Applications of Mathematics

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Applications of Mathematics, Vol. 48 (2003), No. 1, 3--30

Persistent URL: http://dml.cz/dmlcz/134514

COMPUTING THE DIFFERENTIAL OF AN UNFOLDED CONTACT DIFFEOMORPHISM*

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(Received April 28, 2000)

Abstract. Consider a bifurcation problem, namely, its bifurcation equation. There is a diffeomorphism Φ linking the actual solution set with an unfolded normal form of the bifurcation equation. The differential $D\Phi(0)$ of this diffeomorphism is a valuable information for a numerical analysis of the imperfect bifurcation.

The aim of this paper is to construct algorithms for a computation of $D\Phi(0)$. Singularity classes containing bifurcation points with codim $\leqslant 3$, corank $= 1$ are considered.

Keywords: bifurcation points, imperfect bifurcation diagrams, qualitative analysis

MSC 2000: 34A34, 35B32, 65L99, 37C05

1. Introduction

Consider

$$g\colon\ \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \to \mathbb{R}^1, \quad g = g(x,t,z) \tag{1.1}$$

to be a smooth mapping defined on a neighbourhood of the origin. Assume $g(0,0,0) = g_x(0,0,0) = 0$.

We have in mind the particular applications when g comes out from a Ljapunov-Schmidt reduction at a singular point with corank $= 1$, see e.g. [4], [6]; for the

* This work was supported by the Grant of the Deutsche Forschungsgemeinschaft Bo-622/13-3, Grants No. 201/02/0844 and No. 201/02/0595 of the Grant Agency of the Czech Republic and by the projects MSM 113200007, MSM 223400007 of the Czech Ministry of Education.

numerical version of the reduction, see e.g. [7] and for a similar reduction in Banach spaces, see e.g. [1]. In the context of the Theory for Imperfect Bifurcation, see [5], the variables (x, t, z) of g can be interpreted as a (reduced) state $x \in \mathbb{R}^1$, a control $t \in \mathbb{R}^1$ and an imperfection $z \in \mathbb{R}^k$.

Let $h\colon \mathbb{R}^2 \to \mathbb{R}^1$ be defined as the restriction $h(x,t) \equiv g(x,t,0)$. Hence, the solutions to $h(x,t) = 0$ represent the *perfect bifurcation scenario.* Following [6], this scenario can be classified by linking h with a suitable *normal form* $h^*\colon \mathbb{R}^2 \to \mathbb{R}^1$.

We recall, see [5], that h and h^* are *contact equivalent* if there exist a smooth $M\colon \mathbb{R}^2 \to \mathbb{R}^1$ and a local diffeomorphism $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\Psi(x,t) = (\chi(x,t), \tau(t))$, such that

$$\chi = 0, \quad M > 0, \quad \chi_x > 0 \quad \text{and} \quad \tau = 0, \quad \tau_t > 0 \quad \text{at} \quad 0 \in \mathbb{R}^2 \tag{1.2}$$

and

$$h = Mh^* \circ \Psi \tag{1.3}$$

in a neighbourhood of $0 \in \mathbb{R}^2$.

We will call Ψ a *contact diffeomorphism.* It links the solutions of $h(x,t) = 0$ with the roots of $h^*(\chi, \tau) = 0$.

We shall abbreviate (1.2), (1.3) writing $h \sim h^*$; the operation $\sim$ is a well defined equivalence on germs of smooth functions $\mathbb{R}^2 \to \mathbb{R}^1$, see [6], p. 104.

For the list of normal forms considered in this paper see Tab. 2.1. We assume that for a given g there exists a normal form h^* from Tab. 1.1 such that $h \sim h^*$ where $h(x,t) \equiv g(x,t,0)$. Moreover, we assume that k, which is the number of unfolding parameters, is equal to the appropriate codim, see Tab. 1.1; for the notion of *codimension*, see [6].

In Tab. 1.1, *universal unfoldings* of the relevant normal forms are also listed, see [6], p. 196.

In [3], we proved

Lemma 1.1. *Let $M\colon \mathbb{R}^2 \to \mathbb{R}^1$ and a local diffeomorphism $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfy (1.2) and (1.3) in a neighbourhood of $0 \in \mathbb{R}^2$. Let $g^*\colon \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}^1$ be a universal unfolding of h^*. Then there exist a smooth $S\colon \mathbb{R}^{2+k} \to \mathbb{R}^1$ and a smooth mapping*

$$\Phi\colon \mathbb{R}^{2+k} \to \mathbb{R}^{2+k}, \quad \Phi(x,t,z) = (X(x,t,z), T(t,z), Z(z)), \tag{1.4}$$

such that

$$\Phi(\cdot,\cdot,0) = \Psi(\cdot,\cdot), \quad S(\cdot,\cdot,0) = M(\cdot,\cdot), \tag{1.5}$$

Singularity	codim	Normal Form $h^*(x,t)$	Universal Unfolding $g^*(x,t,z)$
Pitchfork Family	$k=2$	px^3+qtx	$px^3+qtx+z_1+z_2x^2$
	$k=3$	px^4+qtx	$px^4+qtx+z_1+z_2x^2+z_3x^3$
	$k\geqslant 2$	$px^{k+1}+qtx$	$px^{k+1}+qtx+z_1+z_2x^2+\ldots+z_kx^k$
Hysteresis Family	$k=0$	px^2+qt	px^2+qt
	$k=1$	px^3+qt	$px^3+qt+zx$
	$k=2$	px^4+qt	$px^4+qt+z_1x+z_2x^2$
	$k=3$	px^5+qt	$px^5+qt+z_1x+z_2x^2+z_3x^3$
	$k\geqslant 1$	$px^{k+2}+qt$	$px^{k+2}+qt+z_1x+z_2x^2+\ldots+z_kx^k$
Asymmetric Cusp Family	$k=1$	px^2+qt^2	px^2+qt^2+z
	$k=2$	px^2+qt^3	$px^2+qt^3+z_1+z_2t$
	$k=3$	px^2+qt^4	$px^2+qt^4+z_1+z_2t+z_3t^2$
	$k\geqslant 1$	px^2+qt^{k+1}	$px^2+qt^{k+1}+z_1+z_2t+\ldots+z_kt^{k-1}$
Winged Cusp	$k=3$	px^3+qt^2	$px^3+qt^2+z_1+z_2x+z_3xt$

Table 1.1. The considered bifurcation singularities of corank $=1$; $|p|=|q|=1$.

hence they satisfy (1.2), *and*

$$g = Sg^* \circ \Phi \tag{1.6}$$

in a neighbourhood of $0 \in \mathbb{R}^{2+k}$.

Assume Φ to be a diffeomorphism in a neighbourhood of the origin, i.e., let $D\Phi(0) \in \mathcal{L}(\mathbb{R}^{2+k},\mathbb{R}^{2+k})$ be a regular matrix. The basic observation is that $g(x,t,z)=0$ if and only if $g^*(X,T,Z)=0$, where $\Phi(x,t,z)=(X(x,t,z),T(t,z),Z(z))$; the statement holds in the obvious local sense. The same applies to singular roots of g and g^* since Φ^{-1} provides a one-to-one link between stratified manifolds of singular points of g and those of g^*. As a rule, singular roots of g^* are easily computable; it is also important that a parametrization of these roots is known.

The diffeomorphism Φ is constructed given a contact diffeomorphism Ψ. We shall say that Φ is an *unfolded contact diffeomorphism.*

In [3], we proposed a postprocessig technique aiming at finding the first order predictors to *all* singular solutions of $g(x,t,z)=0$. The idea was to linearize the unfolded contact diffeomorphism Φ at the origin. For numerical experiments, see case studies [8] and [2].

The aim of this paper is a computation of $D\Phi(0)$ as the data for the above mentioned postprocessing analysis. We present algorithms for the construction of $D\Phi(0)$ provided that g is a singularity from Tab. 1.1. This covers e.g. all "elementary" bifurcation singularities with codim $\leqslant 3$, corank $= 1$ from the classical book [6]. The differential $D\Phi(0)$ depends on selected partial derivatives of g at the origin. We assume that all the partials required are known.

Observe that due to the canonical structure of Φ,

$$D\Phi(0) = \begin{pmatrix} X_x & X_t & X_z \\ 0 & T_t & T_z \\ 0 & 0 & Z_z \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{2+k}, \mathbb{R}^{2+k}), \quad Z_z \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k); \tag{1.7}$$

the required partial derivatives of $X(x,t,z)$, $T(t,z)$ and $Z(z)$ are evaluated at the origin.

Let us recall an equivalent characterization of bifurcation points by means of Defining and Nondegeneracy Conditions, [6], p. 198, Tab. 2.3: Let g be classified by a particular normal form h^* from Tab. 1.1, i.e., $h \sim h^*$ where $h(x,t) \equiv g(x,t,0)$. This is equivalent to the fact that particular partial derivatives of g being evaluated at the origin are equal to zero while the other particular partials of g are nonzero. In Tab. 1.2, these Defining and Nondegeneracy Conditions are listed for the relevant items of Tab. 1.1. This "algebraic classification" becomes later a substantial tool.

The outline of the paper is as follows: For each singularity class, we explain (and justify) computation of $D\Phi(0)$ for a general codimension k. The relevant general formulae are quite nasty. The particular evaluation for $k \leqslant 3$ simplifies $D\Phi(0)$ considerably: We usually give the result skipping the calculation.

2. Pitchfork family

We recall that the normal form of a *pitchfork bifurcation point* is the function

$$h^*(x,t) = px^{k+1} + qtx, \tag{2.1}$$

where codim $= k \geqslant 2$, the constants p, q are normalized so that $|p| = |q| = 1$.

A universal unfolding of (2.1) is of the form

$$g^*(x,t,z) = h^*(x,t) + z_1 + z_2x^2 + z_3x^3 + \ldots + z_{k-1}x^{k-1} + z_kx^k, \tag{2.2}$$

see Tab. 1.1.

