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RECOVERY OF AN UNKNOWN FLUX IN PARABOLIC PROBLEMS WITH NONSTANDARD BOUNDARY CONDITIONS: ERROR ESTIMATES

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Abstract. In this paper, we consider a 2nd order semilinear parabolic initial boundary value problem (IBVP) on a bounded domain $\Omega \subset \mathbb{R}^N$, with nonstandard boundary conditions (BCs). More precisely, at some part of the boundary we impose a Neumann BC containing an unknown additive space-constant $\alpha(t)$, accompanied with a nonlocal (integral) Dirichlet side condition.

We design a numerical scheme for the approximation of a weak solution to the IBVP and derive error estimates for the approximation of the solution u and also of the unknown function α.

Keywords: nonlocal boundary condition, parameter identification, parabolic IBVP

MSC 2000: 35K20, 35B30

1. Introduction

We study a transient IBVP for a semilinear parabolic partial differential equation of the second order of the type

$$\frac{\partial u}{\partial t} - \Delta u = f(u) \quad \text{in } (0,T)\times\Omega \tag{1}$$

along with an initial condition

$$u(0) = u_0 \quad \text{in } \Omega, \tag{2}$$

and the boundary conditions

$$
\begin{aligned}
u &= g_{\mathrm{Dir}} && \text{in} \quad (0,T) \times \Gamma_{\mathrm{Dir}}, \\
-\nabla u \cdot \boldsymbol{\nu} - g_{\mathrm{Rob}} u &= g_{\mathrm{Neu}} && \text{in} \quad (0,T) \times \Gamma_{\mathrm{Neu}}, \\
-\nabla u \cdot \boldsymbol{\nu} &= g_{\mathrm{non}} + \alpha && \text{in} \quad (0,T) \times \Gamma_{\mathrm{non}}, \\
\int_{\Gamma_{\mathrm{non}}} u \,\mathrm{d}\gamma &= w.
\end{aligned} \tag{3}
$$

Here, $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ which is split into three mutually disjoint parts $\Gamma_{\mathrm{Dir}}, \Gamma_{\mathrm{Neu}}$ and Γ_{non}. We assume

$$
|\Gamma_{\mathrm{non}}| > 0, \quad \overline{\Gamma_{\mathrm{non}}} \cap \overline{\Gamma_{\mathrm{Dir}}} = \emptyset, \tag{4}
$$

i.e., Γ_{non} and Γ_{Dir} are not adjacent. Further, we suppose that the function g_{Dir} can be prolonged to the whole domain Ω in such a way that

$$
\begin{gathered}
\tilde{g} \in L_2((0,T), H^1(\Omega)), \quad \frac{\partial \tilde{g}}{\partial t} \in L_2((0,T), L_2(\Omega)), \\
\tilde{g} = \begin{cases} 0 & \text{in } [0,T] \times \Gamma_{\mathrm{non}}, \\ g_{\mathrm{Dir}} & \text{in } [0,t] \times \Gamma_{\mathrm{Dir}}. \end{cases}
\end{gathered} \tag{5}
$$

The right-hand side f and the data functions $g_{\mathrm{Neu}}, g_{\mathrm{non}}, g_{\mathrm{Rob}}$ and w obey

$$
\begin{gathered}
\exists C > 0\colon\ |f(x) - f(y)| \leqslant C|x-y|, \ \ \forall x, y \in \mathbb{R} \\
0 \leqslant g_{\mathrm{Rob}} \leqslant C, \ \ g_{\mathrm{Neu}} \in L_2((0,T), L_2\Gamma_{\mathrm{Neu}})), \ \ g_{\mathrm{non}} \in L_2((0,T), L_2(\Gamma_{\mathrm{non}})), \\
w \in C([0,T]).
\end{gathered} \tag{6}
$$

Thus, the Dirichlet boundary condition is prescribed on Γ_{Dir}, and there is a Robin type BC on Γ_{Neu}. We consider nonstandard boundary conditions on the part Γ_{non}. Here, neither the solution nor the flux are prescribed pointwise. Instead, the time dependent *average* of the solution over Γ_{non} is given and the flux along Γ_{non} has to follow a prescribed shape function g_{non}, apart from an additive (*unknown*) time function α which is independent of the space variable.

Such a type of IBVPs arises from some specific heat transfer problems, where at one part of the boundary the average temperature at each time is prescribed, while from other reasons one knows that the heat flux should follow a given shape up to an additive space constant. The problem consists of finding the solution $u(t)$ and of determining the unknown function $\alpha(t)$ for all $t \in [0,T]$, in order to get the full description of the flux at Γ_{non}. Of course, in a realistic model, the coefficients in the differential equation depend on some material data functions. This is omitted here in order to focus the attention on the nonlocal BC.

Various mathematical models containing nonlocal BCs can be found in literature, e.g., in Friedman [6], p. 520 in the so called *plasma problem*; in the computation of the electromagnetic losses in a lamination of an electric machine—see Van Keer, Dupré and Melkebeek [15]; in Navier-Stokes equations cf. Heywood, Rannacher and Turek [10]; or in the Stokes problem, cf. Bramble, Lee [3]. Further, nonstandard BCs have also been studied in Andreucci and Gianni [1], De Schepper and Slodička [4], [13]; Pao [9], Slodička [12], Van Keer and Slodička [17].

The IBVP (1)–(3) has already been considered by Van Keer and Slodička [16], where the uniqueness of a weak solution has been shown. Moreover, the authors have designed a numerical scheme for the approximation of an exact solution but they did not discuss the existence of solution and the error analysis.

The main purpose of this paper is to show both the convergence of the algorithm and to derive error estimates for the numerical scheme from Van Keer and Slodička [15]. The time discretization is based on Rothe's method, see Kačur [7] or Rektorys [11]. After linearization, we are left with a recurrent system of linear elliptic BVPs at each successive time point t_i of a suitable time partitioning. We use the ideas from De Schepper and Slodička [4] to prove the existence of a weak solution u_i at each time step t_i.

Next, we establish a priori estimates for u_i and α_i, which is the main difficulty because of the fact that the bounds for u_i must be independent of α_i. Later, using the a priori estimates, we prove the convergence of the approximate solution, viz. a Rothe's function constructed in terms of all u_i, to the exact one. The convergence depends clearly on the properties of u_0 and w. We derive formulae for practical computation of α_i which is an approximation of $\alpha(t_i)$.

2. Time discretization

We denote by $(w,z)_M$ the usual L_2 scalar product of any real or vector-valued functions w and z on a set M, i.e., $(w,z)_M = \int_M wz$. The corresponding norm is introduced by $\|w\|_{0,M} = \sqrt{(w,w)_M}$. Let $H^1(\Omega)$ be the first-order Hilbert space equipped with the usual norm

$$\|w\|_{1\Omega}^2 = (w,w)_\Omega + (\nabla w, \nabla w)_\Omega = \|w\|_{0,\Omega}^2 + |w|_{1,\Omega}^2.$$

We introduce the following subspace of $H^1(\Omega)$:

$$V = \{\varphi \in H^1(\Omega);\ \varphi = 0 \text{ on } \Gamma_{\text{Dir}}\},$$

which is endowed with the induced norm $\|\cdot\|_{1,\Omega}$ from the space $H^1(\Omega)$. If $|\Gamma_{\mathrm{Dir}}| > 0$, then one can also use the semi-norm $|\cdot|_{1,\Omega}$ instead of the norm.

