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GALERKIN APPROXIMATIONS FOR THE LINEAR PARABOLIC EQUATION WITH THE THIRD BOUNDARY CONDITION*

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Abstract. We solve a linear parabolic equation in $\mathbb{R}^d$, $d \geqslant 1$, with the third nonhomogeneous boundary condition using the finite element method for discretization in space, and the θ-method for discretization in time. The convergence of both, the semidiscrete approximations and the fully discretized ones, is analysed. The proofs are based on a generalization of the idea of the elliptic projection. The rate of convergence is derived also for variable time step-sizes.

Keywords: linear parabolic equation, third boundary condition, finite element method, semidiscretization, fully discretized scheme, elliptic projection

MSC 2000: 65M60, 65M15

1. Introduction

Initial-boundary value problems of parabolic type serve as mathematical models in many practical applications [1], [5], [6], [10], [11], [13], [14].

Numerical solution of such problems using the finite element discretization in space is presented in papers [4], [7], [8], [9], [20], comprehensive material on the subject is also given in the monograph [18].

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In our work, a numerical solution of a linear parabolic problem with the third nonhomogeneous boundary condition is presented. Such a model, in the 3D-setting with a self-adjoint operator, describes a nonstationary heat conduction in a stator of a motor, see [11]; numerical analysis of convergence of the semidiscrete approximations for this problem is given in [10]. The rates of convergence for both, the semidiscrete and the fully discrete approximations, for this problem in any dimension setting are derived in [8]. In the present paper we extend the results from [8], and estimate the convergence of the semidiscrete and the fully discretized solutions to the weak solution for a problem having more general form.

First, we include the lower order terms into the elliptic part of the problem, which leads to the nonsymmetricity of the corresponding bilinear form. We prove that the method of the elliptic projection, first introduced in works [19], [20], can be also easily applied to such a more general problem. Second, we show the applicability of the whole class of fully discretized schemes. In the existing literature, usually only special cases of discretization in time are considered, and mostly for problems with the Dirichlet boundary condition. One of the main reasons why we analyse the convergence for the whole family of discretization in time ($\theta \in [0.5, 1]$) is, that for non-smooth initial data, the optimal schemes are obtained for values of θ different from 0.5 and 1 (i.e., schemes different from Crank-Nicolson and the backward Euler schemes), see [12], [16]. Third, a more exact estimate for the L^2-convergence of the semidiscrete approximations is presented, and also the rate of convergence in the H^1-norm (for the symmetric case) is derived. Fourth, we analyse the rate of convergence of the fully discretized solutions also for the case of a variable time step-size.

The paper is organized in the following manner. In Section 2, we give all necessary mathematical facts and define the weak formulation of the problem. Section 3 is devoted to the semidiscretization in standard finite element subspaces. In Section 4, we consider the numerical solution of the semidiscrete problem via the θ-method of [15]. We derive the second order of convergence in time for the Crank-Nicolson-Galerkin scheme, while for the other proposed schemes it is of the first order.

2. Mathematical background and notation

We consider the partial differential equation of parabolic type

$$\frac{\partial u}{\partial t} - \operatorname{div}(\mathcal{A} \operatorname{grad} u) + b \cdot \operatorname{grad} u + cu = f \quad \text{in } (0,T) \times \Omega, \tag{2.1}$$

with the third nonhomogeneous boundary condition

$$\alpha u + \nu^T \mathcal{A} \operatorname{grad} u = g \quad \text{on } (0,T) \times \partial\Omega, \tag{2.2}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^d$, $d = 1, 2, \ldots$, with a boundary $\partial\Omega$, ν is the outward unit normal to Ω; $t \in (0,T)$, $T > 0$ is fixed; $\mathcal{A} = \mathcal{A}(x) := (a_{ij}(x))_{i,j=1}^d$ is a symmetric positive definite matrix, i.e., $(\mathcal{A}\eta, \eta) \geqslant C_0\|\eta\|^2$, $b = b(x) := (b_1(x), \ldots, b_d(x))$, $c = c(x)$, where $x \in \Omega$; $\alpha = \alpha(s) \geqslant 0$, where $s \in \partial\Omega$.

The initial condition is given as

$$u(x,0) = u^0(x), \quad x \in \Omega. \tag{2.3}$$

From now on, we employ the following standard notation:

$(\cdot\,,\cdot)$	scalar product in $L^2(\Omega)$,
$\langle\cdot\,,\cdot\rangle$	scalar product in $L^2(\partial\Omega)$,
$\|\cdot\|_{0,\partial\Omega}$	norm in the space $L^2(\partial\Omega)$,
$H^r(\Omega)$	Sobolev space, $r = 1, 2, \ldots,$
$\|v\|_r = \Big(\sum_{\|m\| \leqslant r} \int_\Omega \|D^m v\|^2 \,\mathrm{d}x\Big)^{1/2}$	norm of $v \in H^r(\Omega)$,
$\|v\|_r = \Big(\sum_{\|m\| = r} \int_\Omega \|D^m v\|^2 \,\mathrm{d}x\Big)^{1/2}$	seminorm of $v \in H^r(\Omega)$,
V_h	finite element subspace of $H^1(\Omega)$,
$C^k([0,T], H^r(\Omega))$	space of the k times continuously differentiable mappings from $[0,T]$ into $H^r(\Omega)$,
$H^1((0,T), H^r(\Omega))$	space of measurable mappings from $(0,T)$ into $H^r(\Omega)$, which belong to $H^1((0,T))$,
$H^1((0,T), V_h)$	space of measurable mappings from $(0,T)$ into V_h, which belong to $H^1((0,T))$,
$\mathcal{P}_r(K)$	space of polynomials of the r-th degree over a domain K.

Further, notation C_i stands for generic positive constants, $\|\cdot\|$—for $\|\cdot\|_0$, and for simplicity we do not write explicitly the dependence of functions on variables if it does not lead to misunderstanding.