Pitchfork Family: $h^*(x,t) = px^{k+1} + qtx$		
Codim	Defining Conditions	Nondegeneracy Conditions
$k=2$	$g = g_x = g_t = g_{xx} = 0$	$\operatorname{sgn} g_{xxx} = p, \quad \operatorname{sgn} g_{xt} = q$
$k=3$	$g = g_x = g_t = g_{xx} = g_{xxx} = 0$	$\operatorname{sgn} \frac{\partial^4}{\partial x^4} g = p, \quad \operatorname{sgn} g_{xt} = q$
$k \geqslant 2$	$g = g_x = g_t = g_{xx} = \ldots = \frac{\partial^k}{\partial x^k} g = 0$	$\operatorname{sgn} \frac{\partial^{k+1}}{\partial x^{k+1}} g = p, \quad \operatorname{sgn} g_{xt} = q$
Hysteresis Family: $h^*(x,t) = px^{k+2} + qt$		
Codim	Defining Conditions	Nondegeneracy Conditions
$k=0$	$g = g_x = 0$	$\operatorname{sgn} g_{xx} = p, \quad \operatorname{sgn} g_t = q$
$k=1$	$g = g_x = g_{xx} = 0$	$\operatorname{sgn} g_{xxx} = p, \quad \operatorname{sgn} g_t = q$
$k=2$	$g = g_x = g_{xx} = g_{xxx} = 0$	$\operatorname{sgn} \frac{\partial^4}{\partial x^4} g = p, \quad \operatorname{sgn} g_t = q$
$k=3$	$g = g_x = g_{xx} = g_{xxx} = \frac{\partial^4}{\partial x^4} g = 0$	$\operatorname{sgn} \frac{\partial^5}{\partial x^5} g = p, \quad \operatorname{sgn} g_t = q$
$k \geqslant 0$	$g = g_x = g_{xx} = \ldots = \frac{\partial^{k+1}}{\partial x^{k+1}} g = 0$	$\operatorname{sgn} \frac{\partial^{k+2}}{\partial x^{k+2}} g = p, \quad \operatorname{sgn} g_t = q$
Asymmetric Cusp Family: $h^*(x,t) = px^2 + qt^{k+1}$		
Codim	Defining Conditions	Nondegeneracy Conditions
$k=1$	$g = g_x = g_t = 0$	$\operatorname{sgn} g_{xx} = p, \quad \operatorname{sgn} D_2(g) = pq$
$k=2$	$g = g_x = g_t = D_2(g) = 0$	$\operatorname{sgn} g_{xx} = p, \quad \operatorname{sgn} D_3(g) = q$
$k=3$	$g = g_x = g_t = D_2(g) = D_3(g) = 0$	$\operatorname{sgn} g_{xx} = p, \quad \operatorname{sgn} D_4(g) = pq$
$k \geqslant 1$	$g = g_x = g_t = D_2(g) = \ldots = D_k(g) = 0$	$\operatorname{sgn} g_{xx} = p, \quad \operatorname{sgn} D_{k+1}(g) = p^k q$
where $D_1(g) = g_t, \ D_{j+1}(g) = g_{xx}(D_j(g))_t - g_{xt}(D_j(g))_x$ for $j \geqslant 1$		
Winged Cusp Singularity: $h^*(x,t) = px^3 + qt^2$		
Codim	Defining Conditions	Nondegeneracy Conditions
$k=3$	$g = g_x = g_t = g_{xx} = g_{xt} = 0$	$\operatorname{sgn} g_{xxx} = p, \quad \operatorname{sgn} g_{tt} = q$

Table 1.2. The relevant Defining and Nondegeneracy Conditions; $|p| = |q| = 1$.

The claim that g is contact equivalent to (2.1) can be formulated algebraically:

$$g = g_x = g_t = g_{xx} = \ldots = \frac{\partial^k}{\partial x^k} g = 0 \tag{2.3}$$

and

$$\operatorname{sgn} \frac{\partial^{k+1}}{\partial x^{k+1}} g = p, \quad \operatorname{sgn} g_{xt} = q \tag{2.4}$$

at $0 \in \mathbb{R}^{k+2}$, see Tab. 1.2.

Consider S, Φ satisfying (1.2)–(1.6).

R e m a r k 2 . 1 Let g^* be the particular unfolding (2.2). Taking an arbitrary $c > 0$, we define another pair $\widetilde{S}$, $\widetilde{\Phi}$ satisfying (1.2)–(1.6) as $\widetilde{S} = c^{-k-1}S$, $\widetilde{\Phi} = \mathrm{diag}(c, c^k, c^{k+1}, c^{k-1}, \dots, c^2, c) \cdot \Phi$. In particular, taking $c = (X_x(0))^{-1}$ yields that $\widetilde{X}_x(0) = 1$.

Consequently, we may scale X arbitrarily and consider $X_x(0) = 1$ without loss of generality.

Let us discuss the computation of $D\Phi(0) \in \mathcal{L}(\mathbb{R}^{k+2}, \mathbb{R}^{k+2})$ in this particular case. Partial derivatives of g at the origin are considered as data.

Let

$$\mathbf{B} \equiv \begin{pmatrix} g_x & g_t & g_z \\ g_{xx} & g_{xt} & g_{xz} \\ g_{xt} & g_{tt} & g_{tz} \\ g_{xxx} & g_{xxt} & g_{xxz} \\ \vdots & \vdots & \vdots \\ \frac{\partial^{k+1}}{\partial x^{k+1}} g & \frac{\partial^{k+1}}{\partial x^k \partial t} g & \frac{\partial^{k+1}}{\partial x^k \partial z} g \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{k+2}, \mathbb{R}^{k+2}),$$

where the derivatives are evaluated at the origin. We call $\mathbf{B}$ the gradient of Defining Conditions (2.3). Analogously, we define $\mathbf{B}^* \in \mathcal{L}(\mathbb{R}^{k+2}, \mathbb{R}^{k+2})$ as the gradient of Defining Conditions for a universal unfolding g^* of the normal form (2.1) at the origin:

$$\left(\begin{array}{ccc|ccccc} 0 & 0 & 1 & & & & & \\ 0 & q & 0 & & & \mathbf{0}_{3\times(k-1)} & & \\ q & 0 & 0 & & & & & \\ \hline 0 & 0 & 0 & 2! & & & & \\ \vdots & \vdots & \vdots & & 3! & & & \\ \vdots & \vdots & \vdots & & & \ddots & & \\ 0 & 0 & 0 & & & & (k-1)! & 0 \\ p(k+1)! & 0 & 0 & 0 & & & 0 & k! \end{array}\right).$$

It is possible to verify that $\mathbf{B}$ is related to $D\Phi(0)$ as follows:

$$\mathbf{B} = \mathbf{A}\mathbf{B}^* D\Phi(0), \tag{2.5}$$

where $\mathbf{A} \in \mathcal{L}(\mathbb{R}^{k+2}, \mathbb{R}^{k+2})$,

$$\mathbf{A} \equiv \begin{pmatrix} S & 0 & 0 & 0 & \dots & 0 \\ S_x & SX_x & 0 & 0 & \dots & 0 \\ S_t & SX_t & ST_t & 0 & \dots & 0 \\ S_{xx} & 2S_xX_x + SX_{xx} & 0 & SX_x^2 & \dots & 0 \\ S_{xxx} & 3S_{xx}X_x + 3S_xX_{xx} + SX_{xxx} & 0 & \frac{\partial^3}{\partial x^3}\left(\frac{1}{2}SX^2\right) & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k}{\partial x^k} S & \frac{\partial^k}{\partial x^k}(SX) & 0 & \frac{\partial^k}{\partial x^k}\left(\frac{1}{2}SX^2\right) & \dots & SX_x^k \end{pmatrix}$$

at $0 \in \mathbb{R}^{k+2}$.

Note that the lower triangular matrix $\mathbf{A}$ is regular (due to (1.5), (1.2)); it can be checked directly that also $\mathbf{B}^*$ is regular. Therefore, $D\Phi(0)$ is regular if and only if $\mathbf{B}$ is regular. In the sequel, we assume (apart from (2.3) and (2.4)) that

$$\det \mathbf{B} \neq 0. \tag{2.6}$$

R e m a r k 2.2. The assumption (2.6) is equivalent to the fact that g is a universal unfolding of h, see [6], Proposition 4.4 (for an example), and also Table 3.2 on p. 204.

In the identity (2.5), $\mathbf{B}$ is the data while the elements of $D\Phi(0)$, see (1.7), are to be computed ($k^2 + 2k + 2$ unknowns). Moreover, there are $2k + 1$ additional unknowns $S, S_x, S_t, S_{xx}, \dots, \frac{\partial^k}{\partial x^k} S$ and $X_{xx}, \dots, \frac{\partial^k}{\partial x^k} X$ in $\mathbf{A}$ inherited from the chain rule differentiation. Mind also that the entry g_{xt} appears in $\mathbf{B}$ twice. Therefore, the identity (2.5) represents k^2+4k+3 equations for $(k^2+2k+2)+(2k+1)$ unknowns, which sounds plausible. Unfortunatelly, these equations are *not* independent.

In order to observe this, we recall (2.3); similarly, $g^*_x, g^*_t, g^*_{xx}, \dots, \frac{\partial^k}{\partial x^k} g^*$ are zero at the origin. Consequently, the conditions related to $g_x, g_t, g_{xx}, \dots, \frac{\partial^k}{\partial x^k} g$ in $\mathbf{B}$ on the left-hand side of (2.5) are redundant. For example, the equation related to g_x reads $g_x = S_xg^* + Sg^*_xX_x$. Restricting all functions to the origin, it yields no information on S, S_x and X_x. Therefore, at least $k+1$ conditions are missing in order to be able to determine $D\Phi(0)$ directly from (2.5).

One may try to go on with the differentiation of (1.6) at the origin. Unfortunatelly, due to the chain rule, we have to compute unknowns we are not interested in (namely, higher derivatives of S and Φ) in order to obtain a condition on an element of $D\Phi(0)$. This produces a kind of snowball effect. The remedy was hinted at in [8] and [2]: We seek for an additional information concerning the contact diffeomorphism (1.3). In fact, we will construct Ψ. In what follows, we shall prove step by step Lemma 2.4 and formulate Theorem 2.1.