The variational formulation of the IBVP (1)–(3) reads as:

Problem 1. *Find a couple* (u, α) *such that*

1. $u \in C([0,T], L_2(\Omega)) \cap L_\infty((0,T), H^1(\Omega))$,
2. $\partial u/\partial t \in L_2((0,T), L_2(\Omega))$,
3. $u = g_{\mathrm{Dir}}$ *on* $(0,T) \times \Gamma_{\mathrm{Dir}}$,
4. $u(0) = u_0$,
5. $\alpha \in L_2((0,T))$

and

$$\begin{aligned}(7)\qquad & \Big(\frac{\partial u(t)}{\partial t}, \varphi\Big)_\Omega + (\nabla u(t), \nabla\varphi)_\Omega + (\alpha(t), \varphi)_{\Gamma_{\mathrm{non}}} + (g_{\mathrm{Rob}}(t)u(t), \varphi)_{\Gamma_{\mathrm{Neu}}} \\ &\qquad = (f(u(t)), \varphi)_\Omega - (g_{\mathrm{Neu}}(t), \varphi)_{\Gamma_{\mathrm{Neu}}} - (g_{\mathrm{non}}(t), \varphi)_{\Gamma_{\mathrm{non}}}, \\ &\qquad \int_{\Gamma_{\mathrm{non}}} u(t) = w(t)\end{aligned}$$

holds for all $\varphi \in V$ *and for almost all* $t \in [0,T]$.

We divide the time interval $[0,T]$ into n equidistant subintervals (t_{i-1}, t_i) for $t_i = i\tau$, where $\tau = T/n$. We introduce the notation

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}$$

for any function z. We are left with a recurrent system of linear elliptic BVPs at each successive time point t_i for $i = 1, \ldots, n$:

Problem 2. *Find a couple* $(u_i, \alpha_i) \in H^1(\Omega) \times \mathbb{R}$ *such that* $u_i - \tilde{g}_i \in V$ *and*

$$\begin{aligned}(8)\qquad & (\delta u_i, \varphi)_\Omega + (\nabla u_i, \nabla\varphi)_\Omega + (\alpha_i, \varphi)_{\Gamma_{\mathrm{non}}} + (g_{\mathrm{Rob_i}} u_i, \varphi)_{\Gamma_{\mathrm{Neu}}} \\ &\qquad = (f(u_{i-1}), \varphi)_\Omega - (g_{\mathrm{Neu_i}}, \varphi)_{\Gamma_{\mathrm{Neu}}} - (g_{\mathrm{non_i}}, \varphi)_{\Gamma_{\mathrm{non}}}, \\ &\qquad \int_{\Gamma_{\mathrm{non}}} u_i = w_i\end{aligned}$$

holds for any $\varphi \in V$.

We recall that the initial datum u_0 is given. The existence of a weak solution (u_i, α_i) at each t_i follows from De Schepper and Slodička [4]. We sketch the proof to enhance the readability of the paper.

Consider the following two auxiliary problems at any time step t_i:

Find $v_i \in H^1(\Omega)$ such that $v_i - \tilde{g}_i \in V$,

$$(9)\quad \Big(\frac{v_i}{\tau}, \varphi\Big)_\Omega + (\nabla v_i, \nabla \varphi)_\Omega + (g_{\mathrm{Rob_i}} v_i, \varphi)\Gamma_{\mathrm{Neu}} = (f(u_{i-1}), \varphi)_\Omega - (g_{\mathrm{Neu_i}}, \varphi)_{\Gamma_{\mathrm{Neu}}} - (g_{\mathrm{non_i}}, \varphi)_{\Gamma_{\mathrm{non}}} + \Big(\frac{u_{i-1}}{\tau}, \varphi\Big)_\Omega \quad \forall \varphi \in V.$$

Find $z_i \in H^1(\Omega)$ such that $z_i = 0$ on Γ_{Dir},

$$(10)\qquad \Big(\frac{z_i}{\tau}, \varphi\Big)_\Omega + (\nabla z_i, \nabla \varphi)_\Omega + (g_{\mathrm{Rob_i}} z_i, \varphi)_{\Gamma_{\mathrm{Neu}}} = -(1, \varphi)_{\Gamma_{\mathrm{non}}} \quad \forall \varphi \in V.$$

Both problems admit unique weak solutions v_i, z_i for all $i = 1, \ldots, n$, which follows from the Lax-Milgram lemma.

We define the integral operator $P(h) = \int_{\Gamma_{\mathrm{non}}} h$. We are looking for an α_i satisfying $P(v_i) + \alpha_i P(z_i) = w_i$. Clearly

$$(11)\qquad \alpha_i = \frac{w_i - P(v_i)}{P(z_i)}.$$

This choice gives rise to the solution $(u_i, \alpha_i) = (v_i + \alpha_i z_i, \alpha_i)$ to the BVP 2, which can be obtained by the principle of superposition.

3. A priori estimates

The main goal of this section is to establish suitable a priori estimates for u_i and α_i, which allow us to prove the convergence in some functional spaces of the approximate solution to the exact one. The crucial point in the technique of the proof is a suitable choice of the test functions in the variational setting, which allows us to separate both unknown functions u_i and α_i from each other.

Lemma 1. *Let* (4), (5), (6) *and* $\mathrm{d}w/\mathrm{d}t \in L_2((0,T))$ *be satisfied. Moreover, we assume* $u_0 \in L_2(\Omega)$. *Then there exists a positive constant* C *such that*

(i) $\|u_j\|^2_{0,\Omega} + \sum\limits_{i=1}^{j} \|u_i - u_{i-1}\|^2_{0,\Omega} + \sum\limits_{i=1}^{j} |u_i|^2_{1,\Omega}\tau \leqslant C,$

(ii) $\Big|\sum\limits_{i=1}^{j} \alpha_i \tau\Big| \leqslant C$

for all $j = 1, \ldots, n$.

Proof. (i) Let us fix a function $\Phi \in C^\infty(\overline{\Omega})$ such that

$$(12)\qquad \Phi = \begin{cases} 0 & \text{on } \Gamma_{\mathrm{Dir}}, \\ 1 & \text{on } \Gamma_{\mathrm{non}}. \end{cases}$$

The existence of such a function follows from Friedman [5], Lemma 5.1, because of $\overline{\Gamma_{\rm non}} \cap \overline{\Gamma_{\rm Dir}} = \emptyset$.

