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Alexander Ženíšek

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ON A GENERALIZATION OF NIKOLSKIJ'S EXTENSION THEOREM IN THE CASE OF TWO VARIABLES*

Alexander Ženíšek, Brno

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Dedicated to Professor Karel Rektorys on the occasion of his 80th birthday.

Abstract. A modification of the Nikolskij extension theorem for functions from Sobolev spaces $H^k(\Omega)$ is presented. This modification requires the boundary $\partial\Omega$ to be only Lipschitz continuous for an arbitrary $k \in \mathbb{N}$; however, it is restricted to the case of two-dimensional bounded domains.

Keywords: Whitney's extension, Calderon's extension, Nikolskij's extension, modified Nikolskij's extension in case of 2D-domains with a Lipschitz continuous boundary

MSC 2000: 46E35, 46E39, 46E99

0. Introduction

The extension theorems concerning extensions of functions from Sobolev spaces $H^k(\Omega)$ are very useful in the mathematical theory of the finite element method when analyzing the convergence (or rate of convergence) of FEM in the case of a domain the boundary $\partial\Omega$ of which is not polygonal. In FEM usually the Calderon extension theorem (see, e.g., [6, pp. 77–81]) has been used. This extension has one great advantage: extending functions from an arbitrary Sobolev space $H^k(\Omega)$ ($k \in \mathbb{N}$) into $H^k(\mathbb{R}^N)$ (N is the dimension of Ω) only the Lipschitz continuous boundary $\partial\Omega$ is required. However, the Calderon extension has also a great disadvantage: it is defined by all generalized derivatives $D^\alpha u \in L_2(\Omega)$ ($|\alpha| \leqslant k$) of the extended function u. Thus, the Calderon extension from $H^k(\Omega)$ is not the Calderon extension from $H^m(\Omega)$

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($m < k$; $k, m \in \mathbb{N}_0$) and, using it, rather complicated tricks must be sometimes used (see, for example, [10, proof of Theorem 31.2]). Moreover, we cannot apply the abstract interpolation theory between Banach spaces to the Calderon extensions.

Contrary to Calderon's extensions, the Nikolskij extensions are based on linear combinations of sophisticated "reflections" of function values of the extended function u (see (1.22), (1.23)). These "reflections" were first considered in the Hestenes theory of extensions of functions belonging to $C^k(\Omega)$ (see [4]; this paper is an important modification of the theory introduced in [9]). As only function values of the extended function u are considered in the definition of the extension, the extension from $H^k(\Omega)$ is also an extension from $H^m(\Omega)$ ($m < k$; $m, k \in \mathbb{N}$).

In [6, pp. 75–77] the Nikolskij extension is described and the extension theorem proved for $\Omega \in \mathcal{C}^{k-1,1}$. A similar result is formulated in [7]; however, for $\Omega \in \mathcal{C}^{k+1,1}$ only. In [1] the basic idea introduced in [6, (3.33), (3.34)] is presented with reference to [3, Appendix] where, however, only the Whitney-Hestenes theory of extensions of functions belonging to $C^k(\overline{\Omega})$ is explained. Also in [8, pp. 21–24] a modification of the $C^k(\overline{\Omega})$-extension theory is presented. How to modify the trick described in [8] to the $H^k(\Omega)$-extension theory (and to obtain an extension of Nikolskij-Hestenes type for spaces $H^k(\Omega)$ with $\Omega \in \mathcal{C}^{0,1}$) is the contents of the present paper.

1. The case of a smooth boundary and some auxiliary results

1.1. Definition. Let Ω be a bounded (in the general case multiply connected) domain in $\mathbb{R}^N$, which is considered in a Cartesian coordinate system $x_1, \ldots, x_N$ (the points of this system will be denoted by $X = [x_1, \ldots, x_N]$). Let Ω satisfy the following conditions:

a) There exist positive constants α, β, a finite number m of Cartesian coordinate systems $x_{r1}, \ldots, x_{rN}$ ($r = 1, \ldots, m$) (the points in the r-th system will be denoted by $X_r = [x_{r1}, \ldots, x_{rN}]$), orthogonal mappings $A_r\colon \mathbb{R}^N \to \mathbb{R}^N$ ($r = 1, \ldots, m$) and bounded domains $U_1, \ldots, U_N$ (the form of which will be specified at point c)) such that

$$(1.1) \qquad \bigcup_{r=1}^{m} U_r \supset \partial\Omega, \quad \Lambda_r = U_r \cap \partial\Omega \neq \emptyset \quad (r = 1, \ldots, m);$$

relations (1.1) are considered in the (global) system $x_1, \ldots, x_N$.

b) The points X_r of the r-th local system are connected with the points X of the global system by relations

$$(1.2) \qquad X_r = A_r(X), \quad X = A_r^{-1}(X_r) \quad (r = 1, \ldots, m).$$

The image of the domain Ω in the r-th system is denoted by Ω_r; hence

$$\Omega_r = A_r(\Omega), \quad \Omega = A_r^{-1}(\Omega_r) \quad (r = 1, \ldots, m). \tag{1.3}$$

The image of the domain U_r in the r-th system is denoted by $\widehat{U}_r$; hence

$$\widehat{U}_r = A_r(U_r), \quad U_r = A_r^{-1}(\widehat{U}_r) \quad (r = 1, \ldots, m). \tag{1.4}$$

c) There exist m functions $a_r(X'_r)$ *continuous* in $(N-1)$-dimensional open cubes Δ_r (i.e., intervals in the case of $N = 2$), where

$$\Delta_r = \{X'_r = [x_{r1}, \ldots, x_{rN-1}] \colon |x_{rj}| < \alpha, \ j = 1, \ldots, N-1\}, \tag{1.5}$$

such that

$$\widetilde{\Lambda}_r = \widehat{U}_r \cap \partial\Omega_r = \{[X'_r, a_r(X'_r)] \colon X'_r \in \Delta_r\}, \quad (\widetilde{\Lambda}_r = A_r(\Lambda_r)), \tag{1.6}$$

$$\{X_r \colon X'_r \in \Delta_r, \ a_r(X'_r) < x_{rN} < a_r(X'_r) + \beta\} = \widehat{U}_r \cap \Omega_r = \widehat{V}_r^{+}, \tag{1.7}$$

$$\{X_r \colon X'_r \in \Delta_r, \ a_r(X'_r) - \beta < x_{rN} < a_r(X'_r)\} = \widehat{U}_r \backslash \overline{\Omega}_r = \widehat{V}_r^{-}. \tag{1.8}$$

Thus, we have $\widehat{U}_r = \widehat{V}_r^{+} \cup \widetilde{\Lambda}_r \cup \widehat{V}_r^{-}$, $\widehat{V}_r^{+}$ lies inside of Ω_r and $\widehat{V}_r^{-}$ outside of $\overline{\Omega}_r$ (see Fig. 1).

If all these conditions are satisfied then the domain Ω is called *the domain with a continuous boundary*; briefly we denote it by $\Omega \in \mathcal{C}^{0,0}$ (see Fig. 1 for $N = 2$).

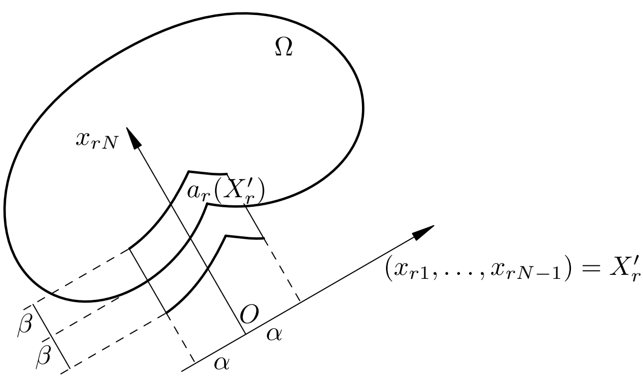

Figure 1.

d) Moreover, if each of functions $a_r(X'_r)$ $(r = 1, \ldots, m)$ is Lipschitz on the cube Δ_r, i.e., there exists a constant L such that for every two points X'_r, Y'_r of this cube we have

$$|a_r(Y'_r) - a_r(X'_r)| \leqslant L\sqrt{(y_{r1} - x_{r1})^2 + \ldots + (y_{rN-1} - x_{rN-1})^2}, \tag{1.9}$$

then we say that Ω is a *domain with a Lipschitz continuous boundary* and write briefly $\Omega \in \mathcal{C}^{0,1}$.

e) Finally, if each of functions $a_r(X_r')$ $(r = 1, \ldots, m)$ has all partial derivatives on the cube Δ_r up to and including the order k which all are Lipschitz continuous then we write $\Omega \in \mathcal{C}^{k,1}$.

The orthogonal transformation $X_r = A_r(X)$ can be written in the form

$$x_{ri} = x_i^0 + \sum_{j=1}^{N} a_{ij} x_j \quad (i = 1, \ldots, N), \tag{1.10}$$

where $x_1^0, \ldots, x_N^0$ are the coordinates of the origin of the global coordinate system in the r-th local coordinate system. The Jacobian of this transformation satisfies

$$J(X) = \frac{D(x_{r1}, \ldots, x_{rN})}{D(x_1, \ldots, x_N)} = \pm 1, \tag{1.11}$$

where the sign plus appears in the case when the coordinate systems are oriented either both positively or both negatively.

In $\mathbb{R}^2$ the transformation (1.10) has the form (we write it in the case when both coordinate systems are oriented positively and the positive direction of the axis x_{r1} makes the angle α with the positive direction of the axis x_1)

$$\begin{aligned} x_{r1} &= x_1^0 + x_1 \cos\alpha + x_2 \sin\alpha, \\ x_{r2} &= x_2^0 - x_1 \sin\alpha + x_2 \cos\alpha. \end{aligned} \tag{1.12}$$

The inverse transformation $X = A_r^{-1}(X_r)$ has in this case the form

$$\begin{aligned} x_1 &= x_{r1}^0 + x_{r1} \cos\alpha - x_{r2} \sin\alpha, \\ x_2 &= x_{r2}^0 + x_{r1} \sin\alpha + x_{r2} \cos\alpha. \end{aligned} \tag{1.13}$$

The following Lemma 1.2 is formulated for greater simplicity only in $\mathbb{R}^2$. The proof is straightforward and thus omitted.

1.2. Lemma. *Let $v \in H^k(\Omega)$, $\Omega \in \mathcal{C}^{0,0}$ and let $v_r(X_r) = v(A_r^{-1}(X_r))$. Then $v_r \in H^k(\Omega_r)$ and we have*

$$D^{(1,0)} v_r = \cos\alpha \frac{\partial v}{\partial x_1} + \sin\alpha \frac{\partial v}{\partial x_2}, \quad D^{(0,1)} v_r = -\sin\alpha \frac{\partial v}{\partial x_1} + \cos\alpha \frac{\partial v}{\partial x_2}$$

and generally for $|\alpha| \leqslant k$

$$D^\alpha v_r = \Bigl(\cos\alpha\frac{\partial}{\partial x_1} + \sin\alpha\frac{\partial}{\partial x_2}\Bigr)^{\alpha_1}\Bigl(-\sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_2}\Bigr)^{\alpha_2} v,$$

where $\alpha = (\alpha_1, \alpha_2)$.

1.3. Theorem. *Let* $G \in \mathcal{C}^{0,0}$, $v \in H^k(G)$, $G_r = A_r(G)$, $v_r(X_r) = v(A_r^{-1}(X_r))$. *Then*

$$\|v_r\|_{k,G_r} = \|v\|_{k,G}. \tag{1.14}$$

Proof. For $N = 2$ Theorem 1.3 is a consequence of the theorem on substitution in the Lebesgue integral, Lemma 1.2 and relation (1.11). In the case of $N \geqslant 3$ the proof is similar. □

The formulation of the following theorem is taken from [6, Theorem 3.9 of Chapter 2].

1.4. Theorem. *Let* $\Omega \in \mathcal{C}^{k-1,1}$. *Then there exists a linear and bounded extension operator* $\mathcal{E}_k\colon H^k(\Omega) \to H^k(\mathbb{R}^N)$, *i.e., we have*

$$\mathcal{E}_k(c_1 v_1 + c_2 v_2) = c_1\mathcal{E}_k(v_1) + c_2\mathcal{E}_k(v_2) \quad \forall\, c_1, c_2 \in \mathbb{R}^1,\ \forall\, v_1, v_2 \in H^k(\Omega), \tag{1.15}$$

$$\mathcal{E}_k(v)(X) = v(X) \quad \forall\, X \in \Omega,\ \forall\, v \in H^k(\Omega), \tag{1.16}$$

$$\|\mathcal{E}_k(v)\|_{k,\mathbb{R}^N} \leqslant C\|v\|_{k,\Omega} \quad \forall\, v \in H^k(\Omega), \tag{1.17}$$

where the constant C *depends only on the domain* Ω. *In addition, we have*

$$\mathcal{E}_k\colon H^m(\Omega) \to H^m(\mathbb{R}^N) \quad \forall\, m \in \mathbb{N}_0,\ m < k, \tag{1.18}$$

i.e., the extension operator from $H^k(\Omega)$ *is also a linear and bounded extension operator from* $H^m(\Omega)$ *for all* $m \in \mathbb{N}_0$, $m < k$.

The proof of Theorem 1.4 is based on Lemmas 1.5 and 1.6.

1.5. Lemma. *Let* F *and* G *be two bounded domains in* $\mathbb{R}^N$ *and let* $T\colon F \to G$ *be a one-to-one and continuous transformation of the domain* F *onto* G. *Let the transformation* T *be* $(k-1)$*-times continuously differentiable on* $\overline{F}$ *and let its all derivatives of order* $k-1$ *be Lipschitz on* $\overline{F}$. *Finally, we assume that* T^{-1} *is a Lipschitz* *transformation defined on* $\overline{G}$. *Let* $u \in H^k(G)$ *and let us set* $v(Y) = u(T(Y))$. *Then* $v \in H^k(F)$ *and we have*

$$\|v\|_{k,F} \leqslant C\|u\|_{k,G}. \tag{1.19}$$

The proof of Lemma 1.5 is not simple and can be found in [6, Lemma 3.4 of Chapter 2] with references to [6, Lemmas 3.1 and 3.2].