First, we recall a simple consequence of the Mean Value Theorem:

Lemma 2.1. *Let $f = f(x,s)$ and $d = d(x)$ be C^k-functions (i.e., k-times continuously differentiable) on $\mathcal{I} \times \mathcal{J}$ and $\mathcal{I}$, respectively, where $\mathcal{I}$ and $\mathcal{J}$ are open intervals in $\mathbb{R}^1$. Let $d\colon \mathcal{I} \to \mathcal{J}$, and let $f(x,d(x)) = 0$ for $x \in \mathcal{I}$. Then there exists a C^{k-1}-function $M = M(x,s)$ on $\mathcal{I} \times \mathcal{J}$ such that*

$$f(x,s) = M(x,s)(s - d(x)), \quad (x,s) \in \mathcal{I} \times \mathcal{J}. \tag{2.7}$$

Proof. Let $v = v(x,z) \equiv f(x, z + d(x))$. Hence v is a C^k-function on $\mathcal{M} \equiv \{(x,z)\colon (x, z + d(x)) \in \mathcal{I} \times \mathcal{J}\}$ and $v(x,0) = 0$. The Mean Value Theorem yields $v(x,z) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} v(x,tz)\,\mathrm{d}t$. Consequently, $v(x,z) = \widetilde{M}(x,z)z$, where $\widetilde{M}(x,z) = \int_0^1 f_s(x,s)\big|_{s:=\,tz+d(x)}\,\mathrm{d}t$. Then $f(x,s) = v(x,z)\big|_{z:=\,s-d(x)} = M(x,s)(s-d(x))$, where $M(x,s) = \widetilde{M}(x, s - d(x))$. □

Let $h(x,t) \equiv g(x,t,0)$, i.e. the germ g without imperfection. Note that the proof of the following lemma mimics the proof of Lemma 2.7 in [5].

Lemma 2.2. *Under the assumptions* (2.3) *and* (2.4), *the germ h factors as*

$$h(x,t) = M(x,t)(x - \psi(t))(x^k\varphi(x) - t), \tag{2.8}$$

where M, ψ and φ are smooth functions defined on a neighbourhood of the origin. Moreover,

$$\operatorname{sgn} M(0,0) = -q, \quad \operatorname{sgn}\varphi(0) = -pq, \quad \psi(0) = 0. \tag{2.9}$$

Proof. In the case of pitchfork bifurcation singularities with $\operatorname{codim} k$ we shall construct two transversal solution sets $t = x^k\varphi$ and $x = \psi(t)$ of $h(x,t) = 0$, and then factor h using Lemma 2.1. Let us elaborate:

It follows from (2.3) and (2.4) that

$$h(x,t) = x^{k+1}a(x,t) + xtb(x,t) + t^2c(x,t),$$

where $\operatorname{sgn} a(0,0) = p$ and $\operatorname{sgn} b(0,0) = q$. This motivates the following scaling: $h(x, x^{k-1}\mu) = x^k G(x,\mu)$, where $G(x,\mu) = xa(x, x^{k-1}\mu) + \mu b(x, x^{k-1}\mu) + x^{k-2}\mu^2 c(x, x^{k-1}\mu)$. Clearly, $G(0,0) = 0$, $\operatorname{sgn} G_\mu(0,0) = q$ and $\operatorname{sgn} G_x(0,0) = p$. Hence, locally, $G(x,\mu) = 0$ iff $\mu = x\varphi(x)$, where $\varphi = \varphi(x)$ is smooth and satisfies $\operatorname{sgn}\varphi(0) = -pq$. We conclude that $h(x, x^k\varphi(x)) \equiv 0$ in a neighbourhood of $x = 0$. By virtue of Lemma 2.1, there exists a smooth $E = E(x,t)$ such that $h(x,t) = E(x,t)(x^k\varphi(x) - t)$.

We shall factorize E. Note that $h(x,0) = x^{k+1}a(x,0) = E(x,0)x^k\varphi(x)$. Consequently, $E(0,0)=0$, $\operatorname{sgn} E_x(0,0) = -q$. Due to the Implicit Function Theorem, there exists a smooth function $\psi = \psi(t)$ such that $\psi(0) = 0$ and, locally, $E(\psi(t),t) \equiv 0$. Hence, in accordance with Lemma 2.1, there exists a smooth function $M = M(x,t)$ such that $E(x,t) = M(x,t)(x-\psi(t))$.

We conclude (2.8) immediately. It was already shown that $\operatorname{sgn}\varphi(0) = -pq$ and $\psi(0) = 0$. Direct computation yields $h_{xt}(0,0) = -M(0,0)$. Hence, $\operatorname{sgn} M(0,0) = -q$. □

Lemma 2.3. *Under the assumptions* (2.3) *and* (2.4), *the germ* h *factors as*

$$h(x,t) = M\cdot(p\chi^{k+1} + q\chi\tau), \tag{2.10}$$

where $M = M(x,t)$, $\chi = \chi(x,t)$, $\tau = \tau(t)$ *are smooth functions in a neighbourhood of the origin. Moreover,* $M(0,0) > 0$ *and*

$$\begin{aligned} \tau(t) &= ct, \quad c > 0, \\ \chi(x,t) &= (x-\psi(t))H(x-\psi(t)), \end{aligned} \tag{2.11}$$

where $\psi = \psi(t)$, $H = H(z)$ *are smooth functions satisfying* $\psi(0) = 0$, $H(0) = 1$.

Proof. Let ψ and φ be defined as in the statement of Lemma 2.2. Let us consider the equation $x^k\varphi(x) - t = 0$. Let us set $\xi = x - \psi(t)$ and substitute for x. Hence, $t = (\xi+\psi(t))^k\varphi(\xi+\psi(t))$. This defines implicitly a smooth function $t = t(\xi)$. It is easy to check that $t(0) = t'(0) = t''(0) = \ldots = t^{(k)}(0) = 0$, $t^{(k+1)}(0) = k!\,\varphi(0)$, i.e., $\operatorname{sgn} t^{(k+1)}(0) = -pq$. Hence, there exists a smooth function $\omega = \omega(\xi)$, $\operatorname{sgn}\omega(0) = -pq$, such that $x^k\varphi(x) - t = 0$ iff $t = \xi^k\omega(\xi)$.

By virtue of Lemma 2.1, $t - \xi^k\omega(\xi)$ factors the function $x^k\varphi(x) - t$, namely, there exists a smooth function $E = E(x,t)$ such that

$$x^k\varphi(x) - t = E(x,t)(\xi^k\omega(\xi) - t),$$

where $\xi = x - \psi(t)$. Hence, taking into account (2.8),

$$h(x,t) = \widetilde{M}(x,t)(x-\psi(t))((x-\psi(t))^k\omega(x-\psi(t)) - t).$$

Note that $h_{xt}(0,0) = -\widetilde{M}(0,0)$ and hence $\operatorname{sgn}\widetilde{M}(0,0) = -q$.

We set $\chi = (x-\psi(t))\big(\frac{\omega(x-\psi(t))}{\omega(0)}\big)^{1/k}$, $\tau = c\cdot t$, where $c = -\frac{pq}{\omega(0)} > 0$ and $M = -\frac{q}{c}\widetilde{M}(x,t)\big(\frac{\omega(0)}{\omega(x-\psi(t))}\big)^{1/k}$. Then it is easy to check that the statement of Lemma 2.3 holds with the above defined χ, τ and M. □

R e m a r k 2.3. The function $H = H(z)$ from the statement of Lemma 2.3 has the following structure:

$$H = H(z) = \Big(\frac{\omega(z)}{\omega(0)}\Big)^{\frac{1}{k}},$$

where $\omega = \omega(\xi)$ is defined in the proof of Lemma 2.3.

We consider (2.11), taking the above H into account. Then a direct computation yields $\tau_{tt} \equiv 0$, $\chi(0) = 0$, $\chi_x(0) = 1$, $\chi_t(0) = -\psi'(0)$, $\chi_{xx}(0) = 2H'(0)$ and $\chi_{xt} = -2H'(0)\psi'(0)$.

These formulae imply that $\chi_{xt} = \chi_t \cdot \chi_{xx}$ at the origin.

Lemma 2.4. *Assuming* (2.3) *and* (2.4), *there exist smooth* $S\colon\ \mathbb{R}^{k+2} \to \mathbb{R}^1$ *and a diffeomorphism* $\Phi\colon\ \mathbb{R}^{k+2} \to \mathbb{R}^{k+2}$, $\Phi(x,t,z) = (X(x,t,z), T(t,z), Z(z))$, *satisfying*

$$X = T = Z = 0, \quad X_x = 1, \quad T_t > 0, \quad S > 0, \tag{2.12}$$

$$T_{tt} = 0, \quad X_{xt} = X_{xx}X_t \tag{2.13}$$

at $0 \in \mathbb{R}^{k+2}$, *and the identity* (1.6) *in a neighbourhood of* $0 \in \mathbb{R}^{k+2}$.

P r o o f. Let us consider the functions M, χ and τ from Lemma 2.3. Let us define $\Psi\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ by setting $\Psi(x,t) = (\chi(x,t), \tau(t))$. Obviously, both (1.2) and (1.3) are satisfied. By virtue of Lemma 1.1, there exist smooth S and a diffeomorphism Φ satisfying (1.5) and (1.6). The condition (2.12) is obviously satisfied.

Both assertions in (2.13) follow from Remark 2.3. □

Theorem 2.1. *Let us consider* S *and* Φ *from Lemma* 2.4. *Then the differential* $D\Phi(0)$ *is uniquely defined. The required data are the following derivatives of* g *at the origin:*

$$\mathbf{B} \in \mathcal{L}(\mathbb{R}^{k+2}, \mathbb{R}^{k+2}),\ g_{xtt},\ \frac{\partial^{k+2}}{\partial x^{k+1}\partial t}\, g,\ \frac{\partial^{k+1+j}}{\partial x^{k+1+j}}\, g, \quad \text{where } j = 1, \dots, k. \tag{2.14}$$

P r o o f. In Tab. 2.1, there is a list of $2(2+3k)$ equations for the unknowns listed in the last column of that table. Note that the number of unknowns equals the number of equations. The partial derivatives on the left-hand side are given, see the data (2.14). Each row in Tab. 2.1 represents the relevant differential of (1.6) evaluated at the origin. Imposing (2.12) and (2.13) simplifies the relevant result considerably.