The given coefficients a_{ij}, b_i, c, and α are assumed to be bounded measurable functions on Ω and $\partial\Omega$, respectively, i.e.,

$$\operatorname*{ess\,sup}_{x,i,j} |a_{ij}(x)|,\ \operatorname*{ess\,sup}_{x,i} |b_i(x)|,\ \operatorname*{ess\,sup}_{x} |c(x)|,\ \operatorname*{ess\,sup}_{s} |\alpha(s)| \leqslant C_1, \tag{2.4 a}$$

and we also assume that

$$f \in C([0,T], L^2(\Omega)),\ \ g \in C([0,T], L^2(\partial\Omega)),\ \ u_0 \in H^1(\Omega) \cap C(\overline{\Omega}). \tag{2.4 b}$$

If $u = u(x,t)$ is the classical solution of the problem (2.1)–(2.3), then it satisfies the relation

$$(u'(t), v) + a(u,v) = F(t; v) \quad \forall v \in H^1(\Omega), \tag{2.5}$$

where $u' = \partial u/\partial t$ denotes the classical time derivative, $a(\cdot\,,\cdot)$ is a bilinear form defined as

$$a(v,w) = (\mathcal{A}\operatorname{grad} v, \operatorname{grad} w) + (b\cdot\operatorname{grad} v + cv, w) + \langle \alpha v, w\rangle, \tag{2.6}$$

and

$$F(t;v) = (f(\cdot\,,t), v) + \langle g(\cdot\,,t), v\rangle. \tag{2.7}$$

For the initial condition (2.3) we have the relation

$$(u(\cdot\,,0), v) = (u^0, v) \quad \forall v \in H^1(\Omega). \tag{2.8}$$

Further, we assume that the weak formulation (2.5)–(2.8) of the problem (2.1)–(2.3) has a unique solution in $C^k([0,T], H^l(\Omega))$ with $k \geqslant 1$, $l \geqslant 2$ in Section 3, and $k \geqslant 3$, $l \geqslant 2$ in Section 4.

In what follows, the form $a(\cdot\,,\cdot)$ is assumed to be positive definite:

$$a(v,v) \geqslant C_2\|v\|_1^2 \quad (C_2 > 0). \tag{2.9}$$

Remark 2.1. We notice here that, in general, the nonsymmetric bilinear form $a(\cdot\,,\cdot)$ cannot satisfy condition (2.9). However, similarly to [3, Chap. 2 and Chap. 5], we can show that it holds, for example, in the case $B^2 < 4C_0\beta$, where $B^2 := \sum_{i=1}^{d} \|b_i\|^2_{L_\infty(\Omega)}$, $\beta := \operatorname{ess\,inf}_x c(x)$ and C_0 is the constant from the condition of positivity of $\mathcal{A}$.

In view of (2.4) we immediately observe that

$$|a(u,v)| \leqslant C_3\|u\|_1\|v\|_1. \tag{2.10}$$

3. Error estimates for semidiscrete Galerkin approximations

In this section we analyse the rate of convergence for the semidiscrete approximations of our problem.

3.1. Formulation of the problem.

Let $\{\mathcal{T}_h\}$ be a family of triangulations of $\overline{\Omega}$ consisting of elements K_i with standard regular properties. We assume to have finite-dimensional subspaces of $H^1(\Omega)$ in the form

$$V_h = \{\chi_h \in H^1(\Omega) \mid \chi_h|_K \in \mathcal{P}_{l-1}(K) \ \ \forall K \in \mathcal{T}_h\}. \tag{3.1}$$

Further, we define the operator of the elliptic projection $P_h\colon\ H^l(\Omega) \to V_h$ so that any given $v \in H^l(\Omega)$ is mapped to $P_h v \in V_h$ such that the relation

$$a(P_h v, \chi_h) = a(v, \chi_h) \quad \forall \chi_h \in V_h \tag{3.2}$$

holds. This mapping is correctly defined (cf. [3], Chap. 2).

Remark 3.1. Usually, one requires (cf. [8], [18]) the property of the finite element subspaces in the form

$$\inf_{\chi_h \in V_h} \{\|v - \chi_h\| + h\| \operatorname{grad}(v - \chi_h)\|\} \leqslant C_4 h^p \|v\|_p \tag{3.3}$$

for $v \in H^p(\Omega)$, $1 \leqslant p \leqslant l$.

We assume here that the given subspaces V_h are such that

$$\|v - P_h v\| + h\| \operatorname{grad}(v - P_h v)\| \leqslant C_5 h^l \|v\|_l \tag{3.4}$$

for $v \in H^l(\Omega)$.

Remark 3.2. The condition (3.4) is a rather typical one for the finite element spaces, we refer e.g. to [17], [19] for the formulation of the conditions when it holds.

Remark 3.3. By our assumption, the weak solution $u(x,t)$ of (2.1)–(2.3) belongs to $H^l(\Omega)$ for any fixed $t \in (0,T)$, i.e., by relation (3.2) we may define the function $P_h u(t)$ which belongs to V_h for any fixed $t \in (0,T)$. Since the coefficients of the bilinear form $a(\cdot\,,\cdot)$ do not depend on t, we observe that under condition (3.4), the estimate

$$\Big\|\frac{\partial}{\partial t}u(t) - \frac{\partial}{\partial t}P_h u(t)\Big\| \leqslant C_6 h^l \Big\|\frac{\partial}{\partial t}u(t)\Big\|_l \tag{3.5}$$

holds.

We also mention here that starting from (3.5) we do not write explicitly the dependence of weak and discrete solutions on space variables when it does not lead to misunderstanding.

Further, we define the semidiscrete Galerkin solution of equation (2.1) as a function $u_h \in H^1((0,T), V_h)$ satisfying the relation

$$(u_h'(t), \chi_h) + a(u_h, \chi_h) = F(t; \chi_h) \quad \forall \chi_h \in V_h \tag{3.6}$$

for a.a. $t \in (0,T)$.