The situation in Section 2 will be such that in the proof of Lemma 2.3 we will be able to use a modification of Lemma 1.5 with stronger assumptions concerning the transformation T. We formulate now this modification in Lemma 1.5a and present its proof. This lemma concerns two-dimensional domains and we will use in it the same notation which is used in Lemma 2.3 and its proof.

1.5a. Lemma. *Let* (ξ_1, ξ_2) *and* (y_1, y_2) *be two Cartesian coordinate systems connected by the transformation*

$$\xi_1 = \varphi_1(y_1, y_2) := y_1 + g(y_2), \quad \xi_2 = \varphi_2(y_1, y_2) := y_2 + f(y_1), \tag{1.20}$$

where $f(t)$, $g(t)$ *are* k*-times continuously differentiable functions. Let* $f(0) = g(0) = f'(0) = 0$. *Let* $\mathcal{S}$ *be a square neighbourhood of the origin* $y_1 = y_2 = 0$ *and let* Σ *be its image in the coordinate system* (ξ_1, ξ_2) *in transformation* (1.20). *Let* $\mathcal{S}$ *be so small that all derivatives* $D^\alpha \varphi_i(y_1, y_2)$ $(|\alpha| \leqslant k,\ i = 1, 2)$ *are bounded. Let* $w \in H^k(\Sigma)$ *and let us set*

$$v(y_1, y_2) = w(\varphi_1(y_1, y_2), \varphi_2(y_1, y_2)). \tag{1.21}$$

Then

$$v \in H^k(\mathcal{S}), \quad \|v\|_{k,\mathcal{S}} \leqslant C\|w\|_{k,\Sigma}. \tag{1.22}$$

P r o o f. The Jacobian of (1.20) is given by the expression

$$J(y_1, y_2) = \begin{vmatrix} 1 & g'(y_2) \\ f'(y_1) & 1 \end{vmatrix} = 1 - f'(y_1)g'(y_2).$$

Thus

$$J(0, 0) = 1. \tag{1.23}$$

Relation (1.23) has the following consequence (see [3, Sections 198, 199]): At some neighbourhood of $\xi_1 = \xi_2 = y_1 = y_2 = 0$ the transformation (2.23) has a unique inverse which is expressed by

$$y_1 = \mu_1(\xi_1, \xi_2), \quad y_2 = \mu_2(\xi_1, \xi_2) \tag{1.24}$$

and the functions μ_1, μ_2 are of class C^k. The Jacobians

$$J(y_1, y_2), \quad J^{-1}(\xi_1, \xi_2) = 1/J(y_1, y_2)$$

are different from zero in the corresponding neighbourhoods ($J^{-1}(\xi_1, \xi_2)$ is the Jacobian of the functions $\mu_1(\xi_1, \xi_2)$, $\mu_2(\xi_1, \xi_2)$).

If $y_2 = 0$ and $y_1 \geqslant 0$ then we obtain $\xi_2 = f(\xi_1)$ and $\xi_1 \geqslant 0$. This means that the first arc $\xi_2 = f(\xi_1)$ corresponds to the positive part of the coordinate axis y_1. Similarly, the second arc corresponds to the positive part of the coordinate axis y_2 (see Fig. 6).

Let the square neighbourhood $\mathcal{S}$ be such a part of the whole neighbourhood that all derivatives $D^\alpha \varphi_i(y_1, y_2)$ ($|\alpha| \leqslant k,\ i = 1, 2$) are bounded in $\mathcal{S}$ and all derivatives $D^\alpha \mu_i(\xi_1, \xi_2)$ ($|\alpha| \leqslant k,\ i = 1, 2$) are bounded in Σ.

First, let us consider $w \in C^\infty(\overline{\Sigma})$. Then the function $v(y_1, y_2)$ given by (1.21) satisfies $v \in C^\infty(\overline{\mathcal{S}})$ and we can write, according to the rule of differentiation of a composite function,

$$\begin{aligned}
\frac{\partial v}{\partial y_i}(y_1, y_2) &= \sum_{r=1}^{2} \frac{\partial w}{\partial \xi_r}(\xi_1, \xi_2) \frac{\partial \varphi_r}{\partial y_i}(y_1, y_2),\\
\frac{\partial^2 v}{\partial y_i \partial y_j}(y_1, y_2) &= \sum_{r=1}^{2} \sum_{s=1}^{2} \frac{\partial^2 w}{\partial \xi_r \partial \xi_s}(\xi_1, \xi_2) \frac{\partial \varphi_s}{\partial y_j}(y_1, y_2) \frac{\partial \varphi_r}{\partial y_i}(y_1, y_2)\\
&\quad + \sum_{r=1}^{2} \frac{\partial w}{\partial \xi_r}(\xi_1, \xi_2) \frac{\partial^2 \varphi_r}{\partial y_i \partial y_j}(y_1, y_2)
\end{aligned}$$

and generally

$$\begin{aligned}
\frac{\partial^{\alpha_1+\alpha_2} v}{\partial y_1^{\alpha_1} \partial y_2^{\alpha_2}}(y_1, y_2) &= \frac{\partial^{\alpha_1+\alpha_2} w}{\partial \xi_1^{\alpha_1+\alpha_2}}(\xi_1, \xi_2) \Big(\frac{\partial \varphi_1}{\partial y_1}(y_1, y_2)\Big)^{\alpha_1} \Big(\frac{\partial \varphi_1}{\partial y_2}(y_1, y_2)\Big)^{\alpha_2} + \dots\\
&\quad + \frac{\partial w}{\partial \xi_2}(\xi_1, \xi_2) \frac{\partial^{\alpha_1+\alpha_2} \varphi_2}{\partial y_1^{\alpha_1} \partial y_2^{\alpha_2}}(y_1, y_2),
\end{aligned}$$

where ξ_1 and ξ_2 appearing as arguments on the right-hand sides of these expressions should be expressed by (1.20). Inserting these relations into the right-hand side of the expression

$$\|v\|^2_{k,\mathcal{S}} = \sum_{|\alpha|\leqslant k} \int_{\mathcal{S}} (D^\alpha v)^2 \,\mathrm{d}Y$$

and using the boundedness of the derivatives of φ_1, φ_2, we obtain after applying the Cauchy inequality for sums (see (1.49)) and the theorem on substitution in the Lebesgue integral

$$\|v\|_{k,\mathcal{S}} \leqslant C\|w\|_{k,\Sigma} \quad \forall\, w \in C^\infty(\overline{\Sigma}). \tag{1.25}$$

Let now $w \in H^k(\Sigma)$. As $\Sigma \in \mathcal{C}^{0,0}$ we can use the theorem on the density of $C^\infty(\overline{\Sigma})$ in $H^k(\Sigma)$ and find a sequence $\{w_j\} \subset C^\infty(\overline{\Sigma})$ such that

$$\lim_{j\to\infty} \|w_j - w\|_{k,\Sigma} = 0. \tag{1.26}$$

Let us set

$$v_j(y_1, y_2) := w_j(\varphi_1(y_1, y_2), \varphi_2(y_1, y_2)).$$

We have $v_j \in C^\infty(\overline{\mathcal{S}})$ and so we can derive (similarly as we obtained (1.25))

$$\|v_l - v_j\|_{k,\mathcal{S}} \leqslant C\|w_l - w_j\|_{k,\Sigma}.$$

According to (1.26), the right-hand side of the last relation tends to zero with $j, l \to \infty$. Thus, $\{v_j\}$ is a Cauchy sequence in $H^k(\mathcal{S})$. The completeness of $H^k(\mathcal{S})$ guarantees the existence of a function $\omega \in H^k(\mathcal{S})$ such that

$$\lim_{j\to\infty} \|v_j - \omega\|_{k,\mathcal{S}} = 0.$$

Further, if $k \geqslant 2$ then $w \in H^k(\Sigma)$ implies $w \in C^0(\overline{\Sigma})$, according to the Sobolev imbedding theorem. Hence $v \in C^0(\overline{\mathcal{S}})$ and

$$\|v_j - v\|_{0,\mathcal{S}} \leqslant C\|w_j - w\|_{0,\Sigma} \to 0.$$

The last two relations and the uniqueness of the limit in $L_2(\mathcal{S})$ imply that $\omega = v$ a.e. in $\mathcal{S}$; hence $v \in H^k(\mathcal{S})$ and we have

$$\lim_{j\to\infty} \|v_j - v\|_{k,\mathcal{S}} = 0. \tag{1.27}$$

The rest of the proof of inequality $(1.22)_2$ is simple when $k \geqslant 2$. By (1.25) we have

$$\|v_j\|_{k,\mathcal{S}} \leqslant C\|w_j\|_{k,\Sigma}.$$

Passing to the limit for $j \to \infty$ in this inequality, we obtain, according to (1.26) and (1.27), the desired inequality $(1.22)_2$ in the case $k \geqslant 2$.

Now, let $k = 1$; thus, $w \in H^1(\Sigma)$ only. In this case we cannot use the Sobolev imbedding theorem. However, this theorem served us only to prove the relation

$$v \in L_2(\mathcal{S}) \tag{1.28}$$

which can be proved also in the following way: The function $v(y_1, y_2)$ is connected with the function $w \in H^1(\Sigma)$ by relation (1.21). Thus, by (1.24), $w(\xi_1, \xi_2) = v(\mu_1(\xi_1, \xi_2), \mu_2(\xi_1, \xi_2))$. Since the derivatives of the functions $\mu_1, \mu_2 \in C^k$ are bounded, we have

$$|J^{-1}(\xi_1, \xi_2)| = \left| \frac{D(\mu_1, \mu_2)}{D(\xi_1, \xi_2)} \right| \leqslant K \quad \forall\, (\xi_1, \xi_2) \in \Sigma.$$

Further, owing to $J^{-1}(0,0) = 1$, we have $J^{-1}(\xi_1, \xi_2) \neq 0 \;\forall\, (\xi_1, \xi_2) \in \Sigma$. Finally, $\mathcal{S}$ is the image of Σ in transformation (1.24). Thus, by means of the theorem on substitution in the Lebesgue integral,

$$\begin{aligned} \int_{\mathcal{S}} [v(y_1, y_2)]^2 \,\mathrm{d}Y &= \int_{\Sigma} [v(\mu_1(\xi_1, \xi_2), \mu_2(\xi_1, \xi_2))]^2 \, |J^{-1}(\xi_1, \xi_2)| \,\mathrm{d}\xi \\ &\leqslant K \int_{\Sigma} [w(\xi_1, \xi_2)]^2 \,\mathrm{d}\xi. \end{aligned}$$

Hence (1.28) follows and Lemma 1.5a is proved. □

1.5b. Remark. As the functions μ_1, μ_2 appearing in (1.24) and defining the inverse transformation to (1.20) are of class C^k with bounded derivatives on Σ up and including to the order k we can prove: Let $v \in H^k(\mathcal{S})$ and let us set

$$w(\xi_1, \xi_2) := v(\mu_1(\xi_1, \xi_2), \mu_2(\xi_1, \xi_2)). \tag{1.29}$$

Then

$$w \in H^k(\Sigma), \quad \|w\|_{k,\Sigma} \leqslant C \|v\|_{k,\mathcal{S}}. \tag{1.30}$$

The proof follows lines similar to the proof of Lemma 1.5a (we have $J^{-1}(0,0) = 1$).

1.6. Lemma. *Let K, K^+ and K^* be three prisms given by the relations*

$$K = \{Y = (Y', y_N) \in \mathbb{R}^N : \ |y_i| < \alpha, \ i = 1, \ldots, N-1, \ 0 < y_N < \beta\}, \tag{1.31}$$

$$K^+ = \{Y = (Y', y_N) \in \mathbb{R}^N : \ |y_i| < \alpha, \ i = 1, \ldots, N-1, \ 0 < y_N < k\beta\}, \tag{1.32}$$

$$K^* = \{Y = (Y', y_N) \in \mathbb{R}^N : \ |y_i| < \alpha, \ i = 1, \ldots, N-1, \ |y_N| < \beta\}. \tag{1.33}$$

Let $\widetilde{u}_r \in H^k(K)$ *with* $\widetilde{u}_r(Y) = 0\ \forall Y \notin M$*, where* $M \subset K$ *is a set of the type indicated in Fig. 2. Let us extend* $\widetilde{u}_r$ *by zero from* K *onto* K^+*, and let us define on* K^* *a function* $\widetilde{u}_r^*$ *by the relations*

$$\widetilde{u}_r^*(Y', y_N) = \begin{cases} \widetilde{u}_r(Y', y_N) & \text{for } y_N \geqslant 0, \\ \sum\limits_{j=1}^{k} \lambda_j \widetilde{u}_r(Y', -jy_N) & \text{for } y_N < 0, \end{cases} \tag{1.34}$$

where the numbers $\lambda_1, \ldots, \lambda_k$ *are uniquely determined as the solution of the system of linear algebraic equations*

$$1 = \sum_{j=1}^{k} (-j)^n \lambda_j, \quad n = 0, 1, \ldots, k-1. \tag{1.35}$$

Then

$$\widetilde{u}_r^* \in H^k(K^*), \quad \|\widetilde{u}_r^*\|_{k,K^*} \leqslant C\|\widetilde{u}_r\|_{k,K} \tag{1.36}$$

and the generalized derivatives are given by the relations

$$D^\alpha \widetilde{u}_r^*(Y) = \begin{cases} D^\alpha \widetilde{u}_r(Y) & \text{for } Y \in K, \\ \sum\limits_{j=1}^{k} (-j)^{\alpha_N} \lambda_j D^\alpha \widetilde{u}_r(Y', -jy_N) & \text{for } Y \in K^* \backslash K. \end{cases} \tag{1.37}$$

Finally,

$$\widetilde{u}_r^*(Y) = 0 \quad \forall Y \in K^* \cap N(\partial K^*), \tag{1.38}$$

where $N(\partial K^*)$ *is a neighbourhood of* ∂K^**.*

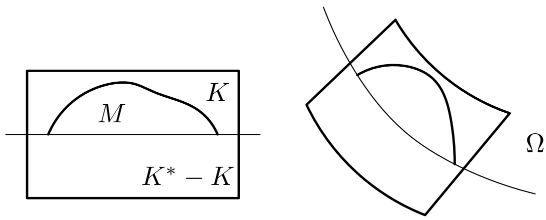

Figure 2.