By virtue of the fact that both $u_i - \tilde{g}_i$ and $w_i \Phi / |\Gamma_{\rm non}|$ belong to the space V, we have $\varphi = u_i - \tilde{g}_i - w_i \Phi/|\Gamma_{\rm non}| \in V$. For such a choice of the test function we get

$$\int_{\Gamma_{\rm non}} \varphi = \left[\int_{\Gamma_{\rm non}} u_i\right] - w_i - \int_{\Gamma_{\rm non}} \tilde{g}_i = 0.$$

Therefore, setting $\varphi = u_i - \tilde{g}_i - w_i \Phi/|\Gamma_{\rm non}|$ in (8a) we obtain

$$\begin{aligned}(13)\quad &(\delta u_i, u_i)_\Omega + (\nabla u_i, \nabla u_i)_\Omega + (g_{{\rm Rob}_{\rm i}} u_i, u_i)_{\Gamma_{\rm Neu}} \\ &= \Bigl(f(u_{i-1}), u_i - \tilde{g}_i - \frac{w_i}{|\Gamma_{\rm non}|}\Phi\Bigr)_\Omega - \Bigl(g_{{\rm Neu}_{\rm i}}, u_i - \tilde{g}_i - \frac{w_i}{|\Gamma_{\rm non}|}\Phi\Bigr)_{\Gamma_{\rm Neu}} \\ &\quad - \Bigl(g_{{\rm non}_{\rm i}}, u_i - \tilde{g}_i - \frac{w_i}{|\Gamma_{\rm non}|}\Phi\Bigr)_{\Gamma_{\rm non}} + \Bigl(\delta u_i, \tilde{g}_i + \frac{w_i}{|\Gamma_{\rm non}|}\Phi\Bigr)_\Omega \\ &\quad + \Bigl(\nabla u_i, \nabla\Bigl[\tilde{g}_i + \frac{w_i}{|\Gamma_{\rm non}|}\Phi\Bigr]\Bigr)_\Omega + \Bigl(g_{{\rm Rob}_{\rm i}} u_i, \tilde{g}_i + \frac{w_i}{|\Gamma_{\rm non}|}\Phi\Bigr)_{\Gamma_{\rm Neu}}.\end{aligned}$$

Let $j \in \{1, \ldots, n\}$. Now we multiply the equation by the time step τ and sum it up for $i = 1, \ldots, j$, i.e., we integrate the equality with respect to the time. Then we estimate the terms on the left-hand side from below, and the ones on the right-hand side from above. We do it in a few steps.

Taking into account the non-negativity of the function $g_{\rm Rob}$ (see (6)), we estimate the left-hand side of (13) from below as follows:

$$\begin{aligned}(14)\qquad &\sum_{i=1}^{j} [(\delta u_i, u_i)_\Omega + (\nabla u_i, \nabla u_i)_\Omega + (g_{{\rm Rob}_{\rm i}} u_i, u_i)_{\Gamma_{\rm Neu}}]\tau \\ &\geqslant \frac{1}{2}\Bigl[\|u_j\|^2_{0,\Omega} - \|u_0\|^2_{0,\Omega} + \sum_{i=1}^{j} \|u_i - u_{i-1}\|^2_{0,\Omega} + \sum_{i=1}^{j} |u_i|^2_{1,\Omega}\tau\Bigr].\end{aligned}$$

We recall that the function f is globally Lipschitz continuous (see (6)). Applying the Cauchy inequality to the term containing the function f, we get

$$\begin{aligned}(15)\qquad &\Bigl|\sum_{i=1}^{j}\Bigl(f(u_{i-1}), u_i - \tilde{g}_i - \frac{w_i}{|\Gamma_{\rm non}|}\Phi\Bigr)_\Omega \tau\Bigr| \\ &\qquad\leqslant C\Bigl[1 + \sum_{i=0}^{j} \|u_i\|^2_{0,\Omega}\tau + \sum_{i=1}^{j} \|\tilde{g}_i\|^2_{0,\Omega}\tau + \sum_{i=1}^{j} w_i^2\tau\Bigr] \\ &\qquad\leqslant C\Bigl[1 + \sum_{i=0}^{j} \|u_i\|^2_{0,\Omega}\tau\Bigr].\end{aligned}$$

Applying the well known inequality for real numbers $|ab| \leqslant \varepsilon a^2 + C_\varepsilon b^2$ (here $\varepsilon \in \mathbb{R}_+$ and $C_\varepsilon = C(1/\varepsilon)$), we deduce

$$\text{(16)}\qquad \Big|\sum_{i=1}^{j}\Big(\nabla u_i, \nabla\Big[\tilde{g}_i + \frac{w_i}{|\Gamma_{\text{non}}|}\Phi\Big]\Big)_\Omega \tau\Big| \leqslant \varepsilon \sum_{i=1}^{j} |u_i|_{1,\Omega}^2 \tau + C_\varepsilon \sum_{i=1}^{j} \big[\|\tilde{g}_i\|_{1,\Omega}^2 + w_i^2\big]\tau$$
$$\leqslant \varepsilon \sum_{i=1}^{j} |u_i|_{1,\Omega}^2 \tau + C_\varepsilon.$$

For the boundary terms we use the Cauchy inequality, the trace theorem and obtain

$$\text{(17)}\qquad \Big|\sum_{i=1}^{j}\Big(g_{\text{Neu}_\text{i}}, u_i - \tilde{g}_i - \frac{w_i}{|\Gamma_{\text{non}}|}\Phi\Big)_{\Gamma_{\text{Neu}}} \tau\Big|$$
$$\leqslant \varepsilon \sum_{i=1}^{j} \Big\|u_i - \tilde{g}_i - \frac{w_i}{|\Gamma_{\text{non}}|}\Phi\Big\|_{0,\Gamma_{\text{Neu}}}^2 \tau + C_\varepsilon \sum_{i=1}^{j} \|g_{\text{Neu}_\text{i}}\|_{0,\Gamma_{\text{Neu}}}^2 \tau$$
$$\leqslant C_\varepsilon + C_\varepsilon \sum_{i=1}^{j} \big[\|\tilde{g}_i\|_{0,\partial\Omega}^2 + w_i^2\big]\tau + \varepsilon \sum_{i=1}^{j} \|u_i\|_{0,\partial\Omega}^2 \tau$$
$$\leqslant \varepsilon \sum_{i=1}^{j} |u_i|_{1,\Omega}^2 \tau + C_\varepsilon \sum_{i=1}^{j} \|u_i\|_{0,\Omega}^2 \tau + C_\varepsilon.$$

In the same way we deduce

$$\text{(18)}\quad \Big|\sum_{i=1}^{j}\Big(g_{\text{non}_\text{i}}, u_i - \tilde{g}_i - \frac{w_i}{|\Gamma_{\text{non}}|}\Phi\Big)_{\Gamma_{\text{non}}} \tau\Big| \leqslant \varepsilon \sum_{i=1}^{j} |u_i|_{1,\Omega}^2 \tau + C_\varepsilon \sum_{i=1}^{j} \|u_i\|_{0,\Omega}^2 \tau + C_\varepsilon$$

and

$$\text{(19)}\quad \Big|\sum_{i=1}^{j}\Big(g_{\text{Rob}_\text{i}} u_i, \tilde{g}_i + \frac{w_i}{|\Gamma_{\text{non}}|}\Phi\Big)_{\Gamma_{\text{Neu}}} \tau\Big| \leqslant \varepsilon \sum_{i=1}^{j} |u_i|_{1,\Omega}^2 \tau + C_\varepsilon \sum_{i=1}^{j} \|u_i\|_{0,\Omega}^2 \tau + C_\varepsilon.$$

