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ON THE CAUCHY PROBLEM FOR THE EQUATIONS  
OF IDEAL COMPRESSIBLE MHD FLUIDS WITH RADIATION\*

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*Abstract.* We consider a system of balance laws describing the motion of an ionized compressible fluid interacting with magnetic fields and radiation effects. The local-in-time existence of a unique smooth solution for the Cauchy problem is proven. The proof follows from the method of successive approximations.

*Keywords:* magnetohydrodynamics, radiation transport, local existence

*MSC 2000:* 35L60, 76N10

1. INTRODUCTION AND MAIN RESULT

We consider the classical equations of compressible magnetohydrodynamics in three dimensional space, i.e. we have in  $\mathbb{R}^3 \times (0, T)$ ,  $T > 0$ :

$$(1) \quad \begin{aligned} \varrho_{,t} + \operatorname{div}(\varrho v) &= 0, \\ (\varrho v)_{,t} + \varrho v \cdot \nabla v + \nabla \cdot \mathbf{P} &= \varrho f, \\ H_{,t} + v \cdot \nabla H - H \cdot \nabla v &= 0, \\ \operatorname{div} H &= 0, \\ \left( \varrho e + \frac{|H|^2}{8\pi} \right)_{,t} + \operatorname{div} \left( (\varrho e + p)v + \frac{1}{4\pi} H \times (v \times H) \right) &= \varrho(f \cdot v + Q_r). \end{aligned}$$

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Here  $\varrho = \varrho(x, t) > 0$  is the density of the fluid and  $e = e(x, t) > 0$  the total (hydrodynamical) energy.  $v = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$  is the velocity vector and  $H = (H_1(x, t), H_2(x, t), H_3(x, t)) \in \mathbb{R}^3$  the magnetic field. The matrix  $\mathbf{P} = \mathbf{P}(x, t) \in \mathbb{R}^{3 \times 3}$  is given by

$$(2) \quad \mathbf{P} = (p_{ij})_{i,j}, \quad p_{ij} = \left( p + \frac{1}{8\pi} |H|^2 \right) \delta_{ij} - \frac{1}{4\pi} H_i H_j.$$

$\delta_{ij}$  is the Kronecker symbol. With the internal energy  $\varepsilon > 0$  the energy splits up into

$$(3) \quad e = \varepsilon + \frac{1}{2} |\vec{v}|^2.$$

To cover a possibly big class of thermodynamical processes we assume the pressure  $p = p(\tau, s) > 0$ , the internal energy  $\varepsilon = \varepsilon(\tau, s) > 0$  and temperature  $\theta = \theta(\tau, s) > 0$  to be functions only of the specific volume  $\tau = \tau(x, t) = 1/\varrho(x, t)$  and the physical entropy  $s = s(x, t) > 0$  satisfying the second law of thermodynamics. In particular, this implies

$$(4) \quad \frac{\partial \varepsilon}{\partial \tau}(\tau, s) = -p(\tau, s) < 0, \quad \frac{\partial \varepsilon}{\partial s}(\tau, s) = \theta(\tau, s) > 0.$$

Additionally we restrict to the case of local thermodynamical equilibrium, meaning that  $\varepsilon$  is a convex function, i.e.

$$(5) \quad \frac{\partial^2 \varepsilon}{\partial s^2} > 0, \quad \frac{\partial^2 \varepsilon}{\partial s^2} \frac{\partial^2 \varepsilon}{\partial \tau^2} - \left( \frac{\partial^2 \varepsilon}{\partial \tau \partial s} \right)^2 > 0.$$

As sources we consider a specific external force field given by a smooth function  $f = (f_1, f_2, f_3)^T: \mathbb{R}^3 \times \mathbb{R}_{>0} \rightarrow \mathbb{R}^3$ . Furthermore we take into account a heat source  $Q_r: \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  which is generated by radiation. It is defined by

$$(6) \quad Q_r = Q_r(x, t) = 4\pi\kappa(\varrho(x, t), \theta(x, t)) \left( \int_{\mathcal{S}_1^2} I(x, t, \mu) d\mu - B(\theta(x, t)) \right).$$

$\kappa = \kappa(\varrho, \theta) \geq 0$  is the absorption coefficient and  $\mathcal{S}_1^2$  denotes the two-dimensional unit sphere. From Planck's law of black body radiation we get for constants  $k, h > 0$

$$B(\theta) = 2\pi h c^2 \frac{1}{\exp\left(\frac{hc}{k\theta}\right) - 1}.$$

$c > 0$  is the speed of light.

The radiation intensity  $I = I(x, t, \mu)$  is not a given function but solution of an additional partial differential equation. For each  $\mu \in \mathcal{S}_1^2$ ,  $\mu$  being the direction of radiation,  $I(\cdot, \cdot, \mu)$  satisfies the transport equation

$$(7) \quad \frac{1}{c} I_{,t} + \mu \cdot \nabla I = -(\sigma(\varrho, \theta) + \kappa(\varrho, \theta)) \varrho I + S$$

in the domain  $\mathbb{R}^3 \times (0, T) \times \mathcal{S}_1^2$ .  $S: \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is given by

$$S = S(x, t) = \sigma(\varrho(x, t), \theta(x, t))\varrho(x, t) \int_{\mathcal{S}_1^2} q(\mu, \mu') I(x, t, \mu') d\mu' \\ + \kappa(\varrho(x, t), \theta(x, t))\varrho(x, t)B(\theta(x, t)),$$

where  $0 \leq q(\mu, \mu') \leq 1$ . The function  $\sigma = \sigma(\varrho, \theta) \geq 0$  describes scattering effects.

Using the state laws (2), (3), and the given relations for  $\varepsilon$ ,  $p$ , and  $\theta$  the differential equations (1), (7) can be written as an at least formally closed  $(9 \times 9)$ -system in the unknowns  $\varrho$ ,  $v$ ,  $s$ ,  $H$  and  $I$ . In the sequel we focus on the Cauchy problem associated to (1) and (7). For  $\varrho_0, s_0: \mathbb{R}^3 \rightarrow \mathbb{R}_+$  and  $v_0, H_0: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $I_0: \mathbb{R}^3 \times \mathcal{S}_1^2 \rightarrow \mathbb{R}_{>0}$  we have

$$(8) \quad \varrho(\cdot, 0) = \varrho_0, \quad v(\cdot, 0) = v_0, \quad s(\cdot, 0) = s_0, \quad H(\cdot, 0) = H_0,$$

$$(9) \quad I(\cdot, 0, \cdot) = I_0.$$

Ignoring the source terms, system (1) is a first order hyperbolic conservation law with divergence constraint for  $H$  when considered in the (conservative) variables  $\varrho$ , the momentum  $m = \varrho v$ ,  $H$ , and the total energy density  $\mathcal{E} = \varrho e + \frac{|H|^2}{8\pi}$ . More information on (1) as an example of a hyperbolic conservation law can be found in the classical book [1]. As long as the initial value problem for a hyperbolic conservation law has a smooth solution the theory of characteristics tells us that its solution can be obtained by solving an ordinary initial value problem along characteristics. The initial values for the latter problem are a *finite* subset of the range of the initial values of the conservation law. This changes for our problem. The radiation transport equation (7) is an advection equation parameterized over *all*  $\mu \in \mathcal{S}_1^2$ . Thus, for each  $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ , the radiation heat source  $Q_r(x, t)$  in (6) which involves integration of  $I(x, t, \mu)$  with respect to  $\mu$  does depend on a (nontrivial) *compact* subset of the range of the initial datum for the conservation law. In other words: the heat source  $Q_r$  introduces more *global* dependencies into the problem.

The system (1), (7) describes the interaction of a plasma with radiation effects ([11]). Applications for physical processes governed by (1), (7) can be found in astrophysics. We refer to the overview article by Mihalas ([10]). In particular it describes the dynamics of the plasma in the sun's upper atmosphere: the photosphere. In this region the main energy transfer is due to radiation transport.

From the mathematical point of view the Riemann problem for (1), (7) has been analyzed in [11]. However, up to the knowledge of the authors no rigorous existence results for general Cauchy problems for (1), (7) have been established so far.

The model with only one direction of radiation has been studied intensively by Kawashima et. al. in the case of instantaneous radiation transport (cf. [8] for a review of results). Then it can be rewritten as a hyperbolic-elliptic system.

Scalar model problems have been derived for this hyperbolic-elliptic system and also for the system (1), (7). For these scalar model problems global existence of weak entropy solutions has been established recently ([2], [6]).