R e m a r k. It is necessary to explain the meaning of the symbol $D^\alpha \widetilde{u}_r$ for the derivative in the expression $D^\alpha \widetilde{u}_r(Y', -jy_n)$: The last component of the function $\widetilde{u}_r^*(Y', y_N)$ is composed in the case of $y_N < 0$. Thus $D^\alpha \widetilde{u}_r$ means the derivative of the function $\widetilde{u}_r$ with respect to the *external* components and the right-hand side of (1.37) is the result of differentiation of a composite function.

In the case of a function in one variable everything is simpler. For example, if $f(x)$ is a given function and we define a function $F(t) := f(at)$ (setting $x = at$) then the meaning of the relations $F'(t_0) = af'(at_0)$ and $F'(t) = af'(at)$ is clear. (The symbol f' denotes the derivative with respect to the external variable $x = at$.) The notation just introduced is an analogue of the case of a function of one variable.

P r o o f of Lemma 1.6. The proof of the assertion of Lemma 1.6 is presented in [6, p. 76] in a rather concise way. As Lemma 1.6 is important for our considerations we prove it in a more detailed way. In part A) inclusion $\widetilde{u}_r^* \in H^k(K^*)$ and relation (1.37) are proved, in part B) inequality $(1.36)_2$ is established.

A) First we note that the determinant of system (1.34) is Vandermond's determinant; thus it is different from zero.

For $y_N < 0$ the function $\widetilde{u}_r^*(Y', y_N)$ is defined as a linear combination of functions belonging to $H^k(K^*\backslash K)$; thus we have $\widetilde{u}_r^* \in H^k(K^*\backslash K)$ and $\widetilde{u}_r^* \in L_2(K^*)$. Now we prove that $D^\alpha \widetilde{u}_r^* \in L_2(K^*)$ for $1 \leqslant |\alpha| \leqslant k$.

Let us assume first that $\widetilde{u}_r \in C^\infty(\overline{K})$. Then by (1.34) we have

$$\widetilde{u}_r^*\big|_{K^*-K} \in C^\infty(\overline{K^*\backslash K}).$$

Let us consider the identity

$$\int_{K^*} \widetilde{u}_r^*(Y) D^\alpha \varphi(Y)\,\mathrm{d}Y = \int_K \widetilde{u}_r(Y) D^\alpha \varphi(Y)\,\mathrm{d}Y + \sum_{j=1}^k \lambda_j \int_{K^*\backslash K} \widetilde{u}_r(Y', -jy_N) D^\alpha \varphi(Y)\,\mathrm{d}Y, \tag{1.39}$$

where $\varphi \in C_0^\infty(\operatorname{int} K^*)$ and the function $\widetilde{u}_r^*$ is given by (1.34). Let $\alpha = (\alpha_1, \ldots, \alpha_N)$, where $|\alpha| \leqslant k$, and let us put $\alpha = \alpha' + \alpha''$, where

$$\alpha' = (\alpha_1, \ldots, \alpha_{N-1}, 0), \quad \alpha'' = (0, \ldots, 0, \alpha_N).$$

If we apply to both integrals on the right-hand side of (1.39) the integration by parts with respect to the multiindex α' we obtain

$$\int_{K^*} \widetilde{u}_r^*(Y) D^\alpha \varphi(Y)\,\mathrm{d}Y = (-1)^{|\alpha'|} \int_K D^{\alpha'} \widetilde{u}_r(Y) \frac{\partial^{\alpha_N}\varphi}{\partial y_N^{\alpha_N}}(Y)\,\mathrm{d}Y + (-1)^{|\alpha'|} \sum_{j=1}^k \lambda_j \int_{K^*\backslash K} D^{\alpha'} \widetilde{u}_r(Y', -jy_N) \frac{\partial^{\alpha_N}\varphi}{\partial y_N^{\alpha_N}}(Y)\,\mathrm{d}Y, \tag{1.40}$$

because $\varphi = 0$ in a neighbourhood of ∂K^* and

$$n(K^*\backslash K) = (0,\dots,0,1), \quad n(K) = (0,\dots,0,-1) \quad \forall Y \in \Delta, \tag{1.41}$$

where Δ is the common face of both the prisms K, $K^*\backslash K$ and $n(G)$ is the outward unit normal to the domain G.

Now we apply to both integrals on the right-hand side of (1.40) the integration by parts with respect to the multiindex α'' and use relations (1.41); we obtain

$$\begin{aligned}
&\int_K D^{\alpha'}\widetilde{u}_r(Y)\frac{\partial^{\alpha_N}\varphi}{\partial y_N^{\alpha_N}}(Y)\,\mathrm{d}Y \\
&\quad = -\int_\Delta D^{\alpha'}\widetilde{u}_r(Y',0)\frac{\partial^{\alpha_N-1}\varphi}{\partial y_N^{\alpha_N-1}}(Y',0)\,\mathrm{d}Y' \\
&\qquad - \int_K \frac{\partial}{\partial y_N}D^{\alpha'}\widetilde{u}_r(Y)\frac{\partial^{\alpha_N-1}\varphi}{\partial y_N^{\alpha_N-1}}(Y)\,\mathrm{d}Y \\
&\quad = -\int_\Delta D^{\alpha'}\widetilde{u}_r(Y',0)\frac{\partial^{\alpha_N-1}\varphi}{\partial y_N^{\alpha_N-1}}(Y',0)\,\mathrm{d}Y' \\
&\qquad + (-1)^2\int_\Delta \frac{\partial}{\partial y_N}D^{\alpha'}\widetilde{u}_r(Y',0)\frac{\partial^{\alpha_N-2}\varphi}{\partial y_N^{\alpha_N-2}}(Y',0)\,\mathrm{d}Y' \\
&\qquad + (-1)^2\int_K \frac{\partial^2}{\partial y_N^2}D^{\alpha'}\widetilde{u}_r(Y)\frac{\partial^{\alpha_N-2}\varphi}{\partial y_N^{\alpha_N-2}}(Y)\,\mathrm{d}Y = \dots \\
&\quad = \sum_{s=1}^{\alpha_N}(-1)^s\int_\Delta \frac{\partial^{s-1}}{\partial y_N^{s-1}}D^{\alpha'}\widetilde{u}_r(Y',0)\frac{\partial^{\alpha_N-s}\varphi}{\partial y_N^{\alpha_N-s}}(Y',0)\,\mathrm{d}Y' \\
&\qquad + (-1)^{\alpha_N}\int_K D^{\alpha''+\alpha'}\widetilde{u}_r(Y)\varphi(Y)\,\mathrm{d}Y,
\end{aligned} \tag{1.42}$$

$$\begin{aligned}
&\int_{K^*\backslash K} D^{\alpha'}\widetilde{u}_r(Y',-jy_N)\frac{\partial^{\alpha_N}\varphi}{\partial y_N^{\alpha_N}}(Y)\,\mathrm{d}Y \\
&\quad = \int_\Delta D^{\alpha'}\widetilde{u}_r(Y',0)\frac{\partial^{\alpha_N-1}\varphi}{\partial y_N^{\alpha_N-1}}(Y',0)\,\mathrm{d}Y' \\
&\qquad + j\int_{K^*\backslash K} D^{(0,\dots,0,1)}D^{\alpha'}\widetilde{u}_r(Y',-jy_N)\frac{\partial^{\alpha_N-1}\varphi}{\partial y_N^{\alpha_N-1}}(Y)\,\mathrm{d}Y \\
&\quad = \int_\Delta D^{\alpha'}\widetilde{u}_r(Y',0)\frac{\partial^{\alpha_N-1}\varphi}{\partial y_N^{\alpha_N-1}}(Y',0)\,\mathrm{d}Y' \\
&\qquad + j\int_\Delta D^{(0,\dots,0,1)}D^{\alpha'}\widetilde{u}_r(Y',0)\frac{\partial^{\alpha_N-2}\varphi}{\partial y_N^{\alpha_N-2}}(Y',0)\,\mathrm{d}Y' \\
&\qquad + j^2\int_{K^*\backslash K} D^{(0,\dots,0,2)}D^{\alpha'}\widetilde{u}_r(Y',-jy_N)\frac{\partial^{\alpha_N-2}\varphi}{\partial y_N^{\alpha_N-2}}(Y)\,\mathrm{d}Y = \dots
\end{aligned} \tag{1.43}$$

$$= \sum_{s=1}^{\alpha_N} j^{s-1} \int_\Delta D^{(0,\dots,0,s-1)} D^{\alpha'} \widetilde{u}_r(Y',0) \frac{\partial^{\alpha_N - s}\varphi}{\partial y_N^{\alpha_N - s}}(Y',0)\,\mathrm{d}Y'$$
$$+ j^{\alpha_N} \int_{K^*\setminus K} D^{\alpha''+\alpha'} \widetilde{u}_r(Y', -jy_N)\varphi(Y)\,\mathrm{d}Y =$$
$$= -\sum_{s=1}^{\alpha_N} (-1)^s (-j)^{s-1} \int_\Delta \frac{\partial^{s-1}}{\partial y_N^{s-1}} D^{\alpha'} \widetilde{u}_r(Y',0) \frac{\partial^{\alpha_N - s}\varphi}{\partial y_N^{\alpha_N - s}}(Y',0)\,\mathrm{d}Y'$$
$$+ (-1)^{\alpha_N}(-j)^{\alpha_N} \int_{K^*\setminus K} D^{\alpha''+\alpha'} \widetilde{u}_r(Y', -jy_N)\varphi(Y)\,\mathrm{d}Y.$$

If we express the right-hand side of (1.40) in terms of (1.42) and (1.43) we get

$$(1.44)\quad \int_{K^*} \widetilde{u}_r^*(Y) D^\alpha \varphi(Y)\,\mathrm{d}Y$$
$$= (-1)^{|\alpha|} \int_K D^\alpha \widetilde{u}_r(Y)\varphi(Y)\,\mathrm{d}Y$$
$$+ (-1)^{|\alpha|} \int_{K^*\setminus K} \left\{ \sum_{j=1}^k (-j)^{\alpha_N} \lambda_j D^\alpha \widetilde{u}_r(Y', -jy_N) \right\} \varphi(Y)\,\mathrm{d}Y$$
$$+ (-1)^{|\alpha'|} \sum_{s=1}^{\alpha_N} (-1)^s \left[1 - \sum_{j=1}^k (-j)^{s-1}\lambda_j \right]$$
$$\times \int_\Delta \frac{\partial^{s-1}}{\partial y_N^{s-1}} D^{\alpha'} \widetilde{u}_r(Y',0) \frac{\partial^{\alpha_N - s}\varphi}{\partial y_N^{\alpha_N - s}}(Y',0)\,\mathrm{d}Y'.$$

The index s satisfies $s \leqslant \alpha_N \leqslant |\alpha| \leqslant k$; hence $s-1 \leqslant k-1$ and the expression in square brackets, which stands in (1.44) in front of the integral over Δ, is equal to zero, according to (1.35). Thus, relation (1.44) implies that the generalized derivatives $D^\alpha \widetilde{u}_r^*$, where $|\alpha| \leqslant k$, exist in K^* and satisfy (1.37).

Till now we have considered $\widetilde{u}_r \in C^\infty(\overline{K})$. The density $C^\infty(\overline{K})$ in $H^k(K)$ and the results just proved imply that the function $\widetilde{u}_r^* \in L_2(K^*)$ defined by relation (1.34), where $\widetilde{u}_r \in H^k(K)$, belongs to $H^k(K^*)$ and its generalized derivatives $D^\alpha \widetilde{u}_r^* \in L_2(K^*)$, where $|\alpha| \leqslant k$, are given by (1.37).

