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Eva Lešanská

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NONSENSITIVENESS REGIONS FOR THRESHOLD ELLIPSOIDS*

Eva Lešanská, Olomouc

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Abstract. The problem is to determine nonsensitiveness regions for threshold ellipsoids within a regular mixed linear model.

Keywords: mixed linear model, power function, threshold ellipsoid, nonsensitiveness region

MSC 2000: 62J05

1. Introduction

Many experiments in agriculture, geography, physics, etc. must be modelled by a linear regression model with inaccurate variance components $\boldsymbol{\vartheta}$, because their true values are not known, must be estimated or are known only approximately. In such cases it is of some interest to know whether and how much the uncertainty in $\boldsymbol{\vartheta}$ influences estimators of unknown parameters, the shape and the position of confidence ellipsoids, the level of statistical tests and their power function.

These problems have been studied in [2], [3], [5], [7] in the case of regularity of the model. In [4] the problem connected with estimators in the universal model with or without constraints is solved.

The aim of this paper is to determine the set of all admissible differences $\delta\boldsymbol{\vartheta}$ of the parameter $\boldsymbol{\vartheta}$, which guarantee that the power of a test on the boundary of the threshold ellipsoid decreases not more than a chosen value ε. Such a set is called the nonsensitiveness region for the threshold ellipsoid.

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2. Definitions and auxiliary statements

Let

$$\text{(2.1)}\quad \boldsymbol{Y} \sim N_n(\boldsymbol{X\beta}, \boldsymbol{\Sigma}(\boldsymbol{\vartheta})),\ \ \boldsymbol{\beta} \in \mathbb{R}^k,\ \ \boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}} = \{\boldsymbol{\vartheta}\colon\ \boldsymbol{\vartheta} \in \mathbb{R}^p,\ \ \vartheta_1 > 0, \ldots, \vartheta_p > 0\},$$

where $\boldsymbol{Y}$ is an n-dimensional random vector (observation vector), $\boldsymbol{X}_{n\times k}$ a known matrix (design matrix), $\boldsymbol{\beta}$ an unknown vector (parameter of the first order), $\boldsymbol{\Sigma}(\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \boldsymbol{V}_i$ a covariance matrix, where $\boldsymbol{\vartheta}$ is an unknown vector (parameter of the second order) and $\boldsymbol{V}_i$, $i = 1, \ldots, p$, are known positively semidefinite matrices of the type $n \times n$.

In the sequel, the mixed linear model (2.1) will be supposed to be regular, i.e. the rank of the matrix $\boldsymbol{X}$ is $r(\boldsymbol{X}) = k < n$ and $\boldsymbol{\Sigma}(\boldsymbol{\vartheta})$ is positively definite for all $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$.

The notation

$$\boldsymbol{C}_H = (\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{X})^{-1}\boldsymbol{H}')^{-1}$$

will be used. Let $\boldsymbol{\vartheta}^*$ be the true value of the parameter $\boldsymbol{\vartheta}$. Let the null hypothesis concerning the parameter $\boldsymbol{\beta}$ be

$$\text{(2.2)}\qquad H_0\colon\ \boldsymbol{H\beta} + \boldsymbol{h} = \boldsymbol{0},$$

where $\boldsymbol{H}_{q\times k}$ is a known matrix with rank $r(\boldsymbol{H}) = q < k$ and $\boldsymbol{h}$ is a known q-dimensional vector. Let the alternative hypothesis be

$$\text{(2.3)}\qquad H_a\colon\ \boldsymbol{H\beta} + \boldsymbol{h} = \boldsymbol{\xi} \neq \boldsymbol{0}.$$

Lemma 2.1. *Let us consider the regular mixed linear model* (2.1) *under the hypotheses* (2.2) *and* (2.3).

(i) *If H_0 is true, then the statistic*

$$\text{(2.4)}\qquad T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) = (\boldsymbol{H}\hat{\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) + \boldsymbol{h})'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1} \times (\boldsymbol{H}\hat{\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) + \boldsymbol{h}),$$

where

$$\hat{\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) = (\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{Y}$$

has the central chi-square distribution with q degrees of freedom ($T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) \sim \chi^2_q(0)$).

(ii) *If* H_0 *is not true, then* $T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^*)$ *has the noncentral chi-square distribution with* q *degrees of freedom* ($T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) \sim \chi^2_q(\delta)$) *and the parameter of its noncentrality is*

$$\delta = (\boldsymbol{H}\boldsymbol{\beta} + \boldsymbol{h})'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1}(\boldsymbol{H}\boldsymbol{\beta} + \boldsymbol{h}).$$

P r o o f. Both statements follow from the second fundamental theorem of the least squares theory given in [8], p. 155. □

Let $\chi^2_q(0, 1-\alpha)$ denote the $(1-\alpha)$-quantile of the central chi-square distribution with q degrees of freedom. The statistic $T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^*)$ has been used for testing the hypothesis H_0 against H_a. If $T_H(\boldsymbol{y}, \boldsymbol{\vartheta}^*) \geqslant \chi^2_q(0, 1-\alpha)$, where $\boldsymbol{y}$ means a realization of $\boldsymbol{Y}$, then H_0 is rejected with the risk α. The power function of this test is

$$\beta(\boldsymbol{\xi}) = P\{\chi^2_q(\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi}) \geqslant \chi^2_q(0, 1-\alpha)\}, \quad \boldsymbol{\xi} = \boldsymbol{H}\boldsymbol{\beta} + \boldsymbol{h}. \tag{2.5}$$

3. Threshold ellipsoids

A threshold ellipsoid is defined in the space of an unknown parameter $\boldsymbol{\beta}$. This region makes it possible to decide which values $\boldsymbol{\xi}$ of an alternative hypothesis are distinguishable from the null hypothesis, i.e. from the value $\mathbf{0}$, with sufficient high chosen power κ_t. The values $\boldsymbol{\beta}$ that cannot be distinguished from a null hypothesis with the chosen probability κ_t on the basis of measurement are inside this region while those distinguishable from a null hypothesis are outside. For details see [6].

Definition 3.1. Let us consider the model (2.1). Let $\boldsymbol{\beta}_0$ be the value of an unknown parameter $\boldsymbol{\beta}$ assumed by the null hypothesis $H_0\colon \boldsymbol{\beta} = \boldsymbol{\beta}_0$ tested against the alternative $H_a\colon \boldsymbol{\beta} \neq \boldsymbol{\beta}_0$ under the risk α. Then the (κ_t, α)-threshold ellipsoid for $\boldsymbol{\beta}$ is

$$\mathcal{T}_{\kappa_t,\alpha}(\boldsymbol{\beta}) = \{\boldsymbol{\beta}\colon \boldsymbol{\beta} \in \mathbb{R}^k,\ (\boldsymbol{\beta} - \boldsymbol{\beta}_0)'T(\boldsymbol{\beta} - \boldsymbol{\beta}_0) \leqslant c^2\}, \quad c \in \mathbb{R}^1, \tag{3.1}$$

where T is a $k \times k$ symmetric matrix and c is a real number, $c > 0$ such that the value of the power function of the used test must be exactly κ_t for the true value $\boldsymbol{\beta}^*$ on the boundary of $\mathcal{T}_{\kappa_t,\alpha}$.