In this paper we will consider classical solutions of the problem (1), (8), (7), (9), i.e. we search for bounded continuously differentiable functions  $\varrho$ ,  $v$ ,  $s$ ,  $H$ ,  $I$  that satisfy (1), (8), (7), (9) pointwise. Furthermore physically relevant solutions have to satisfy the thermodynamical conditions (4), (5). For related work on equations with the property of finite speed of propagation we refer to [7], [9], [12]. The following local-in-time existence result will be proven.

**Theorem 1.1.** *Let  $\varrho_0, s_0: \mathbb{R}^3 \rightarrow \mathbb{R}_{>0}$  and  $v_0, H_0: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $I_0: \mathbb{R}^3 \times \mathcal{S}_1^2 \rightarrow \mathbb{R}_{>0}$  be given. Assume that (4), (5) are satisfied for  $\varrho_0, s_0$ . Furthermore assume that there are constants  $\bar{\varrho}, \bar{s}, \bar{I} \in \mathbb{R}_{>0}$  and  $\bar{v}, \bar{H} \in \mathbb{R}^3$  such that  $\varrho_0 - \bar{\varrho}, s_0 - \bar{s}, I_0(\cdot, \mu) - \bar{I} \in H^3(\mathbb{R}^3, \mathbb{R})$  and  $v_0 - \bar{v}, H_0 - \bar{H} \in H^3(\mathbb{R}^3, \mathbb{R}^3)$  for  $\mu \in \mathcal{S}_1^2$ . Then there exists a constant  $T_* > 0$  such that for  $t \in (0, T_*]$  there exists a classical solution  $(\varrho, v, s, H, I)$  of the problem (1), (8), (7), (9) in  $\mathbb{R}^3 \times [0, t]$  with*

$$\begin{aligned} \varrho - \bar{\varrho}, s - \bar{s} &\in L^\infty(0, t; H^3(\mathbb{R}^3, \mathbb{R})), \\ v - \bar{v}, H - \bar{H} &\in L^\infty(0, t; H^3(\mathbb{R}^3, \mathbb{R}^3)), \\ I(\cdot, \cdot, \mu) - \bar{I} &\in L^\infty(0, t; H^3(\mathbb{R}^3, \mathbb{R})) \quad (\mu \in \mathcal{S}_1^2). \end{aligned}$$

*The classical solution satisfies (4), (5), and  $I > 0$ . Moreover the solution is unique in the class of classical solutions.*

Let us give an outline of the paper. In Section 2 we will put the equations in a symmetric form and reformulate the problem in a more general and analytically more convenient form. This allows us to get a-priori bounds on the solutions of a linearized Cauchy problem in Section 3 using energy estimates and characteristic curves. Finally in Section 4 the method of successive approximations is applied to prove an existence and uniqueness theorem for the reformulated problem from Section 2 (Theorem 4.1). Theorem 1.1 is a direct consequence.

## 2. SYMMETRIZATION OF THE MAGNETOHYDRODYNAMICS EQUATIONS AND REFORMULATION OF THE PROBLEM

To prove local existence of a solution we first rewrite the magnetohydrodynamical system (1) as a symmetric hyperbolic system. Basically we use the results on the symmetrization of constrained hyperbolic balance laws by Friedrichs (cf. [3], [4]) and adjust them to our case.

As mentioned in Section 1 the system (1) is a constrained hyperbolic balance law for the unknown vector  $w = (\varrho, \varrho v_1, \varrho v_2, \varrho v_3, H_1, H_2, H_3, \mathcal{E})^T$ . Recall that we have  $\mathcal{E} = \varrho e + \frac{|H|^2}{8\pi}$ . In other words, for appropriate functions  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3: \mathbb{R}^8 \rightarrow \mathbb{R}^8$  it takes the form

$$(10) \quad w_{,t} + \mathcal{F}_k(w)_{,x_k} = (0, \varrho f_1, \varrho f_2, \varrho f_3, 0, 0, 0, \varrho(f \cdot v + Q_r))^T, \quad \operatorname{div} H = 0.$$

In (10) and henceforth we use the summation rule.

Due to (4) we can invert  $s \mapsto \varepsilon(\tau, s)$  for each  $\tau > 0$  and express the entropy  $s$  as  $s = s(\tau, \varepsilon)$ . It can be checked that the scalar functions  $U, h_1, h_2, h_3$  given by

$$U(w) = -\varrho s \left( \frac{1}{\varrho}, \frac{\mathcal{E}}{\varrho} - \frac{|\varrho v|^2}{2\varrho^2} \right), \quad h_k(w) = -\varrho s \left( \frac{1}{\varrho}, \frac{\mathcal{E}}{\varrho} - \frac{|\varrho v|^2}{2\varrho^2} \right) v_k \quad (k = 1, 2, 3)$$

fulfill

$$(11) \quad \nabla U(w)^T \nabla \mathcal{F}_k(w) = \nabla h_k(w)^T \quad (k = 1, 2, 3).$$

Here  $\nabla$  denotes the total derivative with respect to  $w$ . Then it is well-known from conservation law theory that the compatibility relation (11) implies that a classical solution  $w$  of (10) satisfies an additional balance law for the (mathematical) entropy  $U(w) = -\varrho s$ . In our case we obtain

$$(12) \quad (\varrho s)_{,t} + \operatorname{div}(\varrho s v) = \frac{\varrho}{\theta} Q_r.$$

We now consider the nine equations (1) together with (12) as a  $(10 \times 10)$ -system

$$(13) \quad G_0(u)_{,t} + G_k(u)_{,x_k} = g(u)$$

for the eight unknowns

$$(14) \quad u = (u_1, \dots, u_8)^T \in \mathcal{U} := (\mathbb{R}_{>0} \times \mathbb{R}^6 \times \mathbb{R}_{>0})$$

with

$$\begin{aligned} u_1 = \varrho, \quad u_2 = v_1, \quad u_3 = v_2, \quad u_4 = v_3, \\ u_5 = H_1, \quad u_6 = H_2, \quad u_7 = H_3, \quad u_8 = s. \end{aligned}$$

Introducing  $\Lambda(u) = (\lambda_1(u), \dots, \lambda_{10}(u))^T$  with

$$\begin{aligned} \lambda_1 = -\frac{1}{2}|v|^2 + \varepsilon - s\theta + \frac{p}{\varrho}, \quad \lambda_2 = v_1, \quad \lambda_3 = v_2, \quad \lambda_4 = v_3, \quad \lambda_5 = \frac{1}{4\pi}H_1, \\ \lambda_6 = \frac{1}{4\pi}H_2, \quad \lambda_7 = \frac{1}{4\pi}H_3, \quad \lambda_8 = v \cdot H, \quad \lambda_9 = -1, \quad \lambda_{10} = \theta, \end{aligned}$$

we get

$$(15) \quad \sum_{i=1}^{10} \lambda_i(u) G_{0i}(u),_t = 0, \quad \sum_{i=1}^{10} \lambda_i(u) G_{ki}(u),_{x_k} = 0, \quad k = 1, 2, 3,$$

and therefore  $\sum_{i=1}^{10} \lambda_i(u) \partial_{u_\beta} G_{ki}(u)$  for  $\beta \in \{1, \dots, 8\}$  and by the product rule for  $\alpha \in \{1, \dots, 8\}$

$$(16) \quad \sum_{i=1}^{10} \partial_{u_\alpha} \lambda_i(u) \partial_{u_\beta} G_{ki}(u) = \sum_{i=1}^{10} \lambda_i(u) \partial_{u_\beta u_\alpha}^2 G_{ki}(u).$$

The relation (16) implies that the matrix

$$\mathbf{A}_k(u) := \left( \sum_{i=1}^{10} \partial_{u_\alpha} \lambda_i(u) \partial_{u_\beta} G_{ki}(u) \right)_{\alpha, \beta}$$

is symmetric for  $u \in \mathcal{U}$  and  $k = 0, \dots, 3$ . Note that (16) gives exactly the elements of the matrix  $\mathbf{A}_k(u)$ . Moreover, the matrix  $\mathbf{A}_0$  is given explicitly by

$$\mathbf{A}_0(u) = \begin{pmatrix} \tau^3 \varepsilon_{\tau\tau} & 0 & 0 & 0 & 0 & 0 & 0 & -\tau \varepsilon_{\tau s} \\ 0 & \varrho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varrho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varrho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (4\pi)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (4\pi)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (4\pi)^{-1} & 0 \\ -\tau \varepsilon_{\tau s} & 0 & 0 & 0 & 0 & 0 & 0 & -\varrho \varepsilon_{ss} \end{pmatrix}.$$

It has the triple eigenvalues  $\varrho, (4\pi)^{-1} > 0$  and the two single eigenvalues

$$\frac{\varrho \varepsilon_{ss} + \tau^3 \varepsilon_{\tau\tau}}{2} \pm \sqrt{\frac{(\varrho \varepsilon_{ss} + \tau^3 \varepsilon_{\tau\tau})^2}{4} - \tau^2 (\varepsilon_{ss} \varepsilon_{\tau\tau} - \varepsilon_{\tau s}^2)}.$$

From the second condition in (5) we conclude that  $\mathbf{A}_0(u)$  is positive definite for all  $u \in \mathcal{U}$ .