B) First we prove the estimate

$$(1.45)\qquad \|\widetilde{u}_r^*\|^2_{k,K^*\setminus K} \leqslant C \|\widetilde{u}_r\|^2_{k,K}.$$

According to (1.37), the function $\widetilde{u}_r^*$ given by (1.34) satisfies on $K^*\setminus K$ the relations

$$(1.46)\qquad D^\alpha \widetilde{u}_r^*(Y', y_N) = \sum_{j=1}^k \lambda_j (-j)^{\alpha_N} (D^\alpha \widetilde{u}_r)(Y', -jy_N), \quad y_N < 0.$$

We can write, according to the definition of the norm,

$$\|\widetilde{u}_r^*\|^2_{k,K^*\setminus K} = \sum_{|\alpha|\leqslant k} \int_{K^*\setminus K} [D^\alpha \widetilde{u}_r^*(Y', y_N)]^2 \,\mathrm{d}Y. \tag{1.47}$$

As in the case of $q > 0$ we have

$$\int_{-q}^{0} f(-jy_N)\,\mathrm{d}y_N = -\int_q^0 f(jt)\,\mathrm{d}t = \int_0^q f(jy_N)\,\mathrm{d}y_N,$$

and we get from (1.47) by virtue of (1.46)

$$\|\widetilde{u}_r^*\|^2_{k,K^*\setminus K} = \sum_{|\alpha|\leqslant k} \int_K \Big[\sum_{j=1}^k \lambda_j (-j)^{\alpha_N} (D^\alpha \widetilde{u}_r)(Y', jy_N)\Big]^2 \,\mathrm{d}Y. \tag{1.48}$$

Since

$$|\lambda_j(-j)^{\alpha_N}| \leqslant \max_{j=1,\ldots,k} |\lambda_j| k^k,$$

we can estimate the integrand in (1.48) using the Cauchy inequality

$$\Big(\sum_{s=1}^n a_s\Big)^2 \leqslant n \sum_{s=1}^n a_s^2 \tag{1.49}$$

in the following way:

$$\begin{aligned}\Big[\sum_{j=1}^k \lambda_j(-j)^{\alpha_N}(D^\alpha\widetilde{u}_r)(Y', jy_N)\Big]^2 &\leqslant \big(\max_{j=1,\ldots,k} |\lambda_j| k^k\big)^2 \Big[\sum_{j=1}^k (D^\alpha \widetilde{u}_r)(Y', jy_N)\Big]^2 \\ &\leqslant \big(\max_{j=1,\ldots,k} |\lambda_j| k^k\big)^2 k \sum_{j=1}^k [(D^\alpha\widetilde{u}_r)(Y', jy_N)]^2.\end{aligned}$$

This result and (1.48) imply inequality (1.45) with $C \leqslant k^2\big(\max_{j=1,\ldots,k} |\lambda_j| k^k\big)^2$ because by definition $\widetilde{u}_r(Y) = 0$ for $Y \in K^+ \setminus K$.

Now we prove easily inequality $(1.36)_2$: we have

$$\|\widetilde{u}_r^*\|^2_{k,K^*} = \|\widetilde{u}_r^*\|^2_{k,K^*\setminus K} + \|\widetilde{u}_r\|^2_{k,K}. \tag{1.50}$$

Inequality $(1.36)_2$ follows immediately from (1.45) and (1.50). □

P r o o f of Theorem 1.4. The proof consists of parts A–F.

A) Let $U_1, \ldots, U_m$ be the domains introduced in Definition 1.1 and let U_0 be such a domain that

$$\overline{U}_0 \subset \Omega, \quad \bigcup_{r=0}^{m} U_r \supset \Omega. \tag{1.51}$$

Then the theorem on partition of unity (see [5, Theorem 5.3.8]) implies the existence of such functions $\varphi_r \in C_0^\infty(\mathbb{R}^N)$ $(r = 0, 1, \ldots, m)$ that $\operatorname{supp} \varphi_r \subset U_r$, $0 \leqslant \varphi_r(X) \leqslant 1$ and

$$\sum_{r=0}^{m} \varphi_r(X) = 1, \quad X \in \overline{\Omega}. \tag{1.52}$$

Let $u \in H^k(\Omega)$. Let us define $u(X) = 0$ for $X \notin \Omega$ and let us put

$$u_r(X) := \varphi_r(X)u(X), \quad X \in \mathbb{R}^N. \tag{1.53}$$

It is evident that

$$u_r \in H^k(\Omega), \quad \operatorname{supp} u_r \subset U_r, \quad \operatorname{supp} u_r \subset \Omega. \tag{1.54}$$

Let us consider the domain U_r $(1 \leqslant r \leqslant m)$ and the r-th local Cartesian coordinate system connected with this domain. According to Definition 1.1, the domain is denoted in this system by the symbol $\widehat{U}_r$ and the coordinates in this system by the symbols $x_{r1}, \ldots, x_{rN}$. We use again the notation

$$X_r = (X_r', x_{rN}), \quad X_r' = (x_{r1}, \ldots, x_{rN-1}).$$

Let us consider two prisms K and K^* given by (1.31) and (1.33), respectively, and let us introduce a transformation T_r by the relations

$$X_r' = Y', \quad x_{rN} = y_N + a_r(Y'). \tag{1.55}$$

The inverse transformation T_r^{-1} has the form

$$Y' = X_r', \quad y_N = x_{rN} - a_r(X_r'). \tag{1.56}$$

According to Definition 1.1, we have

$$\widehat{U}_r = T_r(K^*), \quad \widehat{V}_r^+ = T_r(K), \tag{1.57}$$

$$K^* = T_r^{-1}(\widehat{U}_r), \quad K = T_r^{-1}(\widehat{V}_r^+). \tag{1.58}$$

Finally, we denote (as for A_r and A_r^{-1}, see (1.2)–(1.4))

$$\widehat{u}_r(X_r) := u_r(A_r^{-1}(X_r)), \tag{1.59}$$

$$\widetilde{u}_r(Y) := \widehat{u}_r(T_r(Y)). \tag{1.60}$$

The function $\widehat{u}_r(X_r)$ is defined on $\widehat{V}_r^+$ and the function $\widetilde{u}_r(Y)$ on K.

B) According to the assumption $\Omega \in \mathcal{C}^{k-1,1}$ of Theorem 1.4, both transformations T_r a T_r^{-1} given by (1.55) and (1.56), respectively, satisfy the assumptions of Lemma 1.5. As $\widehat{u}_r \in H^k(\widehat{V}_r^+)$, all assumptions of Lemma 1.5 are satisfied with $T = T_r$, $F = K$ and $G = \widehat{V}_r^+$. Hence $\widetilde{u}_r \in H^k(K)$, where $\widetilde{u}_r(Y)$ is given by (1.60), and we have

$$\|\widetilde{u}_r\|_{k,K} \leqslant C\|\widehat{u}_r\|_{k,\widehat{V}_r^+}. \tag{1.61}$$

C) Let us apply Lemma 1.6 on the function $\widetilde{u}_r \in H^k(K)$. We obtain the function $\widetilde{u}_r^*(Y)$ $(Y \in K^*)$—the extension of $\widetilde{u}_r$ on K^*—satisfying relations (1.36). Using (1.56), we set

$$\widehat{u}_r^*(X_r) = \widetilde{u}_r^*(T_r^{-1}(X_r)), \quad Y = T_r^{-1}(X_r) \in K^*, \ X_r \in \widehat{U}_r; \tag{1.62}$$

we have $X_r \in \widehat{U}_r$ in (1.62), according to $(1.58)_1$.

Since the transformation T_r^{-1} (together with T_r) satisfies all assumptions of Lemma 1.5 concerning the transformation T and as relation $(1.58)_1$ holds, we can use Lemma 1.5 with $T = T_r^{-1}$, $F = \widehat{U}_r$, $G = K^*$, $u = \widetilde{u}_r^*$ and $v = \widehat{u}_r^*$. We get

$$\|\hat{u}_r^*\|_{k,\widehat{U}_r} \leqslant C\|\widetilde{u}_r^*\|_{k,K^*}. \tag{1.63}$$

D) Connecting inequalities (1.63), $(1.36)_2$ and (1.61) in the given order, we obtain

$$\|\widehat{u}_r^*\|_{k,\widehat{U}_r} \leqslant C\|\widetilde{u}_r^*\|_{k,K^*} \leqslant C\|\widetilde{u}_r\|_{k,K} \leqslant C\|\widehat{u}_r\|_{k,\widehat{V}_r^+},$$

i.e.,

$$\|\widehat{u}_r^*\|_{k,\widehat{U}_r} \leqslant C\|\widehat{u}_r\|_{k,\widehat{V}_r^+}. \tag{1.64}$$

E) Owing to the definition of $\widehat{u}_r$ and to (1.38), the function $\widehat{u}_r^*$ is equal to zero in $\widehat{U}_r \cap N(\partial\widehat{U}_r)$, where $N(\partial\widehat{U}_r)$ is a neighbourhood of $\partial\widehat{U}_r$. Thus we can define

$$\widehat{u}_r^*(X_r) = 0 \quad \forall\, X_r \notin \widehat{U}_r \quad (r = 0, 1, \ldots, s) \tag{1.65}$$

and obtain thus local extension operators $\mathcal{E}_{k,r}(u)$ by $\mathcal{E}_{k,r}(u)(X) := \widehat{u}_r^*(A_r(X))$. The global extension operator $\mathcal{E}_k(u)$ is defined by

$$\mathcal{E}_k(u) := \sum_{r=0}^{s} \mathcal{E}_{k,r}(u). \tag{1.66}$$

The linearity (1.15) of $\mathcal{E}_k$ follows from the linearity of $\mathcal{E}_{k,r}$. As to (1.16), for $X \in \Omega$ we have

$$\begin{aligned}\mathcal{E}_k(u)(X) &= \sum_{r=0}^{s} \mathcal{E}_{k,r}(u)(X) = \sum_{r=0}^{s} \widehat{u}_r(A_r(X)) = \sum_{r=0}^{s} u_r(X) \\ &= \sum_{r=0}^{s} \varphi_r(X)u(X) = u(X)\sum_{r=0}^{s} \varphi_r(X) = u(X).\end{aligned}$$

Finally,

$$\begin{aligned}\|\mathcal{E}_k(u)\|_{k,\mathbb{R}^N} &= \Big\|\sum_{r=0}^{s} \mathcal{E}_{k,r}(u)\Big\|_{k,\mathbb{R}^N} = \Big\|\sum_{r=0}^{s} \widehat{u}_r^*\Big\|_{k,\mathbb{R}^N} \\ &\leqslant \sum_{r=0}^{s} \|\widehat{u}_r^*\|_{k,\mathbb{R}^N} \leqslant C\sum_{r=0}^{s} \|\widehat{u}_r\|_{k,\Omega_r} = C\sum_{r=0}^{s} \|u_r\|_{k,\Omega} \leqslant (s+1)C\|u\|_{k,\Omega}.\end{aligned}$$

Thus, the boundedness condition (1.17) is verified. (Relation $\|\widehat{u}_r\|_{k,\Omega_r} = \|u_r\|_{k,\Omega}$ follows from Theorem 1.3.)

F) As to the last assertion of Theorem 1.4 (which says that the extension operator $\mathcal{E}_k\colon\ H^k(\Omega) \to H^k(\mathbb{R}^N)$ just described is also a linear and bounded extension operator from $H^m(\Omega)$ into $H^m(\mathbb{R}^N)$ for all $m < k$, $m \in \mathbb{N}_0$), it suffices to prove that the function $\widetilde{u}_r(Y', y_N)$ defined by (1.34), (1.35) (with $\widetilde{u}_r \in H^m(K)$) satisfies for each $m \in \mathbb{N}_0$, $m < k$,

$$\widetilde{u}_r \in H^m(K^*), \quad \|\widetilde{u}_r^*\|_{m,k^*} \leqslant C\|\widetilde{u}_r\|_{m,K}. \tag{1.67}$$

As $m < k$, it follows from (1.35) that

$$1 = \sum_{j=1}^{k} (-j)^n \lambda_j, \quad n = 0, 1, \ldots, m-1. \tag{1.68}$$

Thus repeating the considerations of part A) of the proof of Lemma 1.6 we arrive at the following conclusions: the function $\widetilde{u}_r^* \in L_2(K^*)$ defined by relation (1.34), where $\widetilde{u}_r \in H^m(K)$, belongs to $H^m(K^*)$ and its generalized derivatives $D^\alpha \widetilde{u}_r^* \in L_2(K^*)$, where $|\alpha| \leqslant m$, are given by (1.37). Thus, $(1.67)_1$ is satisfied.

Replacing in part B) of the proof of Lemma 1.6 the symbol k by m, we obtain instead of (1.45) the inequality

$$\|\widetilde{u}_r^*\|_{m,K^*\setminus K}^2 \leqslant C\|\widetilde{u}_r\|_{m,K}^2.$$

This inequality and the identity

$$\|\widetilde{u}_r^*\|_{m,K^*}^2 = \|\widetilde{u}_r^*\|_{m,K^*\setminus K}^2 + \|\widetilde{u}_r\|_{m,K}^2$$

imply the desired estimate $(1.67)_2$ and the proof of Theorem 1.4 is complete. □

It should be noted that for $0 \leqslant m \leqslant k$ the domain $\Omega \in \mathcal{C}^{k-1,1}$ is the same.

In Section 2 we will need the following two-dimensional modification of Lemma 1.6:

1.7. Lemma. *Let $K \subset \mathbb{R}^2$ and $K^* \subset \mathbb{R}^2$ be two prisms given by the relations*

$$K = \{Y = (y_1, y_2)\colon\ |y_1| < \alpha,\ 0 < y_2 < \beta\}, \tag{1.69}$$

$$K^* = \{Y = (y_1, y_2)\colon\ |y_1| < \alpha,\ |y_2| < \beta\}. \tag{1.70}$$

Let $v \in H^q(K)$ $(q \leqslant k)$ and let us define on K^* a function v^* by the relations

$$v^*(y_1, y_2) = \begin{cases} v(y_1, y_2) & \text{for } y_2 \geqslant 0, \\ \sum\limits_{j=1}^{k} \lambda_j v\bigl(y_1, -\frac{1}{j} y_2\bigr) & \text{for } y_2 < 0, \end{cases} \tag{1.71}$$

where the numbers $\lambda_1, \dots, \lambda_k$ are uniquely determined as the solution of the system of linear algebraic equations

$$1 = \sum_{j=1}^{k} \Bigl(-\frac{1}{j}\Bigr)^n \lambda_j, \quad n = 0, 1, \dots, k-1. \tag{1.72}$$

Then

$$v^* \in H^m(K^*), \quad \|v^*\|_{m,K^*} \leqslant C\|v\|_{m,K} \quad (m \leqslant q \leqslant k) \tag{1.73}$$

and the generalized derivatives with $|\alpha| \leqslant m$ are given by the relations

$$D^\alpha v^*(Y) = \begin{cases} D^\alpha v(Y) & \text{for } Y \in K, \\ \sum\limits_{j=1}^{k} \bigl(-\frac{1}{j}\bigr)^{\alpha_2} \lambda_j D^\alpha v\bigl(y_1, -\frac{1}{j} y_2\bigr) & \text{for } Y \in K^* \setminus K. \end{cases} \tag{1.74}$$

Comparing the assumptions of Lemmas 1.6 and 1.7 we see that the integers j with $j = 1, \dots, k$ appearing in Lemma 1.6 are replaced in Lemma 1.7 by the fractions $\frac{1}{j}$

$(j = 1, \dots, k)$. This enables us not to consider the prism K^+ into which the function $\widetilde{u}_r$ is extended by zero. Of course, property (1.38) cannot be now proved. (This will be repaired by another trick in Section 2.)

