end{pmatrix}, \begin{pmatrix} \mathbf{G}_{1,1} & \mathbf{G}_{1,2} & \mathbf{G}_{1,3} \\ \mathbf{G}_{2,1} & \mathbf{G}_{2,2} & \mathbf{G}_{2,3} \\ \mathbf{G}_{3,1} & \mathbf{G}_{3,2} & \mathbf{G}_{3,3} \end{pmatrix}\right],$$

where $\tilde{\tilde{\mathbf{X}}}_{II}$ represents the correction of $\mathbf{X}_{II}$ in the way given in Section 5 with respect to $\hat{\hat{\boldsymbol{\eta}}}_{II}$.

Matrices $\mathbf{G}_{i,j}$ are very easy to obtain as this resulting point field is a linear function of $(\mathbf{Y}'_{II}, \mathbf{X}'_{II})'$.

As we have shortly mentioned in Introduction this procedure cannot be considered optimal in the situations when non-negligible random errors of measurements exist in both coordinate systems. The main problem with this algorithm is that it says about its own results that they are more accurate than they are in reality. So we can gain a false idea about the precision of these results which can consequently make some problems for example when testing hypotheses about the transformation parameters $\boldsymbol{\beta}$ or the coordinate vectors $\boldsymbol{\eta}_{II}$. Accuracy of the final estimators (6.4) and (6.5) can be represented for example by the confidence regions which cover the actual values of unknown parameters $\boldsymbol{\beta}$ or $\boldsymbol{\eta}_{II}$ with probability $(1-\alpha)$—for example $1-\alpha = 0.95$. This probability we usually call the level of the confidence region. But if the construction of such a region is based on the model which neglects the errors in System I the real level of this region is naturally lower. The estimations of these levels can be computed by computer simulation.

During the test we generated the random errors in both systems with the following 16 combinations of dispersions:

$$\sigma_I^2 = (0.1\,\mathrm{m})^2,\ \ (0.3\,\mathrm{m})^2,\ \ (0.5\,\mathrm{m})^2,\ \ (1\,\mathrm{m})^2$$
$$\sigma_{II}^2 = (0.1\,\mathrm{m})^2,\ \ (0.3\,\mathrm{m})^2,\ \ (0.5\,\mathrm{m})^2,\ \ (1\,\mathrm{m})^2.$$

Then 1000 realizations of the confidence region of

a) the chosen grid of the square—for example $\boldsymbol{\eta}_{II,2}$—using the results of standard algorithm (**SA**),

b) the $(\beta_3, \beta_4)'$-representatives of the rotation and the changing of the scale—using **SA**,

c) the $\boldsymbol{\eta}_{II,2}$—using the results of the new algorithm from Section 3 (**NA**)

for each of these combinations on the theoretical level $1-\alpha = 0.95$ were made and subsequently the empirical frequencies of the covering of the actual values of the given vectors by the realized confidence region were counted. 1000 realizations are rather enough to suppose that the acquired frequencies sufficiently approximate the real levels of the confidence regions. The results which we obtained are shown in the following tables:

$\sigma_I^2 \backslash \sigma_{II}^2$	$\boldsymbol{\eta}_{II,2}$ − **SA**			
	0.1^2	0.3^2	0.5^2	1^2
0.1^2	0.882	0.949	0.933	0.936
0.3^2	0.528	0.894	0.918	0.937
0.5^2	0.309	0.775	0.868	0.925
1^2	0.078	0.487	0.704	0.876

Table 1.

$\sigma_I^2 \backslash \sigma_{II}^2$	$(\beta_3, \beta_4)'$ − **SA**			
	0.1^2	0.3^2	0.5^2	1^2
0.1^2	0.772	0.949	0.937	0.941
0.3^2	0.300	0.804	0.892	0.942
0.5^2	0.128	0.616	0.799	0.922
1^2	0.031	0.253	0.498	0.800

Table 2.

$\sigma_I^2 \backslash \sigma_{II}^2$	$\boldsymbol{\eta}_{II,2}$ − **NA**			
	0.1^2	0.3^2	0.5^2	1^2
0.1^2	0.950	0.957	0.944	0.955
0.3^2	0.953	0.949	0.951	0.950
0.5^2	0.946	0.950	0.951	0.960
1^2	0.952	0.954	0.952	0.955

Table 3.