The discussion above shows that the following Cauchy problem in  $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$  is symmetric hyperbolic:

$$(17) \quad \mathbf{A}_0(u) u_{,t} + \mathbf{A}_k(u) u_{,x_k} = \mathbf{F}_1(u) + \mathbf{F}_2(u) \int_{\mathcal{S}_1^2} I(\cdot, \cdot, \mu) d\mu,$$

$$(18) \quad u(\cdot, 0) = u_0.$$

The functions  $\mathbf{F}_1, \mathbf{F}_2: \mathcal{U} \rightarrow \mathbb{R}^8$  are given by

$$\begin{aligned}\mathbf{F}_1(u) &= (4\pi\kappa B(\theta)\varepsilon_{\tau s}\tau/\theta, \varrho f_1, \varrho f_2, \varrho f_3, 0, 0, 0, -4\pi\kappa B(\theta)\varrho\varepsilon_{ss}/\theta)^T, \\ \mathbf{F}_2(u) &= (-4\pi\kappa\varepsilon_{\tau s}\tau/\theta, 0, 0, 0, 0, 0, 0, 4\pi\kappa\varrho\varepsilon_{ss}/\theta)^T.\end{aligned}$$

Note that the inhomogeneous term  $\mathbf{F}_1(u) + \mathbf{F}_2(u) \int_{\mathcal{S}_1^2} I(\cdot, \cdot, \mu) d\mu$  in (17) is equal to  $D\Lambda(u)^T g(u)$  for  $u \in \mathcal{U}$ , where  $D\Lambda$  is the Jacobian of  $\Lambda$ .

The considerations above show that the problem of existence of a classical solution  $u$  of (17), (18) is equivalent to the problem of existence of a classical solution for (1), (8). Therefore it suffices to consider henceforth (17), (18).

We also rewrite the radiation transport equation for  $I = I(x, t, \mu)$  in a more abstract way and obtain for  $\mu \in \mathcal{S}_1^2$

$$(19) \quad I_{,t} + \mu \cdot \nabla I = g_1(u)I + g_2(u) \int_{\mathcal{S}_1^2} q(\mu, \mu') I(\cdot, \cdot, \mu') d\mu' + g_3(u),$$

$$(20) \quad I(\cdot, 0, \mu) = I_0(\cdot, \mu),$$

where

$$g_1(u) = -c(\sigma(\varrho, \theta) + \kappa(\varrho, \theta))\varrho, \quad g_2(u) = c\sigma(\varrho, \theta)\varrho, \quad g_3(u) = c\kappa(\varrho, \theta)\varrho B(\theta).$$

The direction  $\mu$  has been redefined as  $\mu := \mu c$ . Since we do not need the exact form of the coefficients  $\mathbf{F}_1, \mathbf{F}_2, g_1, g_2, g_3$  we can consider general systems of type (17), (19) where only the subsequent conditions are supposed to hold.

**A s s u m p t i o n 2.1.**

- (i) The matrices  $\mathbf{A}_0(u), \dots, \mathbf{A}_3(u)$  are symmetric for  $u \in \mathcal{U}$  and three times continuously differentiable.
- (ii) There exists a continuous function  $\alpha: \mathcal{U} \rightarrow \mathbb{R}_{>0}$  such that

$$v^T \mathbf{A}_0(u)v \geq \alpha(u)v \cdot v \quad (u \in \mathcal{U}, v \in \mathbb{R}^8).$$

- (iii) The functions  $\mathbf{F}_1, \mathbf{F}_2: \mathcal{U} \rightarrow \mathbb{R}^8$  are three times continuously differentiable.

**A s s u m p t i o n 2.2.** The functions  $g_1, g_2, g_3: \mathcal{U} \rightarrow \mathbb{R}$  are three times continuously differentiable.



### 3. A-PRIORI ESTIMATES

#### 3.1. Notations.

Following the notation of Theorem 1.1 let  $\bar{u} \in \mathcal{U}$  be given by  $\bar{u} = (\bar{v}, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{H}_1, \bar{H}_2, \bar{H}_3, \bar{s})^T$ . For a function  $w: \mathbb{R}^3 \times [0, T) \rightarrow \mathbb{R}^8$ , we introduced a function  $\tilde{w}: \mathbb{R}^3 \times [0, T) \rightarrow \mathbb{R}^8$  and define a set of functions  $H_{\bar{u}}^3$  by

$$\begin{aligned} \tilde{w} &= w - \bar{u}, \\ H_{\bar{u}}^3 &= \{w: \mathbb{R}^3 \times [0, T) \rightarrow \mathcal{U} \mid \tilde{w}(\cdot, t) \in H^3(\mathbb{R}^3, \mathbb{R}^8), t \in [0, T)\}. \end{aligned}$$

Analogously we define  $\tilde{I}(\cdot, \cdot, \mu): \mathbb{R}^3 \times [0, T) \rightarrow \mathbb{R}$  and the set  $H_{\bar{I}}^3$  for the constant radiation intensity  $\bar{I} > 0$  and  $\mu \in \mathcal{S}_1^2$ .

In the sequel  $C$  denotes a generic nonnegative constant, and  $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  are monotonely increasing functions that do only depend on the initial datum  $u_0, I_0$  and the data given in Assumptions 2.1, 2.2. In particular they do not depend on  $m \in \mathbb{N}$ , an iteration index to be introduced later.

If not stated differently we denote by  $\|\cdot\|_{L^p}, \|\cdot\|_{H^k}$  the norms of the spaces  $L^p(\mathbb{R}^3, \mathbb{R}^l), H^k(\mathbb{R}^3, \mathbb{R}^l)$  for  $p \geq 1, k, l \in \mathbb{N}$ .

For  $T > 0, m \in \mathbb{N}, \bar{I}, \mu \in \mathcal{S}_1^2$  let  $I_m(\cdot, \cdot, \mu) \in H_{\bar{I}}^3$  and  $I_0 - \bar{I} \in H^3(\mathbb{R}^3, \mathbb{R}_{>0})$  be given. We define  $X_m = X_m(\mu, t)$  and  $X_0 = X_0(\mu)$  by

$$X_m(\mu, t) = \|\tilde{I}_m(\cdot, t, \mu)\|_{H^3} + \left\| \int_{\mathcal{S}_1^2} \tilde{I}_m(\cdot, t, \mu') d\mu' \right\|_{H^3} \quad (t \in [0, T)),$$

and

$$X_0(\mu) = \|I_0(\cdot, \mu) - \bar{I}\|_{H^3} + \left\| \int_{\mathcal{S}_1^2} I_0(\cdot, \mu') - \bar{I} d\mu' \right\|_{H^3}.$$

#### 3.2. The hydrodynamical equations.

For  $T, A > 0, m \in \mathbb{N}, \bar{u} \in \mathcal{U}, \bar{I} > 0$ , and  $\mu \in \mathcal{S}_1^2$  let  $u_m \in L^\infty(0, T; H_{\bar{u}}^3)$  and  $I_m(\cdot, \cdot, \mu) \in L^\infty(0, T; H_{\bar{I}}^3)$  be given bounded smooth functions with

$$(21) \quad \|\tilde{u}_m(\cdot, t)\|_{H^3} + X_m(\mu, t) \leq A \quad (t \in [0, T)).$$

We consider a linear Cauchy problem for  $u_{m+1}$ :

$$(22) \quad \mathbf{A}_0(u_m)u_{m+1,t} + \mathbf{A}_k(u_m)u_{m+1,x_k} = \mathbf{F}_1(u_m) + \mathbf{F}_2(u_m) \int_{\mathcal{S}_1^2} I_m(\cdot, \cdot, \mu') d\mu',$$

$$(23) \quad u_{m+1}(\cdot, 0) = u_0.$$

The existence of a smooth bounded solution  $u_{m+1} \in H_{\bar{u}}^3$  for (22), (23) follows from standard results for linear hyperbolic Cauchy problems ([7], [9]). Note that the

proof of the  $H^3$ -regularity of  $u_{m+1}$  requires even higher regularity of the coefficients, in particular  $u_m$ . In order to do so the initial datum (and the other data of the problem) are mollified and the existence of  $C^\infty$ -solutions is proven. By coupling the mollification parameter with the iteration parameter  $m$  it can be shown that one obtains existence of  $H^3$ -solutions also for initial datum in  $H^3$ . Since this procedure is well-known in the field of first order equations we skip the details and assume that a smooth bounded solution  $u_{m+1} \in H^3_u$  exists from now on.

