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ATTRACTORS FOR GENERAL OPERATORS

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Abstract. In this article we introduce the notion of a minimal attractor for families of operators that do not necessarily form semigroups. We then obtain some results on the existence of the minimal attractor. We also consider the nonautonomous case. As an application, we obtain the existence of the minimal attractor for models of Cahn-Hilliard equations in deformable elastic continua.

Keywords: global attractor, minimal attractor, exponential attractor, weakly coupled system

MSC 2000: 37L30

0. Introduction

The study of the long time behavior of partial differential equations arising from mechanics and physics is a capital issue. It is indeed essential, for practical purposes, to understand and be able to predict the asymptotic behavior of the system under study.

Many parabolic equations, but also some (partially) dissipative wave-type equations, possess a finite dimensional (in the sense of the fractal or the Hausdorff dimension) global attractor, which is a compact invariant set attracting the trajectories as time goes to infinity (see [3], [20], [22] and [33] for extensive reviews on this subject). Since it is the smallest (with respect to the inclusion) set enjoying these properties, it is a suitable object for the study of the long time behavior of the system.

In all these studies, one considers a family of operators $S(t)$, $t \geqslant 0$, acting on a functional space E (a closed set of a Banach space in general) which forms a semigroup (i.e. satisfies $S(0) = \mathrm{Id}$ and $S(t) \circ S(s) = S(t+s)$, $\forall t,\ s \geqslant 0$) and associates with the initial condition the solution at time t (assuming of course that the system is well posed). One then constructs the global attractor associated with this semigroup.

Let us consider a family of operators

$$\mathcal{F}(t)\colon\ E \to F, \quad t \geqslant 0, \tag{0.1}$$

where, typically, $E \neq F$. In that case, we cannot have a semigroup (of course, we can also consider situations in which $E = F$ but the family of operators $\mathcal{F}(t)$ does not form a semigroup (e.g. for multivalued operators, see [30]); this is in particular the case when considering nonautonomous systems, although other problems also occur in that case, see [9], [18], [21], [29], [30], [31], [32] and also below). Furthermore, the notion of a global attractor is no longer valid (in particular, the invariance property no longer makes sense). It is however interesting to study the asymptotic behavior of such systems and to find "good" objects that will characterize this behavior, or, in other words, to find a good notion of an attractor in such situations.

Our investigation was initiated in [7] and [8] and was motivated by the study of the long time behavior of models of Cahn-Hilliard equations in deformable elastic continua (we refer the reader to [4], [5] and [19] for the physical derivation of the classical model and of generalizations of the Cahn-Hilliard equation and to [6], [7], [8], [26] and [27] for the mathematical study of models of Cahn-Hilliard equations in deformable continua). The study of these systems led to the study of systems of the form $\mathcal{F}(t)\colon\ \varrho_0 \mapsto (S(t)\varrho_0, \mathcal{L} \circ S(t)\varrho_0)$, where $S(t)$ is a semigroup, which we called weakly coupled systems (in the sense that the initial conditions for the two components of the system are not independent; we noted in [27] that some models in which the order parameter and the displacement were coupled in the equations as well as in the boundary conditions led, when deriving the variational formulation, to a weakly coupled system as above). Now, one can think of other types of systems that would be interesting to study, e.g. a system of (noncoupled) partial differential equations having the same initial condition.

In [7] and [8], we noted, for a weakly coupled system as above, that the family of operators $\tilde{\mathcal{F}}(t)\colon\ (\varrho_0, u_0) \mapsto (S(t)\varrho_0, \mathcal{L} \circ S(t) \circ \mathcal{L}^{-1}(u_0))$, $t \geqslant 0$, forms, assuming of course that $\mathcal{L}$ is regular enough, a semigroup. We thus called the global attractor associated with the weakly coupled system $\mathcal{F}(t)$ the global attractor for $\tilde{\mathcal{F}}(t)$. The problem is that this notion cannot be generalized to more general families of operators of the form (0.1) (see Section 1 below). Also, this global attractor is, in a sense, too big (we note that the trajectories lie on the manifold $y = \mathcal{L}(x)$).

In this article, we introduce the notion of the minimal (for the inclusion) attractor for a general family of operators of the form (0.1) (we note that the minimality property has been used with success in the context of nonautonomous systems for which we do not have semigroups, although we can often reduce the problem of the research of the minimal attractor to the research of the global attractor for a semigroup on an extended phase space in that case, see [9] and also [21], [30] and [31]). We then obtain some results on the existence of the minimal attractor. In particular, we are able to prove the existence of the minimal attractor for a weakly coupled system and, as an application, for the models of Cahn-Hilliard equations in deformable elastic continua considered in [26].

1. The autonomous case

Let E_0 and G be two closed subsets of Banach spaces $\mathcal{E}_0$ and $\mathcal{G}$. We consider a family of operators

$$\mathcal{F}(t)\colon\ E_0 \to G, \quad t \geqslant 0. \tag{1.1}$$

When $G \neq E_0$, we cannot have a semigroup. We can nonetheless define the notions of trajectories, ω-limit sets, attracting sets and absorbing sets as in the case of semigroups (see for instance [33]; see also [8]). Concerning the notion of the attractor, we give the following definitions (which generalize that given in [7] and [8], where we actually talked of the global attractor; see Remark 1.4 below).

Definition 1.1.

(i) A compact set $\mathcal{A} \subset G$ is an attractor for $\mathcal{F}(t)$ on E_0 if $\forall B \subset E_0$ bounded

$$\lim_{t\to+\infty} \operatorname{dist}_G(\mathcal{F}(t)B, \mathcal{A}) = 0,$$

where $\operatorname{dist}_G(A,B) = \sup_{a\in A}\inf_{b\in B} \|a-b\|_G$ denotes the non-symmetric Hausdorff distance between the sets A and B (in G).

(ii) A compact set $\mathcal{A} \subset G$ is the minimal attractor for $\mathcal{F}(t)$ on E_0 if it is an attractor and if it is minimal (with respect to the inclusion) among the closed sets that attract the bounded sets of E_0.