The nonlinear system is canonically solvable as marked in the second column. Nondegeneracy conditions (2.4) are taken into account. □

Data		$\Rightarrow$
$\frac{\partial^{k+1}}{\partial x^{k+1}}\, g$	$= (k+1)!\, pS$	S
g_{xt}	$= qST_t$	T_t
g_{tt}	$= 2qST_tX_t$	X_t
g_z	$= S(Z_1)_z$	$(Z_1)_z$
g_{xxt}	$= qST_tX_{xx} + 2qT_tS_x$	S_x, X_{xx}
$\frac{\partial^{k+2}}{\partial x^{k+2}}\, g$	$= p\frac{\partial^{k+2}}{\partial x^{k+2}}(SX^{k+1})$	
g_{xz}	$= qST_z + (Z_1)_zS_x$	T_z
g_{xtt}	$= 2qT_t(SX_tX_{xx} + S_xX_t + S_t) + 6pSX_t^2$	S_t
g_{tz}	$= qST_tX_z + qST_zX_t + (Z_1)_zS_t$	X_z
	for $i = 2, \ldots, k$ do	$\frac{\partial^i}{\partial x^i}\, S,\ \frac{\partial^{i+1}}{\partial x^{i+1}}\, X$
$\frac{\partial^{i+2}}{\partial x^{i+1}\partial t}\, g$	$= qT_t\frac{\partial^{i+1}}{\partial x^{i+1}}(SX)$	
$\frac{\partial^{k+1+i}}{\partial x^{k+1+i}}\, g$	$= p\frac{\partial^{k+1+i}}{\partial x^{k+1+i}}(SX^{k+1})$	
$\frac{\partial^{i+1}}{\partial x^i\partial z}\, g$	$= qT_z\frac{\partial^i}{\partial x^i}\, S \quad + \quad \sum_{j=1}^{i} \frac{\partial^i}{\partial x^i}\, (SX^j)(Z_j)_z$	$(Z_i)_z$

Table 2.1. Pitchfork bifurcation analysis, codim $= k$; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^k$.

Finally, consider the formulation of Theorem 2.1 for the pitchforks with codim $= 2$ and codim $= 3$, see Tab. 1.2. The relevant versions of Tab. 2.1 are given in Tab. 2.2 and Tab. 2.3.

3. Hysteresis family

The normal form of a *hysteresis bifurcation point* is

(3.1) $$h^*(x,t) = px^{k+2} + qt, \quad k = \text{codim}, \quad |p| = |q| = 1.$$

The relevant universal unfolding of (3.1) is

(3.2) $$g^*(x,t,z) = h^*(x,t) + z_1x + z_2x^2 + \ldots + z_kx^k,$$

see Tab. 1.1.

Data	$\Rightarrow$
$g_{xxx} = 6pS$	S
$g_{xt} = qST_t$	T_t
$g_z = S(Z_1)_z$	$(Z_1)_z$
$g_{tt} = 2qST_tX_t$	X_t
$g_{xxt} = qSX_{xx}T_t + 2qT_tS_x + 6pSX_t$ $g_{xxxx} = 36pSX_{xx} + 24pS_x$	$S_x,\ X_{xx}$
$g_{xz} = qST_z + S_x(Z_1)_z$	T_z
$g_{xtt} = 2qT_t(SX_tX_{xx} + X_tS_x + S_t) + 6pSX_t^2$	S_t
$g_{tz} = qSX_zT_t + qST_zX_t + (Z_1)_zS_t$	X_z
$g_{xxxt} = qT_t(SX_{xxx} + 3S_xX_{xx} + 3S_{xx})$ $+18pX_t(2SX_{xx} + S_x) + 6pS_t$ $\frac{\partial^5}{\partial x^5} g = 30p(2SX_{xxx} + 3SX_{xx}^2 + 6S_xX_{xx} + 2S_{xx})$	$S_{xx},\ X_{xxx}$
$g_{xxz} = qSX_{xx}T_z + 2qS_xT_z + 6pSX_z + (Z_1)_zS_{xx} + 2S(Z_2)_z$	$(Z_2)_z$

Table 2.2. Pitchfork bifurcation analysis, codim=2; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^2$.

A germ g, see (1.1), is contact equivalent with (3.1) provided that

$$g = g_x = g_{xx} = \ldots = \frac{\partial^{k+1}}{\partial x^{k+1}}\, g = 0 \tag{3.3}$$

and

$$\operatorname{sgn} \frac{\partial^{k+2}}{\partial x^{k+2}}\, g = p, \quad \operatorname{sgn} g_t = q \tag{3.4}$$

at $0 \in \mathbb{R}^{k+2}$, see Table 1.2.

Let S, Φ satisfy (1.2)–(1.6). We can assume without loss of generality that $X_x(0) = 1$:

R e m a r k 3.1. Let $c > 0$. Define

$$\widetilde{S} = c^{-k-2}S, \quad \widetilde{\Phi} = \operatorname{diag}(c, c^{k+2}, c^{k+1}, \ldots, c^2) \cdot \Phi.$$

It can be checked that $\widetilde{S}$, $\widetilde{\Phi}$ satisfy (1.2)–(1.6). Taking $c = (X_x(0))^{-1}$, this yields that $\widetilde{X}_x(0) = 1$.

Data		$\Rightarrow$
$\frac{\partial^4}{\partial x^4}\, g$	$= 4!pS$	S
g_{xt}	$= qST_t$	T_t
g_z	$= S(Z_1)_z$	$(Z_1)_z$
g_{tt}	$= 2qST_tX_t$	X_t
g_{xxt}	$= qST_tX_{xx} + 2qT_tS_x$	$S_x,\ X_{xx}$
$\frac{\partial^5}{\partial x^5}\, g$	$= 5!pS_x + 2\cdot 5!pSX_{xx}$	
g_{xz}	$= qST_z + S_x(Z_1)_z$	T_z
g_{xtt}	$= 2qT_t(SX_tX_{xx} + S_xX_t + S_t) + 6pSX_t^2$	S_t
g_{tz}	$= qST_tX_z + qSX_tT_z + S_t(Z_1)_z$	X_z
g_{xxxt}	$= qT_t(SX_{xxx} + 3S_xX_{xx} + 3S_{xx}) + 24pSX_t$	$S_{xx},\ X_{xxx}$
$\frac{\partial^6}{\partial x^6}\, g$	$= 5!p(3S_{xx} + 12S_xX_{xx} + 9SX_{xx}^2 + 4SX_{xxx})$	
g_{xxz}	$= qST_zX_{xx} + 2qS_xT_z + (Z_1)_zS_{xx} + 2S(Z_2)_z$	$(Z_2)_z$
$\frac{\partial^5}{\partial x^4\partial t}\, g$	$= qT_t(4S_{xxx} + 6S_{xx}X_{xx} + 4S_xX_{xxx} + SX_{xxxx})$ $+4!pS_t + 2\cdot 4!pX_t(2S_x + 5SX_{xx})$	$S_{xxx},\ \frac{\partial^4}{\partial x^4}\, X$
$\frac{\partial^7}{\partial x^7}\, g$	$= 7\cdot 5!p(S_{xxx} + 6S_{xx}X_{xx} + 9S_xX_{xx}^2 + 4S_xX_{xxx}$ $+3SX_{xx}^3 + 6SX_{xx}X_{xxx} + SX_{xxxx})$	
g_{xxxz}	$= qT_z(3S_{xx} + 3S_xX_{xx} + SX_{xxx}) + 4!pSX_z+$ $+S_{xxx}(Z_1)_z + 6(Z_2)_z(S_x + SX_{xx}) + 6S(Z_3)_z$	$(Z_3)_z$

Table 2.3. Pitchfork bifurcation analysis, codim = 3; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^3$.

Applying the chain rule to (1.6), we immediately conclude (2.5) where

$$\mathbf{B} \equiv \begin{pmatrix} g_x & g_t & g_z \\ g_{xx} & g_{xt} & g_{xz} \\ g_{xxx} & g_{xxt} & g_{xxz} \\ \vdots & \vdots & \vdots \\ \frac{\partial^{k+1}}{\partial x^{k+1}}\, g & \frac{\partial^{k+1}}{\partial x^k \partial t}\, g & \frac{\partial^{k+1}}{\partial x^k \partial z}\, g \\ \frac{\partial^{k+2}}{\partial x^{k+2}}\, g & \frac{\partial^{k+2}}{\partial x^{k+1} \partial t}\, g & \frac{\partial^{k+2}}{\partial x^{k+1} \partial z}\, g \end{pmatrix}; \quad \mathbf{B}^* \equiv \begin{pmatrix} 0 & q & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1! & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 2! & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 3! & & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 & k! \\ p(k+2)! & 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

and

$$\mathbf{A} \equiv \begin{pmatrix} S & 0 & 0 & 0 & \dots & 0 \\ S_x & SX_x & 0 & 0 & \dots & 0 \\ S_{xx} & (SX)_{xx} & SX_x^2 & 0 & \dots & 0 \\ S_{xxx} & (SX)_{xxx} & \frac{1}{2}(SX^2)_{xx} & SX_x^3 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{k+1}}{\partial x^{k+1}} S & \frac{\partial^{k+1}}{\partial x^{k+1}}(SX) & \frac{1}{2!}\frac{\partial^{k+1}}{\partial x^{k+1}}(SX^2) & \frac{1}{3!}\frac{\partial^{k+1}}{\partial x^{k+1}}(SX^3) & \dots & SX_x^{k+1} \end{pmatrix}$$

at $0 \in \mathbb{R}^{k+2}$, $\mathbf{B}$, $\mathbf{B}^*$ and $\mathbf{A} \in \mathcal{L}(\mathbb{R}^{k+2}, \mathbb{R}^{k+2})$.

We assume (2.6) for the particular $\mathbf{B}$; for the consequences, see Remark 2.2.

Similarly to Section 2, the matrix identity (2.5) is interpreted as a system of $k^2 + 3k + 3$ equations (mind $k+1$ redundancies) for $k^2 + 2k + 2$ nonzero elements of $D\Phi(0)$ and $2k+2$ additional unknowns due to the chain rule (see $\mathbf{A}$). Therefore, there is lack of at least $k+1$ conditions.

We will try to find additional information concerning the contact diffeomorphism (1.3).