The same argument applies later on for the radiation transport equation.

We have the following estimate.

**Lemma 3.1.** *Suppose that Assumption 2.1 holds and let  $u_0: \mathbb{R}^3 \rightarrow \mathcal{U}$  be a smooth function with  $u_0 - \bar{u} \in H^3(\mathbb{R}^3, \mathbb{R}^8)$ . Furthermore let  $u_m$  and  $I_m$  with 21 be given. Let  $\delta > 0$  be such that*

$$(24) \quad |u_m(x, t) - \bar{u}| + |I_m(x, t, \mu) - \bar{I}| \leq \delta, \quad (x, t, \mu) \in \mathbb{R}^3 \times [0, T] \times \mathcal{S}_1^2.$$

Then there exists a positive constant  $T_* \leq T$  and continuous increasing functions  $\mathcal{C}_1, \mathcal{C}_2: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$  such that we have for all  $t \in [0, T_*]$

$$(i) \quad \begin{aligned} \|\tilde{u}_{m+1}(\cdot, t)\|_{H^3} &\leq \|u_0 - \bar{u}\|_{H^3} \\ &\quad + t\mathcal{C}_1 \left( \sup_{s \in [0, T_*]} \|\tilde{u}_m(\cdot, s)\|_{H^3} \right. \\ &\quad \left. + \sup_{s \in [0, T_*], \mu \in \mathcal{S}_1^2} \|\tilde{I}_m(\cdot, s, \mu)\|_{H^3} + \delta \right) \\ &\quad \times \left( 1 + \sup_{s \in [0, T_*]} \left\| \int_{\mathcal{S}_1^2} \tilde{I}_m(\cdot, s, \mu) d\mu \right\|_{H^3} \right), \end{aligned}$$

$$(ii) \quad \begin{aligned} \|u_{m+1}(\cdot, t) - u_0\|_{L^\infty} &\leq t\mathcal{C}_2 \left( \sup_{s \in [0, T_*]} \|\tilde{u}_m(\cdot, s)\|_{H^3} + \delta \right) \\ &\quad \times \left( 1 + \sup_{s \in [0, T_*]} \left\| \int_{\mathcal{S}_1^2} \tilde{I}_m(\cdot, s, \mu) d\mu \right\|_{H^3} \right). \end{aligned}$$

$T_*, \mathcal{C}_1, \mathcal{C}_2$  depend on  $\delta, \bar{u}, \bar{I}, \mathbf{A}_0, \mathbf{A}_k$  but not on  $m$ .

*Proof.* Recall that  $\mathbf{A}_0$  is symmetric. We multiply  $\mathbf{A}_0(u_m)\tilde{u}_{m+1,t}$  by  $\tilde{u}_{m+1}$  and obtain

$$\mathbf{A}_0(u_m)\tilde{u}_{m+1,t} \cdot \tilde{u}_{m+1} = \frac{1}{2} \frac{d}{dt} [\mathbf{A}_0(u_m)\tilde{u}_{m+1} \cdot \tilde{u}_{m+1}] - T_0^t \cdot \tilde{u}_{m+1},$$

where  $T_0^t = T_0^t(x, t)$  is given by

$$2T_0^t = \mathbf{A}_0(u_m)_{,t} \tilde{u}_{m+1}.$$

In the same way we get for  $T_0^x = T_0^x(x, t) = \sum_k \mathbf{A}_k(u_m(x, t))_{,x_k} \tilde{u}_{m+1}(x, t)$  the equation

$$\sum_{k=1}^3 \mathbf{A}_k(u_m) \partial_{x_k} \tilde{u}_{m+1} \cdot \tilde{u}_{m+1} = \sum_{k=1}^3 \frac{1}{2} \frac{d}{dx_k} [\mathbf{A}_k(u_m) \tilde{u}_{m+1} \cdot \tilde{u}_{m+1}] - T_0^x \cdot \tilde{u}_{m+1}.$$

Since  $\tilde{u}_{m+1}$  is a solution of (22) we get

$$(25) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \mathbf{A}_0(u_m(x, t)) \tilde{u}_{m+1}(x, t) \cdot \tilde{u}_{m+1}(x, t) dx \\ &= 2 \int_{\mathbb{R}^3} \left( \mathbf{F}_1(u_m(x, t)) + \mathbf{F}_2(u_m(x, t)) \right. \\ & \quad \times \left. \int_{S_1^2} I_m(x, t, \mu) d\mu \right) \cdot \tilde{u}_{m+1}(x, t) dx \\ & \quad + \int_{\mathbb{R}^3} T_0^t(x, t) \cdot \tilde{u}_{m+1}(x, t) + T_0^x(x, t) \cdot \tilde{u}_{m+1}(x, t) dx. \end{aligned}$$

Differentiating  $\mathbf{A}_0(u_m)u_{m+1,t}$  with respect to  $x_k$ ,  $k \in \{1, 2, 3\}$  and multiplying by  $\partial_{x_k} \tilde{u}_{m+1}$  yields

$$\begin{aligned} & \sum_{k=1}^3 \frac{d}{dx_k} [\mathbf{A}_0(u_m) \tilde{u}_{m+1,t}] \cdot \partial_{x_k} \tilde{u}_{m+1} \\ &= \sum_{k=1}^3 \frac{1}{2} \frac{d}{dt} [\mathbf{A}_0(u_m) \partial_{x_k} \tilde{u}_{m+1} \cdot \partial_{x_k} \tilde{u}_{m+1}] - T_1^t \cdot \partial_{x_k} \tilde{u}_{m+1}, \end{aligned}$$

where  $T_1^t = T_1^t(x, t)$  is given by

$$T_1^t = \sum_{k=1}^3 \left( \mathbf{A}_0(u_m)_{,x_k} \tilde{u}_{m+1,t} \cdot \partial_{x_k} \tilde{u}_{m+1} - \frac{1}{2} \mathbf{A}_0(u_m)_{,t} \partial_{x_k} \tilde{u}_{m+1} \cdot \partial_{x_k} \tilde{u}_{m+1} \right).$$

With analogous definition of  $T_1^x = T_1^x(x, t)$  we get again from (22)

$$(26) \quad \begin{aligned} & \sum_{k=1}^3 \frac{d}{dt} \int_{\mathbb{R}^3} \mathbf{A}_0(u_m(x, t)) \partial_{x_k} \tilde{u}_{m+1}(x, t) \cdot \partial_{x_k} \tilde{u}_{m+1}(x, t) dx \\ &= 2 \sum_{k=1}^3 \int_{\mathbb{R}^3} \left( \partial_{x_k} \mathbf{F}_1(u_m(x, t)) + \partial_{x_k} \left( \mathbf{F}_2(u_m(x, t)) \right. \right. \\ & \quad \times \left. \left. \int_{S_1^2} I_m(x, t, \mu) d\mu \right) \right) \cdot \partial_{x_k} \tilde{u}_{m+1}(x, t) dx \\ & \quad + \sum_{k=1}^3 \int_{\mathbb{R}^3} T_1^t(x, t) \cdot \partial_{x_k} \tilde{u}_{m+1}(x, t) + T_1^x(x, t) \cdot \partial_{x_k} \tilde{u}_{m+1}(x, t) dx. \end{aligned}$$

By the same procedure as we derived (25) and (26) we derive terms  $T_2^{t/x}, T_3^{t/x}$  for the second and the third order derivatives. In the sequel we omit the index  $k$  and the summation with respect to  $k$  for notational simplicity. Then we obtain

$$\begin{aligned}
 & \frac{d}{dt} \sum_{i=0}^3 \int_{\mathbb{R}^3} \mathbf{A}_0(u_m(x, t)) \partial_x^i \tilde{u}_{m+1}(x, t) \cdot \partial_x^i \tilde{u}_{m+1}(x, t) \, dx \\
 &= \sum_{i=0}^3 \int_{\mathbb{R}^3} \partial_x^i \mathbf{F}_1(u_m(x, t)) \cdot \partial_x^i \tilde{u}_{m+1}(x, t) \, dx \\
 & \quad + \sum_{i=0}^3 \int_{\mathbb{R}^3} \partial_x^i \left( \mathbf{F}_2(u_m(x, t)) \int_{S_1^2} I_m(x, t, \mu) \, d\mu \right) \cdot \partial_x^i \tilde{u}_{m+1}(x, t) \, dx \\
 & \quad + \sum_{i=0}^3 \int_{\mathbb{R}^3} T_i^t(x, t) \cdot \partial_x^i \tilde{u}_{m+1}(x, t) + T_i^x(x, t) \cdot \partial_x^i \tilde{u}_{m+1}(x, t) \, dx \\
 &=: \sum_{i=0}^3 R_i^1(t) + \sum_{i=0}^3 R_i^2(t) + \sum_{i=0}^3 R_i^3(t).
 \end{aligned}$$