Most of the known results on attractors (in the autonomous case) concern attractors associated with semigroups (see [3], [20], [22] and [33]; see [30] for some results in more general situations). It is thus natural, in the first step, to see in which situations we can reduce the problem of the research of (minimal) attractors for $\mathcal{F}(t)$ to the research of (global) attractors for a semigroup (we proceeded in that way in [7] and [8], although the construction considered did not give the minimal attractor). We have the following result:

Theorem 1.1. *We assume that $l = \mathcal{F}(0)$ is invertible, that l and l^{-1} are bounded and that $\mathcal{F}(t+s) = \mathcal{F}(t)\circ l^{-1}\circ\mathcal{F}(s) = \mathcal{F}(s)\circ l^{-1}\circ\mathcal{F}(t)$, $\forall t,\ s \geqslant 0$. Then, if*

$\tilde{\mathcal{F}}(t) = \mathcal{F}(t) \circ l^{-1}$ *possesses the global attractor* $\mathcal{A}$ *on* G, $\mathcal{A}$ *is also the minimal attractor for* $\mathcal{F}(t)$ *on* E_0.

P r o o f. We first note that $\tilde{\mathcal{F}}(t)$ is a semigroup on G. We already have that $\mathcal{A}$ is compact. Let $B_0 \subset E_0$ be a bounded set. Then $B_1 = l(B_0)$ is bounded in G and $\mathcal{A}$ attracts B_1 for $\tilde{\mathcal{F}}(t)$, which is equivalent to $\mathcal{A}$ attracts B_0 for $\mathcal{F}(t)$. Let now $\mathcal{A}' \subset G$ closed be such that $\mathcal{A}'$ attracts the bounded sets of E_0 for $\mathcal{F}(t)$. Let $B_1 \subset G$ be a bounded set. Then $B_0 = l^{-1}(B_1)$ is bounded in E_0. Writing that $\mathcal{A}'$ attracts B_0 for $\mathcal{F}(t)$ is equivalent to writing that $\mathcal{A}'$ attracts B_1 for $\tilde{\mathcal{F}}(t)$. Therefore, $\mathcal{A}'$ attracts the bounded sets of G for $\tilde{\mathcal{F}}(t)$ and, in particular, $\mathcal{A}$. Since $\mathcal{A}$ is invariant by $\tilde{\mathcal{F}}(t)$, it follows that $\mathcal{A} \subset \mathcal{A}'$, hence the minimality property. $\square$

Unfortunately, Theorem 1.1 is too restrictive. It can be applied for instance to systems of the form $(\varrho_0, u_0) \mapsto (S_1(t)\varrho_0, \mathcal{L} \circ S_2(t)u_0)$, where $S_1(t)$ and $S_2(t)$ are two semigroups, $\mathcal{L}$ is invertible and bounded and $\mathcal{L}^{-1}$ is bounded (and more generally to systems of the form $(\varrho_0^1, \ldots, \varrho_0^r) \mapsto (S_1(t)\varrho_0^1, \ldots, S_s(t)\varrho_0^s, \mathcal{L}_{s+1} \circ S_{s+1}(t)\varrho_0^{s+1}, \ldots, \mathcal{L}_r \circ S_r(t)\varrho_0^r))$. We note that in these examples, E_0 and G have in a certain sense the same structure.

If $G = E_0$ and $\mathcal{F}(t)$ is a semigroup, the theorem can also be applied. We note however that if the global attractor is the minimal attractor, the converse is not necessarily true (the minimal attractor need not be invariant; the two definitions thus do not coincide). If we assume in addition that $\mathcal{F}(t)$ is uniformly continuous, we can prove that if $\mathcal{A}$ is the minimal attractor for $\mathcal{F}(t)$ in E_0, then $\mathcal{F}(t_0)\mathcal{A}$ is an attractor, $\forall t_0 \geqslant 0$. Therefore, we have

$$\mathcal{A} \subset \mathcal{F}(t)\mathcal{A}, \ \ \forall t \geqslant 0.$$

In particular, if $\mathcal{F}(t)$ is a uniformly continuous group, we deduce that $\mathcal{A}$ is invariant and, in that case, the two definitions coincide.

R e m a r k 1.1. In [3], the authors introduced the notion of the F-E global attractor for a semigroup $S(t)$. A typical situation arises when $S(t)\colon E \to E$ is well posed only on bounded subsets of $F \subset E$ with compact injection. The F-E global attractor $\mathcal{A}$ is then compact in E, bounded in F, invariant by $S(t)$ and attracts the bounded sets of F in the topology of E. Since $\mathcal{A}$ is invariant by $S(t)$, we easily prove that it is the minimal attractor for $S(t)\colon F \to E$.

It is also natural to look whether the classical proofs of existence of the global attractor (see for instance [33]) can be adapted to our more general setting. We have, in this direction, the following result.

Theorem 1.2. *We assume that $\mathcal{F}(t)$ possesses a bounded absorbing set $\mathcal{B}$ in E_0 and that exists t_0 such that $\bigcup\limits_{t\geqslant t_0} \mathcal{F}(t)\mathcal{B}$ is relatively compact in G. Then the ω-limit set of $\mathcal{B}$,*

$$\mathcal{A} = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} \mathcal{F}(t)\mathcal{B}},$$

is the minimal attractor for $\mathcal{F}(t)$ on E_0.

