Igor Bock; Ján Lovíšek

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ON A RELIABLE SOLUTION OF A VOLTERRA INTEGRAL EQUATION IN A HILBERT SPACE*

Igor Bock, Ján Lovíšek, Bratislava

Abstract. We consider a class of Volterra-type integral equations in a Hilbert space. The operators of the equation considered appear as time-dependent functions with values in the space of linear continuous operators mapping the Hilbert space into its dual. We are looking for maximal values of cost functionals with respect to the admissible set of operators. The existence of a solution in the continuous and the discretized form is verified. The convergence analysis is performed. The results are applied to a quasistationary problem for an anisotropic viscoelastic body made of a long memory material.

Keywords: Volterra integral equation in a Hilbert space, Rothe's method, maximization problem, viscoelastic body

MSC 2000: 45D05, 45N05, 49J22, 65R20, 74D05

0. Introduction

We will deal with the maximum optimization problem connected with a Volterra integral equation in the Hilbert space. We consider a class of operator-functions $t \to A(t)$ appearing in the state integral equation as the admissible set of control parameters. We will use an approach similar to [6], where the maximization problem for the class of coefficients of parabolic problems was considered. Solving the maximum problem with the class of operators appearing in the role of control variables makes it possible to determine the reliability bounds of uncertain coefficients of the coefficients characterizing the long memory viscoelastic structures. The problems of uncertain material functions characterizing the elasto-plastic bodies were solved in [7], [8], [9]. The paper [5] deals with the sensitivity analysis of the uncertain heat conductivity coefficients problems for anisotropic steady-state heat flows.

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In contrast to [6] we start with the abstract formulation of the problem and its approximation in Section 1. We will verify the existence and uniqueness theorem for a certain class of linear continuous operators acting from the Hilbert space into its dual. Applying Rothe's method (see e.g. [10], [11], [14]) we prove a convergence result for the approximated state problem with respect to a time variable and to a sequence of finite-dimensional subspaces modelling the finite element spaces. In Section 2 we state the maximization problem representing the so called "worst scenario", i.e. the worst admissible operators.

The problem formulated in a Hilbert space will be applied to the reliable solution problem for an anisotropic viscoelastic body made of a long memory material. A suitable functional depending on the time and space dependent coefficients is to be maximized. The approximate solution using three dimensional finite elements and the Hermitian interpolation with respect to the time variable is explained.

1. The state problem and its approximation

For any Banach space X and $T > 0$ we introduce the set $L^\infty(0,T;X)$ of all measurable essentially bounded functions $w\colon [0,T] \to X$, the set $C([0,T],X)$ of all continuous functions and the Sobolev space

$$W^{1,\infty}(0,T;X) = \{w \in L^\infty(0,T;X)\colon\ w' \in L^\infty(0,T;X)\}$$

with a derivative w' in the sense of distributions. All sets of functions are Banach spaces with norms

$$\begin{aligned}
\|w\|_{L^\infty(0,T;X)} &= \operatorname*{ess\,sup}_{t\in[0,T]} \|w(t)\|_X,\\
\|w\|_{C([0,T],X)} &= \max_{t\in[0,T]} \|w(t)\|_X,\\
\|w\|_{W^{1,\infty}(0,T;X)} &= \|w\|_{L^\infty(0,T;X)} + \|w'\|_{L^\infty(0,T;X)}.
\end{aligned}$$

We have a continuous imbedding $W^{1,\infty}(0,T;X) \subset C([0,T],X)$. Every element $w \in W^{1,\infty}(0,T;X)$ can be expressed in the form

$$w(t) = w(0) + \int_0^t w'(s)\,\mathrm{d}s, \quad t \in [0,T].$$

Let V be a Hilbert space with a scalar product $((\cdot,))$ and a norm $\|\cdot\|$, V^* its dual space with a norm $\|\cdot\|_*$. We denote by $\langle f, v\rangle$ the duality pairing between the functional $f \in V^*$ and the element $v \in V$.