Lemma 3.1. *Let $h(x,t) \equiv g(x,t,0)$. Under the assumptions* (3.3) *and* (3.4), *the germ h factors as*

$$h(x,t) = M \cdot (p\chi^{k+2} + q\tau), \tag{3.5}$$

where $M = M(x,t)$, $\chi = \chi(x,t)$, $\tau = \tau(t)$ are smooth functions in a neighbourhood of the origin. Moreover, $M(0,0) > 0$, and

$$\tau(t) = c^{k+2} t,$$
$$\chi(x,t) = \chi(x) = cx\omega(x),$$

where $\omega = \omega(x)$ is smooth function satisfying $\omega(0) > 0$ and $c = \frac{1}{\omega(0)}$.

P r o o f. It follows from (3.3), (3.4) that

$$h(x,t) = x^{k+1} a(x,t) + tb(x,t),$$

where $\operatorname{sgn} a(0,0) = p$ and $\operatorname{sgn} b(0,0) = q$. This hints at the following scaling: $h(x, x^{k+1}\mu) = x^{k+1} G(x,\mu)$, where $G(x,\mu) = xa(x, x^{k+1}\mu) + \mu b(x, x^{k+1}\mu)$. Clearly, $G(0,0) = 0$, $\operatorname{sgn} G_\mu(0,0) = q$ and $\operatorname{sgn} G_x(0,0) = p$. Hence, locally, by virtue of the Implicit Function Theorem, $G(x,\mu) = 0$ iff $\mu = x\varphi(x)$, where $\varphi = \varphi(x)$ is smooth and satisfies $\operatorname{sgn}\varphi(0) = -pq$. We conclude that $h(x, x^{k+2}\varphi(x)) \equiv 0$ in a neighbourhood of $x = 0$. By virtue of Lemma 2.1, there exists a smooth $E = E(x,t)$ such that $h(x,t) = E(x,t)(x^{k+2}\varphi(x) - t)$.

Note that $h(x,0) = x^{k+2}a(x,0) = E(x,0)x^{k+2}\varphi(x)$. Consequently, $\operatorname{sgn} E(0,0) = -q$.

We denote $\omega(x) = (-pq\varphi(x))^{\frac{1}{k+2}}$, $c = \omega^{-1}(0)$ and set $\chi = cx\omega(x)$, $\tau = c^{k+2}t$ and $M(x,t) = -qc^{-k-2}E(x,t)$. Then it is easy to check that the statement of Lemma 3.1 holds with the above defined χ, ω, τ and M. $\square$

Remark 3.2. Let us note that $\chi_x = 1$, $\chi_t = 0$, $\tau_t = c^{k+2} \geqslant 0$ and $\tau_{tt} = 0$ at the origin.

Lemma 3.2. *Assuming* (3.3) *and* (3.4), *there exist smooth* $S\colon \mathbb{R}^{k+2} \to \mathbb{R}^1$ *and a diffeomorphism* $\Phi\colon \mathbb{R}^{k+2} \to \mathbb{R}^{k+2}$, $\Phi(x,t,z) = (X(x,t,z), T(t,z), Z(z))$, $z \in \mathbb{R}^k$, *(*$k = \operatorname{codim}$*) satisfying*

$$X = T = Z = 0, \quad X_x = 1, \quad T_t > 0, \quad S > 0, \tag{3.6}$$

$$X_t = 0, \quad T_{tt} = 0 \tag{3.7}$$

at $0 \in \mathbb{R}^{k+2}$, *and the identity* (1.6) *in a neighbourhood of* $0 \in \mathbb{R}^{k+2}$.

Proof. Let us consider the functions M, χ, τ and ω from Lemma 3.1. Let us define $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by setting $\Psi(x,t) = (\chi(x,t), \tau(t))$. Obviously, both (1.2) and (1.3) are satisfied. By virtue of Lemma 1.1, there exist smooth S and a diffeomorphism Φ satisfying (1.5) and (1.6). The conditions (3.6) and (3.7) are also clearly satisfied. $\square$

Theorem 3.1. *Let us consider* S *and* Φ *from Lemma* 3.2. *Then the differential* $D\Phi(0)$ *is uniquely defined. The required data are the following derivatives of* g *at the origin:*

$$\mathbf{B} \in \mathcal{L}(\mathbb{R}^{k+2}, \mathbb{R}^{k+2}), \quad \frac{\partial^{k+2+j}}{\partial x^{k+2+j}}\, g, \quad j = 1, 2, \ldots, k = \operatorname{codim}.$$

Proof. For proof, see Tab. 3.1. This table should be interpreted exactly in the same way as Tab. 2.1 in the proof of Theorem 2.1. $\square$

Finally, consider the formulation of Theorem 3.1 for the hysteresis points with $\operatorname{codim} = 0, \ldots, 4$, see Tab. 1.2. The relevant versions of Tab. 3.1 are given in Tables 3.2–3.5.

Data		$\Rightarrow$
$\frac{\partial^{k+2}}{\partial x^{k+2}}\, g$	$= (k+2)!\, pS$	S
g_t	$= qST_t$	T_t
$\frac{\partial^{j}}{\partial x^{j-1}\partial t}\, g$	$= qT_t \frac{\partial^{j-1}}{\partial x^{j-1}}\, S$ for $j = 2, \ldots, k+1$	$\frac{\partial^{j-1}}{\partial x^{j-1}}\, S$ $j = 2, \ldots, k+1$
g_z	$= qST_z$	T_z
g_{xz}	$= qS_xT_z + S(Z_1)_z$	$(Z_1)_z$
$\frac{\partial^{k+2+j}}{\partial x^{k+2+j}}\, g$	$= p \frac{\partial^{k+2+j}}{\partial x^{k+2+j}}(SX^{k+2})$ for $j = 1, \ldots, k$	$\frac{\partial^{k+2+j}}{\partial x^{k+2+j}}\, X$ $j = 1, \ldots, k$
$\frac{\partial^{k+2}}{\partial x^{k+1}\partial t}\, g$	$= qT_t \frac{\partial^{k+1}}{\partial x^{k+1}}\, S$	$\frac{\partial^{k+1}}{\partial x^{k+1}}\, S$
$\frac{\partial^{j}}{\partial x^{j-1}\partial z}\, g$	$= qT_z \frac{\partial^{j-1}}{\partial x^{j-1}}\, S + \sum\limits_{i=1}^{j-1} \frac{\partial^{j-1}}{\partial x^{j-1}}(SX^i)(Z_i)_z$ for $k \geqslant 2$ and $j = 3, \ldots, k+1$	$(Z_{j-1})_z$ $j = 3, \ldots, k+1$
$\frac{\partial^{k+2}}{\partial x^{k+1}\partial z}\, g$	$= qT_z \frac{\partial^{k+1}}{\partial x^{k+1}}\, S + \sum\limits_{i=1}^{k} \frac{\partial^{k+1}}{\partial x^{k+1}}(SX^i)(Z_i)_z$ $+ pq \frac{\partial^{k+1}}{\partial x^{k+1}}(SX^{k+1})X_z$ for $k \geqslant 1$	X_z

Table 3.1. Hysteresis bifurcation analysis, codim $= k$; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^k$.

Data		$\Rightarrow$
g_{xx}	$= 2pS$	S
g_t	$= qST_t$	T_t

Table 3.2. Limit point bifurcation analysis, codim $= 0$; $|p| = |q| = 1$.

4. Asymmetric cusp family

The normal form of an *asymmetric cusp* with codimension k is

$$h^*(x,t) = px^2 + qt^{k+1}, \quad |p| = |q| = 1. \tag{4.1}$$

As a universal unfolding of (4.1) we use

$$g^*(x,t,z) = h^*(x,t) + z_1 + z_2t + z_3t^2 + \ldots + z_{k-1}t^{k-2} + z_kt^{k-1}, \tag{4.2}$$

see Table 1.1.

Data		$\Rightarrow$
g_{xxx}	$= 3!pS$	S
g_t	$= qST_t$	T_t
g_z	$= qST_z$	T_z
g_{xt}	$= qS_xT_t$	S_x
g_{xz}	$= qS_xT_z + SZ_z$	Z_z
g_{xxxx}	$= 4!pS_x + 6 \cdot 3!pSX_{xx}$	X_{xx}
g_{xxt}	$= qS_{xx}T_t$	S_{xx}
g_{xxz}	$= qS_{xx}T_z + 2S_xZ_z + 3!pSX_z + SZ_zX_{xx}$	X_z

Table 3.3. Hysteresis bifurcation analysis, codim = 1; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^1$.

Data		$\Rightarrow$
g_{xxxx}	$= 4!pS$	S
g_t	$= qST_t$	T_t
g_z	$= qST_z$	T_z
g_{xt}	$= qS_xT_t$	S_x
g_{xz}	$= qS_xT_z + S(Z_1)_z$	$(Z_1)_z$
g_{xxt}	$= qS_{xx}T_t$	S_{xx}
$\frac{\partial^5}{\partial x^5}\, g$	$= 5!pS_x + 2 \cdot 5!pSX_{xx}$	X_{xx}
g_{xxz}	$= qS_{xx}T_z + 2S_x(Z_1)_z + S(Z_1)_zX_{xx} + 2S(Z_2)_z$	$(Z_2)_z$
$\frac{\partial^6}{\partial x^6}\, g$	$= 9 \cdot 5!pSX_{xx}^2 + 4 \cdot 5!pSX_{xxx} + 3 \cdot 5!pS_{xx} + 2 \cdot 6!pS_xX_{xx}$	X_{xxx}
g_{xxxt}	$= qS_{xxx}T_t$	S_{xxx}
g_{xxxz}	$= qS_{xxx}T_z + 3S_{xx}(Z_1)_z + 3S_xX_{xx}(Z_1)_z + 3!S_x(Z_2)_z$ $+4!pSX_z + 3(Z_1)_zX_{xxx} + 3!SX_{xx}(Z_2)_z$	X_z

Table 3.4. Quartic fold bifurcation analysis, codim = 2; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^2$.