P r o o f. We first note that since $\bigcup\limits_{t\geqslant t_0} \mathcal{F}(t)\mathcal{B}$ is relatively compact, $\mathcal{A}$ is a compact and nonempty set. Now, to prove the attraction property, it suffices to prove that $\mathcal{A}$ attracts $\mathcal{B}$ (since $\mathcal{B}$ is a bounded absorbing set). Let us thus assume that $\mathcal{A}$ does not attract $\mathcal{B}$. Then $\exists \delta > 0$ and a sequence $t_n \to +\infty$ such that

$$\operatorname*{dist}_G(\mathcal{F}(t_n)\mathcal{B}, \mathcal{A}) \geqslant \delta > 0, \quad \forall n.$$

Therefore, $\forall n$, $\exists b_n \in \mathcal{B}$ such that

$$\operatorname*{dist}_G(\mathcal{F}(t_n)b_n, \mathcal{A}) \geqslant \frac{\delta}{2} > 0.$$

Furthermore, since $\mathcal{F}(t_n)b_n \in \bigcup\limits_{t\geqslant t_0} \mathcal{F}(t)\mathcal{B}$ if n is large enough, we deduce that, at least for a subsequence, $\mathcal{F}(t_n)b_n \to \beta$ as $n \to +\infty$, where β belongs necessarily to $\mathcal{A} = \omega(\mathcal{B})$, hence a contradiction. Let finally $\mathcal{A}' \subset G$ be a closed set that attracts the bounded sets of E_0 for $\mathcal{F}(t)$. Let y belong to $\mathcal{A}$. Then $\exists t_n \to +\infty$ and $y_n \in \mathcal{B}$ such that $\mathcal{F}(t_n)y_n \to y$ as $n \to +\infty$. Since $\mathcal{A}'$ attracts the bounded sets of E_0, it follows that $\mathrm{dist}_G(\mathcal{F}(t_n)y_n, \mathcal{A}') \to 0$ as $n \to +\infty$. Therefore, $\mathrm{dist}_G(y, \mathcal{A}') = 0$, which yields the minimality property. $\square$

R e m a r k 1.2. Proceeding as in [33], we can adapt the proof of Theorem 1.2 to the case where $\mathcal{F}(t) = \mathcal{F}_1(t) + \mathcal{F}_2(t)$, $\lim\limits_{t\to+\infty} \sup\limits_{\varphi\in C} \|\mathcal{F}_1(t)\varphi\|_G = 0$, $\forall C \subset E_0$ bounded, and $\bigcup\limits_{t\geqslant t_0} \mathcal{F}_2(t)\mathcal{B}$ is relatively compact. Also, we note that if G is uniformly convex and if $\mathcal{F}(t)$ possesses a bounded absorbing set in E_0, then the above decomposition is equivalent to the existence of a compact attracting set and to the asymptotic compactness property (see [33] for more details).

E x a m p l e 1.1. We consider a system

$$\begin{aligned} \mathcal{F}(t)\colon\ E_0 &\to E_0 \times E_0 \\ \varrho_0 &\mapsto (S_0(t)\varrho_0, S_1(t)\varrho_0), \end{aligned}$$

where $S_i(t)$, $i = 0, 1$, are continuous and uniformly compact semigroups on E_0. Such a system can be seen as a system of two noncoupled partial differential equations with the same initial data. We assume that $S_i(t)$ possesses a bounded absorbing set $\mathcal{B}_i \subset E_0$, $i = 0, 1$. Then $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$ is bounded and absorbing for both the semigroups. Let us prove that $\exists t_0$ such that $\bigcup\limits_{t \geqslant t_0} \mathcal{F}(t)\mathcal{B}$ is relatively compact in $E_0 \times E_0$. To do so, it suffices to note that $\bigcup\limits_{t \geqslant \tau} \mathcal{F}(t)\mathcal{B} \subset \bigcup\limits_{t \geqslant \tau} S_0(t)\mathcal{B} \times \bigcup\limits_{t \geqslant \tau} S_1(t)\mathcal{B}$ and that $S_0(t)$ and $S_1(t)$ are uniformly compact. We thus deduce that $\mathcal{F}(t)$ possesses the minimal attractor $\mathcal{A} = \bigcap\limits_{s \geqslant 0} \overline{\bigcup\limits_{t \geqslant s} \mathcal{F}(t)\mathcal{B}}$ on E_0. We note here that the semigroup $S_i(t)$ possesses the global attractor $\mathcal{A}_i$ on E_0, $i = 0, 1$, and we easily prove that $\mathcal{A}_0 \times \mathcal{A}_1$ is an attractor for $\mathcal{F}(t)$ on E_0. Of course, we have $\mathcal{A} \subset \mathcal{A}_0 \times \mathcal{A}_1$. It would be interesting here to see whether this inclusion is strict or not. For instance, when $S_0 = S_1$ (and thus $\mathcal{A}_0 = \mathcal{A}_1$), we can prove that $\mathcal{A} = \{(a_0, a_0),\ a_0 \in \mathcal{A}_0\}$ (see below) and, in that case, the inclusion is strict. One could also consider systems of coupled partial differential equations having, for instance, the same initial condition (e.g. systems of reaction-diffusion equations) and study the existence of the minimal attractor (in that case, we do not have semigroups, either).

We now have the following result:

Theorem 1.3. *We assume that $\mathcal{F}(t)$ possesses an attractor $\mathcal{A}$ on E_0 and that there exists a bounded operator $\Pi\colon G \to E_0$ such that*

$$\mathcal{F}(t) \circ \Pi(\mathcal{A}) = \mathcal{A}, \quad \forall t \geqslant 0. \tag{1.2}$$

Then $\mathcal{A}$ is the minimal attractor for $\mathcal{F}(t)$ on E_0.

P r o o f. Let $\mathcal{A}' \subset G$ closed be such that $\mathcal{A}'$ attracts the bounded sets of E_0 for $\mathcal{F}(t)$. Then

$$\operatorname*{dist}_G(\mathcal{F}(t)[\Pi(\mathcal{A})], \mathcal{A}') \to 0 \quad \text{as} \quad t \to +\infty,$$

which implies, thanks to (1.2), that

$$\operatorname*{dist}_G(\mathcal{A}, \mathcal{A}') = 0,$$

hence the result. $\square$

For instance, when $E_0 = G$ and $S(t)$ is a semigroup, we recover, for $\Pi = \mathrm{Id}$, that the global attractor is the minimal attractor (however, as already noted, the converse is not necessarily true).