It remains to estimate the sum containing $\big(\delta u_i, \tilde{g}_i + w_i\Phi/|\Gamma_{\text{non}}|\big)_\Omega$. First, we apply the summation by parts and get

$$\sum_{i=1}^{j}\Big(\delta u_i, \tilde{g}_i + \frac{w_i}{|\Gamma_{\text{non}}|}\Phi\Big)_\Omega \tau = \Big(u_j, \tilde{g}_j + \frac{w_j}{|\Gamma_{\text{non}}|}\Phi\Big)_\Omega - \Big(u_0, \tilde{g}_0 + \frac{w_0}{|\Gamma_{\text{non}}|}\Phi\Big)_\Omega$$
$$- \sum_{i=1}^{j}\Big(u_{i-1}, \delta\tilde{g}_i + \frac{\delta w_i}{|\Gamma_{\text{non}}|}\Phi\Big)_\Omega \tau.$$

The assumption (5) gives $\tilde{g} \in C([0,T], L_2(\Omega))$, and similarly the fact $\mathrm{d}w/\mathrm{d}t \in L_2((0,T))$ yields $w \in C([0,T])$. Therefore, we get in the standard way

$$(20) \qquad \left|\sum_{i=1}^{j}\Big(\delta u_i, \tilde{g}_i + \frac{w_i}{|\Gamma_{\mathrm{non}}|}\Phi\Big)_\Omega \tau\right| \leqslant \varepsilon \|u_j\|_{0,\Omega}^2 + C_\varepsilon\Big[1 + \sum_{i=1}^{j}\|u_i\|_{0,\Omega}^2\tau\Big].$$

Summarizing (13)–(20) we obtain

$$(1-\varepsilon)\Big[\|u_j\|_{0,\Omega}^2 + \sum_{i=1}^{j}\|u_i - u_{i-1}\|_{0,\Omega}^2 + \sum_{i=1}^{j}|u_i|_{1,\Omega}^2\tau\Big] \leqslant C_\varepsilon\Big[1 + \sum_{i=1}^{j}\|u_i\|_{0,\Omega}^2\tau\Big].$$

Now, we choose $\varepsilon \in (0,1)$ and apply Gronwall's lemma to conclude the proof of part (i).

(ii) Let us return to the identity (8a). We multiply it by the time step τ and sum it up for $i = 1\ldots,j$. We arrive at $(\forall \varphi \in V)$

$$\begin{aligned}(u_j - u_0, \varphi)_\Omega + \Big(\sum_{i=1}^{j}\nabla u_i\tau, \nabla\varphi\Big)_\Omega + \Big(\sum_{i=1}^{j}\alpha_i\tau, \varphi\Big)_{\Gamma_{\mathrm{non}}} + \Big(\sum_{i=1}^{j}g_{\mathrm{Rob_i}}u_i\tau, \varphi\Big)_{\Gamma_{\mathrm{Neu}}} \\ = \Big(\sum_{i=1}^{j}f(u_{i-1})\tau, \varphi\Big)_\Omega - \Big(\sum_{i=1}^{j}g_{\mathrm{Neu_i}}\tau, \varphi\Big)_{\Gamma_{\mathrm{Neu}}} - \Big(\sum_{i=1}^{j}g_{\mathrm{non_i}}\tau, \varphi\Big)_{\Gamma_{\mathrm{non}}}.\end{aligned}$$

Inserting $\varphi = \Phi$ and using Lemma 1 (i), we conclude the proof in a straightforward way. □

Stronger assumptions on the initial data u_0 and on the BCs allow us to prove better a priori estimates than those given in Lemma 1.

Lemma 2. *Let (4), (5), (6) and $\mathrm{d}w/\mathrm{d}t \in L_2((0,T))$ be satisfied. Moreover, we assume $u_0 \in H^1(\Omega)$; $\partial g_{\mathrm{Rob}}/\partial t, \partial g_{\mathrm{Neu}}/\partial t \in L_2((0,T), L_2(\Gamma_{\mathrm{Neu}}))$; $\partial g_{\mathrm{non}}/\partial t \in L_2((0,T), L_2(\Gamma_{\mathrm{non}}))$ and $\partial\tilde{g}/\partial t \in L_2((0,T), H^1(\Omega))$. Then there exists a positive constant C such that*

(i) $\sum_{i=1}^{j}\|\delta u_i\|_{0,\Omega}^2\tau + |u_j|_{1,\Omega}^2 + \sum_{i=1}^{j}|u_i - u_{i-1}|_{1,\Omega}^2 \leqslant C,$

(ii) $\sum_{i=1}^{j}\alpha_i^2\tau \leqslant C$

hold for all $j = 1,\ldots,n$.

Proof. (i) We fix a function Φ satisfying (12). Using the fact that both $u_i - \tilde{g}_i$ and $w_i\Phi/|\Gamma_{\mathrm{non}}|$ belong to the space V, we have

$$\varphi = \Big(\delta u_i - \delta\tilde{g}_i - \frac{\delta w_i}{|\Gamma_{\mathrm{non}}|}\Phi\Big)\tau \in V.$$

For such a choice of the test function we get

$$\int_{\Gamma_{\rm non}} \varphi = \delta\Big(\Big[\int_{\Gamma_{\rm non}} u_i\Big] - w_i\Big)\tau - \int_{\Gamma_{\rm non}} (\tilde{g}_i - \tilde{g}_{i-1}) = 0.$$

Let $j \in \{1,\dots,n\}$. We set $\varphi = \big(\delta u_i - \delta\tilde{g}_i - \delta w_i/|\Gamma_{\rm non}|\big)\tau$ in (8a) and sum it up for $i = 1,\dots,j$. We obtain

$$\begin{aligned}
(21)\quad & \sum_{i=1}^{j} \|\delta u_i\|_{0,\Omega}^2\tau + \sum_{i=1}^{j}(\nabla u_i, \nabla[u_i - u_{i-1}])_\Omega + \sum_{i=1}^{j}(g_{\rm Rob_i} u_i, u_i - u_{i-1})_{\Gamma_{\rm Neu}} \\
&= \sum_{i=1}^{j}(f(u_{i-1}), \delta u_i)_\Omega\tau - \sum_{i=1}^{j}\Big(f(u_{i-1}), \delta\tilde{g}_i + \frac{\delta w_i}{|\Gamma_{\rm non}|}\Phi\Big)_\Omega\tau \\
&\quad - \sum_{i=1}^{j}(g_{\rm Neu_i}, u_i - u_{i-1})_{\Gamma_{\rm Neu}} + \sum_{i=1}^{j}\Big(g_{Neu_i}, \delta\tilde{g}_i + \frac{\delta w_i}{|\Gamma_{\rm non}|}\Phi\Big)_{\Gamma_{\rm Neu}}\tau \\
&\quad - \sum_{i=1}^{j}(g_{\rm non_i}, u_i - u_{i-1})_{\Gamma_{\rm non}} + \sum_{i=1}^{j}\Big(g_{\rm non_i}, \delta\tilde{g}_i + \frac{\delta w_i}{|\Gamma_{non}|}\Phi\Big)_{\Gamma_{\rm non}}\tau \\
&\quad + \sum_{i=1}^{j}\Big(\delta u_i, \delta\tilde{g}_i + \frac{\delta w_i}{|\Gamma_{\rm non}|}\Phi\Big)_\Omega\tau + \sum_{i=1}^{j}\Big(\nabla u_i, \nabla\Big[\delta\tilde{g}_i + \frac{\delta w_i}{|\Gamma_{\rm non}|}\Phi\Big]\Big)_\Omega\tau \\
&\quad + \sum_{i=1}^{j}\Big(g_{\rm Rob_i} u_i, \delta\tilde{g}_i + \frac{\delta w_i}{|\Gamma_{\rm non}|}\Phi\Big)_{\Gamma_{\rm Neu}}\tau.
\end{aligned}$$