For the initial data we assume that $u_h(0)$ is a function from V_h. The most typical way to define $u_h(0)$ is as the best approximation of u^0 in V_h, i.e.,

$$(u_h(0), \chi_h) = (u^0, \chi_h) \quad \forall \chi_h \in V_h. \tag{3.7}$$

We are looking for u_h in the form

$$u_h(x,t) = \sum_{j=1}^{N} z_j(t) v^j(x), \quad t \in (0,T), \;\; x \in \Omega, \tag{3.8}$$

where $N := \dim V_h$ and $v^1, \dots, v^N$ denote the basis functions in V_h. Using (3.6), we get a system of ordinary differential equations of the first order

$$\sum_{j=1}^{N} (v^j, v^i) z_j'(t) + \sum_{j=1}^{N} a(v^j, v^i) z_j(t) = F(t; v^i), \quad i = 1, \dots, N, \tag{3.9}$$

for the unknown functions $z_1, \dots, z_N$. By (3.7) the corresponding initial conditions are determined by

$$\sum_{j=1}^{N} (v^j, v^i) z_j(0) = (u^0, v^i), \quad i = 1, \dots, N. \tag{3.10}$$

Using the notation

$$\begin{aligned} &z(t) = (z_1(t), \dots, z_N(t))^T, \;\; M = ((v^i, v^j))_{i,j=1}^N, \;\; A = (a(v^i, v^j))_{i,j=1}^N, \\ &\mathcal{F}(t) = (F(t; v^1), \dots, F(t; v^N))^T \;\text{ and }\; z^0 = ((u^0, v^1), \dots, (u^0, v^N))^T, \end{aligned}$$

we may rewrite the Cauchy problem (3.9)–(3.10) in the form

$$M z'(t) + A z(t) = \mathcal{F}(t), \quad t \in (0,T), \tag{3.11}$$

$$M z(0) = z^0. \tag{3.12}$$

Since the matrix M is symmetric and positive definite (i.e. invertible), the following corollary holds (cf. [2], Chapter 10]):

Corollary 3.4. *The problem* (2.1)–(2.3) *has a unique semidiscrete solution in* $H^1((0,T), V_h)$ *in the sense of* (3.6)–(3.7).

3.2. Error estimate in the L^2-norm.

Theorem 3.5. *Let u and u_h be solutions of* (2.5) *and* (3.6), *respectively. If we assume that* $\|u(0)-u_h(0)\| \leqslant C_7 h^l$, *then for* $t \in [0,T]$ *the estimate*

$$\|u(t)-u_h(t)\| \leqslant C_8 h^l \Big(1 + \|u(0)\|_l + \|u(t)\|_l + T^{1/2} \sup_{0 \leqslant t \leqslant T} \Big\| \frac{\partial}{\partial t} u(t) \Big\|_l \Big) \tag{3.13}$$

holds.

P r o o f. In view of (2.5), (3.2) and (3.6), we have the following equalities for all $\chi_h \in V_h$:

$$\begin{aligned}\Big(\frac{\partial}{\partial t}[u(t)-P_h u(t)], \chi_h\Big) &= F(t,\chi_h) - a(u(t),\chi_h) - \Big(\frac{\partial}{\partial t} P_h u(t), \chi_h\Big) \\ &= \Big(\frac{\partial}{\partial t} u_h(t), \chi_h\Big) + a(u_h(t),\chi_h) - a(P_h u(t), \chi_h) \\ &\quad - \Big(\frac{\partial}{\partial t} P_h u(t), \chi_h\Big).\end{aligned} \tag{3.14}$$

In the above relation, we take $\chi_h = u_h - P_h u$ (which belongs to V_h for any fixed $t \in [0,T]$), and then we get

$$\begin{aligned}\Big(\frac{\partial}{\partial t}[u - P_h u], u_h - P_h u\Big) &= \Big(\frac{\partial}{\partial t}[u_h - P_h u], u_h - P_h u\Big) \\ &\quad + a(u_h - P_h u, u_h - P_h u).\end{aligned} \tag{3.15}$$

For the left-hand side of (3.15) we use the Cauchy-Schwarz-Buniakovsky inequality and the inequality $|\lambda\mu| \leqslant (4C_2)^{-1}\lambda^2 + C_2\mu^2$ (C_2 is the constant from (2.9)), which yields

$$\begin{aligned}\Big(\frac{\partial}{\partial t}[u - P_h u], u_h - P_h u\Big) &\leqslant \Big\|\frac{\partial}{\partial t}[u - P_h u]\Big\| \|u_h - P_h u\| \\ &\leqslant \frac{1}{4C_2}\Big\|\frac{\partial}{\partial t}[u - P_h u]\Big\|^2 + C_2\|u_h - P_h u\|^2.\end{aligned} \tag{3.16}$$

For the second term on the right-hand side of (3.15) we get, by (2.9), that the inequality

$$a(u_h - P_h u, u_h - P_h u) \geqslant C_2 \|u_h - P_h u\|_1^2 \geqslant C_2 \|u_h - P_h u\|^2 \tag{3.17}$$

holds. Since $(\partial\varphi/\partial t, \varphi) \equiv \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2$, from (3.15), (3.16) and (3.17) we observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_h - P_h u\|^2 \leqslant \frac{1}{2C_2}\Big\|\frac{\partial}{\partial t}[u - P_h u]\Big\|^2. \tag{3.18}$$