Lemma 3.2. *Let us consider the regular mixed linear model* (2.1) *under the hypotheses* (2.2) *and* (2.3). *Then the* (κ_t, α)*-threshold ellipsoid is*

$$\begin{aligned} \mathcal{T}_{\kappa_t,\alpha}(\boldsymbol{\beta}) = \{\boldsymbol{\beta}\colon \boldsymbol{\beta} \in \mathbb{R}^k,\ &(\boldsymbol{H}\boldsymbol{\beta} + \boldsymbol{h})'[\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{X})^{-1}\boldsymbol{H}']^{-1} \\ &\times (\boldsymbol{H}\boldsymbol{\beta} + \boldsymbol{h}) \leqslant \delta_{\mathrm{krit}}\}, \end{aligned} \tag{3.2}$$

where δ_{krit} *is the value of the noncentrality parameter of the noncentral chi-square distribution with* q *degrees of freedom defined by the relation*

$$P\{\chi_q^2(\delta_{\mathrm{krit}}) \geqslant \chi_q^2(0, 1-\alpha)\} = \kappa_t.$$

P r o o f. The statement follows from Section 4b.2 in [8]. □

4. Nonsensitiveness regions for the power of the test

Let $\boldsymbol{\vartheta}^*$ be changed into $\boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}$. We will study how the change $\delta\boldsymbol{\vartheta}$ influences the power of the test. That is why in the following we will suppose H_a to be true.

Lemma 4.1. *Let the regular mixed linear model* (2.1) *and hypotheses* (2.2), (2.3) *be under consideration. Let*

$$\delta T_H = \delta\boldsymbol{\vartheta}' \frac{\partial T_H(\boldsymbol{Y}, \boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}}\bigg|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}^*}.$$

Then

$$\begin{aligned}(4.1)\quad \delta T_H = & -2[\boldsymbol{H}\hat{\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) + \boldsymbol{h}]'\boldsymbol{C}_H\boldsymbol{F}_H\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*)) \\ & - [\boldsymbol{H}\hat{\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) + \boldsymbol{h}]'\boldsymbol{C}_H\boldsymbol{F}_H\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{F}_H'\boldsymbol{C}_H[\boldsymbol{H}\hat{\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) + \boldsymbol{h}],\end{aligned}$$

where

$$\boldsymbol{F}_H = \boldsymbol{H}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*).$$

The mean value of δT_H *is*

$$\begin{aligned}(4.2)\qquad E(\delta T_H|\boldsymbol{\beta}, \boldsymbol{\vartheta}^*) = & -\delta\boldsymbol{\vartheta}'[\mathrm{Tr}(\boldsymbol{U}_H\boldsymbol{V}_1), \ldots, \mathrm{Tr}(\boldsymbol{U}_H\boldsymbol{V}_p)]' \\ & -\delta\boldsymbol{\vartheta}'[\boldsymbol{\xi}'\boldsymbol{Z}_1\boldsymbol{\xi}, \ldots, \boldsymbol{\xi}'\boldsymbol{Z}_p\boldsymbol{\xi}]',\end{aligned}$$

where $\boldsymbol{U}_H = \boldsymbol{F}_H'\boldsymbol{C}_H\boldsymbol{F}_H$ and $\boldsymbol{Z}_i = \boldsymbol{C}_H\boldsymbol{F}_H\boldsymbol{V}_i\boldsymbol{F}_H'\boldsymbol{C}_H$, $i = 1, \ldots, p$. *Here* $\mathrm{Tr}(\boldsymbol{U}_H)$ *means the trace of the matrix* $\boldsymbol{U}_H$.

The variance of δT_H *is*

$$\begin{aligned}(4.3)\quad \mathrm{var}(\delta T_H|\boldsymbol{\beta}, \boldsymbol{\vartheta}^*) = & \, 4\,\mathrm{Tr}\{\boldsymbol{U}_H\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})[\boldsymbol{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\boldsymbol{M}_X]^+\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\} \\ & + 2\,\mathrm{Tr}\{\boldsymbol{U}_H\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{U}_H\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\} \\ & + 4\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{F}_H\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})[\boldsymbol{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\boldsymbol{M}_X]^+\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{F}_H'\boldsymbol{C}_H\boldsymbol{\xi},\end{aligned}$$

where

$$[\boldsymbol{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\boldsymbol{M}_X]^+ = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) - \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\boldsymbol{X}]^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*).$$

P r o o f. Proof can be found in [3]. □

We can use a linear approximation of the statistic T_H

$$T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \approx T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) + \delta T_H,$$

where a random variable δT_H characterizes the change of T_H caused by the shift $\delta\boldsymbol{\vartheta}$ of the parameter $\boldsymbol{\vartheta}^*$. It is necessary to realize that "the dangerous movement" of the test statistic T_H is to the left, which makes its power decrease. The movement of T_H to the right is not interesting since the power of the test increases. Under H_0, however, the movement to the right might change the significance level of the test (for this problem see e.g. [3], [5], [7]).

The mean value $E(\delta T_H)$ depends on $\delta\boldsymbol{\vartheta}$ linearly and the term $t\sqrt{\operatorname{var}(\delta T_H)}$ depends linearly on the norm $\|\delta\boldsymbol{\vartheta}\| = \sqrt{(\delta\boldsymbol{\vartheta})'(\delta\boldsymbol{\vartheta})}$. Let a function $\Phi_\xi(\delta\boldsymbol{\vartheta})$, $\delta\boldsymbol{\vartheta} \in \mathbb{R}^p$, be defined as

$$\Phi_\xi(\delta\boldsymbol{\vartheta}) = -\delta\boldsymbol{\vartheta}'\boldsymbol{a}_\xi - t\sqrt{\delta\boldsymbol{\vartheta}'\boldsymbol{A}_\xi\delta\boldsymbol{\vartheta}}, \tag{4.4}$$

where for $i, j = 1, \ldots, p$

$$\boldsymbol{a}_\xi = [\operatorname{Tr}(\boldsymbol{U}_H\boldsymbol{V}_1), \ldots, \operatorname{Tr}(\boldsymbol{U}_H\boldsymbol{V}_p)]' + [\boldsymbol{\xi}'\boldsymbol{Z}_1\boldsymbol{\xi}, \ldots, \boldsymbol{\xi}'\boldsymbol{Z}_p\boldsymbol{\xi}]', \tag{4.5}$$

$$\begin{aligned}\{\boldsymbol{A}_\xi\}_{i,j} &= 2\operatorname{Tr}(\boldsymbol{U}_H\boldsymbol{V}_i\boldsymbol{U}_H\boldsymbol{V}_j) + 4\operatorname{Tr}(\boldsymbol{U}_H\boldsymbol{V}_i[\boldsymbol{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\boldsymbol{M}_X]^+\boldsymbol{V}_j) \\ &\quad + 4\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{F}_H\boldsymbol{V}_i[\boldsymbol{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\boldsymbol{M}_X]^+\boldsymbol{V}_j\boldsymbol{F}_H'\boldsymbol{C}_H\boldsymbol{\xi}.\end{aligned} \tag{4.6}$$

Definition 4.2. Let

$$\mathcal{H}_{\varepsilon,\xi} = \{\delta\boldsymbol{\vartheta}\colon\ \delta\boldsymbol{\vartheta} \in \mathcal{R}^p,\ \ \Phi_\xi(\delta\boldsymbol{\vartheta}) \geqslant -\delta_{\varepsilon,\xi}\}, \tag{4.7}$$

where $\delta_{\varepsilon,\xi}$ is given by the relation

$$P\{\chi_q^2(\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi}) \geqslant \chi_q^2(0, 1-\alpha) + \delta_{\varepsilon,\xi}\} = \beta(\boldsymbol{\xi}) - \varepsilon.$$

The set $\mathcal{H}_{\varepsilon,\xi}$ is called the nonsensitiveness region for the power of the test at the point $\boldsymbol{\xi}$.