Let $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ be any sequences of real numbers such that all b_i are non-negative. We start with an obvious identity

$$a_i(a_i - a_{i-1}) = \frac{1}{2}[a_i^2 - a_{i-1}^2 + (a_i - a_{i-1})^2],$$

which after summation gives

$$\begin{aligned}
\sum_{i=1}^{j} b_i a_i(a_i - a_{i-1}) &= \frac{1}{2}\sum_{i=1}^{j} b_i[a_i^2 - a_{i-1}^2 + (a_i - a_{i-1})^2] \\
&= \frac{1}{2}\sum_{i=1}^{j} b_i(a_i - a_{i-1})^2 + \frac{1}{2}\sum_{i=1}^{j} b_i(a_i^2 - a_{i-1}^2) \\
&= \frac{1}{2}\sum_{i=1}^{j} b_i(a_i - a_{i-1})^2 + \frac{1}{2}\Big[b_j a_j^2 - b_0 a_0^2 - \sum_{i=1}^{j}\delta b_i\, a_{i-1}\tau\Big] \\
&\geqslant \frac{1}{2}\Big[b_j a_j^2 - b_0 a_0^2 - \sum_{i=1}^{j}\delta b_i\, a_{i-1}\tau\Big].
\end{aligned}$$

Therefore, using the Cauchy inequality, the trace theorem and Lemma 1 (i) we estimate the sum containing the Robin term $(g_{\mathrm{Rob_i}} u_i, u_i - u_{i-1})_{\Gamma_{\mathrm{Neu}}}$ from below as follows:

$$
\begin{aligned}
(22) \quad \sum_{i=1}^{j} (g_{\mathrm{Rob_i}} u_i, u_i - u_{i-1})_{\Gamma_{\mathrm{Neu}}} \geqslant{}& \frac{1}{2}\int_{\Gamma_{\mathrm{Neu}}} g_{\mathrm{Rob_j}} u_j^2 - \frac{1}{2}\int_{\Gamma_{\mathrm{Neu}}} g_{\mathrm{Rob_0}} u_0^2 \\
&- \frac{1}{2}\sum_{i=1}^{j} \|\delta g_{\mathrm{Rob_i}}\|_{0,\Gamma_{\mathrm{Neu}}}^2 \tau - \frac{1}{2}\sum_{i=1}^{j} \|u_{i-1}\|_{0,\Gamma_{\mathrm{Neu}}}^2 \tau \\
\geqslant{}& -C - C\|u_0\|_{0,\Gamma_{\mathrm{Neu}}}^2 - C\sum_{i=0}^{j} \|u_i\|_{0,\Gamma_{\mathrm{Neu}}}^2 \tau \\
\geqslant{}& -C\sum_{i=1}^{j} \|u_i\|_{1,\Omega}^2 \tau - C \geqslant -C.
\end{aligned}
$$

The sums containing $(g_{\mathrm{Neu_i}}, u_i - u_{i-1})_{\Gamma_{\mathrm{Neu}}}$ and $(g_{\mathrm{non_i}}, u_i - u_{i-1})_{\Gamma_{\mathrm{non}}}$ can be estimated in a similar way, thus we demonstrate it for one of them only. We use the summation by parts, the trace theorem, Lemma 1 (i) and get

$$
\begin{aligned}
&\left| \sum_{i=1}^{j} (g_{\mathrm{Neu_i}}, u_i - u_{i-1})\Gamma_{\mathrm{Neu}} \right| \\
&\quad = \left| \frac{1}{2}(g_{\mathrm{Neu_j}}, u_j)\Gamma_{\mathrm{Neu}} - \frac{1}{2}(g_{\mathrm{Neu_0}}, u_0)_{\Gamma_{\mathrm{Neu}}} - \frac{1}{2}\sum_{i=1}^{j} (\delta g_{\mathrm{Neu_i}}, u_{i-1})_{\Gamma_{\mathrm{Neu}}} \tau \right| \\
&\quad \leqslant C_\varepsilon + \varepsilon \|u_j\|_{0,\Gamma_{\mathrm{Neu}}}^2 + C\sum_{i=0}^{j} \|u_i\|_{0,\Gamma_{\mathrm{Neu}}}^2 \tau \\
&\quad \leqslant C_\varepsilon + \varepsilon |u_j|_{1,\Omega}^2 + C\sum_{i=0}^{j} |u_i|_{1,\Omega}^2 \tau \\
&\quad \leqslant C_\varepsilon + \varepsilon |u_j|_{1,\Omega}^2.
\end{aligned}
$$

The rest of the terms in (21) can be estimated in the standard way, thus we omit the details. In the end we arrive at

$$
(1-\varepsilon)\left[\sum_{i=1}^{j} \|\delta u_i\|_{0,\Omega}^2 \tau + |u_j|_{1,\Omega}^2 + \sum_{i=1}^{j} |u_i - u_{i-1}|_{1,\Omega}^2 \right] \leqslant C_\varepsilon.
$$

Choosing a sufficiently small but positive ε, we conclude the proof of part (i).

(ii) We choose $\varphi = \Phi/|\Gamma_{\text{non}}| \in C^\infty(\overline{\Omega})$ in (8a) and get

$$\Big(\delta u_i, \frac{\Phi}{|\Gamma_{\text{non}}|}\Big)_\Omega + \Big(\nabla u_i, \frac{\nabla\Phi}{|\Gamma_{\text{non}}|}\Big)_\Omega + \alpha_i + \Big(g_{\text{Rob}_\text{i}} u_i, \frac{\Phi}{|\Gamma_{\text{non}}|}\Big)_{\Gamma_{\text{Neu}}}$$
$$= \Big(f(u_{i-1}), \frac{\Phi}{|\Gamma_{\text{non}}|}\Big)_\Omega - \Big(g_{\text{Neu}_\text{i}}, \frac{\Phi}{|\Gamma_{\text{non}}|}\Big)_{\Gamma_{\text{Neu}}} - \Big(g_{\text{non}_\text{i}}, \frac{\Phi}{|\Gamma_{\text{non}}|}\Big)_{\Gamma_{\text{non}}}.$$