Data	$\Rightarrow$
$\frac{\partial^5}{\partial x^5} g \quad = 5!pS$	S
$g_t \quad = qST_t$	T_t
$g_z \quad = qST_z$	T_z
$g_{xt} \quad = qS_xT_t$	S_x
$g_{xz} \quad = qS_xT_z + S(Z_1)_z$	$(Z_1)_z$
$g_{xxt} \quad = qS_{xx}T_t$	S_{xx}
$g_{xxxt} \quad = qS_{xxx}T_t$	S_{xxx}
$\frac{\partial^5}{\partial x^4 \partial t} g \quad = qS_{xxxx}T_t$	S_{xxxx}
$\frac{\partial^6}{\partial x^6} g \quad = 6!pS_x + 15 \cdot 5!pSX_{xx}$	X_{xx}
$g_{xxz} \quad = qS_{xx}T_z + 2S_x(Z_1)_z + S(Z_1)_zX_{xx} + 2S(Z_2)_z$	$(Z_2)_z$
$\frac{\partial^7}{\partial x^7} g \quad = 21 \cdot 5!pS_{xx} + 105 \cdot 5!pS_xX_{xx} + 105 \cdot 5!pSX_{xx}^2 + 35 \cdot 5!pSX_{xxx}$	X_{xxx}
$g_{xxxz} \quad = qS_{xxx}T_z + 3S_{xx}(Z_1)_z + 3S_x(Z_1)_zX_{xx} + 6S_x(Z_2)_z$ $+S(Z_1)_zX_{xxx} + 6S(Z_2)_zX_{xx} + 6S(Z_3)_z$	$(Z_3)_z$
$\frac{\partial^8}{\partial x^8} g \quad = 56 \cdot 5!pS_{xxx} + 10 \cdot 7!pS_{xx}X_{xx} + 20 \cdot 7!pS_xX_{xx}^2 + 280 \cdot 5!pS_xX_{xxx}$ $+10 \cdot 7!pSX_{xx}^3 + 560 \cdot 5!pSX_{xx}X_{xxx} + 70 \cdot 5!pSX_{xxxx}$	X_{xxxx}
$\frac{\partial^5}{\partial x^4 \partial z} g \quad = qS_{xxxx}T_z + 4S_{xxx}(Z_1)_z + 6S_{xx}(Z_1)_zX_{xx} + 12S_{xx}(Z_2)_z$ $+4S_xX_{xxx}(Z_1)_z + 24S_xX_{xx}(Z_2)_z + 4!pS_x(Z_3)_z + 5!pSX_z$ $+S(Z_1)_zX_{xxxx} + 6S(Z_2)_zX_{xx}^2 + 8S(Z_2)_zX_{xxx} + 36(Z_3)_zX_{xx}$	X_z

Table 3.5. Hysteresis bifurcation analysis, codim = 3; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^3$.

The germ g is contact equivalent to (4.1) if and only if

$$g = g_x = g_t = D_2(g) = \ldots = D_k(g) = 0 \tag{4.3}$$

and

$$\operatorname{sgn} g_{xx} = p, \quad \operatorname{sgn} D_{k+1}(g) = p^k q \tag{4.4}$$

at $0 \in \mathbb{R}^{k+2}$, where

$$D_1(g) = g_t \quad \text{and} \quad D_{j+1}(g) = g_{xx}(D_j(g))_t - g_{xt}(D_j(g))_x \quad \text{for } j \geqslant 1,$$

see Tab. 1.2.

R e m a r k 4.1. Let $c > 0$. Then define

$$\widetilde{S} = c^{-k-1} S, \quad \widetilde{\Phi} = \operatorname{diag}(c^{\frac{k+1}{2}}, c, c^{k+1}, c^k, c^{k-1}, \dots, c^2) \cdot \Phi.$$

It can be checked that $\widetilde{S}$, $\widetilde{\Phi}$ satisfy (1.6). Taking $c = (X_x(0))^{-1}$, this yields that $\widetilde{X}_x(0) = 1$.

Applying the chain rule to (1.6), we immediately conclude (2.5) where

$$\mathbf{B} \equiv \begin{pmatrix} g_x & g_t & g_z \\ g_{xx} & g_{xt} & g_{xz} \\ g_{xt} & g_{tt} & g_{tz} \\ (D_2(g))_x & (D_2(g))_t & (D_2(g))_z \\ (D_3(g))_x & (D_3(g))_t & (D_3(g))_z \\ \vdots & \vdots & \vdots \\ (D_{k-1}(g))_x & (D_{k-1}(g))_t & (D_{k-1}(g))_z \\ (D_k(g))_x & (D_k(g))_t & (D_k(g))_z \end{pmatrix};$$

$$\mathbf{B}^* \equiv \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 2p & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 2p \cdot 2! & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 2^2 \cdot 3! & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & 2^{k-2} p^k (k-1)! \\ 0 & 2^{k-1}(k+1)!\, p^{k-1} q & 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix};$$

$$\mathbf{A} \equiv \begin{pmatrix} S & 0 & 0 & 0 & \dots & 0 \\ S_x & S & 0 & 0 & \dots & 0 \\ S_t & SX_t & ST_t & 0 & \dots & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & S^2 T_t^2 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k+2,1} & a_{k+2,2} & \dots & \dots & a_{k+2,k+1} & S^k T_t^k \end{pmatrix},$$

$\mathbf{B}$, $\mathbf{B}^*$ and $\mathbf{A} \in \mathcal{L}(\mathbb{R}^{k+2}, \mathbb{R}^{k+2})$. The formulae for the lower triangular elements of the matrix $\mathbf{A}$ are very complicated and we will skip them. Nevertheless, an explicit evaluation of these elements is inevitable to derive Tabs. 4.1–4.6.

We assume (2.6) for the particular $\mathbf{B}$; for the consequences, see Remark 2.2.

The matrix identity (2.5) can be interpreted as a system of $k^2 + 4k + 1$ equations (mind 3 obvious redundancies) for $k^2 + 2k + 2$ nonzero elements of $D\Phi(0)$ and $k^2 + 4k - 2$ additional unknowns due to the chain rule (see $\mathbf{A}$). Therefore, there is lack of at least $k^2 + 2k - 1$ conditions.

We will proceed as in the previous sections trying to find additional information concerning the contact diffeomorphism (1.3).

Lemma 4.1. *Let* $h(x,t) \equiv g(x,t,0)$. *Under the assumptions* (4.3) *and* (4.4), *the germ* h *factors as*

$$h(x,t) = M \cdot (p\chi^2 + q\tau^{k+1}), \tag{4.5}$$

where $M = M(x,t)$, $\chi = \chi(x,t)$, $\tau = \tau(t)$ *are smooth functions in a neighbourhood of the origin. Moreover,* $M(0,0) > 0$ *and*

$$\tau(t) = (pqH(t))^{1/(k+1)}, \tag{4.6}$$

$$\chi(x,t) = x + a(t), \tag{4.7}$$

where $a = a(t)$ *and* $H = H(t)$ *are smooth function satisfying* $a(0) = 0$ *and* $H(0) = H'(0) = 0$, $\operatorname{sgn}(H''(0)) = pq$.

P r o o f. Note that $h = \frac{1}{2}g_{xx}(0,0)(1+\mathcal{O}(x))$. Due to the Malgrange Preparation Theorem, see [4], there exist smooth functions $M = M(x,t)$, $b = b(t)$, $c = c(t)$ such that $b(0) = c(0) = 0$ and

$$h(x,t) = M(x,t)(px^2 + b(t)x + c(t)) \tag{4.8}$$

in a neighbourhood of $0 \in \mathbb{R}^2$. Therefore

$$h(x,t) = pM(x,t)\left(\left(x + \frac{b(t)}{2p}\right)^2 + \frac{4pc(t) - b^2(t)}{4}\right). \tag{4.9}$$

This suggests to define (4.7), where $a(t) = \frac{b(t)}{2p}$. The function M is positive in a neighbourhood of the origin since $h_{xx} = 2pM(0,0)$ and (4.4) holds.

Let us evaluate $D_j(h)$ for $j = 1, \ldots, k$ at the origin: $D_1(h) = Mc_t$, $D_2(h) = M^2(2pc_{tt} - b_t^2)$, $D_3(h) = 2pM^3(2pc_{ttt} - 3b_tb_{tt}), \ldots$, etc. Therefore, due to (4.3), we have $c_t = 0$, $2pc_{tt} - b_t^2 = 0$, $2pc_{ttt} - 3b_tb_{tt} = 0$, …, etc. Going back to (4.9), we easily conclude that

$$H(t) \equiv \frac{1}{4}(4pc(t) - b^2(t)) = Kt^{k+1}(1 + \mathcal{O}(t)) \quad \text{as} \quad t \to 0, \tag{4.10}$$

where

$$K = \frac{1}{(k+1)!}\frac{D_{k+1}(h)(0,0)}{h_{xx}^{k+1}(0,0)};$$

the formal proof can be done by induction. Due to (4.4), $\operatorname{sgn} K = pq$. Finally, let us define a real function $\tau = \tau(t)$ setting

$$\tau = (pqH(t))^{1/k+1}. \tag{4.11}$$

It can be easily verified that τ is smooth in a neighbourhood of $0 \in \mathbb{R}^1$, and $\tau(0) = 0$, $\tau_t(0) = (pqK)^{1/(k+1)} > 0$ and

$$h(x,t) = M(x,t)(p\chi^2 + q\tau^{k+1}) = Mh^* \circ \Psi. \tag{4.12}$$

In (4.6), $c = (pqK)^{1/k+1}$. □

Lemma 4.2. *Assuming* (4.3) *and* (4.4), *there exist smooth* $S\colon \mathbb{R}^{k+2} \to \mathbb{R}^1$ *and a diffeomorphism* $\Phi\colon \mathbb{R}^{k+2} \to \mathbb{R}^{k+2}$, $\Phi(x,t,z) = (X(x,t,z), T(t,z), Z(z))$, $z \in \mathbb{R}^k$ *satisfying*

$$X = T = Z = 0, \quad X_x = 1, \quad T_t > 0, \quad S > 0, \tag{4.13}$$

$$X_{xx} = X_{xt} = X_{xxx} = X_{xxt} = X_{xtt} = \ldots = \frac{\partial^{k+1}}{\partial x^{k+1-i}\partial t^i} X = 0 \quad \textit{for } i = 0, \ldots, k \tag{4.14}$$

at $0 \in \mathbb{R}^{k+2}$, *and the identity* (1.6) *in a neighbourhood of* $0 \in \mathbb{R}^{k+2}$.