Lemma 4.3. *Let the regular mixed linear model* (2.1) *and the hypothesis* (2.3) *be under consideration. Let H_a be true and let $\boldsymbol{a}_\xi$ and $\boldsymbol{A}_\xi$ be given by* (4.5) *and* (4.6)*, respectively. The boundary of the set $\mathcal{H}_{\varepsilon,\xi}$ is*

$$\overline{\mathcal{H}}_{\varepsilon,\xi} = \left\{\delta\boldsymbol{\vartheta}\colon\ \delta\boldsymbol{\vartheta} \in \mathbb{R}^p,\ (\delta\boldsymbol{\vartheta} + \boldsymbol{x}_0)'(t^2\boldsymbol{A}_\xi - \boldsymbol{a}_\xi\boldsymbol{a}_\xi')(\delta\boldsymbol{\vartheta} + \boldsymbol{x}_0) = \frac{\delta_{\varepsilon,\xi}^2 t^2}{t^2 - \boldsymbol{a}_\xi'\boldsymbol{A}_\xi^-\boldsymbol{a}_\xi}\right\}, \tag{4.8}$$

where $\boldsymbol{x}_0 = \frac{\delta_{\varepsilon,\xi}}{t^2 - \boldsymbol{a}_\xi'\boldsymbol{A}_\xi^-\boldsymbol{a}_\xi}\boldsymbol{A}_\xi^-\boldsymbol{a}_\xi$, $\delta_{\varepsilon,\xi} = \chi_q^2(\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi}, 1-\kappa_t+\varepsilon) - \chi_q^2(0, 1-\alpha)$ and ε, t are chosen positive numbers. Here $\boldsymbol{A}_\xi^-$ means g-inverse of the matrix $\boldsymbol{A}_\xi$.

Proof. It follows from the solution of the equation $\Phi_\xi(\delta\boldsymbol{\vartheta}) = -\delta_{\varepsilon,\xi}$ from Definition 4.2 of the nonsensitiveness region. For details see [3]. □

Lemma 4.4. *Let the regular mixed linear model* (2.1) *and the hypothesis* (2.3) *be under consideration. Let H_a be true. Then*

$$\delta\boldsymbol{\vartheta} \in \mathcal{H}_{\varepsilon,\xi} \Rightarrow P\{T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \geqslant \chi_q^2(0, 1-\alpha)\} \geqslant \beta(\boldsymbol{\xi}) - \varepsilon. \tag{4.9}$$

Proof. Proof can be found in [3]. □

5. Nonsensitiveness regions for threshold ellipsoids

From Definition 3.1 we can see that the problem of a nonsensitiveness region for the threshold ellipsoid is closely connected with a nonsensitiveness region for the power of the test. In the case of the power, we are seeking for the region of $\delta\boldsymbol{\vartheta}$ such that the power $\beta(\boldsymbol{\xi})$ decreases by not more than the chosen value ε at the fixed point $\boldsymbol{\xi}$. In the case of the threshold ellipsoid, we are seeking for the region of $\delta\boldsymbol{\vartheta}$ such that the power $\beta(\boldsymbol{\xi}) = \kappa_t$ decreases by not more than ε at all points $\boldsymbol{\xi}$, $\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi} = \delta_{\text{krit}}$. Thus, if we find the nonsensitiveness region for the power of the test independent of $\boldsymbol{\xi}$, it will be also the nonsensitiveness region for the threshold ellipsoid.

At first, let us suppose $r(\boldsymbol{H}) = 1$. Then $r(\boldsymbol{C}_H) = 1$ and the equation $\boldsymbol{C}_H\boldsymbol{\xi}^2 = \delta_{\text{krit}}$ has exactly two solutions $\xi_0 = \pm\sqrt{\frac{\delta_{\text{krit}}}{\boldsymbol{C}_H}}$. Thus the solution $\xi_{\text{krit}} = |\xi_0|$ is unique, since the mean value and the variance of δT_H are functions of ξ^2. Hence, by Lemma 4.4, $\mathcal{H}_{\varepsilon,\xi_{\text{krit}}}$ is the nonsensitiveness region for the threshold region. More precisely, the power of the test $\beta(\xi_{\text{krit}}) = \kappa_t$ decreases by not more than ε at all points $\boldsymbol{\beta}_0 \in \overline{\mathcal{T}}_{\kappa_t,\alpha}$, i.e. at all $\boldsymbol{\beta}_0$, $\boldsymbol{H}\boldsymbol{\beta}_0 + h = \xi_{\text{krit}}$.

Let $r(\boldsymbol{H}) \geqslant 2$. In this case, the set of all solutions $\boldsymbol{\xi}_{\text{krit}}$ of the equation $\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi} = \delta_{\text{krit}}$ is uncountable and for a different $\boldsymbol{\xi}_{\text{krit}}$ we have a different region $\mathcal{H}_{\varepsilon,\xi_{\text{krit}}}$. One possible approach for determining a joint region $\mathcal{H}_{\varepsilon,\delta_{\text{krit}}}$ for all $\boldsymbol{\xi}$, $\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi} = \delta_{\text{krit}}$ is to eliminate the dependence of the mean value and the variance of δT_H on $\boldsymbol{\xi}$. According to the previous section, the problem is to determine the upper bound of the variance and the lower bound of the mean value of the random variable δT_H independent of $\boldsymbol{\xi}$.

Definition 5.1. Let $\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi} = \delta_{\text{krit}}$ and

$$\mathcal{H}_{\varepsilon,\delta_{\text{krit}}} = \{\delta\boldsymbol{\vartheta}\colon\ \delta\boldsymbol{\vartheta} \in \mathbb{R}^p,\ \ \Phi_\xi(\delta\boldsymbol{\vartheta}) \geqslant -\delta_\varepsilon\}, \tag{5.1}$$

where δ_ε is given by the relationship

$$P\{\chi_q^2(\delta_{\text{krit}}) \geqslant \chi_q^2(0, 1-\alpha) + \delta_\varepsilon\} = \kappa_t - \varepsilon.$$

The set $\mathcal{H}_{\varepsilon,\delta_{\text{krit}}}$ is called the nonsensitiveness region for the (κ_t, α)-threshold ellipsoid.

Lemma 5.2. *Let $\boldsymbol{E}_{ij}$, $i,j=1,\ldots,p$, be the $p\times p$ matrix with the (i,j)-th entry equal to 1 and with the other entries equal to 0. Then in the regular mixed model (2.1) we have*

$$\operatorname{var}_{\xi}(\delta T_H|\boldsymbol{\beta},\boldsymbol{\vartheta}^*)\leqslant\delta\boldsymbol{\vartheta}'(\boldsymbol{A}+\boldsymbol{D}_{\delta_{\rm krit}})\delta\boldsymbol{\vartheta}\quad\forall\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi}=\delta_{\rm krit},\tag{5.2}$$

where

$$\begin{aligned}\{\boldsymbol{A}\}_{i,j}&=2\operatorname{Tr}(\boldsymbol{U}_H\boldsymbol{V}_i\boldsymbol{U}_H\boldsymbol{V}_j)+4\operatorname{Tr}(\boldsymbol{U}_H\boldsymbol{V}_i[\boldsymbol{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\boldsymbol{M}_X]^+\boldsymbol{V}_j),\\ \boldsymbol{K}_{ij}&=4\boldsymbol{C}_H\boldsymbol{F}_H\boldsymbol{V}_i[\boldsymbol{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\boldsymbol{M}_X]^+\boldsymbol{V}_j\boldsymbol{F}_H'\boldsymbol{C}_H,\quad i,j=1,\ldots,p,\\ \boldsymbol{D}_{\delta_{\rm krit}}&=\sum_{r=1}^{s}\delta_{\rm krit}\gamma_r\boldsymbol{G}_r\boldsymbol{C}_H\boldsymbol{G}_r',\\ \boldsymbol{G}_r&=\begin{pmatrix}\boldsymbol{g}_{r,1}'\\ \vdots\\ \boldsymbol{g}_{r,p}'\end{pmatrix},\quad r=1,\ldots,s,\\ \sum_{i=1}^{p}\sum_{j=1}^{p}(\boldsymbol{E}_{ij}\otimes\boldsymbol{K}_{ij})&=\sum_{r=1}^{s}\gamma_r\boldsymbol{g}_r\boldsymbol{g}_r'\quad\textit{(the spectral decomposition)},\end{aligned}$$

where $r\left(\sum\limits_{i=1}^{p}\sum\limits_{j=1}^{p}(\boldsymbol{E}_{ij}\otimes\boldsymbol{K}_{ij})\right)=s\leqslant pq$, $\boldsymbol{g}_r\in\mathbb{R}^{pq}$, $\boldsymbol{g}_r'\boldsymbol{g}_s=\begin{cases}1 & \textit{if } r=s,\\ 0 & \textit{otherwise,}\end{cases}$ $\boldsymbol{g}_{r,i}\in\mathbb{R}^{q}$, $\boldsymbol{g}_r=(\boldsymbol{g}_{r,1}',\ldots,\boldsymbol{g}_{r,p}')'$, $i=1,\ldots,p$ *and* $r=1,\ldots,s$. *Here* $\otimes$ *means Kronecker product.*