The proof of Lemma 1.7 follows the same lines as the proof of Lemma 1.6 and thus can be omitted. The difference between Lemma 1.6 (with $N = 2$) and Lemma 1.7 consists in the following: In Lemma 1.7 we cannot prove an analogue of (1.38); however, the function v^* is defined in (1.71) only by the function values of $v(y_1, y_2)$ with $(y_1, y_2) \in K$ (and not also with $(y_1, y_2) \in K^+$ as in Lemma 1.6).

2. The case of a two-dimensional domain with a piecewise smooth boundary

Let $\Omega \in \mathcal{C}^{0,1}$ be a domain with a piecewise smooth boundary $\partial\Omega$ (we assume that its smooth parts are sufficiently smooth). Let $P_1, \dots, P_n \in \partial\Omega$ be the points at which the boundary $\partial\Omega$ is not smooth (see Fig. 3 with $n = 3$).

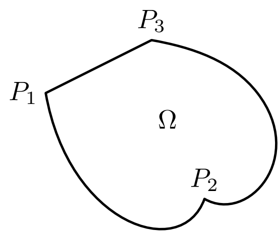

Figure 3.

Let $\mathcal{K}(P_i, r_i)$ be a closed disk with its centre at the point P_i and radius r_i $(i = 1, \dots, n)$. The magnitude of the radii r_i will be specified later.

Let the point $P_i = [x_1^0, x_2^0]$ $(1 \leqslant i \leqslant n)$ be fixed and let us set

$$r := \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2}. \tag{2.1}$$

Let $p_i(r)$ be the Hermite polynomial of degree $2k - 1$ in one variable r uniquely determined by the conditions

$$p_i\Big(\frac{1}{2}r_i\Big) = 1, \quad p_i^{(j)}\Big(\frac{1}{2}r_i\Big) = 0 \quad (j = 1, \dots, k-1),$$
$$p_i^{(j)}(r_i) = 0 \quad (j = 0, 1, \dots, k-1). \tag{2.2}$$

Let $\zeta_i(r)$ be a function defined by the relations

$$\zeta_i(r) = \begin{cases} 1 & \text{for } 0 \leqslant r \leqslant \frac{1}{2}r_i, \\ p_i(r) & \text{for } \frac{1}{2}r_i \leqslant r \leqslant r_i, \\ 0 & \text{for } r \geqslant r_i. \end{cases} \tag{2.3}$$

Then $\zeta_i(r) \in C^{k-1}\langle 0, \infty)$. Let $u \in H^k(\Omega)$ and let us define functions

$$u_i(X) = u(X)\zeta_i(r) \quad (i = 1, \ldots, n)\,. \tag{2.4}$$

Let $\kappa\big(P_i, \frac{1}{2}r_i\big)$ denote a closed disk with its centre at the point P_i and radius $\frac{1}{2}r_i$. Then each function $u_i(X)$ can be expressed in the form

$$u_i(X) = \begin{cases} u(X) & \text{for } X \in \Omega \cap \kappa\big(P_i, \frac{1}{2}r_i\big), \\ u(X)p_i(r) & \text{for } X \in \Omega \cap \big\{\mathcal{K}(P_i, r_i) \backslash \kappa\big(P_i, \frac{1}{2}r_i\big)\big\}, \\ 0 & \text{for } X \in \{\Omega \backslash \mathcal{K}(P_i, r_i)\}. \end{cases} \tag{2.5}$$

2.1. Lemma. *We have*

$$u_i \in H^k(\Omega). \tag{2.6}$$

Before proving Lemma 2.1 we introduce

2.2. Lemma. *Let $v \in H^k(\Omega)$ and let $\{v_n\} \subset C^\infty(\overline{\Omega})$ be such a sequence that $v_n \to v$ in $H^k(\Omega)$. Then for an arbitrary polynomial $p(X)$ we have $pv_n \to pv$ in $H^k(\Omega)$.*

P r o o f. We have

$$\|pv_n - pv\|^2_{k,\Omega} = \sum_{|\alpha| \leqslant k} \int_\Omega [D^\alpha(pv_n - pv)]^2 \, \mathrm{d}X \leqslant C\|p\|^2_{C^k(\overline{\Omega})}\|v_n - v\|^2_{k,\Omega} \to 0.$$

□

P r o o f of Lemma 2.1. We restrict ourselves to the case $k = 2$. Let us denote

$$\widetilde{p}_i(X) = p_i\Big(\sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2}\Big)$$

and let

$$L := \partial\kappa\Big(P_i, \frac{1}{2}r_i\Big) \cap \Omega, \quad C := \partial\mathcal{K}(P_i, r_i) \cap \Omega.$$

Relations (2.2) with $k = 2$ imply

$$\widetilde{p}_i\big|_L = 1, \quad \frac{\partial \widetilde{p}_i}{\partial x_j}\bigg|_L = 0. \tag{2.7}$$

Similarly, on C we have

$$\widetilde{p}_i\big|_C = 0, \quad \frac{\partial \widetilde{p}_i}{\partial x_j}\bigg|_C = 0. \tag{2.8}$$

As concerns the first derivatives we can write

$$\int_\Omega u_i(X)\frac{\partial\varphi}{\partial x_j}(X)\,\mathrm{d}X = \int_{A_i} u_i(X)\frac{\partial\varphi}{\partial x_j}(X)\,\mathrm{d}X + \int_{B_i} u_i(X)\frac{\partial\varphi}{\partial x_j}(X)\,\mathrm{d}X + \int_{\Omega\setminus(A_i\cup B_i)} u_i(X)\frac{\partial\varphi}{\partial x_j}(X)\,\mathrm{d}X,$$

where $\varphi\in C_0^\infty(\Omega)$ and

$$A_i = \kappa\Big(P_i,\frac{1}{2}r_i\Big)\cap\Omega,\quad B_i = \Big(\mathcal{K}(P_i,r_i)\setminus\kappa\Big(P_i,\frac{1}{2}r_i\Big)\Big)\cap\Omega.$$

Considering $u\in C^\infty(\overline{\Omega})$ and using Green's theorem with $(2.7)_1$, $(2.8)_2$, we find after some computation

$$\int_\Omega u_i(X)\frac{\partial\varphi}{\partial x_j}(X)\,\mathrm{d}X = -\int_{A_i}\frac{\partial u}{\partial x_j}(X)\varphi(X)\,\mathrm{d}X - \int_{B_i}\Big(\frac{\partial u}{\partial x_j}\widetilde{p}_i + u\frac{\partial\widetilde{p}_i}{\partial x_j}\Big)(X)\varphi(X)\,\mathrm{d}X.$$

Similarly, using Green's theorem twice, we obtain by virtue of (2.7), (2.8)

$$\int_\Omega u_i(X)\frac{\partial^2\varphi}{\partial x_j\partial x_k}(X)\,\mathrm{d}X = \int_{A_i}\frac{\partial^2 u}{\partial x_j\partial x_k}(X)\varphi(X)\,\mathrm{d}X + \int_{B_i}\Big(\frac{\partial^2 u}{\partial x_j\partial x_k}\widetilde{p}_i + \frac{\partial u}{\partial x_j}\frac{\partial\widetilde{p}_i}{\partial x_k} + \frac{\partial u}{\partial x_k}\frac{\partial\widetilde{p}_i}{\partial x_j} + u\frac{\partial^2\widetilde{p}_i}{\partial x_j\partial x_k}\Big)(X)\varphi(X)\,\mathrm{d}X.$$

The last two relations together with Lemma 2.2 give the assertion. □

Let us set

(2.9) $$u_0(X) = u(X) - \sum_{i=1}^n u_i(X),\quad \forall\, X\in\Omega.$$

Relations (2.5), (2.6) and (2.9) imply

(2.10) $$u_0\in H^k(\Omega),$$

(2.11) $$u_0(X) = \begin{cases} 0 & \text{for } X\in\Big\{\Omega\cap\bigcup_{i=1}^n\kappa\big(P_i,\frac{1}{2}r_i\big)\Big\},\\ u(X)(1-p_i(r)) & \text{for } X\in\Omega\cap\big\{\mathcal{K}(P_i,r_i)\setminus\kappa\big(P_i,\frac{1}{2}r_i\big)\big\},\\ u(X) & \text{for } X\in\Big\{\Omega\setminus\bigcup_{i=1}^n\mathcal{K}(P_i,r_i)\Big\},\end{cases}$$

(2.12) $$u(X) = u_0(X) + \sum_{i=1}^n u_i(X),\quad \forall\, X\in\Omega$$

where $i=1,\ldots,n$ on the second line of (2.11).

Now, let $\Omega_0 \supset \Omega$ be a domain with the following properties:[1]

$$\Omega_0 \in \mathcal{C}^{k,1}, \quad \Omega_0 = \Omega \cup \omega_1 \cup \ldots \cup \omega_n \tag{2.13}$$

where the sets ω_i satisfy (see Fig. 4 and Fig. 5)

$$\omega_i \subset \left\{\kappa\Big(P_i, \frac{1}{2}r_i\Big) - \Omega\right\} \quad (i = 1, \ldots, n). \tag{2.14}$$

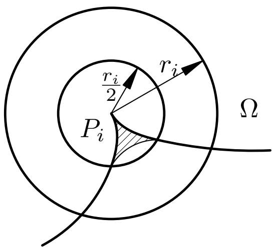

Figure 4.

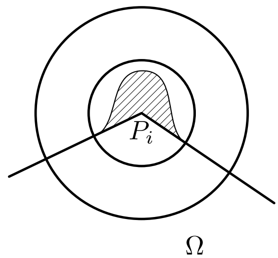

Figure 5.

Let us set

$$u_0(X) = 0, \quad X \in \Omega_0 - \Omega. \tag{2.15}$$

Taking into account (2.10) and $(2.11)_1$, we see that the function $u_0(X)$ defined by relations (2.9) a (2.15) satisfies the relation

$$u_0 \in H^k(\Omega_0). \tag{2.16}$$

As $\Omega_0 \in \mathcal{C}^{k,1}$, the function (2.16) can be extended by means of Theorem 1.4; hence

$$u_0^* \in H^k(\mathbb{R}^2), \quad u_0^*\big|_{\Omega_0} = u_0, \quad \|u_0^*\|_{k,\mathbb{R}^2} \leqslant C\|u_0\|_{k,\Omega_0}. \tag{2.17}$$

2.3. Lemma. *Let r_i be sufficiently small*[2] *and let us consider the linear space L_i^k of functions $u \in H^k(\Omega)$ for which*[3]

$$\begin{aligned} u(X) = 0 \quad &\forall\, X \in \Omega \backslash \mathcal{K}(P_i, r_i), \\ D^\alpha u(X) = 0 \quad &\forall\, |\alpha| \leqslant k-1, \ \forall\, X \in \partial\mathcal{K}(P_i, r_i) \cap \Omega. \end{aligned} \tag{2.18}$$

[1] The definition (2.13) of the domain Ω_0 together with (2.9), $(2.11)_1$, (2.15), (2.16) is one of the basic tricks of this approach.

[2] This requirement will be precisely formulated in the proof of Lemma 2.3.

[3] Lemma 2.3 will be applied to the functions u_i. The symbol u used in Lemma 2.3 is a general symbol; u has nothing in common with the function u considered at the beginning of Chapter 2.

There exists a linear and bounded extension operator $\mathcal{P}_k^i\colon L_i^k \to H^k(\mathbb{R}^N)$, *i.e., we have*

$$\mathcal{P}_k^i(c_1u_1 + c_2u_2) = c_1\mathcal{P}_k^i(u_1) + c_2\mathcal{P}_k^i(u_2) \quad \forall\, c_1, c_2 \in \mathbb{R}^1,\ \forall\, u_1, u_2 \in L_i^k, \tag{2.19}$$

$$\mathcal{P}_k^i(u)(X) = u(X) \quad \forall\, X \in \Omega,\ \forall\, u \in L_i^k, \tag{2.20}$$

$$\|\mathcal{P}_k^i(u)\|_{k,\mathbb{R}^N} \leqslant C\|u\|_{k,\Omega} \quad \forall\, u \in L_i^k, \tag{2.21}$$

where the constant C *depends only on the radius* r_i. *In addition, we have*

$$\mathcal{P}_k^i\colon L_i^m \to H^m(\mathbb{R}^N) \quad \forall\, m \in \mathbb{N}_0,\ m < k, \tag{2.22}$$

i.e, the extension from L_i^k *is also a linear and bounded extension from* L_i^m *into* $H^m(\mathbb{R}^N)$ *for all* $m \in \mathbb{N}_0$, $m < k$.

P r o o f. Let ξ_1, ξ_2 be the positively oriented Cartesian coordinate system with the origin $\xi_1 = \xi_2 = 0$ at the point P_i and let the non-negative part of the axis ξ_1 be tangent to one of the smooth curves which meet at the point P_i (see Fig. 6).