Therefore, using the Cauchy inequality and the trace theorem we obtain

$$|\alpha_i| \leqslant C(1 + \|\delta u_i\|_{0,\Omega} + \|u_i\|_{1,\Omega} + \|u_{i-1}\|_{0,\Omega} + \|g_{\text{Neu}_\text{i}}\|_{0,\Gamma_{\text{Neu}}} + \|g_{\text{non}_\text{i}}\|_{0,\Gamma_{\text{non}}}).$$

Hence

$$\sum_{i=1}^{j} \alpha_i^2 \tau \leqslant C\Big(1 + \sum_{i=1}^{j} \|\delta u_i\|_{0,\Omega}^2 \tau + \sum_{i=1}^{j} |u_i\|_{1,\Omega}^2 \tau\Big) \leqslant C$$

takes place for any $j = 1, \ldots, n$. □

4. Convergence of the scheme

Now, let us introduce piecewise linear in time function

$$u_n(0) = u_0,$$
$$u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i \quad \text{for } t \in (t_{i-1}, t_i],$$

and step functions $\overline{\alpha}_n, \overline{u}_n$

$$\overline{\alpha}_n(0) = \alpha_0, \quad \overline{\alpha}_n(t) = \alpha_i,$$
$$\overline{u}_n(0) = u_0, \quad \overline{u}_n(t) = u_i, \quad \text{for } t \in (t_{i-1}, t_i].$$

Exactly in the same way we also define step functions $\overline{g}_{\text{Neu}_\text{n}}$, $\overline{g}_{\text{non}_\text{n}}$ and $\overline{g}_{\text{Rob}_\text{n}}$. Using this notation we rewrite (8) into the form

$$\text{(23)} \quad \Big(\frac{\partial u_n(t)}{\partial t}, \varphi\Big)_\Omega + (\nabla \overline{u}_n(t), \nabla\varphi)_\Omega + (\overline{\alpha}_n(t), \varphi)_{\Gamma_{\text{non}}} + (\overline{g}_{\text{Rob}_\text{n}}(t)\overline{u}_n(t), \varphi)_{\Gamma_{\text{Neu}}}$$
$$= (f(\overline{u}_n(t-\tau)), \varphi)_\Omega - (\overline{g}_{\text{Neu}_\text{n}}(t), \varphi)_{\Gamma_{\text{Neu}}} - (\overline{g}_{\text{non}_\text{n}}(t), \varphi)_{\Gamma_{\text{non}}},$$
$$\int_{\Gamma_{\text{non}}} \overline{u}_n(t) = \overline{w}_n(t).$$

A priori estimates from Lemmas 1 and 2 rewritten in terms of u_n, $\overline{u}_n$ and $\overline{\alpha}_n$ assume the form

$$\text{(24)} \qquad \|u_n(t)\|_{1,\Omega} + \int_0^T \Big\|\frac{\partial u_n}{\partial t}\Big\|_{0,\Omega}^2 + \int_0^T \overline{\alpha}_n^2 \leqslant C \quad \forall t \in [0,T].$$

The existence of a weak solution to the IBVP 1 is guaranteed by the next theorem.

Theorem 1 (convergence). *Let the assumptions of Lemma* 2 *be fulfilled. Then there exists a solution to the* IBVP 1.

Proof. A priori estimates (24) together with Lemma 1.3.13 from Kačur [7] imply the existence of a function $u \in C([0,T], L_2(\Omega)) \cap L_\infty((0,T), H^1(\Omega))$ obeying $\partial u/\partial t \in L_2((0,T), L_2(\Omega))$ and a subsequence of $\{u_n\}$ (which we again denote by the same symbol) for which

$$
\begin{aligned}
u_n &\to u && \text{in } C([0,T], L_2(\Omega)),\\
\frac{\partial u_n}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} && \text{in } L_2((0,T), L_2(\Omega)),\\
\overline{u}_n(t) &\rightharpoonup u(t) && \text{in } H^1(\Omega) \text{ for all } t \in [0,T].
\end{aligned} \tag{25}
$$

The inequality (see, e.g., Nečas [8], (I.1.10))

$$
\|w\|^2_{0,\partial\Omega} \leqslant \varepsilon \|w\|^2_{1,\Omega} + C_\varepsilon \|w\|^2_{0,\Omega} \tag{26}
$$

holds for arbitrary small positive ε. Thus, according to (25), we deduce

$$
u_n, \overline{u}_n \to u \quad \text{in} \quad L_2((0,T), L_2(\partial\Omega)). \tag{27}
$$

The reflexivity of the function space $L_2((0,T))$ and the relation (24) imply the existence of such an $\alpha \in L_2((0,T))$ that

$$
\overline{\alpha}_n \rightharpoonup \alpha \quad \text{in } L_2((0,T))
$$

for a subsequence of $\overline{\alpha}_n$ which we denote by the same symbol as before.

Integrating (23) over the time interval $(0,t)$ for any $t \in [0,T]$ and $\varphi \in V$, we obtain

$$
\begin{gathered}
\int_0^t \Big(\frac{\partial u_n}{\partial t}, \varphi\Big)_\Omega + \int_0^t (\nabla \overline{u}_n, \nabla\varphi)_\Omega + \int_0^t (\overline{\alpha}_n, \varphi)_{\Gamma_{\rm non}} + \int_0^t (\overline{g}_{\rm Rob_n} \overline{u}_n, \varphi)_{\Gamma_{\rm Neu}} \\
= \int_0^t (f(\overline{u}_n(s-\tau)), \varphi)_\Omega \, {\rm d}s - \int_0^t (\overline{g}_{\rm Neu_n}, \varphi)_{\Gamma_{\rm Neu}} - \int_0^t (\overline{g}_{non_n}, \varphi)_{\Gamma_{\rm non}}, \\
\int_0^t \int_{\Gamma_{\rm non}} \overline{u}_n = \int_0^t \overline{w}_n.
\end{gathered} \tag{28}
$$

Now, we let n approach ∞ in this equation. To do this we use the a priori estimates, (25) and (27). We arrive at

$$\int_0^t \Big(\frac{\partial u}{\partial t}, \varphi\Big)_\Omega + \int_0^t (\nabla u, \nabla\varphi)_\Omega + \int_0^t (\alpha, \varphi)_{\Gamma_{\rm non}} + \int_0^t (g_{\rm Rob} u, \varphi)_{\Gamma_{\rm Neu}}$$
$$= \int_0^t (f(u), \varphi)_\Omega - \int_0^t (g_{\rm Neu}, \varphi)_{\Gamma_{\rm Neu}} - \int_0^t (g_{\rm non}, \varphi)_{\Gamma_{\rm non}},$$
$$\int_0^t \int_{\Gamma_{\rm non}} u = \int_0^t w.$$

Differentiating this with respect to the time t we obtain (7). Therefore, the couple (u, α) is a solution to the IBVP 1.

5. Error estimates

The constructive character of Rothe's method allows also to establish some error estimates for the time discretization. The technique of the proof for the derivation of the rate of convergence is more or less standard, but we must proceed carefully by the choice of the appropriate test function, in order to split the coupling of the solution u and the unknown function α. Therefore, we point out only the most important steps in the proof.