Proof. We will use a procedure analogous to that in [4]. It is true that

$$\begin{aligned}&4\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{F}_H\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})[\boldsymbol{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}^*)\boldsymbol{M}_X]^+\boldsymbol{\Sigma}(\delta\boldsymbol{\vartheta})\boldsymbol{F}_H'\boldsymbol{C}_H\boldsymbol{\xi}\\ &\qquad=\delta\boldsymbol{\vartheta}'\begin{pmatrix}\boldsymbol{\xi}'\boldsymbol{K}_{11}\boldsymbol{\xi},\ldots,\boldsymbol{\xi}'\boldsymbol{K}_{1p}\boldsymbol{\xi}\\ \ldots\\ \boldsymbol{\xi}'\boldsymbol{K}_{p1}\boldsymbol{\xi},\ldots,\boldsymbol{\xi}'\boldsymbol{K}_{pp}\boldsymbol{\xi}\end{pmatrix}\delta\boldsymbol{\vartheta}\\ &\qquad=(\delta\boldsymbol{\vartheta}'\otimes\boldsymbol{\xi}')\sum_{i=1}^{p}\sum_{j=1}^{p}(\boldsymbol{E}_{ij}\otimes\boldsymbol{K}_{ij})(\delta\boldsymbol{\vartheta}\otimes\boldsymbol{\xi}),\end{aligned}$$

since

$$\begin{aligned}(\delta\boldsymbol{\vartheta}'\otimes\boldsymbol{\xi}')\sum_{i=1}^{p}\sum_{j=1}^{p}(\boldsymbol{E}_{ij}\otimes\boldsymbol{K}_{ij})(\delta\boldsymbol{\vartheta}\otimes\boldsymbol{\xi})&=\sum_{i=1}^{p}\sum_{j=1}^{p}(\delta\boldsymbol{\vartheta}'\boldsymbol{E}_{ij}\otimes\boldsymbol{\xi}'\boldsymbol{K}_{ij})(\delta\boldsymbol{\vartheta}\otimes\boldsymbol{\xi})]\\ &=\sum_{i=1}^{p}\sum_{j=1}^{p}(\delta\boldsymbol{\vartheta}'\boldsymbol{E}_{ij}\delta\boldsymbol{\vartheta}\otimes\boldsymbol{\xi}'\boldsymbol{K}_{ij}\boldsymbol{\xi})\\ &=\sum_{i=1}^{p}\sum_{j=1}^{p}(\delta\vartheta_i\delta\vartheta_j\boldsymbol{\xi}'\boldsymbol{K}_{ij}\boldsymbol{\xi}).\end{aligned}$$

Let $\sum_{r=1}^{s}\gamma_r \boldsymbol{g}_r\boldsymbol{g}_r'$ be the spectral decomposition of $\sum_{i=1}^{p}\sum_{j=1}^{p}(\boldsymbol{E}_{ij}\otimes\boldsymbol{K}_{ij})$, where $\boldsymbol{g}_r\in\mathbb{R}^{pq}$, $\boldsymbol{g}_r'\boldsymbol{g}_s=\begin{cases}1 & \text{if } r=s,\\ 0 & \text{otherwise.}\end{cases}$ Thus we can divide the vector $\boldsymbol{g}_r$ into p subvectors of dimension q, i.e. $\boldsymbol{g}_r=(\boldsymbol{g}_{r,1}',\ldots,\boldsymbol{g}_{r,p}')'$, $\boldsymbol{g}_{r,i}\in\mathbb{R}^q$, $i=1,\ldots,p$ and $r=1,\ldots,s$. Let us denote

$$\boldsymbol{G}_r=\begin{pmatrix}\boldsymbol{g}_{r,1}'\\ \vdots\\ \boldsymbol{g}_{r,p}'\end{pmatrix}.$$

Then using the Schwarz inequality with the seminorm $\|\boldsymbol{x}\|_{C_H}=\sqrt{\boldsymbol{x}'\boldsymbol{C}_H\boldsymbol{x}}$ we get

$$\begin{aligned}
&(\delta\boldsymbol{\vartheta}'\otimes\boldsymbol{\xi}')\sum_{i=1}^{p}\sum_{j=1}^{p}(\boldsymbol{E}_{ij}\otimes\boldsymbol{K}_{ij})(\delta\boldsymbol{\vartheta}\otimes\boldsymbol{\xi})\\
&\quad=(\delta\boldsymbol{\vartheta}'\otimes\boldsymbol{\xi}')\sum_{r=1}^{s}\gamma_r\boldsymbol{g}_r\boldsymbol{g}_r'(\delta\boldsymbol{\vartheta}\otimes\boldsymbol{\xi})\\
&\quad=\sum_{r=1}^{s}\gamma_r(\delta\boldsymbol{\vartheta}'\boldsymbol{G}_r\boldsymbol{\xi})^2\\
&\quad\leqslant\sum_{r=1}^{s}\gamma_r\Big(\sqrt{\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi}}\sqrt{\delta\boldsymbol{\vartheta}'\boldsymbol{G}_r\boldsymbol{C}_H\boldsymbol{G}_r'\delta\boldsymbol{\vartheta}}\Big)^2\\
&\quad=\delta\boldsymbol{\vartheta}'\left(\sum_{r=1}^{s}\delta_{\mathrm{krit}}\gamma_r\boldsymbol{G}_r\boldsymbol{C}_H\boldsymbol{G}_r'\right)\delta\boldsymbol{\vartheta}\\
&\quad\equiv\delta\boldsymbol{\vartheta}'\boldsymbol{D}_{\delta_{\mathrm{krit}}}\delta\boldsymbol{\vartheta}
\end{aligned}$$

and the proof is complete. □

Lemma 5.3. *In the regular mixed model* (2.1) *we have*

$$E_\xi(\delta T_H|\boldsymbol{\beta},\boldsymbol{\vartheta}^*)\geqslant-\delta\boldsymbol{\vartheta}'\begin{bmatrix}\mathrm{Tr}(\boldsymbol{U}_H\boldsymbol{V}_1)+k_1\delta_{\mathrm{krit}}\\ \vdots\\ \mathrm{Tr}(\boldsymbol{U}_H\boldsymbol{V}_p)+k_p\delta_{\mathrm{krit}}\end{bmatrix},\quad\forall\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi}=\delta_{\mathrm{krit}},\tag{5.3}$$

where

$$k_i=\max\left\{\lambda_j\colon\ \boldsymbol{C}_H^{-\frac12}\boldsymbol{Z}_i\boldsymbol{C}_H^{-\frac12}=\sum_{j=1}^{r(\boldsymbol{Z}_i)}\lambda_j\boldsymbol{f}_j\boldsymbol{f}_j'\right\},\quad i=1,\ldots,p.$$