Let us define

$$(27) \quad S = \sup_{s \in [0, T_*]} \|\tilde{u}_m(\cdot, s)\|_{H^3} + \sup_{s \in [0, T_*], \mu \in S_1^2} \|\tilde{I}_m(\cdot, s, \mu)\|_{H^3}.$$

For  $R_0^1, R_1^1, R_2^1, R_3^1, R_0^2, R_1^2, R_2^2, R_3^2$  there is an increasing continuous function  $\mathcal{C}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$  such that

$$\begin{aligned}
 (28) \quad & \sum_{i=0}^3 |R_i^1(t)| + \sum_{i=0}^3 |R_i^2(t)| \\
 & \leq \mathcal{C}(S + \delta) \left( 1 + \left\| \int_{S_1^2} \sup_{s \in [0, T]} \tilde{I}_m(\cdot, s, \mu) \, d\mu \right\|_{H^3} \right) \sup_{s \in [0, T]} \|\tilde{u}_{m+1}(\cdot, s)\|_{H^3}
 \end{aligned}$$

holds in  $[0, T)$ . As the proof of (28) is quite similar for all terms we present only the (most complex) case of  $R_3^2$ .

$$\begin{aligned}
 |R_3^2(t)| &= \left| \int_{\mathbb{R}^3} \partial_x^3 \left( \mathbf{F}_2(u_m(x, t)) \int_{S_1^2} I_m(x, t, \mu) \, d\mu \right) \cdot \partial_x^3 \tilde{u}_{m+1}(x, t) \right| dx \\
 &\leq \left\| \partial_x^3 \left( \mathbf{F}_2(u_m(\cdot, t)) \int_{S_1^2} I_m(\cdot, t, \mu) \, d\mu \right) \right\|_{L^2} \|\tilde{u}_{m+1}(\cdot, t)\|_{H^3} \\
 &\leq \left( \|\partial_x^3 \mathbf{F}_2(u_m(\cdot, t))\|_{L^2} \left\| \int_{S_1^2} I_m(\cdot, t, \mu) \, d\mu \right\|_{L^\infty} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \|\partial_x^2 \mathbf{F}_2(u_m(\cdot, t))\|_{L^4} \left\| \partial_x \int_{\mathcal{S}_1^2} I_m(\cdot, t, \mu) \, d\mu \right\|_{L^4} \\
& + \|\partial_x \mathbf{F}_2(u_m(\cdot, t))\|_{L^4} \left\| \partial_x^2 \int_{\mathcal{S}_1^2} I_m(\cdot, t, \mu) \, d\mu \right\|_{L^4} \\
& + \|\mathbf{F}_2(u_m(\cdot, t))\|_{L^\infty} \left\| \int_{\mathcal{S}_1^2} \tilde{I}_m(\cdot, t, \mu) \, d\mu \right\|_{H^3} \Big) \|\tilde{u}_{m+1}(\cdot, t)\|_{H^3}.
\end{aligned}$$

Using Assumption 2.1 (iii) and the Sobolev imbedding estimate for  $L^4(\mathbb{R}^3)$  we obtain

$$\begin{aligned}
|R_3^2(t)| & \leq \mathcal{C}(\|u_m(\cdot, t)\|_{L^\infty} + \sup_{s \in [0, T_*], \mu \in \mathcal{S}_1^2} \|\tilde{I}_m(\cdot, s, \mu)\|_{L^\infty}) (\|\tilde{u}_m(\cdot, t)\|_{H^3}^3 + 1) \\
& \times \left( \left\| \int_{\mathcal{S}_1^2} \tilde{I}_m(\cdot, t, \mu) \, d\mu \right\|_{H^3} + 1 \right) \|\tilde{u}_{m+1}(\cdot, t)\|_{H^3}.
\end{aligned}$$

Since  $\tilde{u}_m(\cdot, t) \in H^3(\mathbb{R}^3)$  for  $t \in [0, T)$  we can apply the Sobolev imbedding theorem to obtain for some constant  $C > 0$  by (24) the estimate  $\|u_m(\cdot, t)\|_{L^\infty} \leq C(\|\tilde{u}_m(\cdot, t)\|_{H^3} + \delta)$ . From (24) and the definition of  $\tilde{u}_m$  we conclude that

$$|R_3^2(t)| \leq \mathcal{C}(S + \delta) \left( \left\| \int_{\mathcal{S}_1^2} \tilde{I}_m(\cdot, t, \mu) \, d\mu \right\|_{H^3} + 1 \right) \|\tilde{u}_{m+1}(\cdot, t)\|_{H^3},$$

which proves (28) for  $R_3^2$ .

We proceed by estimating the terms  $R_0^3, R_1^3, R_2^3, R_3^3$ .

Since  $\mathbf{A}_0$  is invertible by Assumption 2.1 we get from (22)

$$u_{m+1,t} = [\mathbf{A}_0(u_m)]^{-1} \left( -\mathbf{A}_k(u_m)u_{m+1,x_k} + \mathbf{F}_1(u_m) + \mathbf{F}_2(u_m) \int_{\mathcal{S}_1^2} I_m(\cdot, \cdot, \mu) \, d\mu \right).$$

If we substitute  $u_{m+1,t}$  in  $R_i^3, i = 1, \dots, 3$  by the expression above we obtain with the same techniques as for  $R_i^{1/2}, i = 1, \dots, 3$  the estimate

$$\begin{aligned}
(29) \quad \sum_{i=0}^3 |R_i^3(t)| & \leq \mathcal{C}(S + \delta) \sup_{s \in [0, T)} \|\tilde{u}_{m+1}(\cdot, s)\|_{H^3} \\
& \times \left( 1 + \sup_{s \in [0, T)} \|\tilde{u}_{m+1}(\cdot, s)\|_{H^3} + \sup_{s \in [0, T)} \left\| \int_{\mathcal{S}_1^2} \tilde{I}_m(\cdot, s, \mu) \, d\mu \right\|_{H^3} \right)
\end{aligned}$$

in  $[0, T)$ .

Altogether we obtain from (28) and (29) after integration with respect to time

$$\begin{aligned} & \sum_{i=0}^3 \int_{\mathbb{R}^3} \mathbf{A}_0(u_m(x, t)) \partial_x^i \tilde{u}_{m+1}(x, t) \cdot \partial_x^i \tilde{u}_{m+1}(x, t) \, dx \\ & \leq \sum_{i=0}^3 \int_{\mathbb{R}^3} \mathbf{A}_0(u_0(x)) \partial_x^i \tilde{u}_0(x) \cdot \partial_x^i \tilde{u}_0(x) \, dx \\ & \quad + t\mathcal{C}(S + \delta) \sup_{s \in [0, T]} \|\tilde{u}_{m+1}(\cdot, s)\|_{H^3} \\ & \quad \times \left( 1 + \sup_{s \in [0, T]} \|\tilde{u}_{m+1}(\cdot, s)\|_{H^3} + \sup_{s \in [0, T]} \left\| \int_{S_1^2} \tilde{I}_m(\cdot, s, \mu) \, d\mu \right\|_{H^3} \right). \end{aligned}$$

Now using Assumption 2.1, in particular the continuity of  $\alpha$ , and the bound (21) we find a time  $T_*$  such that the estimate (i) holds for  $t \leq T_*$ .