It is evident that the real levels of the confidence regions in the case of the standard algorithm strongly depend on the relation between σ_I^2 and $\sigma_I I^2$.

- If $\sigma_I^2 < \sigma_{II}^2$, then the empirical level is getting near the value 0.95. The higher is the ratio between the given dispersions the closer is this level to 0.95.
- If $\sigma_I^2 > \sigma_{II}^2$, then these empirical levels are very low with a strong tendency to fall as the ratio between the two dispersions increases. So in these situations the accuracy of the final results is essentially lower than this algorithm asserts.

On the contrary, we can see—the third table—that the empirical levels of confidence regions in the case of our new algorithm are on the theoretical values $1 - \alpha = 0.95$, i.e. theoretical accuracy given by this algorithm nicely corresponds to the real one. This can be a strong reason for using this algorithm instead of the standard one.

Distance between two nonidentical points from different coordinate systems.

At the beginning of this section we have simulated the measurements of one nonidentical point in both systems $\mathbf{X}_I$ and $\mathbf{X}_{II}$. Now we are interested in the estimation of their mutual distance or better to say in the estimation of the vector which represents the difference between these two points and its characteristics.

To solve this problem we have to transform the nonidentical point from System I into System II and then to use the information from

1. the resulting point field (4.3) or,
2. the resulting point field (5.2) or,
3. the resulting point field (6.6).

Let's denote

$$\vec{\mathbf{d}}_1 = \mathbf{X}_{II} - \mathbf{1} \otimes \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} + \left[\mathbf{I} \otimes \begin{pmatrix} \hat{\beta}_3 & \hat{\beta}_4 \\ -\hat{\beta}_4 & \hat{\beta}_3 \end{pmatrix} \right] \mathbf{X}_I,$$
$$\vec{\mathbf{d}}_2 = \tilde{\mathbf{X}}_{II} - \mathbf{1} \otimes \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} + \left[\mathbf{I} \otimes \begin{pmatrix} \hat{\beta}_3 & \hat{\beta}_4 \\ -\hat{\beta}_4 & \hat{\beta}_3 \end{pmatrix} \right] \tilde{\mathbf{X}}_I,$$
$$\vec{\mathbf{d}}_3 = \tilde{\tilde{\mathbf{X}}}_{II} - \mathbf{1} \otimes \begin{pmatrix} \hat{\hat{\beta}}_1 \\ \hat{\hat{\beta}}_2 \end{pmatrix} + \left[\mathbf{I} \otimes \begin{pmatrix} \hat{\hat{\beta}}_3 & \hat{\hat{\beta}}_4 \\ -\hat{\hat{\beta}}_4 & \hat{\hat{\beta}}_3 \end{pmatrix} \right] \boldsymbol{\xi}_I.$$

These vectors are the estimators of the vector distance which we are looking for. If the vector $\vec{\mathbf{d}}$ represents the actual value of this vector distance, i.e.

$$\vec{\mathbf{d}} = \boldsymbol{\xi}_{II} - \mathbf{1} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \left[\mathbf{I} \otimes \begin{pmatrix} \beta_3 & \beta_4 \\ -\beta_4 & \beta_3 \end{pmatrix} \right] \boldsymbol{\xi}_I,$$

then it follows from (4.3), (5.2) and (6.6) that

$$\begin{aligned} &\vec{\mathbf{d}}_1 \sim (\vec{\mathbf{d}} - \mathbf{R}_{\mathbf{X}_I}, \mathbf{C}_{2,2} + \mathbf{C}_{3,3} - \mathbf{C}_{2,3} - \mathbf{C}_{3,2}), \\ &\vec{\mathbf{d}}_2 \sim (\vec{\mathbf{d}} - \mathbf{R}_{\tilde{\mathbf{X}}_I}, \mathbf{D}_{2,2} + \mathbf{D}_{3,3} - \mathbf{D}_{2,3} - \mathbf{D}_{3,2}), \\ &\vec{\mathbf{d}}_3 \sim (\vec{\mathbf{d}}, \mathbf{G}_{2,2} + \mathbf{G}_{3,3} - \mathbf{G}_{2,3} - \mathbf{G}_{3,2}). \end{aligned} \tag{6.7}$$

All the above mentioned terms are at our disposal.