P r o o f. The problem is to minimize $E_\xi(\delta T_H)$, i.e. to maximize $[\boldsymbol{\xi}'\boldsymbol{Z}_1\boldsymbol{\xi},\ldots,\boldsymbol{\xi}'\boldsymbol{Z}_p\boldsymbol{\xi}]'$ subject to the condition $\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi}=\delta_{\mathrm{krit}}$. Using the method of Lagrangian multipliers, we get for $i = 1, \ldots, p$

$$\Phi(\boldsymbol{\xi}) = \boldsymbol{\xi}'\boldsymbol{Z}_i\boldsymbol{\xi} - \lambda(\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi} - \delta_{\text{krit}}),$$
$$\frac{\partial\Phi(\boldsymbol{\xi})}{\partial\boldsymbol{\xi}} = 2\boldsymbol{Z}_i\boldsymbol{\xi} - 2\lambda\boldsymbol{C}_H\boldsymbol{\xi}.$$

Thus

$$0 = \det(\boldsymbol{Z}_i - \lambda\boldsymbol{C}_H) = \det(\boldsymbol{C}_H^{-\frac{1}{2}}\boldsymbol{Z}_i\boldsymbol{C}_H^{-\frac{1}{2}} - \lambda\boldsymbol{I}).$$

Let $\lambda_1 \geqslant \ldots \geqslant \lambda_{r(Z_i)} \geqslant 0$ be eigenvalues of the matrix $\boldsymbol{C}_H^{-\frac{1}{2}}\boldsymbol{Z}_i\boldsymbol{C}_H^{-\frac{1}{2}}$. Then for all $\boldsymbol{\xi}$, $\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi} = \delta_{\text{krit}}$ we have

$$\boldsymbol{\xi}'\boldsymbol{Z}_i\boldsymbol{\xi} \leqslant k_i\delta_{\text{krit}}$$

and the proof is complete. □

Theorem 5.4. *Let the regular mixed linear model* (2.1) *and hypotheses* (2.2), (2.3) *be under consideration. Let matrices* $\boldsymbol{A}$ *and* $\boldsymbol{D}_{\delta_{\text{krit}}}$ *be defined as in Lemma* 5.2. *Let*

$$[k_1, \ldots, k_p]' \in \mathcal{M}(\boldsymbol{A} + \boldsymbol{D}_{\delta_{\text{krit}}}), \tag{5.4}$$

where $\mathcal{M}(\boldsymbol{A} + \boldsymbol{D}_{\delta_{\text{krit}}}) = \{\boldsymbol{u}\colon\ \boldsymbol{u} \in \mathbb{R}^p,\ \exists\boldsymbol{x} \in \mathbb{R}^p,\ (\boldsymbol{A} + \boldsymbol{D}_{\delta_{\text{krit}}})\boldsymbol{x} = \boldsymbol{u}\}$. *The boundary of the nonsensitiveness region* $\mathcal{H}_{\varepsilon,\delta_{\text{krit}}}$ *for the threshold ellipsoid* $\mathcal{T}_{\kappa_t,\alpha}(\boldsymbol{\beta})$ *is*

$$\overline{\mathcal{H}}_{\varepsilon,\delta_{\text{krit}}} = \left\{\delta\boldsymbol{\vartheta}\colon\ (\delta\boldsymbol{\vartheta} + \boldsymbol{x}_1)'(t^2\boldsymbol{A}_{\delta_{\text{krit}}} - \boldsymbol{a}\boldsymbol{a}')(\delta\boldsymbol{\vartheta} + \boldsymbol{x}_1) = \frac{\delta_\varepsilon^2 t^2}{t^2 - \boldsymbol{a}'\boldsymbol{A}^-_{\delta_{\text{krit}}}\boldsymbol{a}}\right\}, \tag{5.5}$$

where δ_ε *is given by the relation*

$$P\{\chi_q^2(\delta_{\text{krit}}) \geqslant \chi_q^2(0, 1-\alpha) + \delta_\varepsilon\} = \kappa_t - \varepsilon$$

and

$$\boldsymbol{A}_{\delta_{\text{krit}}} = \boldsymbol{A} + \boldsymbol{D}_{\delta_{\text{krit}}},$$
$$\boldsymbol{a} = [\text{Tr}(\boldsymbol{U}_H\boldsymbol{V}_1), \ldots, \text{Tr}(\boldsymbol{U}_H\boldsymbol{V}_p)]' + [k_1, \ldots, k_p]'\delta_{\text{krit}},$$
$$\boldsymbol{x}_1 = \frac{\delta_\varepsilon}{t^2 - \boldsymbol{a}'\boldsymbol{A}^-_{\delta_{\text{krit}}}\boldsymbol{a}}\boldsymbol{A}^-_{\delta_{\text{krit}}}\boldsymbol{a}.$$

P r o o f. The proof follows from Lemma 4.3, Lemma 5.2 and Lemma 5.3. □

R e m a r k 5.5. The presumption (5.4) in Theorem 5.4 cannot be omitted. The boundary of the nonsensitiveness region $\mathcal{H}_{\varepsilon,\delta_{\rm krit}}$ is derived from the boundary of the nonsensitiveness region $\mathcal{H}_{\varepsilon,\xi}$ for the power of the test (cf. Lemma 4.3). In the case of the power, the presumption $\boldsymbol{a}_\xi \in \mathcal{M}(\boldsymbol{A}_\xi)$ is always fulfilled. Hence, if the presumption (5.4) is not fulfilled, then we cannot apply the expression from Lemma 4.3 and the boundary of the nonsensitiveness region $\mathcal{H}_{\varepsilon,\delta_{\rm krit}}$ is given by the general quadratic form

$$\overline{\mathcal{H}}_{\varepsilon,\delta_{\rm krit}} = \Big\{\delta\boldsymbol{\vartheta}\colon\ (\delta\boldsymbol{\vartheta}_0 + \boldsymbol{x}_2)'(t^2\boldsymbol{A}_{\delta_{\rm krit}} - \boldsymbol{c}_0\boldsymbol{c}_0')(\delta\boldsymbol{\vartheta}_0 + \boldsymbol{x}_2) - \delta\boldsymbol{\vartheta}_1'\boldsymbol{c}_1\boldsymbol{c}_1'\delta\boldsymbol{\vartheta}_1 + 2\delta\boldsymbol{\vartheta}_1'\boldsymbol{c}_1\delta_\varepsilon = \frac{\delta_\varepsilon^2 t^2}{t^2 - \boldsymbol{c}_0'\boldsymbol{A}^-_{\delta_{\rm krit}}\boldsymbol{c}_0}\Big\},$$

where

$$\boldsymbol{x}_2 = \frac{\delta_\varepsilon}{t^2 - \boldsymbol{c}_0'\boldsymbol{A}^-_{\delta_{\rm krit}}\boldsymbol{c}_0}\boldsymbol{A}^-_{\delta_{\rm krit}}\boldsymbol{c}_0,$$

$$\boldsymbol{a} = \boldsymbol{c}_0 + \boldsymbol{c}_1,\ \ \boldsymbol{c}_0 \in \mathcal{M}(\boldsymbol{A}_{\delta_{\rm krit}}),\ \ \boldsymbol{c}_0 \perp \boldsymbol{c}_1,\ \ (\boldsymbol{c}_0,\ \boldsymbol{c}_1 \text{ are orthogonal}),$$

$$\delta\boldsymbol{\vartheta} = \delta\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}_1,\ \ \delta\boldsymbol{\vartheta}_0 \in \mathcal{M}(\boldsymbol{A}_{\delta_{\rm krit}}),\ \ \delta\boldsymbol{\vartheta}_0 \perp \delta\boldsymbol{\vartheta}_1.$$