To obtain (ii) we note that for  $(x, t) \in \mathbb{R}^3 \times [0, T)$

$$u_{m+1}(x, t) - u_0(x) = \int_0^t \partial_t u_{m+1}(x, s) \, ds,$$

and deduce

$$\begin{aligned} & \|u_{m+1}(\cdot, t) - u_0\|_{L^\infty} \\ & \leq t \sup_{s \in [0, T]} \|\partial_t u_{m+1}(\cdot, s)\|_{L^\infty} \\ & \leq t \sup_{s \in [0, T]} \left\| -[\mathbf{A}_0(u_m(\cdot, s))]^{-1} \left( \mathbf{A}_k(u_m(\cdot, s)) u_{m+1, x_k}(\cdot, s) \right. \right. \\ & \quad \left. \left. + \mathbf{F}_1(u_m(\cdot, s)) + \mathbf{F}_2(u_m(\cdot, s)) \int_{S_1^2} I_m(\cdot, s, \mu) \, d\mu \right) \right\|_{L^\infty} \\ & \leq t\mathcal{C} \left( \sup_{s \in [0, T]} \|u_m(\cdot, s)\|_{L^\infty} \right) \\ & \quad \times \left( \sup_{s \in [0, T]} \|\partial_x u_{m+1}(\cdot, s)\|_{L^\infty} + 1 + \sup_{s \in [0, T]} \left\| \int_{S_1^2} I_m(\cdot, s, \mu) \, d\mu \right\|_{L^\infty} \right). \end{aligned}$$

Since  $\tilde{u}_m(\cdot, s) \in H^3(\mathbb{R}^3)$  for  $s \in [0, T)$  we can use again the Sobolev imbedding theorem to obtain for some constant  $C > 0$  by (24) the estimate  $\|u_m(\cdot, s)\|_{L^\infty} \leq C(\|\tilde{u}_m(\cdot, s)\|_{H^3} + \delta)$ . Then the estimate (i) and (21) show that there is a  $T_* > 0$  such that for  $t \leq T_*$

$$\begin{aligned} \|u_{m+1}(\cdot, t) - u_0\|_{L^\infty} & \leq t\mathcal{C} \left( \sup_{s \in [0, T_*]} \|\tilde{u}_m(\cdot, s)\|_{H^3} + \delta \right) \\ & \quad \times \left( 1 + \sup_{s \in [0, T_*]} \left\| \int_{S_1^2} \tilde{I}_m(\cdot, s, \mu) \, d\mu \right\|_{H^3} \right). \end{aligned}$$

□

### 3.3. The radiation transport equation.

We start with the same setting as in Subsection 3.2, in particular let smooth functions  $u_m$  and  $I_m$  satisfying condition (21) be given.

With  $\mu \in \mathcal{S}_1^2$  we consider the following linear Cauchy problem for  $I_{m+1} = I_{m+1}(x, t, \mu)$ :

$$(30) \quad I_{m+1,t} + \mu \cdot \nabla I_{m+1} = g_1(u_m)I_m + g_2(u_m) \int_{\mathcal{S}_1^2} q(\mu, \mu') I_m(\mu') d\mu' g_3(u_m),$$

$$(31) \quad I_m(\cdot, 0, \mu) = I_0(\cdot, \mu).$$

The existence of a smooth solution  $I_{m+1}(\cdot, \cdot, \mu) \in H_T^3$  for this Cauchy problem follows from the method of characteristics and will be assumed henceforth.

**Lemma 3.2.** *Suppose that Assumption 2.2 holds. For  $\mu \in \mathcal{S}_1^2$ , let  $I_0(\cdot, \mu): \mathbb{R}^3 \rightarrow \mathbb{R}_{>0}$  with  $I_0(\cdot, \mu) - \bar{I} \in H^3(\mathbb{R}^3, \mathbb{R})$  be a given smooth function. Furthermore let smooth functions  $I_m$  and  $u_m$  with (21) be given. Let  $\delta > 0$  be such that*

$$|u_m(x, t) - \bar{u}| + |I_m(x, t, \mu) - \bar{I}| \leq \delta, \quad (x, t, \mu) \in \mathbb{R}^3 \times [0, T] \times \mathcal{S}_1^2.$$

Then there exists a positive constant  $T_* \leq T$  and continuous increasing functions  $\mathcal{C}_3, \mathcal{C}_4: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$  such that we have for all  $t \in [0, T_*)$

$$(i) \quad \begin{aligned} X_{m+1}(t, \mu) &\leq X_0(\mu) + t\mathcal{C}_3 \left( \sup_{s \in [0, T_*)} \|\tilde{u}_m(\cdot, s)\|_{H^3} \right. \\ &\quad \left. + \sup_{s \in [0, T_*], \mu' \in \mathcal{S}_1^2} \|\tilde{I}_m(\cdot, s, \mu')\|_{H^3} + \delta \right) \\ &\quad \times \left( \sup_{s \in [0, T_*]} X_m(\mu, s) + 1 \right), \end{aligned}$$

$$(ii) \quad \begin{aligned} \|I_m(\cdot, t, \mu) - I_0(\cdot, \mu)\|_{L^\infty} &\leq t\mathcal{C}_4 \left( \sup_{s \in [0, T_*]} \|\tilde{u}_m(\cdot, s)\|_{H^3} \right. \\ &\quad \left. + \sup_{s \in [0, T_*], \mu' \in \mathcal{S}_1^2} \|\tilde{I}_m(\cdot, s, \mu')\|_{H^3} + \delta \right) \\ &\quad \times \left( 1 + \sup_{s \in [0, T_*]} \left\| \int_{\mathcal{S}_1^2} \tilde{I}_m(\cdot, s, \mu') d\mu' \right\|_{H^3} \right). \end{aligned}$$

$T_*, \mathcal{C}_3, \mathcal{C}_4$  depend on  $\delta, \bar{u}, \bar{I}, \mathbf{A}_0, \mathbf{A}_k$  but not on  $m$ .

**Proof.** The characteristics  $y = y(x; t, \mu)$  of (30) are given by solutions of the simple differential equation

$$\frac{dy}{dt} = \mu, \quad y(0) = x \in \mathbb{R}.$$

This implies

$$(32) \quad y(t) = \mu t + x \equiv y(x; t, \mu).$$

On the characteristic curves the Cauchy problem (30), (31) can be written as

$$\begin{aligned} \frac{d}{dt}(I_{m+1}(y(x; t, \mu), t, \mu)) &= R(x, t, \mu) \\ &:= g_1(u_m(y(x; t, \mu), t))I_m(y(x; t, \mu), t, \mu) \\ &\quad + g_2(u_m(y(x; t, \mu), t)) + g_3(u_m(y(x; t, \mu), t)) \\ &\quad \times \int_{S_1^2} q(\mu, \mu') I_m(y(x; t, \mu), t, \mu') d\mu', \\ I_{m+1}(y(x; 0, \mu), 0, \mu) &= I_{m+1}(x, 0, \mu) = I_0(x, \mu). \end{aligned}$$

Integrating this initial value problem with respect to time  $t$  yields a representation formula for  $\tilde{I}_{m+1}$ :

$$(33) \quad \tilde{I}_{m+1}(y(x; t, \mu), t, \mu) = I_0(x, \mu) - \bar{I} + \int_0^t R(x, s, \mu) ds.$$

As in the proof of Lemma 3.1 we obtain for  $t \in [0, T)$

$$\begin{aligned} &\sum_{i=0}^3 \left| \int_{\mathbb{R}^3} \partial_x^i R(x, t, \mu) \partial_x^i + \tilde{I}_{m+1}(x, t, \mu) \right| dx \\ &\leq \sum_{i=0}^3 \|\partial_x^i R(\cdot, t, \mu)\|_{L^2} \|\partial_x^i \tilde{I}_{m+1}(\cdot, t, \mu)\|_{L^2} \\ &\leq \mathcal{C}(S + \delta)(X_m(t, \mu) + 1) \|\tilde{I}_{m+1}(\cdot, t, \mu)\|_{H^3}. \end{aligned}$$

$S$  is defined as in (27).

If we differentiate (33)  $i$ -times, multiply the result by  $\partial_x^i \tilde{I}_{m+1}$  for  $i = 0, \dots, 3$ , respectively, and integrate with respect to space we get for  $t$  smaller than some positive number  $T_*$  the following estimate by the last inequality.

$$\|\tilde{I}_{m+1}(\cdot, \mu, t)\|_{H^3} \leq \|I_0(\cdot, \mu) - \bar{I}\|_{H^3} + t\mathcal{C}(S + \delta) \left( \sup_{s \in [0, T_*)} X_m(s, \mu) + 1 \right).$$

Integrating (33) with respect to  $\mu$  and repeating the arguments above we obtain the same estimate for  $\|\int_{S_1^2} \tilde{I}_{m+1}\|_{H^3}$ . Part (i) of the lemma is proven, and part (ii) follows in the same way as the corresponding statement in Lemma 3.1.  $\square$



#### 4. CONVERGENCE BY SUCCESSIVE APPROXIMATIONS

We compute the sequences of functions  $\{u_m\}_{m \in \mathbb{N}}$ ,  $\{I_m\}_{m \in \mathbb{N}}$  by solving iteratively the problems (22), (23) in  $\mathbb{R}^3 \times [0, T]$  and (30), (31) in  $\mathbb{R}^3 \times [0, T] \times \mathcal{S}_1^2$  where we start for  $m = 1$  with

$$u_1(x, t) \equiv u_0(x), \quad I_1(x, \mu, t) \equiv I_0(x, \mu) \quad ((x, t, \mu) \in \mathbb{R}^3 \times [0, T] \times \mathcal{S}_1^2).$$