Integrating (3.18) over the interval $(0,t)$ we obtain

$$\|u_h(t) - P_h u(t)\|^2 \leqslant \|u_h(0) - P_h u(0)\|^2 + \frac{T}{2C_2} \sup_{0\leqslant t\leqslant T} \Big\| \frac{\partial}{\partial t}[u - P_h u] \Big\|^2, \quad 0 \leqslant t \leqslant T,$$

and therefore

$$\|u_h(t) - P_h u(t)\| \leqslant \|u_h(0) - P_h u(0)\| + \Big(\frac{T}{2C_2}\Big)^{1/2} \sup_{0\leqslant t\leqslant T} \Big\| \frac{\partial}{\partial t} u(t) - \frac{\partial}{\partial t} P_h u(t) \Big\|. \tag{3.19}$$

Clearly, the triangle inequality results in

$$\|u(t) - u_h(t)\| \leqslant \|u(t) - P_h u(t)\| + \|u_h(t) - P_h u(t)\|, \tag{3.20}$$

and from (3.4) we have $\|u(t) - P_h u(t)\| \leqslant C_5 h^l \|u(t)\|_l$. Consider now the second term on the right-hand side of (3.20), which is estimated in (3.19). Since

$$\begin{aligned} \|u_h(0) - P_h u(0)\| &\leqslant \|u_h(0) - u(0)\| + \|u(0) - P_h u(0)\| \\ &\leqslant \|u_h(0) - u(0)\| + C_5 h^l \|u(0)\|_l, \end{aligned} \tag{3.21}$$

then, using the assumption of the theorem on the initial condition, we estimate the first term on the right-hand side of (3.19) as

$$\|u_h(0) - P_h u(0)\| \leqslant C_9 h^l (1 + \|u(0)\|_l). \tag{3.22}$$

The second term on the right-hand side of (3.19) can be now estimated by (3.5). Summing up all the necessary estimates, we complete the proof. □

Remark 3.6. If $u_h(0)$ is chosen by (3.7) then the assumption of Theorem 3.5 on the initial condition is fulfilled.

Remark 3.7. Theorem 3.5 shows that the semidiscretization in V_h results in the optimal order of the approximation in $L^2(\Omega)$.

The following theorem provides us with more precise information on the behaviour in time of the error of the semidiscretization.

Theorem 3.8. *Let u and u_h be solutions of* (2.5) *and* (3.6), *respectively. Then for $t \in [0,T]$ the estimate*

$$\begin{aligned} \|u(t) - u_h(t)\| \leqslant C_{10} h^l \|u(t)\|_l &+ \mathrm{e}^{-C_2 t}(\|u(0) - u_h(0)\| + C_{11} h^l \|u(0)\|_l) \\ &+ C_{12} h^l \int_0^t \mathrm{e}^{-C_2(t-s)} \Big\| \frac{\partial u}{\partial t}(s) \Big\|_l \,\mathrm{d}s \end{aligned} \tag{3.23}$$

holds.

P r o o f. From (3.14) we have

$$\Big(\frac{\partial}{\partial t}[u - P_h u], \chi_h\Big) = \Big(\frac{\partial}{\partial t}[u_h - P_h u], \chi_h) + a(u_h - P_h u, \chi_h\Big) \quad \forall \chi_h \in V_h. \tag{3.24}$$

Let $\chi_h = u_h - P_h u$ (as in (3.15)), then (3.24) implies

$$\begin{aligned} &\Big(\frac{\partial}{\partial t}[u - P_h u], u_h - P_h u\Big) \\ &\qquad = \Big(\frac{\partial}{\partial t}[u_h - P_h u], u_h - P_h u\Big) + a(u_h - P_h u, u_h - P_h u). \end{aligned} \tag{3.25}$$

From (3.17) we get

$$a(u_h - P_h u, u_h - P_h u) \geqslant C_2 \|u_h - P_h u\|^2.$$

Also, the obvious identity

$$\Big(\frac{\partial}{\partial t}(u_h - P_h u), u_h - P_h u\Big) \equiv \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|u_h - P_h u\|^2$$

holds. Hence from (3.25) we observe that

$$\Big\|\frac{\partial}{\partial t}[u - P_h u]\Big\| \cdot \|u_h - P_h u\| \geqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|u_h - P_h u\|^2 + C_2\|u_h - P_h u\|^2. \tag{3.26}$$

Since $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|u_h - P_h u\|^2 = \|u_h - P_h u\| \cdot \frac{\mathrm{d}}{\mathrm{dt}}\|u_h - P_h u\|$, it follows from (3.26) that

$$\Big\|\frac{\partial}{\partial t}[u - P_h u]\Big\| \geqslant \frac{\mathrm{d}}{\mathrm{dt}}\|u_h - P_h u\| + C_2\|u_h - P_h u\|. \tag{3.27}$$

We note that

$$\mathrm{e}^{C_2 t}\frac{\mathrm{d}}{\mathrm{dt}}\|u_h - P_h u\| + C_2\mathrm{e}^{C_2 t}\|u_h - P_h u\| = \frac{\mathrm{d}}{\mathrm{dt}}(\mathrm{e}^{C_2 t}\|u_h - P_h u\|). \tag{3.28}$$

From (3.28) and (3.27) we get

$$\mathrm{e}^{C_2 t}\Big\|\frac{\partial}{\partial t}[u - P_h u]\Big\| \geqslant \frac{\mathrm{d}}{\mathrm{dt}}(\mathrm{e}^{C_2 t}\|u_h - P_h u\|). \tag{3.29}$$

Further, integrating (3.29) over the interval $(0, t)$, we immediately get

$$\begin{aligned} \|u_h(t) - P_h u(t)\| &\leqslant \mathrm{e}^{-C_2 t}\|u_h(0) - P_h u(0)\| \\ &\quad + \int_0^t \mathrm{e}^{-C_2(t-s)}\Big\|\frac{\partial}{\partial t}[u - P_h u]\Big\|\,\mathrm{d}s. \end{aligned} \tag{3.30}$$

Using the estimates (3.30), (3.21) and (3.4) for the representation (3.20) we get (3.23). □