Theorem 5.6. *Let the regular mixed linear model* (2.1) *and hypothesis* (2.3) *be under consideration. Let H_a be true. Let $\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi} = \delta_{\rm krit}$, where $\boldsymbol{H\beta} + \boldsymbol{h} = \boldsymbol{\xi}$. If $\delta\boldsymbol{\vartheta} \in \mathcal{H}_{\varepsilon,\delta_{\rm krit}}$, then*

$$P\{T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}) \geqslant \chi^2_q(0, 1-\alpha)|\boldsymbol{\xi}\} \geqslant \kappa_t - \varepsilon.$$

P r o o f. It is an obvious consequence of Lemma 4.4. □

R e m a r k 5.7. With respect to the Chebyshev inequality it seems that the proper value of the parameter t lies in the interval [3,5], since

$$t = 5:\quad P\big\{|\delta T_H - E(\delta T_H)| \geqslant 5\sqrt{\mathrm{var}(\delta T_H)}\big\} \leqslant 0.04.$$

If δT_H is approximately normally distributed, then

$$t = 3:\quad P\big\{|\delta T_H - E(\delta T_H)| \geqslant 3\sqrt{\mathrm{var}(\delta T_H)}\big\} \approx 0.003.$$

In the case that we want to find the optimum value of the parameter t, we must determine the distribution of δT_H. The optimum value t^* which maximizes the size of the nonsensitiveness region is $t^* = \max\{t_{\delta\vartheta}\colon\ \|\delta\boldsymbol{\vartheta}\| = 1\}$ subject to the condition

$$E(\delta T_H|\delta\boldsymbol{\vartheta}) + t_{\delta\vartheta}\sqrt{\mathrm{var}(\delta T_H|\delta\boldsymbol{\vartheta})} = q(1-\alpha),$$

where $q(1-\alpha)$ is the $(1-\alpha)$-quantile of the distribution of δT_H with sufficiently small α. It was found out that in some cases the sufficiently large value of t can be smaller than 3; for details cf. [7].

R e m a r k 5.8. If sets $\overline{\mathcal{H}}_{\varepsilon,\xi}$ and $\overline{\mathcal{H}}_{\varepsilon,\delta_{\text{krit}}}$ are surfaces of ellipsoids, then nonsensitiveness regions $\mathcal{H}_{\varepsilon,\xi}$ and $\mathcal{H}_{\varepsilon,\delta_{\text{krit}}}$ are unions of $\overline{\mathcal{H}}_{\varepsilon,\xi}$ and $\overline{\mathcal{H}}_{\varepsilon,\delta_{\text{krit}}}$ and their interiors, respectively. If $\overline{\mathcal{H}}_{\varepsilon,\xi}$ or $\overline{\mathcal{H}}_{\varepsilon,\delta_{\text{krit}}}$ is not characterized as an ellipsoid, the change of $\boldsymbol{\vartheta}^*$ can be arbitrarily large in some direction.

In practice, if right-hand sides in the expressions of $\overline{\mathcal{H}}_{\varepsilon,\xi}$ (cf. (4.8)) and $\overline{\mathcal{H}}_{\varepsilon,\delta_{\text{krit}}}$ (cf. (5.5)) are positive, we get ellipsoids by replacing the negative eigenvalues of matrices $t^2\boldsymbol{A}_\xi - \boldsymbol{a}_\xi\boldsymbol{a}'_\xi$ and $t^2\boldsymbol{A}_{\delta_{\text{krit}}} - \boldsymbol{a}\boldsymbol{a}'$ by their absolute values.

If the right-hand side is negative, it is necessary to find a suitable subset of $\mathcal{H}_{\varepsilon,\xi}$, $\mathcal{H}_{\varepsilon,\delta_{\text{krit}}}$ including the point $\delta\boldsymbol{\vartheta} = \mathbf{0}$ (e.g. an ellipsoid, a sphere, a cube).

R e m a r k 5.9. The boundary $\overline{\mathcal{H}}_{\varepsilon,\delta_{\text{krit}}}$ of the nonsensitiveness region for the threshold ellipsoid in Theorem 5.4 is determined for the worst situation, since we consider the maximum possible variance and the minimum possible mean value of the correction term δT_H. This can make the region $\mathcal{H}_{\varepsilon,\delta_{\text{krit}}}$ in some situation so small that any permitted differences from the true value $\boldsymbol{\vartheta}^*$ are negligible and thus values of the parameter $\boldsymbol{\vartheta}$ must be known more precisely.

6. Numerical demonstration

E x a m p l e 6.1. Let a straight line be given in the plane. We have four measurements at points $x = 1, 2, 3, 4$. The accuracy of measurement is characterized by the standard deviation $\sigma_1^* = 0.004$ (at points $x = 1, 2$ and $x = 2, 3$ in an experiment I and II, respectively) and $\sigma_2^* = 0.001$ (at points $x = 3, 4$ and $x = 1, 4$ in an experiment I and II, respectively). Let the null hypothesis be "the coefficients of the straight line are equal to one" and the alternative hypothesis be "the coefficients of the straight line are not equal to one".

Let two different designs of an experiment be under consideration. The process of measurement if the error vector is assumed to be normally distributed can be modelled by

$$\boldsymbol{Y} \sim N_4[\boldsymbol{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}_i(\boldsymbol{\vartheta}^*)], \quad i = I, II,$$

where

$$\boldsymbol{X} = \begin{pmatrix} 1, & 1 \\ 1, & 2 \\ 1, & 3 \\ 1, & 4 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

and

$$\boldsymbol{\Sigma}_I(\boldsymbol{\vartheta}^*) = \begin{pmatrix} 16\cdot 10^{-6}, & 0, & 0, & 0 \\ 0, & 16\cdot 10^{-6}, & 0, & 0 \\ 0, & 0, & 1\cdot 10^{-6}, & 0 \\ 0, & 0, & 0, & 1\cdot 10^{-6} \end{pmatrix},$$

$$\boldsymbol{\Sigma}_{II}(\boldsymbol{\vartheta}^*) = \begin{pmatrix} 1\cdot 10^{-6}, & 0, & 0, & 0 \\ 0, & 16\cdot 10^{-6}, & 0, & 0 \\ 0, & 0, & 16\cdot 10^{-6}, & 0 \\ 0, & 0, & 0, & 1\cdot 10^{-6} \end{pmatrix}.$$

The null hypothesis is

$$H_0: \quad \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{0}$$

and the alternative hypothesis is

$$H_a: \quad \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boldsymbol{\xi} \neq 0.$$

It is obvious that under H_0

$$T_H(\boldsymbol{Y}, \boldsymbol{\vartheta}^*) \sim \chi^2_2(0).$$

Let the risk of the test be $\alpha = 0.05$. For the given power $\kappa_t = 0.99$ we determine the critical value of the noncentrality parameter δ_{krit} by solving the equation

$$P\{\chi^2_2(\delta_{\text{krit}}) \geqslant \chi^2_2(0, 0.95)\} = 0.99.$$

Using the approximation of the noncentral chi-square distribution by the central distribution (cf. [1], p. 27)

$$\chi^2_q(\delta) \approx \frac{q+2\delta}{q+\delta}\chi^2_{\frac{(q+\delta)^2}{q+2\delta}}(0)$$

we get $\delta_{\text{krit}} \doteq 19.31$. Hence

$$\mathcal{T}_{0.99,0.05}(\boldsymbol{\beta}) = \left\{\boldsymbol{\beta}\colon\ \boldsymbol{\beta} \in \mathbb{R}^2,\ \left[\boldsymbol{\beta} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]' \boldsymbol{C}_H \left[\boldsymbol{\beta} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] \leqslant 19.31\right\},$$

where $\boldsymbol{C}_H = \boldsymbol{X}'\boldsymbol{\Sigma}_i(\boldsymbol{\vartheta}^*)\boldsymbol{X}$, $i = I, II$.