Let us now consider a more interesting case, which actually motivated our study (as well as [7] and [8]). We assume that $\mathcal{G}$ is of the form $\mathcal{G} = \mathcal{E}_0 \times \mathcal{E}_1$, where $\mathcal{E}_1$ is a

Banach space, and that G is of the form $G = E_0 \times E_1$, where E_1 is a closed subset of $\mathcal{E}_1$. We then consider a system of the form

$$(1.3) \qquad \begin{aligned} \mathcal{F}(t)\colon\ E_0 &\to E_0 \times E_1 \\ \varrho_0 &\mapsto (S(t)\varrho_0, \mathcal{L} \circ S(t)\varrho_0), \end{aligned}$$

$t \geqslant 0$, where $S(t)$ is a continuous semigroup on E_0 and $\mathcal{L}\colon\ E_0 \to E_1$ is a bounded and uniformly continuous operator. We have said in [7] and [8] that (1.3) defines a weakly coupled dynamical system, in the sense that the initial conditions for the two components of the system are not independent.

Let us assume that $S(t)$ possesses the global attractor $\mathcal{A}_0$ on E_0. We set

$$(1.4) \qquad \mathcal{A} = \{(a_0, \mathcal{L}(a_0)),\ \ a_0 \in \mathcal{A}_0\}.$$

We have

Theorem 1.4. *The set $\mathcal{A}$ is the minimal attractor for the family of operators $\mathcal{F}(t)$ defined by* (1.3) *on E_0.*

Proof. We first prove that $\mathcal{A}$ is an attractor for $\mathcal{F}(t)$ on E_0. Since the mapping $x \mapsto (x, \mathcal{L}(x))$ is continuous, $\mathcal{A}$ is compact. Let now $B \subset E_0$ be a bounded set. We have

$$\lim_{t \to +\infty} \operatorname*{dist}_{E_0}(S(t)B, \mathcal{A}_0) = 0.$$

Let $\varepsilon > 0$ be fixed. Then there exists $\eta > 0$ such that if $\|x - y\|_{E_0} < \eta$, $\|\mathcal{L}(x) - \mathcal{L}(y)\|_{E_1} < \varepsilon$. Let us fix such an η (which we can choose, without loss of generality, strictly less than ε). Then there exists $t_0 \geqslant 0$ such that if $t \geqslant t_0$ is fixed then

$$\operatorname*{dist}_{E_0}(S(t)B, \mathcal{A}_0) < \eta.$$

Therefore, $\forall b > 0$, $\exists a_0 \in \mathcal{A}_0$ such that

$$\|S(t)b - a_0\|_{E_0} < \eta,$$

which implies

$$\|\mathcal{L} \circ S(t)b - \mathcal{L}(a_0)\|_{E_1} < \varepsilon.$$

In particular, we deduce that $\forall b \in B$, $\exists a_0 \in \mathcal{A}_0$ such that

$$\|\mathcal{F}(t)b - (a_0, \mathcal{L}(a_0))\|_{E_0 \times E_1} < \varepsilon$$

and, by the definition of $\mathcal{A}$, we conclude that

$$\operatorname*{dist}_{E_0\times E_1}(\mathcal{F}(t)B,\mathcal{A})\leqslant\varepsilon,$$

hence the attraction property. In order to prove the minimality property, we first note that, since $\mathcal{A}_0$ is invariant by $S(t)$, we have

$$\mathcal{F}(t)\mathcal{A}_0=\mathcal{A},\quad\forall t\geqslant 0. \tag{1.5}$$

Therefore

$$\mathcal{F}(t)\circ\Pi_0(\mathcal{A})=\mathcal{A},\quad\forall t\geqslant 0, \tag{1.6}$$

where

$$\Pi_0\colon\ E_0\times E_1\to E_0$$

is the projector on the first component (indeed, we easily prove that $\Pi_0\mathcal{A}=\mathcal{A}_0$ and $\Pi_1\mathcal{A}=\mathcal{L}(\mathcal{A}_0)$, Π_1 denoting the projector on the second component). We then conclude the proof by Theorem 1.3. □

Remark 1.3. If we assume that $S(t)$ possesses a bounded absorbing set $\mathcal{B}_0$ in E_0 and if $S(t)$ is uniformly compact, then we can also use Theorem 1.2 to prove the existence of the minimal attractor for $\mathcal{F}(t)$ on E_0. However, we have a more precise description of the minimal attractor in Theorem 1.4.

Remark 1.4. Properties (1.5) and (1.6) (and, more generally, (1.2)) can be seen as generalized (or partial) invariance properties for $\mathcal{F}(t)$ and thus we could also use the term global attractor in such a context. Actually, in [7] and [8], we used the term global attractor for weakly coupled systems of the form (1.3). However, in these references, the global attractor for $\mathcal{F}(t)$ was the set $\tilde{\mathcal{A}}=\mathcal{A}_0\times\mathcal{A}_1$, where $\mathcal{A}_1=\mathcal{L}(\mathcal{A}_0)$ is the global attractor for the semigroup $\tilde{S}(t)=\mathcal{L}\circ S(t)\circ\mathcal{L}^{-1}$ (assuming that $\mathcal{L}$ is regular enough) on E_1. The problem is that this set is too big (in particular, it does not lie on the manifold $y=\mathcal{L}(x)$ as could be expected here). Furthermore, it loses, in a sense, the fact that the system is coupled via the boundary conditions.

Remark 1.5. If $\mathcal{A}_0$ has a finite (fractal or Hausdorff) dimension and if $\mathcal{L}$ is Lipschitz, then $\mathcal{A}$ has also a finite dimension and $\dim\mathcal{A}\leqslant\dim\mathcal{A}_0$. Furthermore, if $\mathcal{L}$ is bi-Lipschitz (onto its image), then $\dim\mathcal{A}=\dim\mathcal{A}_0$. We note that for more general operators of the form (1.1), the study of the finite dimensionality of the minimal attractor is in general very difficult, if not impossible, the main reason being the lack of invariance. However, when the generalized invariance property (1.2) holds, such a study can be carried out by using, for instance, the Lyapunov exponents (see for example [33] for more details and for the assumptions needed).