R e m a r k 3.9. If $u_h(0)$ is a suitable approximation of the given initial function $u^0 = u(0)$ in V_h, i.e.,

$$\|u(0) - u_h(0)\| \leqslant C_{13} h^l \|u(0)\|_l$$

(for instance, $u_h(0)$ can be chosen by (3.7)), then (3.23) can be rewritten in the form

$$\tag{3.31} \begin{aligned} &\|u(t) - u_h(t)\| \\ &\leqslant C_{14} h^l \Big[\|u(t)\|_l + \mathrm{e}^{-C_2 t}(1 + \|u(0)\|_l) + \int_0^t \mathrm{e}^{-C_2(t-s)} \Big\| \frac{\partial u}{\partial t}(s) \Big\|_l \, \mathrm{d}s \Big]. \end{aligned}$$

□

R e m a r k 3.10. The estimate (3.23) shows that for large time levels the error estimate does not depend strongly on the quality of the approximation of the initial values.

3.3. Error estimate in the H^1-norm.

In this subsection, we derive an error estimate in the H^1-norm under the condition that the bilinear form $a(\cdot\,,\cdot)$ is symmetric.

Let now $\chi_h = \frac{\partial}{\partial t}[u_h - P_h u]$ in (3.24). Then we get

$$\tag{3.32} \begin{aligned} &\Big(\frac{\partial}{\partial t}[u - P_h u], \frac{\partial}{\partial t}[u_h - P_h u] \Big) \\ &\quad = \Big\| \frac{\partial}{\partial t}[u_h - P_h u] \Big\|^2 + a\Big(u_h - P_h u, \frac{\partial}{\partial t}[u_h - P_h u] \Big). \end{aligned}$$

Since $a(\cdot\,,\cdot)$ is assumed to be symmetric, we obviously have

$$\tag{3.33} a\Big(v, \frac{\partial}{\partial t} v\Big) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} a(v, v).$$

Then, with the choice $v := u_h - P_h u$, (3.32) results in

$$\begin{aligned} \Big\| \frac{\partial}{\partial t}(u_h - P_h u) \Big\|^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} a(v, v) &\leqslant \Big\| \frac{\partial}{\partial t}(u - P_h u) \Big\| \cdot \Big\| \frac{\partial}{\partial t}(u_h - P_h u) \Big\| \\ &\leqslant \frac{1}{2} \Big\| \frac{\partial}{\partial t}(u - P_h u) \Big\|^2 + \frac{1}{2} \Big\| \frac{\partial}{\partial t}(u_h - P_h u) \Big\|^2. \end{aligned}$$

Therefore, we have

$$\tag{3.34} \frac{\mathrm{d}}{\mathrm{dt}} a(v, v) \leqslant \Big\| \frac{\partial}{\partial t}(u - P_h u) \Big\|^2.$$

Integrating (3.34) over the interval $(0,t)$, we get

$$a(v,v)(t) - a(v,v)(0) \leqslant \int_0^t \left\| \frac{\partial}{\partial t}(u - P_h u) \right\|^2 \mathrm{d}s,$$

i.e.,

$$a(v,v)(t) \leqslant a(v,v)(0) + \int_0^t \left\| \frac{\partial}{\partial t}(u - P_h u) \right\|^2 \mathrm{d}s.$$

Since $C_{15}\|v\|_1^2 \geqslant a(v,v) \geqslant C_2\|v\|_1^2$, it follows that

$$\|u_h(t) - P_h u(t)\|_1^2 \leqslant \frac{1}{C_2}\left[C_{15}\|u_h(0) - P_h u(t)\|_1^2 + \int_0^t \left\| \frac{\partial}{\partial t}(u - P_h u) \right\|^2 \mathrm{d}s\right]. \tag{3.35}$$

Since

$$\|u_h - P_h u\|_1 \leqslant \|u_h - u\|_1 + \|u - P_h u\|_1,$$

we have

$$\|u_h(0) - P_h u(0)\|_1 \leqslant \|u_h(0) - u(0)\|_1 + \|u(0) - P_h u(0)\|_1.$$

That is,

$$\|u_h(0) - P_h u(0)\|_1^2 \leqslant 2\left(\|u_h(0) - u(0)\|_1^2 + \|u(0) - P_h u(0)\|_1^2\right).$$

Further,

$$\begin{aligned} &\|u_h(t) - P_h(t)u\|_1^2 \\ &\quad\leqslant C_{16}\left[\|u_h(0) - u(0)\|_1^2 + \|u(0) - P_h u(0)\|_1^2 + \int_0^t \left\| \frac{\partial}{\partial t}(u - P_h u) \right\|^2 \mathrm{d}s\right]. \end{aligned} \tag{3.36}$$

Now (3.36) yields

$$\begin{aligned} \|u_h(t) - P_h u(t)\|_1 &\leqslant C_{17}\|u_h(0) - u(0)\|_1 \\ &\quad + C_{18}\left[\|u(0) - P_h u(0)\|_1^2 + \int_0^t \left\| \frac{\partial}{\partial t}(u - P_h u) \right\|^2 \mathrm{d}s\right]^{1/2}. \end{aligned} \tag{3.37}$$

Due to

$$\|u - u_h\|_1 \leqslant \|u - P_h u\|_1 + \|u_h - P_h u\|_1,$$

from (3.37) we get

$$\begin{aligned} \|u(t) - u_h(t)\|_1 &\leqslant \|u(t) - P_h u(t)\|_1 + C_{17}\|u_h(0) - u(0)\|_1 \\ &\quad + C_{18}\left[\|u(0) - P_h u(0)\|_1^2 + \int_0^t \left\| \frac{\partial}{\partial t}(u - P_h u) \right\|^2 \mathrm{d}s\right]^{1/2}. \end{aligned} \tag{3.38}$$

Finally, using (3.4) for (3.38), we get the following theorem.