We shall deal with the set of operator functions $t \to A(t)$ with values in the Banach space $\mathcal{B} = \mathcal{L}(V, V^*)$ of all linear bounded operators $A\colon V \to V^*$. We assume moreover that $A \in \mathcal{U}$, where $\mathcal{U} = W^{1,\infty}(0, T; \mathcal{B})$. The operator $A(0)\colon V \to V^*$ is assumed to be positive definite, i.e.

$$(1) \qquad \langle A(0)v, v\rangle \geqslant \alpha_0 \|v\|^2 \quad \forall v \in V, \ \alpha_0 > 0.$$

We introduce a norm in $\mathcal{U}$ equivalent to the original norm in $W^{1,\infty}(0, T; \mathcal{B})$ by

$$\|A\|_{\mathcal{U}} = \|A(0)\|_{\mathcal{B}} + \operatorname{ess}\sup_{t\in[0,T]} \|A'(t)\|_{\mathcal{B}}.$$

Let $f\colon [0, T] \to V^*$, $(A' * u)(t) = \int_0^t A'(t-s)u(s)\,\mathrm{d}s$. We consider

The state problem:

To find $u\colon [0, T] \to V$ fulfilling

$$(2) \qquad A(0)u(t) + (A' * u)(t) = f(t), \quad t \in [0, T].$$

Theorem 1.1. *Let $f \in C([0, T], V^*)$. Then there exists a unique solution $u \in C([0, T], V)$ of the equation* (2).

P r o o f. Due to the Lax-Milgram theorem there exists an inverse operator $A(0)^{-1} \in \mathcal{L}(V^*, V)$. The equation (2) is then equivalent to the Volterra integral equation in the Banach space V:

$$(3) \qquad u(t) + (B * u)(t) = q(t), \quad t \in [0, T],$$

with $B \in L^\infty(0, T; \mathcal{L}(V, V))$, $q \in C([0, T], V)$ defined by $B(t) = A(0)^{-1}A'(t)$, $q(t) = A(0)^{-1}f(t)$, $t \in [0, T]$.

The equation (3) can be expressed in the form

$$(4) \qquad u = \mathcal{A}(u),$$

where $\mathcal{A}\colon C([0, T], V) \to C([0, T], V)$ is defined by $\mathcal{A}(u) = q - B * u$.

It can be seen easily that there exists an integer $n \equiv n(B, T)$ such that the operator $\mathcal{A}^n$ is contractive in the Banach space $C([0, T], V)$. More precisely, we have

$$\|\mathcal{A}^n u - \mathcal{A}^n v\| \leqslant \frac{T^n \|B\|^n_{L^\infty(0,T;\mathcal{L}(V,V))}}{n!} \|u - v\|, \quad n = 1, 2, \ldots$$

and hence there exist $n_0 \in \mathbb{N}$ and $\kappa \in (0, 1)$ such that

$$\|\mathcal{A}^n u - \mathcal{A}^n v\| \leqslant \kappa \|u - v\| \quad \forall u, v \in V, \ n \geqslant n_0.$$

Applying the Banach fixed point theorem we obtain the existence and uniqueness of a solution of (4) which is also a unique solution $u \in C([0, T], V)$ of (3) and (2). $\square$

We will continue with a full discretization of the problem (2). Let us assume a family of finite-dimensional subspaces $\{V_h\}$, $V_h \subset V$, $h \in (0, h_0)$ such that for any $v \in V$ there exist $v_h \in V_h$, $h \in (0, h_0)$ fulfilling

$$v_h \to v \quad \text{in } V \text{ as } h \to 0+\,. \tag{5}$$

Let $A^h \in W^{1,\infty}(0,T;\mathcal{B})$, $h \in (0, h_0)$ be approximating operators satisfying

$$\langle A^h(0)u, u\rangle \geqslant \alpha_0 \|u\|^2 \quad \forall u \in V \text{ with } \alpha_0 > 0, \tag{6}$$

$$A^h \to A \quad \text{in } \mathcal{U} \text{ as } h \to 0+\,. \tag{7}$$

Further we assume for $\tau \in (0, \tau_0)$ the division of the interval $[0,T]$ by

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T, \quad t_i = i\tau,\ i = 1, \ldots, N \equiv N(\tau).$$