E x a m p l e. The actual vector of distance between $\boldsymbol{\xi}_I$ and $\boldsymbol{\xi}_{II}$ from the beginning of this section is $\vec{\mathbf{d}} = (-11.9615, -99.2820)'$. Now let's apply the estimators $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$ and $\vec{\mathbf{d}}_3$. The results, i.e. the final estimators with their covariance matrices, for some combinations of the dispersions σ_I^2 and σ_{II}^2 are given in the following three tables:

$\sigma_I^2 = (0.2\,\mathrm{m})^2,\ \sigma_{II}^2 = (0.1\,\mathrm{m})^2$					
$\vec{\mathbf{d}}_1$		$\vec{\mathbf{d}}_2$		$\vec{\mathbf{d}}_3$	
$(-11.7807, -99.7473)'$		$(-11.7526, -99.7313)'$		$(-11.8147, -99.5187)'$	
$\mathrm{var}(\vec{\mathbf{d}}_1)$		$\mathrm{var}(\vec{\mathbf{d}}_2)$		$\mathrm{var}(\vec{\mathbf{d}}_3)$	
0.0524	0.0048	0.0449	0.0027	0.0143	−0.0007
0.0048	0.0275	0.0027	0.0267	−0.0007	0.0043

Table 4.

$\sigma_I^2 = (0.1\,\mathrm{m})^2,\ \sigma_{II}^2 = (0.2\,\mathrm{m})^2$		
$\vec{\mathbf{d}}_1$	$\vec{\mathbf{d}}_2$	$\vec{\mathbf{d}}_3$
$(-11.7900, -99.3816)'$	$(-11.7068, -99.3296)'$	$(-11.9958, -99.2057)'$
$\mathrm{var}(\vec{\mathbf{d}}_1)$	$\mathrm{var}(\vec{\mathbf{d}}_2)$	$\mathrm{var}(\vec{\mathbf{d}}_3)$
$\begin{matrix} 0.0844 & 0.0064 \\ 0.0064 & 0.0246 \end{matrix}$	$\begin{matrix} 0.0630 & -0.0018 \\ -0.0018 & 0.0225 \end{matrix}$	$\begin{matrix} 0.0573 & -0.0027 \\ -0.0027 & 0.0172 \end{matrix}$

Table 5.

$\sigma_I^2 = (0.01\,\mathrm{m})^2,\ \sigma_{II}^2 = (0.01\,\mathrm{m})^2$		
$\vec{\mathbf{d}}_1$	$\vec{\mathbf{d}}_2$	$\vec{\mathbf{d}}_3$
$(-11.9779,\ -99.2797)'$	$(-11.9616,\ -99.2744)'$	$(-11.9668,\ -99.2850)'$
$\mathrm{var}(\vec{\mathbf{d}}_1) \times 10^{-3}$	$\mathrm{var}(\vec{\mathbf{d}}_2) \times 10^{-3}$	$\mathrm{var}(\vec{\mathbf{d}}_3) \times 10^{-3}$
$\begin{matrix} 0.2733 & 0.0229 \\ 0.0229 & 0.1038 \end{matrix}$	$\begin{matrix} 0.2097 & 0.0021 \\ 0.0021 & 0.0977 \end{matrix}$	$\begin{matrix} 0.1432 & -0.0069 \\ -0.0069 & 0.0430 \end{matrix}$

Table 6.

In all the mentioned cases $\mathbf{R}_{\mathbf{X}_I}$ and $\mathbf{R}_{\tilde{\mathbf{X}}_I}$ are of order 10^{-4} or lower and that's why they don't play practically any role in this example.

R e m a r k 6.1. If we compare the covariance matrices of $\vec{\mathbf{d}}_1$, $\vec{\mathbf{d}}_2$ and $\vec{\mathbf{d}}_3$ from the previous tables we can see that the estimator $\vec{\mathbf{d}}_2$ looks better than the estimator $\vec{\mathbf{d}}_1$ which indicates that the idea about the correction of coordinates of the nonidentical points from Section 5 was valid. The estimator $\vec{\mathbf{d}}_3$ looks like the best from these three but it is necessary to remember that this estimator was derived from $\hat{\hat{\boldsymbol{\eta}}}_{II}$. In this section we have shown that the theoretical precision of the estimator $\hat{\hat{\boldsymbol{\eta}}}_{II}$ does not correspond to the real one and this is why the same problem naturally proceeds to the estimator $\vec{\mathbf{d}}_3$.

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