In the case of semigroups (and one would expect a similar situation here for the minimal attractor), the global attractor presents two major drawbacks for practical purposes. Indeed, it can attract the trajectories at a slow rate and it is very sensitive to perturbations. In order to have a more stable (and perhaps a more realistic) object, the notion of an exponential attractor was introduced in [11]. An exponential attractor is a compact semi-invariant set which contains the global attractor, has a finite fractal dimension and attracts the trajectories at an exponential rate. Also, the global attractor may be trivial (say, reduced to one point only) and thus may fail to capture important transient behaviors. Again, in such situations, an exponential attractor may be a more suitable object.

Exponential attractors are now as general as the global attractor (see [2], [11], [12], [13], [14], [17], [23], [24], [25], [26], [27] and the references therein). We note that the classical constructions of exponential attractors make an essential use of projectors with finite rank (in order to prove the so-called squeezing property) and are thus valid in Hilbert spaces only (see [2], [11], [17], [24] and [25]). In [12], however, we proposed a construction that is valid in Banach spaces and that no longer requires to prove the squeezing property.

For the more general operators (1.1), we give the following definition:

Definition 1.2. A compact set $\mathcal{M} \subset G$ is an exponential attractor for $\mathcal{F}(t)$ on a closed subset X of E_0 if it contains the minimal attractor for $\mathcal{F}(t)$ on X, has a finite fractal dimension and satisfies the attraction property

$$\operatorname*{dist}_G(\mathcal{F}(t)B, \mathcal{M}) \leqslant c_1 \mathrm{e}^{-c_2 t}, \ \ \forall t \geqslant 0,$$

$\forall B \subset X$ bounded, where c_1 and c_2 are strictly positive constants that depend only on B.

The known constructions of exponential attractors (see [2], [10], [11] and [12]) cannot be adapted to general operators of the form (1.1), the reason being again the lack of (semi) invariance.

Let us however consider the particular case of weakly coupled systems of the form (1.3). Let us assume that $\mathcal{L}$ is Lipschitz and that $S(t)$ possesses the global attractor $\mathcal{A}_0$ on X and an exponential attractor $\mathcal{M}_0$ on X. Then $\mathcal{F}(t)$ possesses the minimal attractor

$$\mathcal{A} = \{(a_0, \mathcal{L}(a_0)), \ \ a_0 \in \mathcal{A}_0\}$$

on X. We set

$$\mathcal{M} = \{(a_0, \mathcal{L}(a_0)), \ \ a_0 \in \mathcal{M}_0\}.$$

We have

Theorem 1.5. *The set $\mathcal{M}$ is an exponential attractor for $\mathcal{F}(t)$ on X.*

P r o o f. We easily prove that $\mathcal{M}$ is compact and contains the minimal attractor. Furthermore, since the mapping $L = (\mathrm{Id}, \mathcal{L})$ is Lipschitz and since $\mathcal{M} = L(\mathcal{M}_0)$, we deduce that $\mathcal{M}$ has a finite fractal dimension and that

$$\dim_F \mathcal{M} \leqslant \dim_F \mathcal{M}_0.$$

Let now $B \subset X$ be a bounded set. Then

$$\operatorname*{dist}_{E_0}(S(t)B, \mathcal{M}_0) \leqslant c_1 \mathrm{e}^{-c_2 t}, \ \ \forall t \geqslant 0.$$

Since $\mathcal{L}$ is Lipschitz (say, with a constant k), we deduce that

$$\operatorname*{dist}_{E_1}(\mathcal{L} \circ S(t)B, \mathcal{L}(\mathcal{M}_0)) \leqslant k c_1 \mathrm{e}^{-c_2 t}, \ \ \forall t \geqslant 0,$$

which yields, by the definition of $\mathcal{M}$, that

$$\operatorname*{dist}_{E_0 \times E_1}(\mathcal{F}(t)B, \mathcal{M}) \leqslant \mathrm{Max}(c_1, k c_1) \mathrm{e}^{-c_2 t}, \ \ \forall t \geqslant 0,$$

which completes the proof of the theorem. □

R e m a r k 1.6. We note that, in Definition 1.2, we no longer have the semi-invariance property for an exponential attractor. We note however that, for a weakly coupled dynamical system of the form (1.3), we have the following partial semi-invariance property:

$$\mathcal{F}(t)\mathcal{M}_0 \subset \mathcal{M}, \ \ \forall t \geqslant 0, \tag{1.7}$$

or, equivalently

$$\mathcal{F}(t) \circ \Pi_0(\mathcal{M}) \subset \mathcal{M}, \ \ \forall t \geqslant 0. \tag{1.8}$$

E x a m p l e 1.2. Let $\Omega = (0, L_1) \times (0, L_2)$. We consider the following problem:

Find $(\varrho, u)\colon\ [0, T] \to V \times W$, $T > 0$, such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}[(\varrho, q) + (\varrho, d \cdot \nabla q) + (\tilde{B} \nabla \varrho, \nabla q)] + \alpha(\nabla B^{1/2} \nabla \varrho, \nabla B^{1/2} \nabla q) \\
&\qquad + e^2 \operatorname{Tr}(CI)(B \nabla \varrho, \nabla q) - \frac{e}{2}(B \nabla \operatorname{Tr} C(\nabla u + {}^t\nabla u), \nabla q) \\
&\qquad + (B \nabla f'(\varrho), \nabla q) = 0, \quad \forall q \in V,
\end{aligned} \tag{1.9}$$

$$(C(\nabla u + {}^t\nabla u), \nabla v) = 2e(\varrho C I, \nabla v), \quad \forall v \in W, \tag{1.10}$$