P r o o f. Let us consider the functions M, χ, τ from Lemma 4.1. Let us define $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by setting $\Psi(x,t) = (\chi(x,t), \tau(t))$. Obviously, both (1.2) and (1.3) are satisfied. By virtue of Lemma 1.1, there exist smooth S and a diffeomorphism Φ satisfying (1.5) and (1.6). Then the conditions (4.13) and (4.14) are clearly satisfied. □

Theorem 4.1. *Let* S *and* Φ *satisfy* (1.6) *and the conditions* (4.13) *and* (4.14). *Then the differential* $D\Phi(0)$ *is uniquely defined by the set of nonlinear equations. The required data are the following derivatives of* g *at the origin:*

for codim $= k = 1$*:*

$$\mathbf{B} \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2), \quad \frac{\partial^3}{\partial x^3} g, \quad \frac{\partial^3}{\partial x^2 \partial t} g \tag{4.15}$$

for codim $= k \geqslant 2$*:*

$$\mathbf{B} \in \mathcal{L}(\mathbb{R}^{k+1} \mathbb{R}^{k+1}), \quad \frac{\partial^3}{\partial x^3} g, \quad \frac{\partial^3}{\partial x^2 \partial t} g, \tag{4.16}$$

$$\frac{\partial^{j+2}}{\partial x^{j+2-i} \partial t^i} g \quad \textit{for} \quad j = 2, \ldots, k; \quad i = 0, \ldots, j, \tag{4.17}$$

and

$$\frac{\partial^{k+2}}{\partial x \partial t^{k+1}} g, \quad \frac{\partial^{k+2}}{\partial t^{k+2}}. \tag{4.18}$$

P r o o f. The proof consists in directly expressing the desired unknowns from the listed derivatives. The particular formulae for the asymmetric cusp with codim $k \leqslant 3$ are given in the Tabs. 4.1–4.3. $\square$

Data			$\Rightarrow$
g_{xx}	$=$	$2pS$	S
g_z	$=$	SZ_z	Z_z
g_{xt}	$=$	$2pSX_t$	X_t
g_{tt}	$=$	$2pSX_t^2 + 2qST_t^2$	T_t
g_{xxx}	$=$	$6pS_x$	S_x
g_{xz}	$=$	$S_xZ_z + 2pSX_z$	X_z
g_{xxt}	$=$	$4pS_xX_t + 2pS_t$	S_t
g_{tz}	$=$	$S_tZ_z + 2pSX_tX_z + 2qST_tT_z$	T_z

Table 4.1. Simple bifurcation point and isola center, codim $= 1$; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^1$.

5. Winged cusp

The normal form of a *winged cusp* is the function

$$h^*(x,t) = px^3 + qt^2, \tag{5.1}$$

where the constants p, q are normalised so that $|p| = |q| = 1$. The codimension k of (5.1) equals 3, i.e. we set $k = 3$ in this section. As a universal unfolding of (5.1) we use

$$g^*(x,t,z) = h^*(x,t) + z_1 + z_2x + z_3xt, \tag{5.2}$$

see [6], p. 203, Tab. 3.1.

The claim that g is contact equivalent to (5.1) can be formulated algebraically:

$$g = g_x = g_t = g_{xx} = g_{xt} = 0 \tag{5.3}$$

and

$$\operatorname{sgn} g_{xxx} = p, \quad \operatorname{sgn} g_{tt} = q \tag{5.4}$$

at $0 \in \mathbb{R}^5$.

Data	$\Rightarrow$
$g_{xx} = 2pS$	S
$g_{xt} = 2pSX_t$	X_t
$g_z = S(Z_1)_z$	$(Z_1)_z$
$g_{xxx} = 6pS_x$	S_x
$g_{xz} = 2pSX_z + S_x(Z_1)_z$	X_z
$g_{xxt} = 4pS_xX_t + 2pS_t$	S_t
$(D_2(g))_x = 4S^2X_{tt}$	X_{tt}
$(D_2(g))_t = 4S^2X_tX_{tt} + 12pqS^2T_t^3$	T_t
$g_{tz} = 2pSX_tX_z + S_t(Z_1)_z + ST_t(Z_2)_z$	$(Z_2)_z$
$g_{xxxx} = 12pS_{xx}$	S_{xx}
$g_{xxxt} = 6pS_{xx}X_t + 6pS_{xt}$	S_{xt}
$g_{xxtt} = 2pS_{xx}X_t^2 + 8pS_{xt}X_t + 4pS_xX_{tt} + 2pS_{tt}$	S_{tt}
$g_{xttt} = 6pS_{xt}X_t^2 + 6pS_{tt}X_t + 6pS_xX_tX_{tt} + 6qS_xT_t^3 + 6pS_tX_{tt} + 2pSX_{ttt}$	X_{ttt}
$g_{tttt} = 12pS_{tt}X_t^3 + 24pS_tX_tX_{tt} + 24qS_tT_t^3 + 6pSX_{tt}^2 + 8pSX_tX_{ttt} + 30qST_t^2T_{tt}$	T_{tt}
$(D_2(g))_z = 12pqS^2T_t^2T_z + 2pS(Z_1)_z(X_t^2S_{xx} + S_{tt} - 2X_tS_{xt}) + 2pS(Z_2)_z(2T_tS_t - 2S_xX_tT_t + ST_{tt}) + 4S^2X_zX_{tt}$	T_z

Table 4.2. Asymmetric cusp bifurcation analysis, codim = 2; $|p| = |q| = 1$ $X_x = 1$, $z \in \mathbb{R}^2$.

Remark 5.1. Assume that S, Φ exist. Let g^* be the particular unfolding (5.2). Taking an arbitrary $c > 0$, we define another pair $\widetilde{S}$, $\widetilde{\Phi}$ that satisfies (1.6), namely, $\widetilde{S} = c^{-6}S$, $\widetilde{\Phi} = \operatorname{diag}(c^2, c^3, c^6, c^4, c)\Phi$. Then $Sg^* \circ \Phi = \widetilde{S}g^* \circ \widetilde{\Phi}$. Consequently, we may again consider $X_x(0) = 1$ without loss of generality.

The direct differentiation of (1.6) in a neighbourhood of $0 \in \mathbb{R}^5$ yields for a wing cusp bifurcation with codimension $k = 3$:

$$\mathbf{B} = \mathbf{A}\mathbf{B}^* D\Phi(0), \tag{5.5}$$

where

$$\mathbf{B} \equiv \begin{pmatrix} g_x & g_t & g_z \\ g_{xx} & g_{xt} & g_{xz} \\ g_{xt} & g_{tt} & g_{tz} \\ g_{xxx} & g_{xxt} & g_{xxz} \\ g_{xxt} & g_{xtt} & g_{xtz} \end{pmatrix}; \quad \mathbf{B}^* = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2q & 0 & 0 & 0 \\ 6p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Data	$\Rightarrow$
$g_{xx} = 2pS$	S
$g_{xt} = 2pSX_t$	X_t
$g_z = S(Z_1)_z$	$(Z_1)_z$
$(D_2(g))_x = 4S^2X_{tt}$	X_{tt}
$g_{xxx} = 6pS_x$	S_x
$g_{xz} = 2pSX_z + S_x(Z_1)_z$	X_z
$g_{xxt} = 4pS_xX_t + 2pS_t$	S_t
$(D_3(g))_x = 8pS^2(SX_{ttt} + S_tX_{tt} - S_xX_tX_{tt}$	X_{ttt}
$(D_3(g))_t = 96qS^3T_t^4 + 8pS^2X_tX_{tt}(S_t - S_xX_t) + 8pS^3(X_{tt}^2 + X_tX_{ttt})$	T_t
$g_{tz} = 2pSX_tX_z + S_t(Z_1)_z + ST_t(Z_2)_z$	$(Z_2)_z$
$g_{xxxx} = 12pS_{xx}$	S_{xx}
$g_{xxxt} = 6pS_{xx}X_t + 6pS_{xt}$	S_{xt}
$g_{xxtt} = 2pS_{xx}X_t^2 + 8pS_{xt}X_t + 4pS_xX_{tt} + 2pS_{tt}$	S_{tt}
$\frac{\partial^5}{\partial x^5}\,g = 20pS_{xxx}$	S_{xxx}
$\frac{\partial^5}{\partial x^4 \partial t}\,g = 8pS_{xxx}X_t + 12pS_{xxt}$	S_{xxt}
$\frac{\partial^5}{\partial x^3 \partial t^2}\,g = 2pS_{xxx}X_t^2 + 12pS_{xxt}X_t + 6pS_{xtt} + 6pS_{xx}X_{tt}$	S_{xtt}
$\frac{\partial^5}{\partial x^2 \partial t^3}\,g = 6pS_{xxt}X_t^2 + 12pS_{xtt}X_t + 6pS_{xx}X_tX_{tt} + 12pS_{xt}X_{tt}$ $+2pS_{ttt} + 4pS_xX_{ttt}$	S_{ttt}
$\frac{\partial^5}{\partial x \partial t^4}\,g = 8pS_{ttt}X_t + 12pS_{xtt}X_t^2 + 12pS_{tt}X_{tt} + 24pS_{xt}X_tX_{tt}$ $+8pS_tX_{ttt} + 6pS_xX_{tt}^2 + 8pS_xX_tX_{ttt} + 24qS_xT_t^4 + 2pS\frac{\partial^4}{\partial t^4}\,X$	$\frac{\partial^4}{\partial t^4}\,X$
$\frac{\partial^5}{\partial t^5}\,g = 20pS_{ttt}X_t^2 + 60pS_{tt}X_tX_{tt} + 30pS_tX_{tt}^2 + 40pS_tX_tX_{ttt}$ $+20pSX_{tt}X_{ttt} + 10pSX_t\frac{\partial^4}{\partial t^4}X + 120qS_tT_t^4 + 240qST_t^3T_{tt}$	T_{tt}
$(D_2(g))_z = 2pS(Z_1)_z(X_t^2S_{xx} + S_{tt} - 2X_tS_{xt}) + 4pS^2T_t^2(Z_3)_z$ $+2pS(Z_2)_z(2T_tS_t - 2S_xX_tT_t + ST_{tt}) + 4S^2X_zX_{tt}$	$(Z_3)_z$
$(D_3(g))_z = 96qS^3T_t^3T_z + 16pSS_xX_t^2(S_xX_tX_z - S_xX_tT_z - 2S_tX_z)$ $+8pS^2(SX_zX_{ttt} + S_tX_zX_{tt} - S_xX_tX_zX_{tt})$ $+4S(Z_1)_z[S(S_{ttt} - 3S_{xtt}X_t + 3S_{xxt} + X_t^2 - S_{xxx}X_t^3)$ $+S_{xx}X_t(3SX_{tt} + S_tX_t - S_xX_t^2) + S_{tt}(S_t - S_xX_t)$ $+S_{xt}(2S_xX_t^2 - 2S_tX_t - 3SX_{tt})]$ $+4S(Z_2)_z[3ST_t(S_{xx}X_t^2 + S_{tt} - 2S_{xt}X_t)$ $+2T_t(X_t^2S_x^2 + S_t^2 - 2S_xS_tX_t)$ $+S(4S_tT_{tt} - 4S_xX_tT_{tt} - 3S_xX_{tt}T_t) + S^2T_{ttt}]$ $+8(Z_3)_zS^2T_t(3ST_{tt} + 4S_tT_t - 4S_xX_tT_t)$	T_z

Table 4.3. Asymmetric cusp bifurcation analysis, codim = 3; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^3$.

and

$$\mathbf{A} = \begin{pmatrix} S & 0 & 0 & 0 & 0 \\ S_x & S & 0 & 0 & 0 \\ S_t & SX_t & ST_t & 0 & 0 \\ S_{xx} & SX_{xx} + 2S_x & 0 & S & 0 \\ S_{xt} & S_xX_t + S_t + SX_{xt} & S_xT_t & SX_t & ST_t \end{pmatrix},$$

$\mathbf{B}$, $\mathbf{B}^*$ and $\mathbf{A} \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^5)$.