From the preceding section we conclude

**Lemma 4.1.**

Let the assumptions of Lemmata 3.1 and 3.2 be valid. For  $\mu \in \mathcal{S}_1^2$ ,  $\bar{u} \in \mathcal{U}$ ,  $\bar{I} > 0$  assume that  $I_0(\cdot, \mu): \mathbb{R}^3 \rightarrow \mathbb{R}_{>0}$  and  $u_0: \mathbb{R}^3 \rightarrow \mathcal{U}$  are smooth functions with  $u_0 - \bar{u} \in H^3(\mathbb{R}^3, \mathbb{R}^8)$ ,  $I_0(\cdot, \mu) - \bar{\mu} \in H^3(\mathbb{R}^3, \mathbb{R})$ . Let  $\delta > 0$  be such that

$$|u_0(x) - \bar{u}| + |I_0(x, \mu) - \bar{I}| \leq \delta, \quad (x, \mu) \in \mathbb{R}^3 \times \mathcal{S}_1^2.$$

Then there are positive constants  $T_*, C > 0$  (independent of  $m$ ) such that for all  $m \in \mathbb{N}$  and  $t \in [0, T_*)$  the estimates

$$(34) \quad \|\tilde{u}_m(\cdot, t)\|_{H^3} + X_m(\mu, t) \leq Ct$$

and

$$(35) \quad \sup_{x \in \mathbb{R}^3} |u_m(x, t) - u_0(x)| + \sup_{x \in \mathbb{R}^3} |I_m(x, t, \mu) - I_0(x, \mu)| \leq Ct$$

hold. The constants  $T_*, C$  depend on  $\delta, \bar{u}, \bar{I}, \mathbf{A}_0, \mathbf{A}_k$  but not on  $m$ .

*P r o o f.* The proof follows directly from Lemmata 3.1 and 3.2 by induction with respect to  $m$ . Note that (21) is satisfied for  $m = 0$  with  $A = \|\tilde{u}_0\|_{H^3} + \sup_{\mu \in \mathcal{S}_1^2} X_0(\mu)$ . □

Finally we obtain

**Theorem 4.1.** Suppose that Assumptions 2.1 and 2.2 hold. For  $\mu \in \mathcal{S}_1^2$ , let  $u_0: \mathbb{R}^3 \rightarrow \mathcal{U}$  and  $I_0(\cdot, \mu): \mathbb{R}^3 \rightarrow \mathbb{R}_{>0}$  be functions with  $u_0 - \bar{u} \in H^3(\mathbb{R}^3, \mathbb{R}^8)$  and  $I_0(\cdot, \mu) - \bar{I} \in H^3(\mathbb{R}^3, \mathbb{R})$ .

Then there is a constant  $T_* > 0$  such that there exists a classical solution  $u: \mathbb{R}^3 \times [0, T_*) \rightarrow \mathcal{U}$ ,  $I: \mathbb{R}^3 \times [0, T_*) \times \mathcal{S}_1^2 \rightarrow \mathbb{R}_{>0}$  of (17), (18), (19), (20) with

$$(36) \quad \sup_{t \in [0, T_*)} \|\tilde{u}(\cdot, t)\|_3 + \sup_{t \in [0, T_*)} \|\tilde{I}(\cdot, \mu, t)\|_3 + \sup_{t \in [0, T_*)} \left\| \int_{\mathcal{S}_1^2} \tilde{I}(\cdot, \mu', t) d\mu' \right\|_3 < \infty,$$

for  $\mu \in \mathcal{S}_1^2$ . The solution is unique within the class of classical solutions.

P r o o f. Let as first assume that the initial datum  $u_0, I_0$  is smooth. We introduce for  $m \in \mathbb{N}$  the differences

$$\begin{aligned} v_{m+1} &= u_{m+1} - u_m, \\ J_{m+1} &= I_{m+1} - I_m. \end{aligned}$$

From (22) we deduce that  $v_{m+1}$  satisfies  $v_{m+1}(\cdot, 0) = 0$  and

$$\begin{aligned} (37) \quad \mathbf{A}_0(u_m)v_{m+1,t} + \mathbf{A}_k(u_m)v_{m+1,x_k} & \\ &= (\mathbf{A}_0(u_m) - \mathbf{A}_0(u_{m-1}))u_{m,t} + (\mathbf{A}_k(u_m) - \mathbf{A}_k(u_{m-1}))u_{m,x_k} \\ &\quad + \mathbf{F}_1(u_m) - \mathbf{F}_1(u_{m-1}) \\ &\quad + \mathbf{F}_2(u_m) \int_{S_1^2} I_m(\cdot, \cdot, \mu) \, d\mu - \mathbf{F}_2(u_{m-1}) \int_{S_1^2} I_{m-1}(\cdot, \cdot, \mu) \, d\mu. \end{aligned}$$

We now proceed exactly as in the proof of Lemma 3.1 replacing  $\tilde{u}_{m+1}$  by  $v_{m+1}$ . Then we get

$$\begin{aligned} &\frac{d}{dt} \sum_{i=0}^2 \int_{\mathbb{R}^3} \mathbf{A}_0(u_m(x, t)) \partial_x^i v_{m+1}(x, t) \cdot \partial_x^i v_{m+1}(x, t) \, dx \\ &= \sum_{i=0}^2 \int_{\mathbb{R}^3} \partial_x^i ((\mathbf{A}_0(u_m(x, t)) - \mathbf{A}_0(u_{m-1}(x, t)))u_{m,t}(x, t)) \cdot \partial_x^i v_{m+1}(x, t) \, dx \\ &\quad + \sum_{i=0}^2 \int_{\mathbb{R}^3} \partial_x^i (\mathbf{A}_k(u_m(x, t)) - \mathbf{A}_k(u_{m-1}(x, t)))u_{m,x_k}(x, t) \cdot \partial_x^i v_{m+1}(x, t) \, dx \\ &\quad + \sum_{i=0}^2 \int_{\mathbb{R}^3} \partial_x^i (\mathbf{F}(u_m(x, t)) - \mathbf{F}(u_{m-1}(x, t))) \cdot \partial_x^i v_{m+1}(x, t) \, dx \\ &\quad + \sum_{i=0}^2 \int_{\mathbb{R}^3} \partial_x^i \left( f_1(u_m(x, t)) \int_{S_1^2} I_m(x, t, \mu) \, d\mu \right) \cdot \partial_x^i v_{m+1}(x, t) \\ &\quad - \sum_{i=0}^2 \int_{\mathbb{R}^3} \partial_x^i \left( f_1(u_{m-1}(x, t)) \int_{S_1^2} I_{m-1}(x, t, \mu) \, d\mu \right) \cdot \partial_x^i v_{m+1}(x, t) \, dx \\ &\quad + \sum_{i=0}^2 \int_{\mathbb{R}^3} T_i^t(x, t) \cdot \partial_x^i v_{m+1}(x, t) + T_i^x(x, t) \cdot \partial_x^i v_{m+1}(x, t) \, dx. \end{aligned}$$

With the same techniques as used to prove (28) or (29) in the proof of Lemma 3.1 we obtain for  $t \in [0, T)$

$$\begin{aligned}
& \sum_{i=0}^2 \int_{\mathbb{R}^3} \mathbf{A}_0(u_m(x, t)) \partial_x^i v_{m+1}(x, t) \cdot \partial_x^i v_{m+1}(x, t) \, dx \\
& \leq t \mathcal{C} \left( \sup_{s \in [0, T)} \|\tilde{u}_m(\cdot, s)\|_{H^3} + \sup_{s \in [0, T)} \|u_m(\cdot, s)\|_{L^\infty} \right) \\
& \quad \times \sup_{s \in [0, T)} \left( \|v_{m+1}(\cdot, s)\|_{H^2} + \|v_m(\cdot, s)\|_{H^2} + \left\| \int_{\mathcal{S}_1^2} J_m(\cdot, s, \mu) \, d\mu \right\|_{H^2} \right) \\
& \quad \times \sup_{s \in [0, T)} \|v_{m+1}(\cdot, s)\|_{H^2}.
\end{aligned}$$

Note that the third factor in the last product does not contain an additive constant 1 as in (28) or (29) since  $v_m(\cdot, t) \in L^2(\mathbb{R}^3, \mathbb{R}^8)$ .