where $V = H^2_{\mathrm{per}}(\Omega)$ and $W = \{v \in H^1_{\mathrm{per}}(\Omega)^2,\ m(v) \equiv \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega v \,\mathrm{d}x = 0\}$. We assume here that α and e are strictly positive constants, B and $\tilde{B}$ are symmetric and positive definite matrices (the matrix B is called the mobility tensor) and C is a constant, symmetric and positive definite linear transformation that maps symmetric matrices onto symmetric matrices (C is called the elasticity tensor). Furthermore, we assume that f is a polynomial of degree $2p+2$ with a strictly positive leading coefficient, $p \geqslant 1$. Finally, Tr denotes the trace operator and I the identity matrix. These equations were derived in [26] and correspond to generalizations of the Cahn-Hilliard equation that take into account the work of internal microforces, the anisotropy of the material and the deformations of the material (see also [19], [6] and [27]); the unknown ϱ corresponds to the order parameter and u corresponds to the displacement. We restrict ourselves to two space dimensions here; we could as well consider the three dimensional case, except that we would then have restrictions on the degree of f (see [26]). We set, for $u, v \in W$ and $\varrho \in L^2(\Omega)$

$$E(u,v) = (C(\nabla u + {}^t\nabla u), \nabla v), \tag{1.11}$$

$$l_\varrho(v) = 2e(\varrho C I, \nabla v). \tag{1.12}$$

We proved in [26] that l_ϱ is a continuous and linear form on W and that E is a continuous, coercive and bilinear form on W. We thus see that (1.9)–(1.10) could be uncoupled and we first have to solve the following problem:

Find $\varrho\colon [0,T] \to V$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}[(\varrho, q) + (\varrho, d \cdot \nabla q) + (\tilde{B}\nabla\varrho, \nabla q)] &+ \alpha(\nabla B^{1/2}\nabla\varrho, \nabla B^{1/2}\nabla q) \\ &+ e^2 \operatorname{Tr}(CI)(B\nabla\varrho, \nabla q) + (B\nabla\mathcal{G}(\varrho), \nabla q) \\ &+ (B\nabla f'(\varrho), \nabla q) = 0,\ \ \forall q \in V. \end{aligned} \tag{1.13}$$

We then set

$$u(t) = \mathcal{L}(\varrho(t)), \quad t \in [0,T], \tag{1.14}$$

where $\mathcal{L}$ is defined by (1.10) (it is a linear and Lipschitz mapping from $L^2(\Omega)$ onto W; it is bi-Lipschitz (onto its image) if CI is positive definite) and

$$\mathcal{G}(\varrho) = -\frac{e}{2} \operatorname{Tr} C(\nabla\mathcal{L}(\varrho) + {}^t\nabla\mathcal{L}(\varrho)). \tag{1.15}$$

We proved in [26] that the problem is well posed and that the semigroup

$$\begin{aligned} S(t)\colon\ H_\delta &\to H_\delta \\ \varrho_0 &\mapsto \varrho(t), \end{aligned}$$

where ϱ is the solution of (1.13) with initial data ϱ_0 and $H_\delta = \{q \in H^1_{\rm per}(\Omega),\ |m(q)| \leqslant \delta\}$, $\delta > 0$, is continuous and uniformly compact when e is small enough. It thus possesses the global attractor $\mathcal{A}^1_\delta$ on H_δ (we have here the existence of bounded absorbing sets for e small enough). Furthermore, setting

$$X_\delta = \overline{\bigcup_{t \geqslant t_1} S(t)\mathcal{B}^1_\delta},$$

where $\mathcal{B}^1_\delta$ is a bounded absorbing set on H_δ and t_1 is such that $t \geqslant t_1$ implies that $S(t)\mathcal{B}^1_\delta$ belongs to a bounded absorbing set on $H_\delta \cap V$, we obtained the existence of an exponential attractor $\mathcal{M}^1_\delta$ on X_δ (we note that $\mathcal{A}^1_\delta \subset X_\delta$). Thus, thanks to the above results, the weakly coupled system

$$\begin{aligned} \mathcal{F}(t)\colon\ H_\delta &\to H_\delta \times \mathcal{L}(H_\delta) \\ \varrho_0 &\mapsto (S(t)\varrho_0, \mathcal{L} \circ S(t)\varrho_0), \end{aligned}$$

possesses the minimal attractor $\mathcal{A}_\delta = \{(a_0, \mathcal{L}(a_0)),\ \ a_0 \in \mathcal{A}^1_\delta\}$ on H_δ and an exponential attractor $\mathcal{M}_\delta = \{(a_0, \mathcal{L}(a_0)),\ \ a_0 \in \mathcal{M}^1_\delta\}$ on X_δ. In particular, we deduce that $\mathcal{A}_\delta$ has a finite fractal dimension with $\dim_F \mathcal{A}_\delta \leqslant \dim_F \mathcal{A}^1_\delta$ and $\dim_F \mathcal{A}_\delta = \dim_F \mathcal{A}^1_\delta$ if CI is positive definite.

Remark 1.7. In [16], the authors introduced the notion of an inertial manifold, which is a smooth (at least Lipschitz; we note that an exponential attractor is not regular in general) hyperbolic (and thus stable, contrary to the global attractor) finite dimensional manifold which contains the global attractor and attracts exponentially the trajectories. Furthermore, it yields an interaction law between the small and large structures of the flow. This object is thus particularly interesting for the study of the long time behavior of the system. In the case of a system of the form (1.1), we can give a definition similar to Definition 1.2 for an inertial manifold. Furthermore, for a weakly coupled system of the form (1.3), we can prove a result similar to Theorem 1.5 (in that case, the regularity of the inertial manifold constructed will depend on the regularity of the inertial manifold for $S(t)$ and on the regularity of $\mathcal{L}$). Unfortunately, all the known constructions of inertial manifolds are based on a very restrictive condition, namely the so-called spectral gap condition (see [16]). Consequently, the existence of inertial manifolds is not known for many physically important equations (e.g. the Navier-Stokes equations, even in two space dimensions) and a nonexistence result has even been obtained for a reaction-diffusion equation in higher space dimensions (see [28]), hence the interest (and an additional motivation) for the notion of the exponential attractor. Actually, an exponential attractor may be viewed as an intermediate object between the two ideal objects that the global attractor and an inertial manifold are.