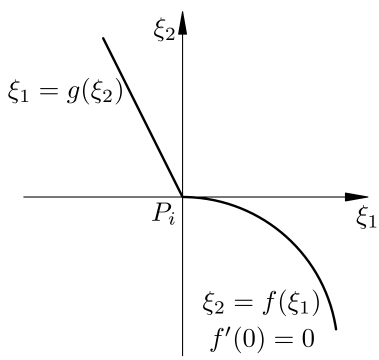

Figure 6.

Let both curves be expressed explicitly in the form

$$\xi_2 = f(\xi_1), \quad \xi_1 = g(\xi_2),$$

let the functions f, g be (for simplicity) k-times continuously differentiable and let $f'(0) = 0$ (see again Fig. 6). Under these conditions, which can be easily satisfied in applications, let us introduce the transformation

$$\xi_1 = \varphi_1(y_1, y_2) := y_1 + g(y_2), \quad \xi_2 = \varphi_2(y_1, y_2) := y_2 + f(y_1). \tag{2.23}$$

The Jacobian of (2.23) is given by the expression

$$J(y_1, y_2) = \begin{vmatrix} 1 & g'(y_2) \\ f'(y_1) & 1 \end{vmatrix} = 1 - f'(y_1)g'(y_2).$$

Thus

$$J(0,0) = 1. \tag{2.24}$$

Relation (2.24) has the following consequence (see [3, Sections 198, 199]): In the neighbourhood of $\xi_1 = \xi_2 = y_1 = y_2 = 0$ the transformation (2.23) has a unique inverse which is expressed by

$$y_1 = \mu_1(\xi_1, \xi_2), \quad y_2 = \mu_2(\xi_1, \xi_2) \tag{2.25}$$

and the functions μ_1, μ_2 are of class C^k. The Jacobians

$$J(y_1, y_2), \quad J^{-1}(\xi_1, \xi_2) = 1/J(y_1, y_2)$$

are different from zero in the corresponding neighbourhoods ($J^{-1}(\xi_1, \xi_2)$ is the Jacobian of the functions $\mu_1(\xi_1, \xi_2)$, $\mu_2(\xi_1, \xi_2)$).

If $y_2 = 0$ and $y_1 \geqslant 0$ then we obtain $\xi_2 = f(\xi_1)$ and $\xi_1 \geqslant 0$. This means that the first arc $\xi_2 = f(\xi_1)$ corresponds to the positive part of the coordinate axis y_1. Similarly, the second arc corresponds to the positive part of the coordinate axis y_2.

Let $\mathcal{S}$ be a square neighbourhood of the origin $y_1 = y_2 = 0$ (a part of the whole neighbourhood) and let Σ be its image in the coordinate system ξ_1, ξ_2 in transformation (2.23). This means that Σ is a neighbourhood of the point P_i. We choose the radius r_i so small that

$$\mathcal{K}(P_i, r_i) \subset \Sigma. \tag{2.26}$$

The magnitude of r_i depends also on k (we first explain the situation in the case of $k = 2$ which is the most important in applications): We choose r_i such that $\mathcal{K}(P_i, r_i) \subset \Sigma$ and the image $\widetilde{\mathcal{G}}$ of $\mathcal{G} = \mathcal{K}(P_i, r_i) \cap \Omega$ in transformation (2.25) has the property

$$\operatorname{dist}(\widetilde{\mathcal{G}}, \partial\mathcal{S}) > \frac{a}{2} + \varepsilon, \tag{2.27}$$

where $2a$ is the length of the side of the square $\mathcal{S}$ lying in the plane y_1, y_2 ($\mathcal{S}$ has the vertices $D_1, \ldots, D_4$) and $\varepsilon > 0$ is a small number (see Figs. 7 and 8).

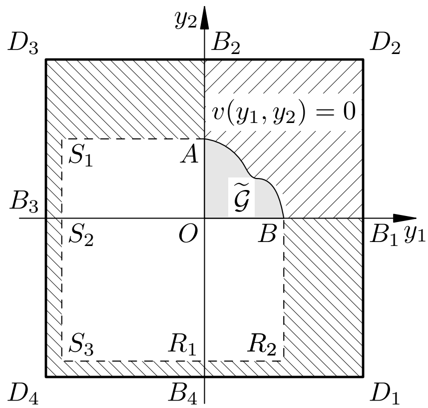

Figure 7.

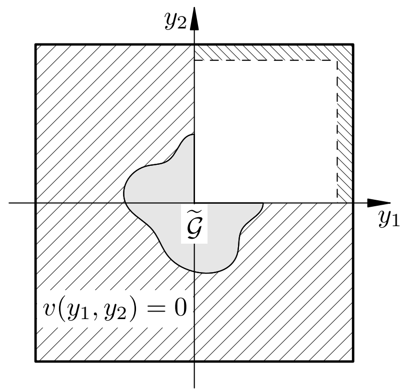

Figure 8.

We see from Figs. 7 and 8 that there are two situations possible: A) the image $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ lies in $\mathcal{S}_I$, B) the image $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ lies in $\mathcal{S}_{\mathrm{II}}\cup\mathcal{S}_{\mathrm{III}}\cup\mathcal{S}_{\mathrm{IV}}$, where $\mathcal{S}_I,\dots,\mathcal{S}_{\mathrm{IV}}$ are the parts of the square $\mathcal{S}$ lying in the quadrants I, …, IV of the Cartesian coordinate system y_1, y_2. We shall discuss both situations separately in parts A) and B).

A) Let X_i be the points of the local Cartesian coordinate system ξ_1, ξ_2 connected with $\mathcal{K}(P_i,r_i)$ and described at the beginning of this proof. In accordance with Definition 1.1, let X denote the points of the global coordinate system x_1, x_2 and let A_i be such a transformation that

$$X_i=A_i(X),\quad X=A_i^{-1}(X_i). \tag{2.28}$$

Let $u(X)$ satisfy the assumptions of Lemma 2.3 (i.e., $u\in H^k(\Omega)$ and (2.18) holds). Let us denote

$$w(\xi_1,\xi_2)\equiv w(X_i):=u(A_i^{-1}(X_i)), \tag{2.29}$$

$$v(y_1,y_2)=w(\varphi_1(y_1,y_2),\varphi_2(y_1,y_2))=w(y_1+g(y_2),y_2+f(y_1)). \tag{2.30}$$

The function $v(y_1,y_2)$ is defined on $\mathcal{S}_I$ and is equal to zero on $\mathcal{S}_I-\widetilde{\mathcal{G}}$ (see Fig. 7). By (2.18) and (2.29), (2.30), this function is not only equal to zero on the arc AB (see Fig. 7) but it also satisfies[4]

$$D^\alpha v(Y)=0\quad\forall\,|\alpha|\leqslant k-1\quad\forall\,Y\in\operatorname{arc}AB\quad\text{in the sense of traces.} \tag{2.31}$$

[4] We can see from (2.31) that the Hermite polynomial $p_i(r)$ appearing in the definition of the function u_i is the optimal one: Functions belonging to $H^k(\widetilde{\mathcal{G}})$ has as traces all derivatives up to and including the order $k-1$. Thus it has no sense to choose degree of $p_i(r)$ greater than $2k-1$.

In our case $k = 2$. Let us define the function

$$(2.32) \qquad \widehat{v}(y_1, y_2) = \begin{cases} v(y_1, y_2) & \text{for } y_2 \geqslant 0, \\ -3v(y_1, -y_2) + 4v\big(y_1, -\frac{1}{2}y_2\big) & \text{for } y_2 < 0. \end{cases}$$

According to Lemma 1.7 (with $m = q = k = 2$), the function $\widehat{v} \in H^2(\mathcal{S}_{\mathrm{I}\cup\mathrm{IV}})$ is an extension of $v \in H^2(\mathcal{S}_I)$, where we set $\mathcal{S}_{\mathrm{I}\cup\mathrm{IV}} = \mathcal{S}_I \cup \mathcal{S}_{\mathrm{IV}}$ for brevity. (Similar notation will be used in the following text.) Further, we define

$$(2.33) \qquad v^*(y_1, y_2) = \begin{cases} \widehat{v}(y_1, y_2) & \text{for } y_1 \geqslant 0, \\ -3\widehat{v}(-y_1, y_2) + 4\widehat{v}\big(-\frac{1}{2}y_1, y_2\big) & \text{for } y_1 < 0. \end{cases}$$

According to Lemma 1.7, the function $v^*(y_1, y_2) \in H^2(\mathcal{S})$ is an extension of $\widehat{v} \in H^2(\mathcal{S}_{\mathrm{I}\cup\mathrm{IV}})$. Since according to (2.33) and (2.32),

$$(2.34) \qquad v^*(y_1, y_2) = v(y_1, y_2) \quad \forall\, (y_1, y_2) \in \mathcal{S}_I,$$

the function $v^* \in H^2(\mathcal{S})$ is an extension of $v \in H^2(\mathcal{S}_I)$ onto the whole square $\mathcal{S}$. The linearity of this extension is clear. Now, we verify its boundedness. By Lemma 1.7 we have

$$\|\widehat{v}\|_{2,\mathcal{S}_{\mathrm{I}\cup\mathrm{IV}}} \leqslant C_1 \|v\|_{2,\mathcal{S}_I}, \quad \|v^*\|_{2,\mathcal{S}} \leqslant C_2 \|\widehat{v}\|_{2,\mathcal{S}_{\mathrm{I}\cup\mathrm{IV}}}.$$

Hence

$$(2.35) \qquad \|v^*\|_{2,\mathcal{S}} \leqslant C \|v\|_{2,\mathcal{S}_I}$$

with $C = C_1 C_2$.

At the end of part A) we prove (2.40). On $\mathcal{S}_{\mathrm{IV}}$, the function $v^*(y_1, y_2)$ is expressed, according to (2.32) and (2.33), by

$$(2.36) \qquad v^*(y_1, y_2) = -3v(y_1, -y_2) + 4v\Big(y_1, -\frac{1}{2}y_2\Big), \quad y_1 \geqslant 0,\ y_2 < 0.$$

Similarly we obtain that the function $v^*(y_1, y_2)$ is expressed on $\mathcal{S}_{\mathrm{II}}$ by

$$(2.37) \qquad v^*(y_1, y_2) = -3v(-y_1, y_2) + 4v\Big(-\frac{1}{2}y_1, y_2\Big), \quad y_1 < 0,\ y_2 \geqslant 0$$

and on $\mathcal{S}_{\mathrm{III}}$ by

$$(2.38) \qquad \begin{aligned} v^*(y_1, y_2) = & -3\Big[-3v(-y_1, -y_2) + 4v\Big(-\frac{1}{2}y_1, -\frac{1}{2}y_2\Big)\Big] \\ & + 4\Big[-3v\Big(-\frac{1}{2}y_1, -y_2\Big) + 4v\Big(-\frac{1}{2}y_1, -\frac{1}{2}y_2\Big)\Big] \\ = {} & 9v(-y_1, -y_2) - 12v\Big(-\frac{1}{2}y_1, -y_2\Big) \\ & + 4v\Big(-\frac{1}{2}y_1, -\frac{1}{2}y_2\Big), \quad y_1 < 0,\ y_2 < 0. \end{aligned}$$

The right-hand sides of relations (2.36)–(2.38) are defined only by the function values of the original function $v \in H^2(\mathcal{S}_I)$; thus they are well-defined.

Now, let $(\widehat{y}_1, \widehat{y}_2)$ be an arbitrary point lying in the interior of $\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}}$ outside the closed domain $\mathcal{D}$ of an L-shape which has the vertices O, A, S_1, S_3, R_2, B (see Fig. 7). We choose $\mathcal{D}$ such that

$$0 < \operatorname{dist}(\mathcal{D}, \partial\mathcal{S}) < \varepsilon. \tag{2.39}$$

As the function $v \in H^2(\mathcal{S}_I)$ is equal to zero outside $\widetilde{\mathcal{G}}$ and on the arc AB and relation (2.27) holds, we have $v^*(\widehat{y}_1, \widehat{y}_2) = 0$. Hence,

$$v^*(y_1, y_2) = 0 \quad \forall\, (y_1, y_2) \in \mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}} \setminus \mathcal{D}. \tag{2.40}$$

B) Let now the image $\widetilde{\mathcal{G}}$ of $\mathcal{G} = \mathcal{K}(P_i, r_i) \cap \Omega$ lie in $\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}}$. The transformed function $v(y_1, y_2)$ is given again by (2.30). We have (see Fig. 8) $v \in H^2(\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}})$ in analogy with (2.31). The function $v(y_1, y_2)$ is equal to zero in $\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}} \setminus \widetilde{\mathcal{G}}$. We recall that we choose the radius r_i of $\mathcal{K}(P_i, r_i)$ so small that (2.27) holds.