From Lemma 4.1 we know that  $\|u_m(\cdot, t)\|_{L^\infty}$ ,  $\|\tilde{u}_m(\cdot, t)\|_{H^3}$  and  $X_m(\mu, t)$  are bounded uniformly with respect to  $m$  if we choose  $t \geq 0$  small enough, say  $t \leq T_*$ . Using Assumption 2.1 on  $\mathbf{A}_0$  we obtain that there is a continuous function  $d_1: [0, T_*) \rightarrow \mathbb{R}_{\geq 0}$  with  $d_1(0) = 0$  such that for  $t \in [0, T_*)$

$$(38) \quad \|v_{m+1}(\cdot, t)\|_{H^2} d_1(T_*) \left( \sup_{s \in [0, T_*)} \|v_m(\cdot, s)\|_{H^2} + \sup_{s \in [0, T_*)} \left\| \int_{\mathcal{S}_1^2} J_m(\cdot, s, \mu) \, d\mu \right\|_{H^2} \right).$$

Turning to the radiation equation we see that  $J_{m+1}$  satisfies

$$\begin{aligned}
(39) \quad J_{m+1,t} + \mu \cdot \nabla J_{m+1} &= g_1(u_m) J_m + g_2(u_m) \int_{\mathcal{S}_1^2} q(\mu, \mu') J_m(\mu') \, d\mu' \\
& \quad + g_3(u_m) - g_3(u_{m-1}) + (g_1(u_m) - g_1(u_{m-1})) I_{m-1} \\
& \quad + (g_2(u_m) - g_2(u_{m-1})) \int_{\mathcal{S}_1^2} q(\mu, \mu') I_{m-1}(\mu') \, d\mu',
\end{aligned}$$

and  $J_{m+1}(\cdot, 0, \mu) = 0$ . Analogously as in the proof of Lemma 3.2 we conclude using the uniform estimates on  $\|u_m(\cdot, t)\|_{L^\infty}$ ,  $\|\tilde{u}_m(\cdot, t)\|_{H^3}$ ,  $X_m(\mu, t)$  from Lemma 4.1 that there is a continuous function  $d_2: [0, T_*) \rightarrow \mathbb{R}_{\geq 0}$  with  $d_2(0) = 0$  such that for  $t \in [0, T_*)$

$$\begin{aligned}
(40) \quad \|J_{m+1}(\cdot, t, \mu)\|_{H^2} &+ \left\| \int_{\mathcal{S}_1^2} J_{m+1}(\cdot, t, \mu) \, d\mu \right\|_{H^2} \\
&\leq d_2(T_*) \left( \sup_{s \in [0, T_*)} \|v_m(\cdot, s)\|_{H^2} + \sup_{s \in [0, T_*)} \|J_m(\cdot, s, \mu)\|_{H^2} \right. \\
&\quad \left. + \sup_{s \in [0, T_*)} \left\| \int_{\mathcal{S}_1^2} J_m(\cdot, s, \mu) \, d\mu \right\|_{H^2} \right).
\end{aligned}$$

In particular, since  $d_1(t), d_2(t) \rightarrow 0$  for  $t \rightarrow 0$ , we can choose  $T_*$  so small that we obtain a contraction for  $\{v_m(\cdot, t)\}$  and  $\{J_m(\cdot, t, \mu)\}$  in  $H^2$  by the estimates (38) and (40) for each  $t \leq T_*$  and  $\mu \in \mathcal{S}_1^2$ . The sequences  $\{\tilde{u}_m\}$ ,  $\{\tilde{I}_m(\cdot, \cdot, \mu)\}$ , and  $\{\int_{\mathcal{S}_1^2} \tilde{I}_m\}$  then converge strongly in  $L^\infty(0, T_*; H^2)$  for all  $\mu \in \mathcal{S}_1^2$ . Furthermore Lemma 4.1 implies that  $\{\tilde{u}_m\}$ ,  $\{\tilde{I}_m(\cdot, \cdot, \mu)\}$ ,  $\{\int_{\mathcal{S}_1^2} \tilde{I}_m\}$  converge weakly-\* in  $L^\infty(0, T_*; H^3)$ . Therefore the limit functions  $\tilde{u}$ ,  $\tilde{I}(\cdot, \cdot, \mu)$  are functions in  $L^\infty(0, T_*; H^3)$  and by embedding in  $L^\infty(0, T_*; C^1)$ . Additionally the sequences converge almost everywhere. Using the invertibility of  $\mathbf{A}_0$  we can show that  $u$  and  $I$  are weakly, in fact classically, differentiable with respect to time.

The functions  $u_m$  and  $I_m$  satisfy for  $\varphi \in C^\infty(\mathbb{R}^3 \times (0, T_*))$  and  $\mu \in \mathcal{S}_1^2$

$$\begin{aligned} & \int_0^{T_*} \int_{\mathbb{R}^3} \left( \mathbf{A}_0(u_{m-1}(x, t))u_{m,t}(x, t) + \mathbf{A}_k(u_{m-1}(x, t))u_{m,x_k}(x, t) \right. \\ & \quad \left. - \mathbf{F}_1(u_{m-1}(x, t)) - \mathbf{F}_2(u_{m-1}(x, t)) \int_{\mathcal{S}_1^2} I_{m-1}(x, t, \mu') d\mu' \right) \varphi(x, t) dx dt = 0, \\ & \int_0^{T_*} \int_{\mathbb{R}^3} \left( I_{m,t}(x, t, \mu) + \mu \cdot \nabla I_m(x, t, \mu) - g_1(u_{m-1}(x, t))I_{m-1}(x, t, \mu) \right. \\ & \quad \left. - g_2(u_{m-1}(x, t)) - \int_{\mathcal{S}_1^2} q(\mu, \mu') I_{m-1}(x, t, \mu') d\mu' \right. \\ & \quad \left. - g_3(u_{m-1}(x, t)) \right) \varphi(x, t) dx dt = 0. \end{aligned}$$

Since we can pass to the limit in the above identities,  $u$  and  $I$  are weak solutions of the problem. The embedding theorem tells us that they are also classical solutions. The estimate (36) is clear.

Furthermore we can deduce from the continuity of the classical solutions,  $u_0 \in \mathcal{U}$ ,  $I_0(\cdot, \mu) > 0$  and the estimates (35) that  $u \in \mathcal{U}$  and  $I(\cdot, \cdot, \mu) > 0$  holds for  $t$  small.

It remains to show the uniqueness of the solutions  $u$  and  $I$ . Let  $u'$ ,  $I'$  be some other classical solutions to the same initial datum. We obtain

$$\begin{aligned} & \mathbf{A}_0(u)(u - u')_{,t} + \mathbf{A}_k(u)(u - u')_{,x_k} \\ & = - (\mathbf{A}_0(u) - \mathbf{A}_0(u'))u'_t - (\mathbf{A}_k(u) - \mathbf{A}_k(u'))u'_{,x_k} \\ & \quad + \mathbf{F}_1(u) - \mathbf{F}_1(u') \\ & \quad + \mathbf{F}_2(u) \int_{\mathcal{S}_1^2} I(\cdot, \cdot, \mu) d\mu - \mathbf{F}_2(u') \int_{\mathcal{S}_1^2} I'(\cdot, \cdot, \mu) d\mu \end{aligned}$$

and for  $\mu \in \mathcal{S}_1^2$

$$\begin{aligned} (I - I')_{,t} + \mu \cdot \nabla(I - I') &= g_1(u)(I - I') \\ &+ g_2(u) \int_{\mathcal{S}_1^2} q(\mu, \mu')(I - I')(\cdot, \cdot, \mu') \, d\mu' \\ &+ g_3(u) - g_3(u') + (g_1(u) - g_1(u'))I' \\ &+ (g_2(u) - g_2(u')) \int_{\mathcal{S}_1^2} q(\mu, \mu')I'(\cdot, \cdot, \mu') \, d\mu'. \end{aligned}$$

These equations are of the same type as (37) and (39), respectively. Using the uniform bounds for  $u, u', I, I'$  from (36) we obtain for  $t \in [0, T_*)$  and  $\mu \in \mathcal{S}_1^2$

$$\begin{aligned} \|(u - u')(\cdot, t)\|_{H^2} + \|(I - I')(\cdot, t, \mu)\|_{H^2} &+ \left\| \int_{\mathcal{S}_1^2} (I - I')(\cdot, t, \mu') \, d\mu' \right\|_{H^2} \\ &< 2(d_1(T_*) + d_2(T_*)) \\ &\times \left( \sup_{s \in [0, T_*)} \|(u - u')(\cdot, s)\|_{H^2} + \sup_{s \in [0, T_*)} \|(I - I')(\cdot, s, \mu)\|_{H^2} \right. \\ &\left. + \sup_{s \in [0, T_*)} \left\| \int_{\mathcal{S}_1^2} (I - I')(\cdot, s, \mu') \, d\mu' \right\|_{H^2} \right). \end{aligned}$$

Choosing  $T_*$  small implies  $u = u'$  and  $I = I'$  in  $[0, T_*)$  since  $d_1(T_*), d_2(T_*) \rightarrow 0$  for  $T_* \rightarrow 0$ .

Finally we can get rid of the smoothness restriction on the initial datum and pass to initial datum in  $H^3$  since the space of smooth functions is dense in  $H^3$ .  $\square$

*Proof of Theorem 1.1.* The proof follows as a corollary from Theorem 4.1.  $\square$

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