2. The nonautonomous case

We consider in this section operators that are, typically, associated with partial differential equations in which the time appears explicitly (in the operators and/or in the forcing terms). Furthermore, we shall consider the approach initiated in [21] and further generalized and developed in [9] which consists in studying actually a whole family of equations. Of course, we could very well take one equation only; we would then be in the general framework developed in Section 1 (see also [30], [31] and Remark 2.2 below). We refer the interested reader to [18], [29] and [32] for other approaches to nonautonomous systems.

We consider in this section a family of operators depending on a parameter $\sigma \in \mathbb{T}^k$, $\mathbb{T}^k$ being the k-dimensional torus

$$\mathcal{F}_\sigma(t,\tau)\colon\ E_0 \to G, \quad t \geqslant \tau, \quad \tau \in \mathbb{R}, \tag{2.1}$$

where the spaces are as in the previous section, and we give the following definitions (inspired by [9] and [21]):

Definition 2.1.

(i) A compact set $\mathcal{A}_{\mathbb{T}^k} \subset G$ is a uniform (with respect to σ) attractor for $\mathcal{F}_\sigma(t,\tau)$ on E_0 if for every $B \subset E_0$ bounded

$$\lim_{t\to+\infty} \sup_{\sigma\in\mathbb{T}^k} \operatorname*{dist}_G(\mathcal{F}_\sigma(t,\tau)B, \mathcal{A}_{\mathbb{T}^k}) = 0, \ \ \forall \tau \in \mathbb{R}.$$

(ii) A compact set $\mathcal{A}_{\mathbb{T}^k} \subset G$ is the minimal uniform attractor for $\mathcal{F}_\sigma(t,\tau)$ on E_0 if it is a uniform attractor and if it is minimal among the closed sets that attract uniformly (with respect to σ) the bounded sets of E_0 for $\mathcal{F}_\sigma(t,\tau)$.

We have the following result, whose proof is exactly the same as those of Proposition 5.1 and Theorem 5.2 of [9] (see also [21]):

Theorem 2.1. *We assume that $\mathcal{F}_\sigma(t,\tau)$ possesses a compact uniformly (with respect to σ) attracting set P on E_0. Then the set*

$$\mathcal{A}_{\mathbb{T}^k} = \overline{\bigcup_{\tau\in\mathbb{R}} \bigcup_{n\in\mathbb{N}} \omega_{\tau,\mathbb{T}^k}(B_n)},$$

where

$$B_n = \{x \in E_0,\ \ \|x\|_{E_0} \leqslant n\}, \quad n \in \mathbb{N},$$

and

$$\omega_{\tau,\mathbb{T}^k}(B_n) = \bigcap_{t\geqslant\tau} \overline{\bigcup_{\sigma\in\mathbb{T}^k} \bigcup_{s\geqslant t} \mathcal{F}_\sigma(s,\tau)B_n},$$

is the minimal uniform attractor for $\mathcal{F}_\sigma(t,\tau)$ on E_0.

We now consider a weakly coupled dynamical system of the form

$$\begin{aligned} \mathcal{F}_\sigma(t,\tau)\colon\ & E_0 \to E_0 \times E_1 \\ & \varrho_0 \mapsto (U_\sigma(t,\tau)\varrho_0, \mathcal{L}\circ U_\sigma(t,\tau)\varrho_0), \end{aligned} \tag{2.2}$$

$\sigma \in \mathbb{T}^k$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, where E_0, E_1 and $\mathcal{L}$ are as in Section 2 and $U_\sigma(t,\tau)$ is a family of processes on E_0 (see [9]). Let us assume that $\mathcal{L}$ is Lipschitz (actually, it suffices to have $\mathcal{L}$ uniformly continuous) and that $U_\sigma(t,\tau)$ possesses a compact uniformly attracting set P_0 on E_0. Then we easily prove that $P = P_0 \times \mathcal{L}(P_0)$ is a compact uniformly attracting set for $\mathcal{F}_\sigma(t,\tau)$ on E_0. Thus, according to Theorem 2.1, $\mathcal{F}_\sigma(t,\tau)$ possesses the minimal uniform attractor $\mathcal{A}_{\mathbb{T}^k}$ on E_0. Now, $U_\sigma(t,\tau)$ possesses the uniform attractor (in the sense of [9]) $\mathcal{A}^0_{\mathbb{T}^k}$ on E_0 and we easily prove that the set

$$\tilde{A}_{\mathbb{T}^k} = \{(a_0, \mathcal{L}(a_0)),\ \ a_0 \in \mathcal{A}^0_{\mathbb{T}^k}\}$$

is a uniform attractor for $\mathcal{F}_\sigma(t,\tau)$ on E_0. However, contrary to the autonomous case, we are not able to prove that $\mathcal{A}_{\mathbb{T}^k} = \tilde{A}_{\mathbb{T}^k}$ in general (of course, we have $\mathcal{A}_{\mathbb{T}^k} \subset \tilde{A}_{\mathbb{T}^k}$). Let us now further assume that $U_{as+\sigma}(t,\tau) = U_\sigma(t+s,\tau+s)$, $\forall t \geqslant \tau \in \mathbb{R}$, $\forall s \in \mathbb{R}$, $\forall \sigma \in \mathbb{T}^k$, where $a = (a_1,\dots,a_k) \in \mathbb{R}^k$ is such that the a_i are rationally independent. It is proved in [9] that

$$\mathcal{A}^0_{\mathbb{T}^k} = \bigcup_{\sigma\in\mathbb{T}^k} \{u(0),\ \ u \text{ is a bounded complete trajectory of } U_\sigma(t,\tau)\}.$$

Consequently, we have

$$\tilde{\mathcal{A}}_{\mathbb{T}^k} = \bigcup_{\sigma\in\mathbb{T}^k} \{(u(0),\mathcal{L}(u(0))),\ \ u \text{ is a bounded complete trajectory of } U_\sigma(t,\tau)\}.$$