The rate of convergence clearly depends on the properties of the data functions appearing in the IBVP setting and also on the regularity of the initial data u_0. We needed $u_0 \in H^1(\Omega)$ in order to establish the existence of a solution of the IBVP 1. Rothe's method for standard semilinear parabolic problems gives the approximation rate $\mathcal{O}(\tau^{1/2})$ in the $L_2(\Omega)$-norm for $u_0 \in H^1(\Omega)$. We will get the same rate of convergence for the nonstandard BCs, too. Moreover, we establish error estimate for the approximation of the function α.

Theorem 2. *Let the assumptions of Lemma 2 be fulfilled. Assume that the functions $g_{\rm Neu}$, $g_{\rm non}$, $g_{\rm Rob}$, $\tilde g$ and w are globally Lipschitz continuous in the time variable. Then,*

(i) $\|u(t) - u_n(t)\|^2_{0,\Omega} + \int_0^t |u(s) - u_n(s)|^2_{1,\Omega}\,{\rm d}s = \mathcal{O}(\tau)$,

(ii) $\big|\int_0^t [\alpha - \overline{\alpha}_n]\big| = \mathcal{O}(\tau^{1/2})$

take place for any $t \in [0, T]$.

Proof. (i) Let the function Φ obey (12). By the fact that both $u - \tilde g - w\Phi/|\Gamma_{\rm non}|$ and $u_n - \tilde g_n - w_n\Phi/|\Gamma_{\rm non}|$ belong to the space V, also their difference is in V. First, we subtract (23a) from (7a), then, we set

$$\varphi = u - u_n - \tilde g + \tilde g_n - (w - w_n)\frac{\Phi}{|\Gamma_{\rm non}|},$$

and in the end we integrate the equation over $(0,t)$ for any $t \in (0,T)$. We recall that for such a choice of the test function we have $\int_{\Gamma_{\mathrm{non}}} \varphi = 0$. The result can be rewritten in the form

$$\begin{aligned}
&\int_0^t \Big(\frac{\partial u}{\partial t} - \frac{\partial u_n}{\partial t}, u - u_n\Big)_\Omega + \int_0^t (\nabla[u-u_n], \nabla[u-u_n])_\Omega \\
&\quad + \int_0^t (g_{\mathrm{Rob}}[u-u_n], u-u_n)_{\Gamma_{\mathrm{Neu}}} \\
&= \int_0^t \Big(f(u(s)) - f(\overline{u}_n(s-\tau)), u - u_n - \tilde{g} + \tilde{g}_n - (w-w_n)\frac{\Phi}{|\Gamma_{\mathrm{non}}|}\Big)_\Omega \\
&\quad + \int_0^t \Big(\overline{g}_{\mathrm{Neu_n}} - g_{\mathrm{Neu}}, u - u_n - \tilde{g} + \tilde{g}_n - (w-w_n)\frac{\Phi}{|\Gamma_{\mathrm{non}}|}\Big)_{\Gamma_{\mathrm{Neu}}} \\
&\quad + \int_0^t \Big(\overline{g}_{\mathrm{non_n}} - g_{\mathrm{non}}, u - u_n - \tilde{g} + \tilde{g}_n - (w-w_n)\frac{\Phi}{|\Gamma_{\mathrm{non}}|}\Big)_{\Gamma_{\mathrm{non}}} \\
&\quad + \int_0^t \Big(\frac{\partial u}{\partial t} - \frac{\partial u_n}{\partial t}, \tilde{g} - \tilde{g}_n + (w-w_n)\frac{\Phi}{|\Gamma_{\mathrm{non}}|}\Big)_\Omega \\
&\quad + \int_0^t \Big(\nabla[u-\overline{u}_n], \nabla\Big[\tilde{g} - \tilde{g}_n + (w-w_n)\frac{\Phi}{|\Gamma_{\mathrm{non}}|}\Big]\Big)_\Omega \\
&\quad + \int_0^t (\nabla[\overline{u}_n - u_n], \nabla[u-u_n])_\Omega \\
&\quad + \int_0^t \Big(g_{\mathrm{Rob}}[u-u_n], \tilde{g} - \tilde{g}_n + (w-w_n)\frac{\Phi}{|\Gamma_{\mathrm{non}}|}\Big)_{\Gamma_{\mathrm{Neu}}} \\
&\quad + \int_0^t \Big(g_{\mathrm{Rob}}[\overline{u}_n - u_n], u - u_n - \tilde{g} + \tilde{g}_n - (w-w_n)\frac{\Phi}{|\Gamma_{\mathrm{non}}|}\Big)_{\Gamma_{\mathrm{Neu}}} \\
&\quad + \int_0^t \Big([\overline{g}_{\mathrm{Rob_n}} - g_{\mathrm{Rob}}]\overline{u}_n, u - u_n - \tilde{g} + \tilde{g}_n - (w-w_n)\frac{\Phi}{|\Gamma_{\mathrm{non}}|}\Big)_{\Gamma_{\mathrm{Neu}}}.
\end{aligned}$$

We have chosen such a long form because it will be more convenient for the estimation. Now, using the Cauchy and Young inequalities, the trace theorem and the a priori estimates we arrive in a straightforward way at

$$\begin{aligned}
(29)\qquad &\|u(t) - u_n(t)\|_{0,\Omega}^2 + (1-\varepsilon)\int_0^t |u-u_n|_{1,\Omega}^2 \\
&\leqslant C_\varepsilon \tau^2 + C_\varepsilon \int_0^t \|u-u_n\|_{0,\Omega}^2 + C_\varepsilon \int_0^t |\overline{u}_n - u_n|_{1,\Omega}^2 \\
&\quad + C_\varepsilon \int_0^t \Big(\frac{\partial u}{\partial t} - \frac{\partial u_n}{\partial t}, \tilde{g} - \tilde{g}_n + (w-w_n)\frac{\Phi}{|\Gamma_{\mathrm{non}}|}\Big)_\Omega.
\end{aligned}$$

The last two summands on the right-hand side can be estimated using Lemma 2 (i) by $C\tau$. Therefore, fixing a sufficiently small positive ε and applying Gronwall's lemma we conclude the proof of part (i).

(ii) We subtract (23a) from (7a), then we set $\varphi = \Phi/|\Gamma_{\text{non}}|$, where the function Φ obeys (12), and in the end we integrate the equation over $(0,t)$ for any $t \in (0,T)$. We recall that for this choice of the test function we have $\int_{\Gamma_{\text{non}}} \varphi = 1$. We obtain

$$\begin{aligned}
\int_0^t [\alpha - \overline{\alpha}_n] = {} & \Big(u_n(t) - u(t), \frac{\Phi}{|\Gamma_{\text{non}}|}\Big)_\Omega + \int_0^t \Big(\nabla[\overline{u}_n - u], \frac{\nabla\Phi}{|\Gamma_{\text{non}}|}\Big)_\Omega \\
& + \int_0^t \Big(f(u(s)) - f(\overline{u}_n(s-\tau), \frac{\Phi}{|\Gamma_{\text{non}}|}\Big)_\Omega \,\mathrm{d}s \\
& + \int_0^t \Big(\overline{g}_{\text{Neu}_\text{n}} - g_{\text{Neu}}, \frac{\Phi}{|\Gamma_{\text{non}}|}\Big)_{\Gamma_{\text{Neu}}} \\
& + \int_0^t \Big(\overline{g}_{\text{non}_\text{n}} - g_{\text{non}}, \frac{\Phi}{|\Gamma_{\text{non}}|}\Big)_{\Gamma_{\text{non}}}.
\end{aligned} \tag{30}$$

The rest of the proof can be easily obtained using Theorem 2 (i). □

6. Numerical experiments

In this section we present two examples of numerical realization the approximate method which has been described above. The first is a linear problem while the second has a nonlinear right-hand side.