Assuming that $\mathbf{B}$ is known, the system (5.5) represents 23 conditions (note that g_{xt} and g_{xxt} appear twice in $\mathbf{B}$) for 17 unknowns in $D\Phi(0)$ (taking into account that $X_x = 1$ has already been fixed) and 7 additional unknowns S, S_x, S_t, S_{xx}, S_{xt}, X_{xx}, X_{xt} inherited from the chain rule differentiation. Hence, five conditions are missing to determine $D\Phi(0)$ from (5.5).

Lemma 5.1. *Under the assumptions* (5.3) *and* (5.4), *the germ h factors as*

$$h(x,t) = M \cdot (p\chi^3 + q\tau^2), \tag{5.6}$$

where $M = M(x,t)$, $\chi = \chi(x,t)$, $\tau = \tau(t)$ are smooth functions in a neighbourhood of the origin. Moreover, $M(0,0) > 0$ and

$$\tau(t) = (pq\beta(t))^{1/2}, \tag{5.7}$$

$$\chi(x,t) = \big((x + a(t))((x + a(t))^2 + \alpha(t))\big)^{1/3}, \tag{5.8}$$

where $a = a(t)$, $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are smooth functions satisfying $a(0) = 0$, $\alpha(0) = \alpha'(0) = 0$ and $\beta(0) = \beta'(0) = 0$, $\operatorname{sgn}(\beta''(0)) = pq$.

P r o o f. We recall the assumptions (5.3) and (5.4). Due to the Malgrange Preparation Theorem, see [4], there exist smooth functions $M = M(x,t)$, $b = b(t)$, $c = c(t)$ and $d = d(t)$ such that $b(0) = c(0) = d(0) = 0$ and

$$h(x,t) = M(x,t), (px^3 + b(t)x^2 + c(t)x + d(t)) \tag{5.9}$$

in a neighbourhood of $0 \in \mathbb{R}^2$.

Differentiating (5.9) at the origin we arrive at the following conclusions: $h_{xxx} = 6pM$ which implies $M > 0$; $c_t = 0$, $d_t = 0$ and $\operatorname{sgn}(d_{tt}) = q \neq 0$.

We rewrite the second factor on the right-hand side of (5.9) as

$$\begin{aligned} &px^3 + b(t)x^2 + c(t)x + d(t) \\ &= p\bigg(\Big(x + \frac{b(t)}{3p}\Big)^3 + \Big(x + \frac{b(t)}{3p}\Big)\Big(pc(t) - \frac{b^2(t)}{3}\Big) - \frac{1}{3}b(t)c(t) + 2p\frac{b^3(t)}{27} + pd(t)\bigg). \end{aligned} \tag{5.10}$$

We set

$$w = x + a(t), \quad \text{where} \quad a(t) = \frac{1}{3}pb(t).$$

The equation (5.10) can be rewritten in the form

$$px^3 + b(t)x^2 + c(t)x + d(t) = w^3 + \alpha(t)w + \beta(t),$$

where $w = w(x,t)$,

$$\alpha(t) = pc(t) - 3a^2(t), \quad \beta(t) = 2a^3(t) - pa(t)c(t) + pd(t).$$

The Taylor expansion at $t = 0$ yields

$$\beta(t) = Kt^2(1 + \mathcal{O}(t)) \quad \text{as} \quad t \to 0,$$

where

$$K = \frac{1}{2}pd_{tt}(0), \tag{5.11}$$
$$\operatorname{sgn} K = pq \neq 0.$$

Thus, we can define a smooth function $\tau = \tau(t)$,

$$\tau = (pq\beta(t))^{1/2} \tag{5.12}$$

for all t from a neighbourhood of $0 \in \mathbb{R}^1$.

We set

$$\chi(x,t) = \big((x + a(t))((x + a(t))^2 + \alpha(t))\big)^{1/3}. \tag{5.13}$$

Obviously $\chi_x(0,0) = 1$, $\chi_{xx}(0,0) = \chi_{xt}(0,0) = 0$, $\tau(0) = 0$, $\tau_t(0) = (pqK)^{1/2} > 0$ and

$$h(x,t) = M(x,t)(p\chi^3 + q\tau^2) = M\, h^* \circ \Psi.$$

$\square$

Lemma 5.2. *Assuming* (5.3) *and* (5.4), *there exist a smooth mapping* $S\colon \mathbb{R}^5 \to \mathbb{R}^1$ *and a diffeomorphism* $\Phi\colon \mathbb{R}^5 \to \mathbb{R}^5$, $\Phi(x,t,z) = (X(x,t,z), T(t,z), Z(z))$ *satisfying*

$$X = T = Z = 0, \quad X_x = 1, \quad T_t > 0, \quad S > 0, \tag{5.14}$$
$$X_{xt} = X_{xx} = X_{xxx} = 0 \tag{5.15}$$

at $0 \in \mathbb{R}^5$ *and the identity* (1.6) *in a neighbourhood of* $0 \in \mathbb{R}^5$.

P r o o f. Let us consider the functions M, χ and τ from Lemma 5.1. Let us define $\Psi\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ by setting $\Psi(x,t) = (\chi(x,t), \tau(t))$. Obviously, both (1.2) and (1.3) are satisfied. By virtue of Lemma 1.1, there exist a smooth S and a diffeomorphism Φ satisfying (1.5) and (1.6). The condition (5.14) is obviously satisfied.

The remaining conditions $X_{xt} = X_{xx} = X_{xxx} = 0$ at the origin are also clearly satisfied. □

Theorem 5.1. *Let us consider S and Φ satisfying* (5.14), (5.15) *and* (1.6). *Then the differential $D\Phi(0)$ is uniquely defined. The required data are the following derivatives of g at the origin:*

$$\mathbf{B} \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^5), \quad g_{xxxxx},\ g_{xxxt},\ g_{xxtt},\ g_{xttt},\ g_{ttt}. \tag{5.16}$$

P r o o f. For proof, see the Tab. 5.1. In this table the particular nonlinear equations and the appropriate computable unknowns for the case of the wing cusp bifurcation are listed for S and Φ which satisfy (5.14), (5.15). □

Data	$\Rightarrow$
$g_{xxx} = 6pS$	S
$g_{xxt} = 6pSX_t$	X_t
$g_{tt} = 2qST_t^2$	T_t
$g_{xtt} = 2qS_xT_t^2 + 6pSX_t^2$	S_x
$g_z = S(Z_1)_z$	$(Z_1)_z$
$g_{xz} = S_x(Z_1)_z + S(Z_2)_z$	$(Z_2)_z$
$g_{xxxt} = 18pS_xX_t + 6pS_t$	S_t
$g_{ttt} = 6pSX_t^3 + 6qS_tT_t^2 + 6qST_tT_{tt}$	T_{tt}
$g_{tz} = 2qST_tT_z + S_t(Z_1)_z$	T_z
$\frac{\partial^5}{\partial x^5} g = 60pS_{xx}$	S_{xx}
$g_{xxz} = 6pSX_z + (Z_1)_zS_{xx} + 2S_x(Z_2)_z$	X_z
$g_{xxtt} = 2qS_{xx}T_t^2 + 12pS_xX_t^2 + 12pS_tX_t + 6pSX_{tt}$	X_{tt}
$g_{xttt} = 6qS_{xt}T_t^2 + 6pS_xX_t^3 + 18pS_tX_t^2 + 18pSX_tX_{tt} + 6qS_xT_tT_{tt}$	S_{xt}
$g_{xtz} = 2qS_xT_tT_z + 6pSX_tX_z + (Z_1)_zS_{xt} + (Z_2)_z(S_xX_t + S_t) + ST_t(Z_3)_z$	$(Z_3)_z$

Table 5.1. Wing cusp bifurcation analysis, codim $= 3$; $|p| = |q| = 1$, $X_x = 1$, $z \in \mathbb{R}^3$.

6. Conclusions

The main results of the paper are algorithms for computing the differential $D\Phi(0)$ of the unfolded contact diffeomorphism Φ at the organizing center. Singularity classes containing singular points with codim $\leqslant 3$, corank $= 1$ are assumed, see [6].

The algorithms are presented in a form of tables (see Tab. 2.1–Tab. 5.1) for each particular singularity: there, *Data* are represented by selected partial derivatives of g computed at the origin. The table itself can be interpreted as a system of nonlinear equations in the variables listed in the last column. Note that among the variables there are all elements of $D\Phi(0)$, see (1.7). The canonical solution of the nonlinear system mentioned is also hinted at.

The crucial step towards a justification of the main results are Lemmas 2.3, 3.1, 4.1 and 5.1. They represent a *constructive* solution of Recognition Problem; see [6] for the formulation of this problem.

The extension of the results presented to problems with corank larger then 1 seems to be difficult. At least, we have not succeeded in any kind of constructive solution of the above mentioned Recognition Problem.

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