Let $\boldsymbol{\vartheta}^*$ be changed into $\boldsymbol{\vartheta}^* + \delta\boldsymbol{\vartheta}$. Let us look at the nonsensitiveness region $\mathcal{H}_{\varepsilon,\xi}$ for the power of the test and $\mathcal{H}_{\varepsilon,\delta_{\text{krit}}}$ for the threshold ellipsoid in more detail. We will concentrate on their behavior, properties and correlations. In the case of the power, we restrict to $\beta(\boldsymbol{\xi}) = \kappa_t$, i.e. to directions $\boldsymbol{\xi}$ subject to the condition $\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi} = \delta_{\text{krit}}$, which we will denote by $\boldsymbol{\xi}_{\text{krit}}$. Then

$$\delta_{\varepsilon,\xi_{\text{krit}}} = \delta_\varepsilon = \chi_2^2(\delta_{\text{krit}}, 1-\kappa_t+\varepsilon) - \chi_2^2(0, 1-\alpha).$$

Hence $\delta_{0.05} = \chi_2^2(19.31, 0.06) - \chi_2^2(0, 0.95) = 3.23$.

In what follows, only the boundary of each nonsensitiveness region will be shown, since we are in the situation with a negative right-hand side (cf. Remark 5.8). In our case boundaries are characterized as hyperbolas.

First, we will engage in a power. Let $\boldsymbol{C}_H = \lambda_1\boldsymbol{f}_1\boldsymbol{f}_1' + \lambda_2\boldsymbol{f}_2\boldsymbol{f}_2'$ be the spectral decomposition. Hence, some interesting directions $\boldsymbol{\xi}_{\text{krit}}$ are for example

$$\begin{aligned}
\boldsymbol{\xi}_1 &= \boldsymbol{f}_1\sqrt{\frac{\delta_{\text{krit}}}{\lambda_1}},\\
\boldsymbol{\xi}_2 &= \boldsymbol{f}_2\sqrt{\frac{\delta_{\text{krit}}}{\lambda_2}},\\
\boldsymbol{\xi}_3 &= \left(\frac{\boldsymbol{f}_1}{\sqrt{\lambda_1}} + \frac{\boldsymbol{f}_2}{\sqrt{\lambda_2}}\right)\sqrt{\frac{\delta_{\text{krit}}}{2}},\\
\boldsymbol{\xi}_4 &= \left(\frac{\boldsymbol{f}_1}{\sqrt{\lambda_1}} - \frac{\boldsymbol{f}_2}{\sqrt{\lambda_2}}\right)\sqrt{\frac{\delta_{\text{krit}}}{2}},\\
\boldsymbol{\xi}_5 &= \frac{\boldsymbol{f}_1}{\sqrt{\lambda_1}} + \frac{\boldsymbol{f}_2}{\sqrt{\lambda_2}}\left(\sqrt{\delta_{\text{krit}}} - 1\right),\\
\boldsymbol{\xi}_6 &= \frac{\boldsymbol{f}_2}{\sqrt{\lambda_2}} + \frac{\boldsymbol{f}_1}{\sqrt{\lambda_1}}\left(\sqrt{\delta_{\text{krit}}} - 1\right)
\end{aligned}$$

(the boundary of the threshold ellipse).

The dependence of $\overline{\mathcal{H}}_{\varepsilon,\xi_{\text{krit}}}$ on the chosen direction $\boldsymbol{\xi}_{\text{krit}}$ is given for $\varepsilon = 0.05$ in Figs. 6.1, 6.2. Designs with covariance matrices $\boldsymbol{\Sigma}_I$ and $\boldsymbol{\Sigma}_{II}$ are used in Figs. 6.2 and 6.1, respectively. Each nonsensitiveness region is the set around the origin of the coordinate system bounded by the branches of the proper hyperbola. As we can see, the design of the experiment plays an important role for the behavior of these regions (for details see [5]). From Fig. 6.1, when we have a more precise measurement at outer points of the straight line (at points $x = 1, 4$), it follows that $\delta\vartheta_1$ can be arbitrarily large, i.e. it depends only on the instrument with σ_2^*. On the other hand, from Fig. 6.2 we see that both instruments should have the true value of the standard deviation approximately equal to σ_1^*, σ_2^*. For instance, let us consider direction $\boldsymbol{\xi}_6$. In the case $\boldsymbol{\Sigma}_I$ (Fig. 6.2) shifts $\delta\vartheta_1$ are admissible in the interval $(-1.6\cdot10^{-5}, 0.3\cdot10^{-5})$ (the lower bound follows from the assumption $\vartheta_1 > 0$) if $\delta\vartheta_2 = 0$. Shifts $\delta\vartheta_2$ are admissible in the interval $(-0.5\cdot10^{-6}, 0.2\cdot10^{-6})$ if $\delta\vartheta_1 = 0$. If $\delta\vartheta_1 > 0$, the interval of admissible shifts $\delta\vartheta_2$ is smaller and vice versa. In the case $\boldsymbol{\Sigma}_{II}$ (Fig. 6.1) shifts of $\delta\vartheta_1$

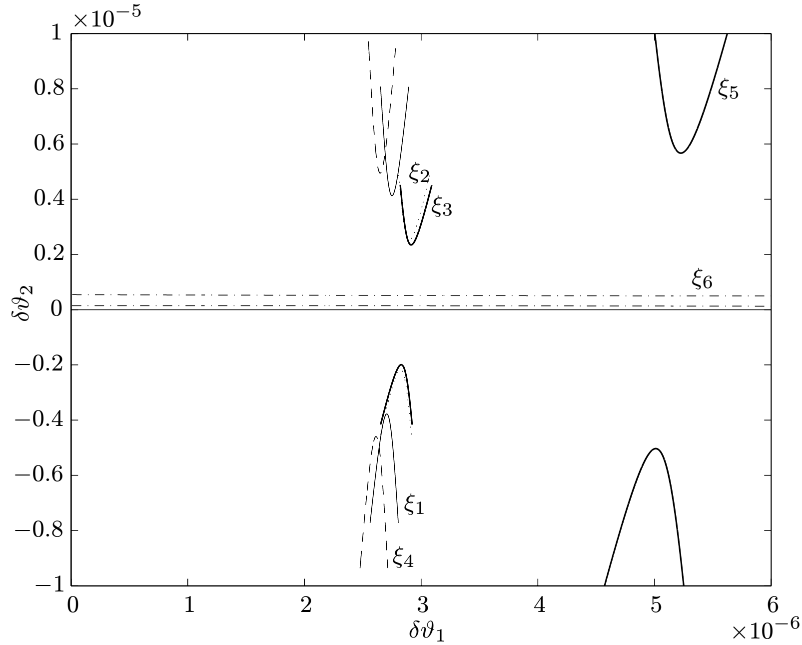

Figure 6.1. The boundary $\overline{\mathcal{H}}_{\varepsilon,\xi_{\mathrm{krit}}}$ for $\mathbf{\Sigma}_{II}$, $\kappa_t = 0.99$, $\alpha = 0.05$, $\varepsilon = 0.05$, $t = 4$.