We define the approximation $u^{h\tau} \in C([0,T], V_h)$ of a solution u of (2) by

$$u^{h\tau}(t) = u^{h\tau}_{i-1} + \frac{t - t_{i-1}}{\tau}(u^{h\tau}_i - u^{h\tau}_{i-1}), \quad t \in [t_{i-1}, t_i], \tag{8}$$

where $\{u^{h\tau}_i\}$ are unique solutions (due to the Lax-Milgram theorem) of stationary problems

$$\langle A^h_0 u^{h\tau}_0, v\rangle = \langle f_0, v\rangle \quad \forall v \in V_h, \tag{9}$$

$$\left\langle A^h_0 u^{h\tau}_i + \sum_{j=0}^{i-1} (A^h_{i-j} - A^h_{i-j-1}) u^{h\tau}_j, v \right\rangle = \langle f^h_i, v\rangle \quad \forall v \in V_h,\ i = 1, \ldots, N \tag{10}$$

with $A^h_i = A^h(t_i)$, $f^h_i = f^h(t_i)$, $i = 0, 1, \ldots, N$. We have introduced $f^h\colon [0,T] \to V^*$—approximating functionals of f.

In order to ensure the convergence of the scheme we impose a smoothness condition on the right-hand side f.

Theorem 1.2. *Let $f \in W^{1,\infty}([0,T], V^*)$, $f^h \in W^{1,\infty}([0,T], V^*)$, $h \in (0, h_0)$ be such that*

$$f^h \to f \quad \text{in } W^{1,\infty}([0,T], V^*) \text{ as } h \to 0+\,. \tag{11}$$

Then

$$u^{h\tau} \rightharpoonup^* u \quad \text{in } W^{1,\infty}([0,T], V) \text{ as } h \to 0+,\ \tau \to 0+, \tag{12}$$

where $u \in W^{1,\infty}(0,T;V)$ is a unique solution of the equation (2) *and $u^{h\tau}$ is defined by* (8)–(10).

If a solution u fulfils the condition

$$\pi_h(u) \to u \quad \text{in } C([0,T],V) \text{ as } h \to 0+, \tag{13}$$

where $\pi_h(u)(t) \in V_h$, $t \in [0,T]$ is the orthogonal projection of $u(t)$ onto the subspace V_h, then

$$u^{h\tau} \to u \quad \text{in } C([0,T],V) \text{ as } h \to 0+,\ \tau \to 0+. \tag{14}$$

Proof. Using the uniform coercivity (6) and the convergence (7) we obtain from (10) the inequalities

$$\alpha_0 \|u_i^{h\tau}\|^2 \leqslant \left\langle -\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (A^h)'(t_i - s)\,\mathrm{d}s\, u_j^{h\tau} + f_i^h, u_i^{h\tau} \right\rangle$$

and

$$\|u_i^{h\tau}\| \leqslant \alpha_0^{-1}(\|A\|_{\mathcal{U}} + \varepsilon) \sum_{j=0}^{i-1} \tau \|u_j^{h\tau}\| + \alpha_0^{-1}\|f_i^h\|_*, \quad i = 1,\dots,N,\ h \in (0,h_0).$$

Applying the convergence (7), (11) and a discrete form of Gronwall's lemma ([10]) with respect to $\{u_i^{h\tau}\}$ we obtain an a priori estimate

$$\|u_i^{h\tau}\| \leqslant C_1(T), \quad i = 0,1,\dots,N(\tau),\ h \in (0,h_0),\ \tau \in (0,\tau_0). \tag{15}$$

Let us denote $\delta w_i = \tau^{-1}(w_i - w_{i-1})$, $w_i = w(t_i)$ for any function $w \in C([0,T],X)$ with values in a Banach space X. Setting $i-1$ instead of i into (10) and subtracting from (10) we obtain the relations

$$\begin{aligned}
\langle A_0^h \delta u_i^{h\tau}, v\rangle &= \left\langle -\tau \sum_{j=0}^{i-2} \delta A_{i-j-1}^h \delta u_j^{h\tau} - \delta A_i^h u_0^{h\tau} + \delta f_i^h, v \right\rangle \\
&= \left\langle -\sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} (A^h)'(t_{i-1} - s)\,\mathrm{d}s\, \delta u_j^{h\tau} - \frac{1}{\tau}\int_{t_{i-1}}^{t_i} (A^h)'(s)\,\mathrm{d}s\, u_0^{h\tau} + \delta f_i^h, v \right\rangle \\
&\quad \forall v \in V_h.
\end{aligned}$$