On the square $\mathcal{S}$ let us define the function

$$v_1(y_1, y_2) = \begin{cases} v(y_1, y_2) & \text{for } y_1 \leqslant 0, \\ -3v(-y_1, y_2) + 4v\big(-\tfrac{1}{2}y_1, y_2\big) & \text{for } y_1 > 0 \end{cases} \tag{2.41}$$

and on the quadrilateral $\mathcal{S}_{\mathrm{III}\cup\mathrm{IV}}$ the function

$$\psi(y_1, y_2) = v(y_1, y_2) - v_1(y_1, y_2), \quad y_2 \leqslant 0. \tag{2.42}$$

We have

$$\psi(y_1, y_2) = 0 \quad \forall\, (y_1, y_2) \in \mathcal{S}_{\mathrm{III}}. \tag{2.43}$$

The inclusion $v \in H^2(\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}})$, Lemma 1.7 and relation (2.41) imply

$$v_1 \in H^2(\mathcal{S}); \tag{2.44}$$

hence

$$\psi \in H^2(\mathcal{S}_{\mathrm{III}\cup\mathrm{IV}}). \tag{2.45}$$

Now, let us define on $\mathcal{S}$ the function

$$\psi_1(y_1, y_2) = \begin{cases} \psi(y_1, y_2) & \text{for } y_2 \leqslant 0, \\ -3\psi(y_1, -y_2) + 4\psi\big(y_1, -\tfrac{1}{2}y_2\big) & \text{for } y_2 > 0. \end{cases} \tag{2.46}$$

Taking into account (2.43), we see that

$$\psi_1(y_1, y_2) = 0 \quad \forall\, (y_1, y_2) \in \mathcal{S}_{\mathrm{II}\cup\mathrm{III}}. \tag{2.47}$$

Lemma 1.7 and relations (2.45), (2.46) imply

$$\psi_1 \in H^2(\mathcal{S}). \tag{2.48}$$

Finally, let us define on $\mathcal{S}$ the function

$$v^*(y_1, y_2) = \psi_1(y_1, y_2) + v_1(y_1, y_2) \quad \forall\, (y_1, y_2) \in \mathcal{S}. \tag{2.49}$$

By (2.44), (2.48) and (2.49),

$$v^* \in H^2(\mathcal{S}). \tag{2.50}$$

Now we prove (2.53). By (2.41),

$$v_1(y_1, y_2) = v(y_1, y_2) \quad \forall\, (y_1, y_2) \in \mathcal{S}_{\mathrm{II}\cup\mathrm{III}}.$$

This relation and (2.47), (2.49) give

$$v^*(y_1, y_2) = v(y_1, y_2) \quad \forall\, (y_1, y_2) \in \mathcal{S}_{\mathrm{II}\cup\mathrm{III}}. \tag{2.51}$$

Further, by (2.46) and (2.42),

$$\psi_1(y_1, y_2) = v(y_1, y_2) - v_1(y_1, y_2) \quad \forall\, (y_1, y_2) \in \mathcal{S}_{\mathrm{IV}}.$$

This fact and (2.49) yield

$$v^*(y_1, y_2) = v(y_1, y_2) \quad \forall\, (y_1, y_2) \in \mathcal{S}_{\mathrm{IV}}. \tag{2.52}$$

Relations (2.51) and (2.52) imply

$$v^*(y_1, y_2) = v(y_1, y_2) \quad \forall\, (y_1, y_2) \in \mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}}. \tag{2.53}$$

Thus, $v^* \in H^2(\mathcal{S})$ is an extension of $v \in H^2(\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}})$. Its linearity is clear. Now we prove its boundedness. By Lemma 1.7 and (2.41) we have

$$\|v_1\|_{2,\mathcal{S}} \leqslant C_1 \|v\|_{2,\mathcal{S}_{\mathrm{II}\cup\mathrm{III}}}. \tag{2.54}$$

Lemma 1.7, (2.46) and then (2.47) give

$$\|\psi_1\|_{2,\mathcal{S}} \leqslant C_2\|\psi\|_{2,\mathcal{S}_{\mathrm{III}\cup\mathrm{IV}}} = C_2\|\psi\|_{2,\mathcal{S}_{\mathrm{IV}}}.$$

This result, (2.49) and (2.54) yield

$$\begin{aligned} \|v^*\|_{2,\mathcal{S}} = \|\psi_1 + v_1\|_{2,\mathcal{S}} &\leqslant \|\psi_1\|_{2,\mathcal{S}} + \|v_1\|_{2,\mathcal{S}} \\ &\leqslant C_1\|v\|_{2,\mathcal{S}_{\mathrm{II}\cup\mathrm{III}}} + C_2\|\psi\|_{2,\mathcal{S}_{\mathrm{IV}}}. \end{aligned} \tag{2.55}$$

By (2.42) and then (2.54) (because $\|v_1\|_{2,\mathcal{S}_{\mathrm{IV}}} \leqslant \|v_1\|_{2,\mathcal{S}}$) we conclude

$$\begin{aligned} \|\psi\|_{2,\mathcal{S}_{\mathrm{IV}}} &= \|v - v_1\|_{2,\mathcal{S}_{\mathrm{IV}}} \leqslant \|v\|_{2,\mathcal{S}_{\mathrm{IV}}} + \|v_1\|_{2,\mathcal{S}_{\mathrm{IV}}} \\ &\leqslant \|v\|_{2,\mathcal{S}_{\mathrm{IV}}} + C_1\|v\|_{2,\mathcal{S}_{\mathrm{II}\cup\mathrm{III}}} \leqslant (1 + C_1)\|v\|_{2,\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}}}. \end{aligned} \tag{2.56}$$

Relations (2.55) and (2.56) give the desired result:

$$\|v^*\|_{2,\mathcal{S}} \leqslant C\|v\|_{2,\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}}} \tag{2.57}$$

with $C \leqslant C_1 + C_2 + C_1C_2$.

At the end of part B) we prove (similarly as in part A)) that $v^* = 0$ in some neighbourhood of $\partial\mathcal{S}$. By virtue of the inclusions $v \in H^2(\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}})$, (2.44), (2.45), (2.48), (2.50) and Sobolev's imbedding theorem, the functions v, v_1, ψ, ψ_1, v^* are continuous on the corresponding domains. By (2.41) we have

$$v_1(y_1, y_2) = -3v(-y_1, y_2) + 4v\Big(-\frac{1}{2}y_1, y_2\Big) \quad \forall\, (y_1, y_2) \in \mathcal{S}_I. \tag{2.58}$$

By (2.42) and (2.41) we have

$$\psi(y_1, y_2) = v(y_1, y_2) + 3v(-y_1, y_2) - 4v\Big(-\frac{1}{2}y_1, y_2\Big), \quad y_1 > 0,\ y_2 < 0.$$

This result and (2.46) yield

$$\begin{aligned} \psi_1(y_1, y_2) = &- 3v(y_1, -y_2) - 9v(-y_1, -y_2) + 12v\Big(-\frac{1}{2}y_1, -y_2\Big) \\ &+ 4v\Big(y_1, -\frac{1}{2}y_2\Big) + 12v\Big(-y_1, -\frac{1}{2}y_2\Big) - 16v\Big(-\frac{1}{2}y_1, -\frac{1}{2}y_2\Big) \\ &\forall\, (y_1, y_2) \in \mathcal{S}_I. \end{aligned} \tag{2.59}$$

Inserting (2.58) and (2.59) into (2.49), we obtain

$$\begin{aligned} v^*(y_1, y_2) = &- 3v(-y_1, y_2) + 4v\Big(-\frac{1}{2}y_1, y_2\Big) - 3v(y_1, -y_2) \\ &- 9v(-y_1, -y_2) + 12v\Big(-\frac{1}{2}y_1, -y_2\Big) + 4v\Big(y_1, -\frac{1}{2}y_2\Big) \\ &+ 12v\Big(-y_1, -\frac{1}{2}y_2\Big) - 16v\Big(-\frac{1}{2}y_1, -\frac{1}{2}y_2\Big) \quad \forall\, (y_1, y_2) \in \mathcal{S}_I. \end{aligned} \tag{2.60}$$

The right-hand side of relation (2.60) is defined only by the function values of the original function $v \in H^2(\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}})$; thus it is well-defined.

Now, let $(\widehat{y}_1, \widehat{y}_2)$ be an arbitrary point lying in the interior of $\mathcal{S}_I$ outside the closed square $\mathcal{F}$ which has two sides on the axes y_1, y_2 and satisfies (see Fig. 8)

$$0 < \operatorname{dist}(\mathcal{F}, \partial\mathcal{S}) < \varepsilon.$$

As the function $v \in H^2(\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}})$ is equal to zero outside $\widetilde{\mathcal{G}}$ and relation (2.27) holds, we have, taking into account (2.60), $v^*(\widehat{y}_1, \widehat{y}_2) = 0$. Hence

$$v^*(y_1, y_2) = 0 \quad \forall\, (y_1, y_2) \in \mathcal{S}_I \setminus \mathcal{F}. \tag{2.61}$$

C) We return to the coordinate system ξ_1, ξ_2 and then to the coordinate system x_1, x_2. By (2.30) and (2.25) we have

$$w(\xi_1, \xi_2) := v(\mu_1(\xi_1, \xi_2), \mu_2(\xi_1, \xi_2)). \tag{2.62}$$

We can set

$$w^*(\xi_1, \xi_2) := v^*(\mu_1(\xi_1, \xi_2), \mu_2(\xi_1, \xi_2)). \tag{2.63}$$

As

$$v^*(y_1, y_2) = 0 \quad \forall\, (y_1, y_2) \in N(\partial\mathcal{S}),$$

where $N(\partial\mathcal{S}) \subset \mathcal{S}$ is a neighbourhood of $\partial\mathcal{S}$, we have

$$\begin{aligned} w(\xi_1, \xi_2) = 0 &\quad \forall\, (\xi_1, \xi_2) \in \Omega_i \cap N(\partial\Sigma), \\ w^*(\xi_1, \xi_2) = 0 &\quad \forall\, (\xi_1, \xi_2) \in N(\partial\Sigma), \end{aligned}$$

where $\Omega_i = A_i(\Omega)$ and $N(\partial\Sigma) \subset \Sigma$ is a neighbourhood of $\partial\Sigma$. Thus (taking into consideration (2.26) and $(2.5)_3$) we can set

$$w(\xi_1, \xi_2) = 0 \quad \forall\, (\xi_1, \xi_2) \in \Omega_i \setminus A_i(\mathcal{K}(P_i, r_i)), \tag{2.64}$$

$$w^*(\xi_1, \xi_2) = 0 \quad \forall\, (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \Sigma \tag{2.65}$$

and we have

$$w \in H^2(\Omega_i), \quad \|w\|_{2,\Omega_i} = \|w\|_{2,\Sigma\cap\Omega_i}, \tag{2.66}$$

$$w^* \in H^2(\mathbb{R}^2), \quad \|w^*\|_{2,\mathbb{R}^2} = \|w^*\|_{2,\Sigma}. \tag{2.67}$$

Lemma 1.5a implies in the case of A)

$$\|v\|_{2,\mathcal{S}_I} \leqslant C\|w\|_{2,\Omega_i\cap\Sigma} \tag{2.68}$$

and in the case of B)

$$\|v\|_{2,\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}}} \leqslant C\|w\|_{2,\Omega_i\cap\Sigma}. \tag{2.69}$$

Remark 1.5b and (2.50), (2.63) yield

$$\|w^*\|_{2,\Sigma} \leqslant C\|v^*\|_{2,\mathcal{S}}. \tag{2.70}$$

Using $(2.67)_2$, (2.70), (2.35) (or (2.57)), (2.68) (or (2.69)) and $(2.66)_2$ in the given order, we obtain

$$\|w^*\|_{2,\mathbb{R}^2} \leqslant C\|w\|_{2,\Omega_i}. \tag{2.71}$$

The last step in section C) consists in returning to the global coordinate system x_1, x_2. Using (2.28) and (2.29) together with Theorem 1.3, we obtain

$$\|w^*\|_{2,\mathbb{R}^2} = \|u^*\|_{2,\mathbb{R}^2}, \quad \|w\|_{2,\Omega_i} = \|u\|_{2,\Omega},$$

where $u^*(X) = w^*(A_i(X))$. These two equalities together with (2.71) give in the Cartesian coordinate system x_1, x_2

$$\|u^*\|_{2,\mathbb{R}^2} \leqslant C\|u\|_{2,\Omega} \quad \forall\, u \in L_i^k. \tag{2.72}$$

Setting $\mathcal{P}_k^i(u) := u^*$ (here $k = 2$), we get inequality (2.21). Relations (2.19), (2.20) follow immediately from parts A) and B) of this proof.

D) It remains to prove the property of $\mathcal{P}_k^i$ connected with (2.22), i.e., that $\mathcal{P}_k^i$ is a linear and bounded extension operator from L_i^m in $H^m(\mathbb{R}^2)$ for $m < k$ with $m \in \mathbb{N}_0$. (In our case $k = 2$ we shall consider $m = 1$.) This means not only to prove

$$\|\mathcal{P}_2^i(u)\|_{1,\mathbb{R}^2} \leqslant C\|u\|_{1,\Omega} \quad \forall\, u \in L_i^2 \tag{2.73}$$

but also to derive

$$\mathcal{P}_2^i(c_1u_1 + c_2u_2) = c_1\mathcal{P}_2^i(u_1) + c_2\mathcal{P}_2^i(u_2) \quad \forall\, c_1, c_2 \in \mathbb{R}^1,\ \forall\, u_1, u_2 \in L_i^1, \tag{2.74}$$

$$\mathcal{P}_2^i(u)(X) = u(X) \quad \forall\, X \in \Omega,\ \forall\, u \in L_i^1, \tag{2.75}$$

$$\|\mathcal{P}_2^i(u)\|_{1,\mathbb{R}^2} \leqslant C\|u\|_{1,\Omega} \quad \forall\, u \in L_i^1. \tag{2.76}$$

Properties (2.74) and (2.75) are evident. It remains to verify (2.73) and (2.76). As for relation (2.73), in case A) we have by Lemma 1.7

$$\|\widehat{v}\|_{1,\mathcal{S}_{\mathrm{I}\cup\mathrm{IV}}} \leqslant C\|v\|_{1,\mathcal{S}_{\mathrm{I}}}, \quad \|v^*\|_{1,\mathcal{S}} \leqslant C\|\widehat{v}\|_{1,\mathcal{S}_{\mathrm{I}\cup\mathrm{IV}}}. \tag{2.77}$$

Hence

$$\|v^*\|_{1,\mathcal{S}} \leqslant C\|v\|_{1,\mathcal{S}_{\mathrm{I}}}. \tag{2.78}$$

The rest of part A) follows in the case $m = 1$ the same lines as in the case $k = 2$. In case B) we easily modify the derivation of (2.57) (using again Lemma 1.7) and obtain

$$\|v^*\|_{1,\mathcal{S}} \leqslant C\|v\|_{1,\mathcal{S}_{\mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}}}. \tag{2.79}$$

The rest of part B) follows in the case $m = 1$ the same lines as in case $k = 2$.