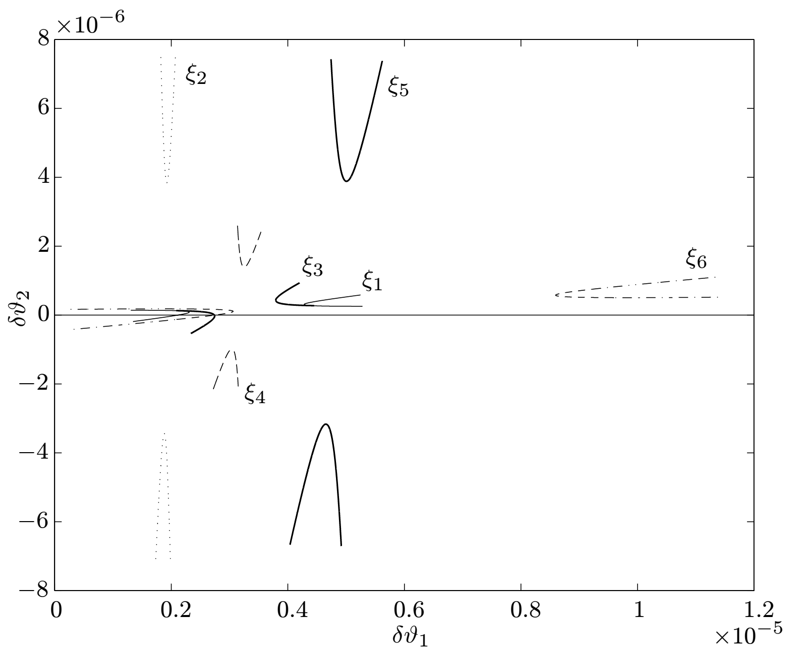

Figure 6.2. The boundary $\overline{\mathcal{H}}_{\varepsilon,\xi_{\mathrm{krit}}}$ for $\mathbf{\Sigma}_{I}$, $\kappa_t = 0.99$, $\alpha = 0.05$, $\varepsilon = 0.05$, $t = 4$.

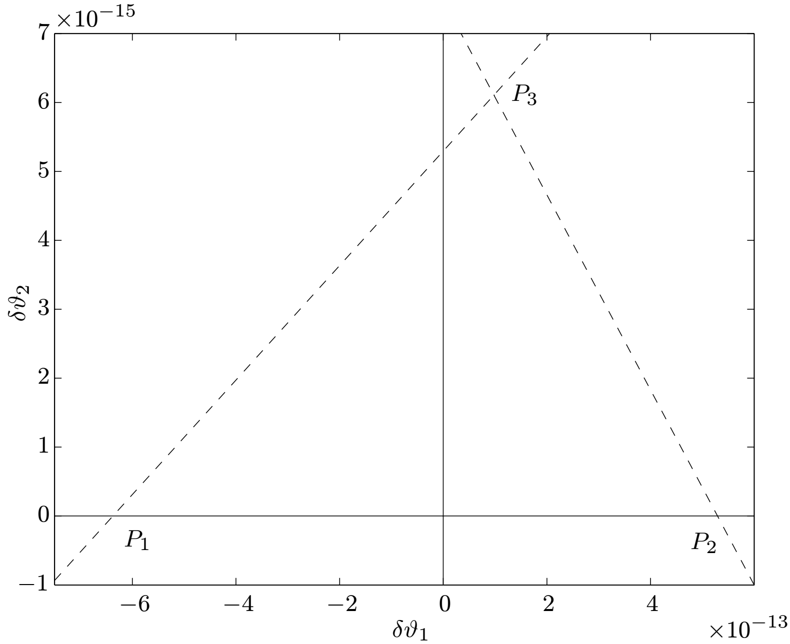

Figure 6.3. Asymptotes for $\overline{\mathcal{H}}_{\varepsilon,\delta_{\mathrm{krit}}}$ for $\mathbf{\Sigma}_I$, $\kappa_t = 0.99$, $\alpha = 0.05$, $\varepsilon = 0.05$, $t = 4$.

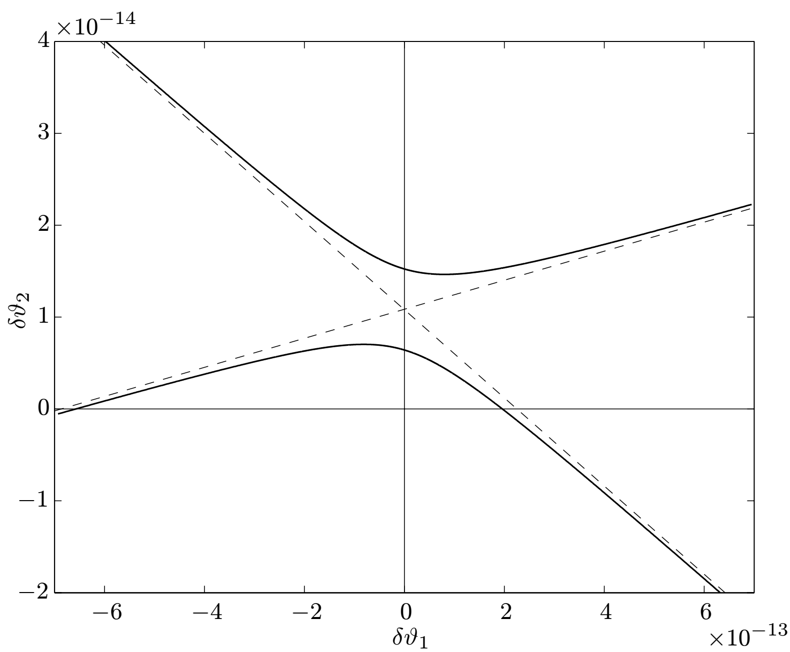

Figure 6.4. The boundary $\overline{\mathcal{H}}_{\varepsilon,\delta_{\mathrm{krit}}}$ for $\mathbf{\Sigma}_{II}$, $\kappa_t = 0.99$, $\alpha = 0.05$, $\varepsilon = 0.05$, $t = 4$.

can be arbitrarily large. Shifts $\delta\vartheta_2$ are admissible in the interval $(-10^{-6}, 1.4\cdot 10^{-7})$ if $\delta\vartheta_1 = 1.6\cdot 10^{-6}$. If $\delta\vartheta_1$ is greater or lower, the maximum tolerable shift $\delta\vartheta_2$ is lower.

The joint nonsensitiveness region for all directions $\boldsymbol{\xi}'\boldsymbol{C}_H\boldsymbol{\xi} = \delta_{\text{krit}}$, i.e. nonsensitiveness regions for the threshold ellipsoid, are given in Figs. 6.3 and 6.4. Figs. 6.3 and 6.4 correspond to the covariance matrices $\boldsymbol{\Sigma}_I$ and $\boldsymbol{\Sigma}_{II}$, respectively.

As it was said, the nonsensitiveness region is a set around the origin of coordinate system bounded by the branches of the hyperbola. Hence, in the case $\boldsymbol{\Sigma}_{II}$ (Fig. 6.4), $\delta\vartheta_1$ can be arbitrarily large. However, a shift in the direction of $\delta\vartheta_2$ must be very small (it is to be remembered that $\vartheta_1 > 0$, $\vartheta_2 > 0$).

In the case $\boldsymbol{\Sigma}_I$ (Fig. 6.3), from graphical purposes asymptotes of the hyperbola $\overline{\mathcal{H}}_{\varepsilon,\delta_{\text{krit}}}$ are given only. Points of intersection of axes and asymptotes are as follows: $P_1 = [-6.36\cdot 10^{-13}; 0]'$, $P_2 = [5.29\cdot 10^{-13}; 0]'$ and $P_3 = [0.99\cdot 10^{-13}; 6.11\cdot 10^{-15}]'$. For graphical reasons, a part of the nonsensitiveness region for $\delta\vartheta_2 \geqslant 0$ is shown only. Under this assumption, the nonsensitiveness region is approximately equal to the triangle given by points P_1, P_2, P_3. It is obvious how the remaining part of the nonsensitiveness region for $\delta\vartheta_2 < 0$ will look like. Hence, the movement in both directions $\delta\vartheta_i$, $i = 1, 2$ must be very small.

Till now no experience is available on the nonsensitiveness region for the threshold ellipsoid. In our example regions for a fixed $\boldsymbol{\xi}$ can be used in practice only, since the region $\mathcal{H}_{\varepsilon,\delta_{\text{krit}}}$ is very small.

An investigation of the region $\mathcal{H}_{\varepsilon,\delta_{\text{krit}}}$ is the aim of a further research.

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Author's address: *E. Lešanská*, Department of Mathematical Analysis and Applied Mathematics, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `lesanske@aix.upol.cz`.