Theorem 3.11. *Let u and u_h be solutions of* (2.5) *and* (3.6)*, respectively. Then, under the condition of symmetry of the bilinear form $a(\cdot,\cdot)$, the estimate*

$$\begin{aligned}(3.39)\qquad \|u-u_h\|_1(t) \leqslant C_{17}\|u_h(0)-u(0)\|_1 \\ + C_{19}h^{l-1}\Biggl[\|u(t)\|_l + \|u(0)\|_l + \Biggl(\int_0^t \Bigl\|\frac{\partial u}{\partial t}(s)\Bigr\|_l^2\,\mathrm{d}s\Biggr)^{1/2}\Biggr]\end{aligned}$$

holds for $t\in[0,T]$.

4. Discretization in time

In this section we consider the fully discretized schemes, obtained via the θ-method (see e.g. [15]), to solve numerically the semidiscrete (Cauchy) problem (3.9)–(3.10).

4.1. Equidistant time-step size.

We introduce some notation: let τ be the equidistant time step, U^n the approximation of $u(t)$ in V_h at $t_n=n\tau$, $n=0,1,2,\ldots$, and let the finite difference operator $\bar{\partial}_t$ be defined as

$$\bar{\partial}_t U^n = \frac{1}{\tau}(U^n - U^{n-1}). \tag{4.1}$$

We assume that $\theta\in[0,1]$ is any fixed parameter and $t_{n,\theta}=t_{n-1}+\theta\tau$. Then the θ-method, applied to (3.6), yields the relations

$$(\bar{\partial}_t U^n,\chi_h)+a(\theta U^n+(1-\theta)U^{n-1},\chi_h) = F(t_{n,\theta},\chi_h)\quad \forall\chi_h\in V_h,\ \ n=1,2,\ldots, \tag{4.2}$$

where

$$U^0 = u_h(0). \tag{4.3}$$

Obviously, this defines U^n implicitly by means of U^{n-1} for all $\theta\neq 0$.

R e m a r k 4.1. Our goal is to obtain unconditionally stable schemes (see [15]), therefore we restrict our considerations only to the case $\theta\in[0.5,1]$ in what follows.

Recall that in this section we require more smoothness with respect to time from the exact solution: we assume that $u\in C^3((0,T),H^l(\Omega))$. In order to estimate the global error $U^n-u(t_n)$ we split the error into two parts

$$U^n - u(t_n) = (U^n - P_h u(t_n)) + (P_h u(t_n) - u(t_n)) := \sigma^n + \varrho^n. \tag{4.4}$$

Due to (3.4), we have the estimate

$$\|\varrho^n\| \leqslant C_5 h^l \|u(t_n)\|_l. \tag{4.5}$$

In the sequel, we consider the term σ^n. Let L denote the elliptic part of the parabolic operator, i.e.,

$$Lu = \operatorname{div}(\mathcal{A}(x)\operatorname{grad} u) - b\cdot\operatorname{grad} u - cu, \tag{4.6}$$

defined on the set of functions satisfying (2.2).

Then we have

$$(-Lu, v) = a(u,v) - \langle g, v\rangle \quad \forall v \in H^1(\Omega). \tag{4.7}$$

Using the definition of σ^n, (4.2), (3.2), (2.6) and (4.7), we obtain

$$\begin{aligned}
(\overline{\partial}_t\sigma^n, \chi_h) &+ a(\theta\sigma^n + (1-\theta)\sigma^{n-1}, \chi_h)\\
&= (\overline{\partial}_t U^n, \chi_h) - (\overline{\partial}_t P_h u(t_n), \chi_h) + a(\theta U^n + (1-\theta)U^{n-1}, \chi_h)\\
&\quad - a(\theta P_h u(t_n), \chi_h) - a((1-\theta)P_h u(t_{n-1}), \chi_h)\\
&= F(t_{n,\theta}, \chi_h) - (\overline{\partial}_t P_h u(t_n), \chi_h) - \theta a(u(t_n), \chi_h) - (1-\theta)a(u(t_{n-1}), \chi_h)\\
&= (u'(t_{n,\theta}), \chi_h) + a(u(t_{n,\theta}), \chi_h) - (\overline{\partial}_t P_h u(t_n), \chi_h)\\
&\quad - \theta a(u(t_n), \chi_h) - (1-\theta)a(u(t_{n-1}), \chi_h)\\
&= (u'(t_{n,\theta}), \chi_h) - (Lu(t_{n,\theta}), \chi_h) + \langle g, \chi_h\rangle - (\overline{\partial}_t P_h u(t_n), \chi_h)\\
&\quad + \theta[(Lu(t_n), \chi_h) - \langle g, \chi_h\rangle] + (1-\theta)[(Lu(t_{n-1}), \chi_h) - \langle g, \chi_h\rangle]\\
&= (u'(t_{n,\theta}), \chi_h) - (\overline{\partial}_t P_h u(t_n), \chi_h)\\
&\quad - (L(u(t_{n,\theta}) - \theta u(t_n) - (1-\theta)u(t_{n-1})), \chi_h).
\end{aligned}$$

Consequently, using the notation

$$\begin{aligned}
\omega^n &:= [(P_h - I)\overline{\partial}_t u(t_n)] + [\overline{\partial}_t u(t_n) - u'(t_{n,\theta})]\\
&\quad + [L(u(t_{n,\theta}) - \theta u(t_n) - (1-\theta)u(t_{n-1}))]\\
&=: \omega_1^n + \omega_2^n + \omega_3^n
\end{aligned} \tag{4.8}$$

(here I is the identity operator), we get the relation

$$(\overline{\partial}_t\sigma^n, \chi_h) + a(\theta\sigma^n + (1-\theta)\sigma^{n-1}, \chi_h) = -(\omega^n, \chi_h) \quad \forall \chi_h \in V_h. \tag{4.9}$$