The domain common for both examples is the unit square $\Omega = (0,1) \times (0,1)$ and the time interval is $[0,1]$. The boundary $\partial\Omega$ is split into three parts Γ_{Dir} (right), Γ_{Neu} (top and bottom) and Γ_{non} (left part of $\partial\Omega$).

For the time discretization we have applied the method described in the previous sections. For the numerical solution of the linear elliptic equation at each time step we have used the mixed nonconforming finite element formulation. This is equivalent to the mixed-hybrid method (see Arnold and Brezzi [2]). We explain very briefly the main idea of this approximation.

Let us consider a regular triangulation $\mathcal{T}_h$ (h denotes the mesh diameter) of the domain Ω. On each element $\mathcal{T} \in \mathcal{T}_h$ we define three linear basis functions associated with the edges of $\mathcal{T}$, i.e., a basis function has the value 1 at the midpoint of one edge and 0 at the midpoints of the other edges of one triangle. Further, we define a bubble function on $\mathcal{T}$, which is a polynomial function of the third order vanishing on the boundary $\partial\mathcal{T}$ whose integral average value on $\mathcal{T}$ is 1. In this way we have enriched the standard linear nonconforming space by bubbles, and we solve the linear elliptic problem in this space replacing the velocity field $\boldsymbol{q}$ by its projection on the Raviart-Thomas space RT_0. For more details see Arnold and Brezzi [2].

We have chosen the time step $\tau = 0.01$ and a fixed uniform mesh consisting of 9800 triangles for all computations.

6.1. Example 1.

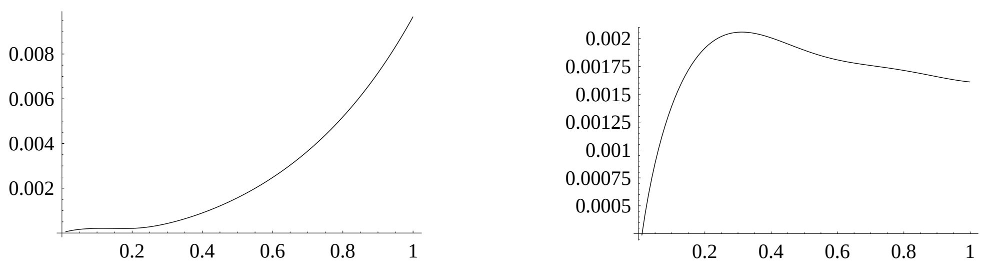
Let us consider the semilinear IBVP

$$\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= u^2 - v^2 + \frac{\partial v}{\partial t} - \Delta v && \text{in } (0,1) \times \Omega \\
u &= v && \text{in } (0,1) \times \Gamma_{\text{Dir},} \\
-\nabla u \cdot \boldsymbol{\nu} &= -\nabla v \cdot \boldsymbol{\nu} && \text{in } (0,1) \times \Gamma_{\text{Neu}}, \\
-\nabla u(t) \cdot \boldsymbol{\nu} &= -\nabla v(t) \cdot \boldsymbol{\nu} - 1 - t^2 + \alpha(t) && \text{in } (0,1) \times \Gamma_{\text{non}}, \\
\int_{\Gamma_{\text{non}}} u(t)\,\mathrm{d}\gamma &= \frac{t}{2} + t^2 && \text{in } (0,1) \times \Gamma_{\text{non}}, \\
u(0) &= v(0) && \text{in } \Omega,
\end{aligned}$$

where the couple (v, α) stands for the exact solution

$$\begin{aligned}
v(t,x,y) &= \sin(\pi x)\sin(\pi y) + ty + t^2, \\
\alpha(t) &= 1 + t^2.
\end{aligned}$$

The behavior of errors for α_n and $u_n(t)$ is depicted in Fig. 1.



$\alpha_n(t)$-error $\qquad$ $L_2(\Omega)$-error of $u_n(t)$

Figure 1. Example 1: The behaviour of errors for $t \in [0,1]$.

6.2. Example 2.

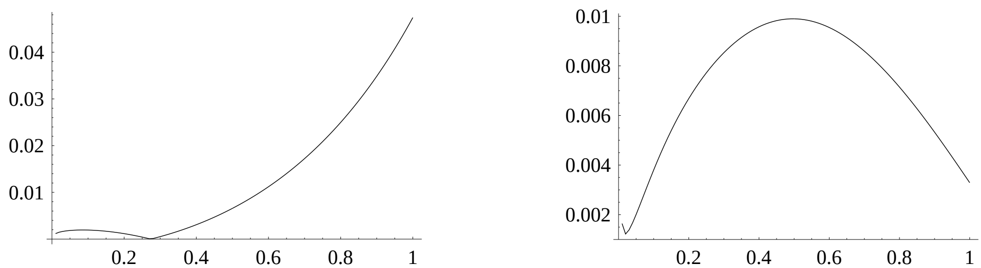
We consider the semilinear evolution problem

$$\begin{aligned}
(1+t)^2 \frac{\partial u}{\partial t} - \Delta u &= \mathrm{e}^u - \mathrm{e}^v + (1+t)^2 \frac{\partial v}{\partial t} - \Delta v && \text{in } (0,1) \times \Omega, \\
u &= v && \text{in } (0,1) \times \Gamma_{\text{Dir}}, \\
-\nabla u \cdot \boldsymbol{\nu} &= -\nabla v \cdot \boldsymbol{\nu} && \text{in } (0,1) \times \Gamma_{\text{Neu}}, \\
-\nabla u(t) \cdot \boldsymbol{\nu} &= -\nabla v(t) \cdot \boldsymbol{\nu} - \sin(\pi t) + \alpha(t) && \text{in } (0,1) \times \Gamma_{\text{non}}, \\
\int_{\Gamma_{\text{non}}} u(t)\,\mathrm{d}\gamma &= \frac{1 - \cos(\pi(1+t))}{\pi(1+t)} && \text{in } (0,1) \times \Gamma_{\text{non}}, \\
u(0) &= v(0) && \text{in } \Omega,
\end{aligned}$$

with the exact solution (v, α) given as

$$v(t, x, y) = \cos(\pi(1+t)x)\sin(\pi(1+t)y),$$
$$\alpha(t) = \sin(\pi t).$$

The α_n-error and the $L_2(\Omega)$-error for the approximate solution $u_n(t)$ on $[0,1]$ for different time steps is shown in Fig. 2.



$\alpha_n(t)$-error $\qquad$ $L_2(\Omega)$-error of $u_n(t)$

Figure 2. Example 2: The behaviour of errors for $t \in [0,1]$.

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