Let $\mathcal{A}'_{\mathbb{T}^k} \subset E_0 \times E_1$ be a closed set that attracts uniformly the bounded sets of E_0 and let u be a bounded complete trajectory of $U_\sigma(t,\tau)$, $\sigma \in \mathbb{T}^k$ being given. We have $u(0) = U_\sigma(0,-n)u(-n) = U_{T(n)\sigma(-n)}(0,-n)u(-n) = U_{\sigma(-n)}(n,0)u(-n)$, where $T(s)\sigma = \sigma(s) = as + \sigma \pmod{\mathbb{T}^k}$, $\forall s \in \mathbb{R}$, $\forall \sigma \in \mathbb{T}^k$. Therefore, $(u(0),\mathcal{L}(u(0))) = \mathcal{F}_{\sigma(-n)}(n,0)u(-n)$. We set $B = \{u(-n),\ \ n \in \mathbb{N}\}$. We easily see that B is bounded in E_0 and we have

$$\begin{aligned} \operatorname*{dist}_{E_0\times E_1}((u(0),\mathcal{L}(u(0))),\mathcal{A}'_{\mathbb{T}^k}) &= \operatorname*{dist}_{E_0\times E_1}(\mathcal{F}_{\sigma(-n)}(n,0)u(-n),\mathcal{A}'_{\mathbb{T}^k}) \\ &\leqslant \sup_{\sigma\in\mathbb{T}^k} \operatorname*{dist}_{E_0\times E_1}(\mathcal{F}_\sigma(n,0)B,\mathcal{A}'_{\mathbb{T}^k}), \end{aligned}$$

which yields that $(u(0),\mathcal{L}(u(0))) \in \mathcal{A}'_{\mathbb{T}^k}$, hence the minimality property for $\tilde{\mathcal{A}}_{\mathbb{T}^k}$. We thus deduce that $\mathcal{A}_{\mathbb{T}^k} = \tilde{\mathcal{A}}_{\mathbb{T}^k}$.

R e m a r k 2.1. We have implicitly considered only nonautonomous systems with a quasiperiodic dependence on time here (by taking $\mathbb{T}^k$ as the symbol space). Of course, we can consider a more general time dependence as in [9]. However, we are able to obtain finite dimensional attractors only in the case of a quasiperiodic dependence (see [9] for more details).

R e m a r k 2.2. Of course, for every fixed $\sigma \in \mathbb{T}^k$, we can construct, assuming for instance that the assumptions of Theorem 2.1 are satisfied, the (nonuniform) minimal attractor $\mathcal{A}_\sigma$ (in the sense of Definition 1.1). We easily prove that $\bigcup_{\sigma \in \mathbb{T}^k} \mathcal{A}_\sigma \subset \mathcal{A}_{\mathbb{T}^k}$. However, this inclusion can be strict, see [21]; the two sets coincide when $k = 1$ (i.e. when considering periodic time dependence; actually, in that case, $\mathcal{A}_\sigma$ does not depend on $\sigma \in \mathbb{T}^1$, see [21] and [31]). Also, it is proved in [31] that, in the periodic case (and also in the asymptotically periodic case), one can use the approach of [18] to prove the finite dimensionality of $\mathcal{A}_\sigma$. More generally, one interest of studying a whole family of equations instead of a single one is that, for a family of processes (and thus for a family of weakly coupled systems), one can generally reduce the problem of the research of the uniform attractor to that of the research of the global attractor for a suitable semigroup. This then allows to study the finite dimensionality of the minimal attractor (see [9] for more details). Here, by considering the family of operators

$$\begin{aligned} \overline{\mathcal{F}}(t)\colon\ & E_0 \times \mathbb{T}^k \to G \times \mathbb{T}^k \\ & (\varrho_0, \sigma) \mapsto (\mathcal{F}_\sigma(t,0)\varrho_0, at + \sigma \pmod{\mathbb{T}^k}), \end{aligned}$$

$t \geqslant 0$, we can, in the spirit of [9], transform the nonautonomous problem into an autonomous one on an extended phase space (again, we see the interest of studying a family of operators). The next step would then be, if $\mathcal{A}$ denotes the minimal attractor for $\overline{\mathcal{F}}(t)$, to prove that $\mathcal{A}_{\mathbb{T}^k} = \Pi_0\mathcal{A}$, Π_0 denoting the projector on the first component, is the minimal uniform attractor for $\mathcal{F}_\sigma(t,\tau)$. Unfortunately, for general operators, we are only able to prove that $\mathcal{A}_{\mathbb{T}^k}$ is a uniform attractor. Indeed, to proceed as in [9], we would need, in order to prove the minimality property, to prove that $\mathcal{A}$ is made of all the bounded complete trajectories of $\overline{\mathcal{F}}(t)$. Now, in the case of the global attractor, this property is a consequence of the invariance.

R e m a r k 2.3. As in Section 1, we can define a notion of (uniform in that case) exponential attractor for a family of operators of the form (2.1) (we refer the interested reader to [1], [14], [15] and [23] for constructions of exponential attractors for nonautonomous (quasiperiodic) systems). Furthermore, for a weakly coupled system of the form (2.2), we can prove, as in Theorem 1.5, the existence of a uniform (with respect to σ) exponential attractor (of course, we shall not have partial semi-invariance properties in that case). We note that in the nonautonomous case, the attractor, and a fortiori an exponential attractor, has in general infinite dimension (see [9] and also Remark 2.1), even for physically relevant situations (e.g. a cascade system constructed on an equation in $\mathbb{R}^3$, see [13]). We proposed in [13] a generalization of the notion of an exponential attractor by using the notion of the Kolmogorov ε-entropy in that case (we recall that, by definition, an exponential attractor has a finite fractal dimension).

Remark 2.4. We can obtain results similar to those obtained in Example 1.2 in the nonautonomous case by considering (quasiperiodic) external actions (external mass supply and external microforces, see [27]).

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