Now, choosing $\chi_h = \theta\sigma^n + (1-\theta)\sigma^{n-1}$ in (4.9) and employing the ellipticity of the bilinear form a (see (2.9)), we obtain

$$(\overline{\partial}_t\sigma^n, \theta\sigma^n + (1-\theta)\sigma^{n-1}) \leqslant \|\omega^n\|(\theta\|\sigma^n\| + (1-\theta)\|\sigma^{n-1}\|). \tag{4.10}$$

First let us consider the Crank-Nicolson scheme, i.e., $\theta = 0.5$. Then (4.10) implies

$$\|\sigma^n\|^2 - \|\sigma^{n-1}\|^2 \leqslant \tau\|\omega^n\|(\|\sigma^n\| + \|\sigma^{n-1}\|),$$

i.e.,

$$\|\sigma^n\| \leqslant \|\sigma^{n-1}\| + \tau\|\omega^n\|. \tag{4.11}$$

Consequently, we have the estimate

$$\|\sigma^n\| \leqslant \|\sigma^0\| + \tau\sum_{j=1}^{n}(\|\omega_1^j\| + \|\omega_2^j\| + \|\omega_3^j\|). \tag{4.12}$$

Obviously, from the triangle inequality and (3.4) we have (cf. (3.21))

$$\|\sigma^0\| = \|u_h(0) - P_h u(0)\| \leqslant \|u_h(0) - u(0)\| + C_5 h^l\|u(0)\|_l. \tag{4.13}$$

Moreover, the relation

$$\omega_1^j = (P_h - I)\frac{1}{\tau}\int_{t_{j-1}}^{t_j} u'(s)\,\mathrm{d}s = \frac{1}{\tau}\int_{t_{j-1}}^{t_j}(P_h - I)u'(s)\,\mathrm{d}s \tag{4.14}$$

holds. Hence, using (3.4), we find that

$$\tau\sum_{j=1}^{n}\|\omega_1^j\| \leqslant \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} C_5 h^l\|u'(s)\|_l\,\mathrm{d}s = C_{20}h^l\int_0^{t_n}\|u'(s)\|_l\,\mathrm{d}s. \tag{4.15}$$

Further, using a simple integral equality, we get

$$\begin{aligned}\|\omega_2^j\| &= \|\overline{\partial}_t u(t_j) - u'(t_{j,\frac{1}{2}})\| \\ &= \frac{1}{2\tau}\left\|\int_{t_{j-1}}^{t_{j,\frac{1}{2}}}(s - t_{j-1})^2 u'''(s)\,\mathrm{d}s + \int_{t_{j,\frac{1}{2}}}^{t_j}(s - t_j)^2 u'''(s)\,\mathrm{d}s\right\| \\ &\leqslant C_{21}\tau\int_{t_{j-1}}^{t_j}\|u'''(s)\|\,\mathrm{d}s.\end{aligned} \tag{4.16}$$

Similarly, using the Taylor expansion, we obtain

$$\|\omega_3^j\| = \left\|L\Big(u(t_{j,\frac{1}{2}}) - \frac{1}{2}u(t_j) - \frac{1}{2}u(t_{j-1})\Big)\right\| \leqslant C_{22}\tau\int_{t_{j-1}}^{t_j}\|Lu''(s)\|\,\mathrm{d}s. \tag{4.17}$$

Finally, substituting (4.13), (4.15), (4.16) and (4.17) into (4.12) and using (4.5), we can summarize our results as follows.

Theorem 4.2. *For the global error of the fully discretized Crank-Nicolson-Galerkin method the estimate*

$$\begin{aligned}\|U^n - u(t_n)\| &\\ \leqslant C_{23}\Big(\|u_h(0) - u(0)\| &+ h^l(\|u(0)\|_l + \|u(t_n)\|_l + \int_0^{t_n} \|u'(s)\|_l \, ds)\Big) \\ &+ C_{24}\tau^2 \int_0^{t_n} (\|u'''(s)\| + \|Lu''(s)\|) \, \mathrm{d}s\end{aligned} \tag{4.18}$$

holds.

R e m a r k 4.3. If $u_h(0)$ is a suitable approximation of $u(0)$ then (4.18) turns into the estimate

$$\begin{aligned}\|U^n - u(t_n)\| \leqslant C_{25} h^l \Big(1 + \|u(0)\|_l + \|u(t_n)\|_l + \int_0^{t_n} \|u'(s)\|_l \, \mathrm{d}s\Big) &\\ + C_{26}\tau^2 \int_0^{t_n} (\|u'''(s)\| + \|Lu''(s)\|) \, \mathrm{d}s.&\end{aligned} \tag{4.19}$$

Let us consider now the case $\theta \in (0.5, 1]$. For the left-hand side of (4.10) we have the identity

$$(\overline{\partial}_t \sigma^n, \theta\sigma^n + (1-\theta)\sigma^{n-1}) = \frac{1}{\tau}(\theta\|\sigma^n\|^2 - (1-\theta)\|\sigma^{n-1}\|^2 + (1-2\theta)(\sigma^{n-1}, \sigma^n)).$$

Using the Cauchy-Schwarz-Buniakovsky inequality for the above equality, we may rewrite (4.10) in the following form:

$$\begin{aligned}\theta\|\sigma^n\|^2 - (1-\theta)\|\sigma^{n-1}\|^2 + (1-2\theta)\|\sigma^{n-1}\| \, \|\sigma^n\| &\\ \leqslant \tau\|\omega^n\|(\theta\|\sigma^n\| + (1-\theta)\|\sigma^{n-1}\|).&\end{aligned} \tag{4.20}$$