If $u \in L_i^1$ only, the definition of $\mathcal{P}_2^i$ remains the same, i.e., $\mathcal{P}_2^i$ is constructed by means of the polynomial $p_i(r)$ of degree $2k-1$, by means of functions $\xi_2 = f(x_1)$, $\xi_1 = g(x_2)$ (which are of the class C^k—here C^2) and by means of (2.32), (2.33) in case A) and (2.41), (2.46), (2.49) in case B). Derivation of (2.78) and (2.79) is the same as in the case just considered. Considerations of part C) depend on Lemma 1.5a and Remark 1.5b which are valid for an arbitrary $k \in \mathbb{N}$. Lemma 2.3 is proved. □

Now we are ready to prove the main result of the paper.

2.4. Theorem. *Let $\Omega \in \mathcal{C}^{0,1}$ be a bounded two-dimensional domain. Let the number of points at which the boundary $\partial\Omega$ is not smooth be finite. Let the smooth parts of the boundary $\partial\Omega$ be of the class $C^{k,1}$. Then there exists a linear and bounded extension operator $\mathcal{E}_k\colon H^k(\Omega) \to H^k(\mathbb{R}^2)$ which is also a linear and bounded extension operator from $H^m(\Omega)$ into $H^m(\mathbb{R}^2)$, where $m < k$, $m \in \mathbb{N}_0$.*

Proof. By Lemma 2.3 we have

$$\|u_i^*\|_{k,\mathbb{R}^2} \leqslant C\|u_i\|_{k,\Omega}, \tag{2.80}$$

where $u_i^* = \mathcal{E}_k^i(u_i)$ and u_i is defined by (2.2)–(2.5). In accordance with (2.12) let us set

$$u^*(X) = u_0^*(X) + \sum_{i=1}^{n} u_i^*(X) \quad \forall\, X \in \mathbb{R}^2, \tag{2.81}$$

where n is the number of points P_i at which the boundary $\partial\Omega$ is not smooth and the function $u_0(X)$ is given by (2.9) and (2.15) with $u_0 \in H^k(\Omega_0)$. We recall that owing to $\Omega_0 \in C^{k,1}$, the function $u_0 \in H^k(\Omega_0)$ can be extended by means of Theorem 1.4 and relations (2.17) hold.

First we prove

$$\|u^*\|_{k,\mathbb{R}^2} \leqslant C\|u\|_{k,\Omega}. \tag{2.82}$$

Using the triangular inequality and relations (2.81), $(2.17)_3$, (2.80), we find

$$\|u^*\|_{k,\mathbb{R}^2} \leqslant \|u_0^*\|_{k,\mathbb{R}^2} + \sum_{i=1}^{n} \|u_i^*\|_{k,\mathbb{R}^2} \leqslant C\Big(\|u_0\|_{k,\Omega_0} + \sum_{i=1}^{n} \|u_i\|_{k,\Omega}\Big). \tag{2.83}$$

By (2.15) and (2.9) we have

$$\|u_0\|_{k,\Omega_0} = \|u_0\|_{k,\Omega} \leqslant \|u\|_{k,\Omega} + \sum_{i=1}^{n} \|u_i\|_{k,\Omega}. \tag{2.84}$$

Now we estimate $\|u_i\|_{k,\Omega}$ by $\|u\|_{k,\Omega}$. We can write

$$\|u_i\|_{k,\Omega}^2 = \|u\|_{k,\kappa_i}^2 + \|up_i\|_{k,\tau_i}^2, \tag{2.85}$$

where we set for brevity

$$\kappa_i = \Omega \cap \kappa\Big(P_i, \frac{1}{2}r_i\Big), \quad \tau_i = \Omega \cap \Big\{\mathcal{K}(P_i, r_i) \setminus \kappa\Big(P_i, \frac{1}{2}r_i\Big)\Big\}.$$

According to [2, (4.1.42)], for $u \in H^k(G)$ and $v \in W^{k,\infty}(G)$ we have

$$|uv|_{m,G} \leqslant C \sum_{j=0}^{m} |u|_{j,G} |v|_{m-j,\infty,G} \quad (0 \leqslant m \leqslant k). \tag{2.86}$$

As p_i is a polynomial it belongs to $W^{k,\infty}(\tau_i)$. Consequently,

$$\|up_i\|_{k,\tau_i}^2 = \sum_{j=0}^{m} |up_i|_{j,\tau_i}^2.$$

This relation and (2.86) yield

$$\|up_i\|_{k,\tau_i} \leqslant CM_i\|u\|_{k,\tau_i}, \tag{2.87}$$

where

$$M_i = \|p_i\|_{C^k(\tau_i)}.$$

Relations (2.85) and (2.87) imply

$$\|u_i\|_{k,\Omega} \leqslant \|u\|_{k,\kappa_i} + CM_i\|u\|_{k,\tau_i};$$

hence

$$\sum_{i=1}^{n} \|u_i\|_{k,\Omega} \leqslant CM\|u\|_{k,\Omega} \tag{2.88}$$

with

$$M = \max_{i=1,\dots,n} M_i.$$

Relations (2.83), (2.84) and (2.88) give the expected estimate (2.82) in the form

$$\|u^*\|_{k,\mathbb{R}^2} \leqslant CM\|u\|_{k,\Omega}.$$

It should be noted that if $\min\limits_{i=1,\dots,n} r_i$ is small then the constant M is a great number.

Relations (2.12) and (2.81) yield the basic property of an extension

$$u^*\big|_{\Omega} = u. \tag{2.89}$$

Linearity of the extension $\mathcal{E}_k$ follows from Theorem 1.4, Lemma 2.3 and relation (2.89).

The last property of $\mathcal{E}_k$ (i.e., that $\mathcal{E}_k$ is also a linear bounded extension operator from $H^m(\Omega)$ into $H^m(\mathbb{R}^2)$ with $m < k$) also follows from Theorem 1.4 and Lemma 2.3. □

2.5. Remark. The above presented results can be extended without any difficulty to the case of spaces $H^{k,p}(\Omega)$ ($k \in \mathbb{N}$, $1 \leqslant p < \infty$).

3. One special case

In [11, Lemma 6], the following theorem was formulated with reference to [8, pp. 20–22]:

3.1. Theorem. *Let Ω be a two-dimensional bounded domain with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 being circles with radii R_1 and $R_2 = R_1 + \varrho_0$ ($\varrho_0 > 0$), respectively (see Fig. 9). We assume that the circles Γ_1, Γ_2 have the same center S_0 and that*

$$R_1 \gg \varrho_0. \tag{3.1}$$

Let Γ_0 be the circle with a center S_0 and radius $R_0 = R_1 - \varrho_0$ and let Ω^* be a bounded domain such that $\partial\Omega^* = \Gamma_0 \cup \Gamma_2$. Then there exists a linear and bounded extension operator (of the Nikolskij–Whitney type) $\mathcal{E}_2\colon\ H^2(\Omega) \to H^2(\Omega^*)$ with the property $\mathcal{E}_2\colon\ H^1(\Omega) \to H^1(\Omega^*)$ and such that the constant C appearing in the inequality

$$\|\mathcal{E}_2(v)\|_{2,\Omega^*} \leqslant C\|v\|_{2,\Omega} \quad \forall\, v \in H^2(\Omega) \tag{3.2}$$

does not depend on R_1/ϱ_0.

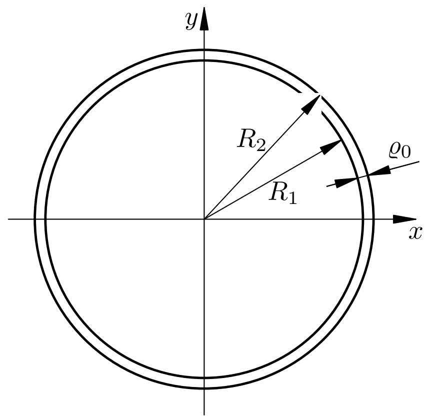

Figure 9.

However, the way from [8, pp. 20–22] to Theorem 3.1 is not straightforward; we sketch, therefore, the proof of Theorem 3.1.

P r o o f of Theorem 3.1. A) The transformation

$$x = (R_1 + \varrho)\cos\varphi, \quad y = (R_1 + \varrho)\sin\varphi, \quad (\varrho, \varphi) \in M, \tag{3.3}$$

maps one-to-one the rectangle

$$M = \langle 0, \varrho_0 \rangle \times \langle 0, 2\pi) \tag{3.4}$$

onto Ω. The inverse transformation to (3.3) has the form

$$\varrho = \sqrt{x^2 + y^2} - R_1, \quad \varphi = \arctan\frac{y}{x}, \quad (x, y) \in \Omega. \tag{3.5}$$

The Jacobian of transformation (3.3) is given by

$$J(\varrho, \varphi) = \frac{D(x, y)}{D(\varrho, \varphi)} = R_1 + \varrho \tag{3.6}$$

and the Jacobian of transformation (3.5) by

$$J^{-1}(x,y) = \frac{D(\varrho,\varphi)}{D(x,y)} = \frac{1}{R_1+\varrho}. \tag{3.7}$$

B) Let $u = u(x,y) \in H^2(\Omega)$ be an arbitrary but fixed function. Let us define on M a function

$$\widetilde{u}(\varrho,\varphi) = u((R_1+\varrho)\cos\varphi, (R_1+\varrho)\sin\varphi), \quad (\varrho,\varphi) \in M. \tag{3.8}$$

We have

$$u(x,y) = \widetilde{u}\Big(\sqrt{x^2+y^2} - R_1, \arctan\frac{y}{x}\Big), \quad (x,y) \in \Omega. \tag{3.9}$$

By Lemma 1.5 (with T defined by (3.3), $F = M$, $G = \Omega$, $v = \widetilde{u}$ (see (3.8)) and $u \in H^2(\Omega)$)

$$\widetilde{u} \in H^2(M), \quad \|\widetilde{u}\|_{2,M} \leqslant C_1\|u\|_{2,\Omega}, \tag{3.10}$$

where the constant $C_1 > 0$ does not depend on $u \in H^2(\Omega)$.

C) In the case $k = 2$ the system of equations (1.35) has the form

$$1 = \lambda_1 + \lambda_2, \quad 1 = -\lambda_1 - \frac{1}{2}\lambda_2$$

with the solution $\lambda_1 = -3$, $\lambda_2 = 4$. Let us set

$$M^* = \langle -\varrho_0, \varrho_0\rangle \times \langle 0, 2\pi) \tag{3.11}$$

and let us define on M^* a function $\widetilde{u}^*(\varrho,\varphi)$ by the relations

$$\widetilde{u}^*(\varrho,\varphi) = \begin{cases} \widetilde{u}(\varrho,\varphi) & \text{for } \varrho \geqslant 0, \\ -3\widetilde{u}(-\varrho,\varphi) + 4\widetilde{u}\big(-\frac{1}{2}\varrho,\varphi\big) & \text{for } \varrho < 0. \end{cases} \tag{3.12}$$

The function $\widetilde{u}^*(\varrho,\varphi)$ is by definition an extension of the function $\widetilde{u}(\varrho,\varphi)$ from M onto M^*. By a modification of Lemma 1.7 (which consists only of a different definition of the rectangles K and K^*) we have

$$\widetilde{u}^* \in H^2(M^*), \quad \|\widetilde{u}^*\|_{2,M^*} \leqslant C_2\|\widetilde{u}\|_{2,M}, \tag{3.13}$$

where the constant $C_2 > 0$ does not depend on $\widetilde{u}$.

D) The extension $u^*(x,y)$ of the function $u(x,y)$ from Ω on Ω^* is given by the relation

$$u^*(x,y) = \widetilde{u}^*\Bigl(\sqrt{x^2+y^2} - R_1, \arctan\frac{y}{x}\Bigr), \quad (x,y)\in\Omega^*. \tag{3.14}$$

Using $(3.13)_1$, we can apply Lemma 1.5 with T given by (3.5) and $F = \Omega^*$, $G = M^*$, $u = \widetilde{u}^*$, $v = u^*$ to obtain

$$u^* \in H(\Omega^*), \quad \|u^*\|_{2,\Omega^*} \leqslant C_3\|\widetilde{u}^*\|_{2,M^*}, \tag{3.15}$$

where the constant C_3 does not depend on $\widetilde{u}^*$.

E) Using inequalities (3.15), (3.13) and (3.10) in the given order, we obtain inequality (3.2) with $\mathcal{E}_2(v) = v^*$ and $C = C_1C_2C_3$. The proof of $\mathcal{E}_2\colon H^1(\Omega)\to H^1(\Omega^*)$ is the same as in Section 2. □

3.2. Remark. A rough estimate of the constant C appearing in inequality (3.2) is

$$C \leqslant 136(R_1+\varrho_0)^2. \tag{3.16}$$

This estimate can be obtained if we compute (using transformations (3.3) and (3.5)) the norms standing on the left-hand sides of inequalities (3.10), (3.13) and (3.15) and transform them by means of the theorem on substitution in the Lebesgue integral. In estimating the obtained expressions we bound all absolute values of trigonometric functions by one and use only the Cauchy inequality for sums.

4. A remark to the three-dimensional case

The result described in this paper can be generalized to the case of three-dimensional bounded domains which can be obtained by means of a continuous deformation of a cube (or a parallelepiped). The result of such a deformation is a bounded domain with six smooth faces which can be curved. This will be proved in detail in a subsequent paper.

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Author's address: *Alexander Ženíšek*, Department of Mathematics, Technical University Brno, Technická 2, 616 69 Brno, Czech Republic, e-mail: `zenisek@um.fme.vutbr.cz`.