Further, we observe that

$$\begin{aligned}\theta\|\sigma^n\|^2 - (1-\theta)\|\sigma^{n-1}\|^2 + (1-2\theta)\|\sigma^{n-1}\| \, \|\sigma^n\| &\\ = (\|\sigma^n\| - \|\sigma^{n-1}\|)(\theta\|\sigma^n\| + (1-\theta)\|\sigma^{n-1}\|),&\end{aligned}$$

therefore (4.20) results in the relation (4.11). Thus, we can directly apply the proof of Theorem 4.2. Clearly, the estimates (4.13)–(4.14) hold. For the term ω_2^j, using the Taylor expansion with the integral remainder, we get

$$\|\omega_2^j\| \leqslant C_{27} \int_{t_{j-1}}^{t_j} \|u''(s)\| \, \mathrm{d}s. \tag{4.21}$$

For the term ω_3^j we similarly have

$$\|\omega_3^j\| \leqslant C_{28} \int_{t_{j-1}}^{t_j} \|Lu'(s)\| \,\mathrm{d}s. \tag{4.22}$$

Using all these estimates, we can summarize our result as follows.

Theorem 4.4. *For the global error of the fully discretized Galerkin method with $\theta \in (0.5, 1]$ the estimate*

$$\begin{aligned}\|U^n - u(t_n)\| \leqslant C_{29}\Big(\|u_h(0) - u(0)\| + h^l\Big(\|u(0)\|_l + \|u(t_n)\|_l \\ + \int_0^{t_n} \|u'(s)\|_l \,\mathrm{d}s\Big)\Big) + C_{30}\tau \int_0^{t_n} (\|u''(s)\| + \|Lu'(s)\|) \,\mathrm{d}s\end{aligned} \tag{4.23}$$

holds.

R e m a r k 4.5. If $u_h(0)$ is a suitable approximation then (4.23) can be rewritten as

$$\begin{aligned}\|U^n - u(t_n)\| \leqslant C_{31}h^l\Big(1 + \|u(0)\|_l + \|u(t_n)\|_l + \int_0^{t_n} \|u'(s)\|_l \,\mathrm{d}s\Big) \\ + C_{32}\tau \int_0^{t_n} (\|u''(s)\| + \|Lu'(s)\|) \,\mathrm{d}s.\end{aligned} \tag{4.24}$$

4.2. Variable time-step size.

The results in Section 4.1 are formulated for the equidistant time step. We consider now also the case of variable time step:

$$0 =: t_0 < t_1 < \ldots < t_n, \quad \tau_j := t_j - t_{j-1}.$$

The following theorem can be proved.

Theorem 4.6. *Under a suitable choice of the initial function $u_h(0)$, for the global error of the fully discretized Crank-Nicolson-Galerkin method with a variable time step size, the estimate*

$$\begin{aligned}\|U^n - u(t_n)\| \leqslant C_{33}h^l\Big(1 + \|u(t_n)\|_l + \|u(0)\|_l + \int_0^{t_n} \|u'(s)\|_l \,\mathrm{d}s\Big) \\ + C_{34} \sum_{j=1}^{n} \tau_j^2 \int_{t_{j-1}}^{t_j} (\|u'''(s)\| + \|Lu''(s)\|) \,\mathrm{d}s\end{aligned} \tag{4.25}$$

holds.

P r o o f. Similarly to the previous proofs we can derive (4.11) in the form

$$\|\sigma^n\| \leqslant \|\sigma^{n-1}\| + \tau_n\|\omega^n\| \tag{4.26}$$

then, as before, we obtain

$$\|\sigma^n\| \leqslant \|\sigma^0\| + \sum_{j=1}^{n}\tau_j\|\omega^j\| \leqslant \|\sigma^0\| + \sum_{j=1}^{n}\tau_j(\|\omega_1^j\| + \|\omega_2^j\| + \|\omega_3^j\|). \tag{4.27}$$

Due to (4.14) and (3.4) we get

$$\tau_j\|\omega_1^j\| \leqslant \int_{t_{j-1}}^{t_j}\|(P_h - I)u'(s)\|\,\mathrm{d}s \leqslant C_{35}h^l\int_{t_{j-1}}^{t_j}\|u'(s)\|_l\,\mathrm{d}s,$$

i.e.

$$\sum_{j=1}^{n}\tau_j\|\omega_1^j\| \leqslant C_{36}h^l\int_0^{t_n}\|u^{'}(s)\|_l\,\mathrm{d}s. \tag{4.28}$$

Now, by virtue of (4.16) and (4.17) combined with (4.28) the inequality (4.27) implies (4.25). □

Obviously, in the same manner we can prove the next theorem.

Theorem 4.7. *For the global error of the fully discretized Galerkin method (with $\theta \in (0.5, 1]$) with a variable time step size, under a suitable choice of the initial function $u_h(0)$, the estimate*

$$\begin{aligned}\|U^n - u(t_n)\| \leqslant C_{37}h^l\Big(1 + \|u(t_n)\|_l + \|u(0)\|_l + \int_0^{t_n}\|u'(s)\|_l\,\mathrm{d}s\Big)\\ + C_{38}\sum_{j=1}^{n}\tau_j\int_{t_{j-1}}^{t_j}(\|u''(s)\| + \|Lu'(s)\|)\,\mathrm{d}s\end{aligned} \tag{4.29}$$

holds.

R e m a r k 4.8. If $\theta = 1$ (the backward Euler scheme) then $\omega_3^j = 0$. For this case (4.29) has the form

$$\begin{aligned}\|U^n - u(t_n)\| \leqslant C_{39}h^l\Big(1 + \|u(t_n)\|_l + \|u(0)\|_l + \int_0^{t_n}\|u'(s)\|_l\,\mathrm{d}s\Big)\\ + C_{40}\sum_{j=1}^{n}\tau_j\int_{t_{j-1}}^{t_j}\|u''(s)\|\,\mathrm{d}s.\end{aligned} \tag{4.30}$$

Consequently, for the backward Euler scheme we have to use smaller τ_j on those intervals where $\|u''(t)\|$ is large.

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