Again using the convergence (7), (11) and a discrete form of Gronwall's lemma we arrive at the estimate

$$\|\delta u_i^{h\tau}\| \leqslant C_2(T), \quad i = 0,1,\dots,N(\tau),\ h \in (0,h_0),\ u \in (0,\tau_0). \tag{16}$$

Let us define step functions $\overline{u}^{h\tau}, \tilde{u}^{h\tau}\colon [0,T] \to V$ by

$$\begin{aligned} &\overline{u}^{h\tau}(0) = u_0^{h\tau}, \quad \overline{u}^{h\tau}(t) = u_i^{h\tau}, \qquad t \in (t_{i-1}, t_i], \\ &\tilde{u}^{h\tau}(0) = 0, \qquad \tilde{u}^{h\tau}(t) = u_{i-1}^{h\tau}, \quad t \in (t_{i-1}, t_i], \ \ i = 1, \dots, N. \end{aligned}$$

The equation (10) can be expressed in the form

$$\begin{aligned} \text{(17)} \qquad &\Big\langle A^h(0)\overline{u}^{h\tau}(t) + (A^h)' * \tilde{u}^{h\tau}(t) + \int_t^{t_i} (A^h)'(t_i - s)\tilde{u}^{h\tau}(s)\,\mathrm{d}s, v\Big\rangle \\ &= \Big\langle \int_0^t [(A^h)'(t-s) - (A^h)'(t_i - s)]\tilde{u}^{h\tau}(s)\,\mathrm{d}s + \overline{f}^{h\tau}(t), v\Big\rangle \\ &\text{for all } \ v \in V_h, \ \ t \in (t_{i-1}, t_i], \ \ i = 1, \dots, N. \end{aligned}$$

The a priori estimates (15), (16) imply the estimate

$$\text{(18)} \qquad \|u^{h\tau}\|_{W^{1,\infty}(0,T;V)} \leqslant C_3(T), \quad h \in (0, h_0), \ \ \tau \in (0, \tau_0)$$

and the existence of a function $w \in W^{1,\infty}(0,T;V)$ and a sequence $\{h_n, \tau_n\}$, $h_n > 0$, $\tau_n > 0$ fulfilling

$$\text{(19)} \qquad h_n \to 0, \quad \tau_n \to 0, \quad u^{h_n\tau_n} \rightharpoonup^* w \ \text{ in } \ W^{1,\infty}(0,T;V).$$

Simultaneously we have inequalities

$$\text{(20)} \qquad \|u^{h\tau}(t) - \overline{u}^{h\tau}(t)\| \leqslant \tau C_2(T),$$

$$\text{(21)} \qquad \|u^{h\tau}(t) - \tilde{u}^{h\tau}(t)\| \leqslant \tau C_2(T).$$

Applying the assumptions (5), (7), (11), the a priori estimate (15) and the estimates (20), (21) we obtain from the equation (17) that the limiting function w from (19) is a solution of the state equation (2). We have $w \equiv u$ due to the uniqueness of the solution and hence the convergence (12) holds. We remark that we have used the relation

$$\lim_{\tau \to 0} \|F(t+\tau) - F(t)\|_{L^p(0,T;X)} = 0, \quad 1 \leqslant p < \infty$$

for any function $F \in L^p(0,T;X)$ extended by $F(t+\tau) = 0$ if $t + \tau \notin [0,T]$ in performing the limit in the integral on the right-hand side of the relation (17).

It remains to prove the uniform convergence (14). Let $u_h(t) := \pi_h u(t) \in V_h$, $t \in [0,T]$ be the orthogonal projection onto V_h. Due to the assumption (13) it fulfils the uniform convergence

$$\text{(22)} \qquad \lim_{h \to 0+} \|u - u_h\|_{C([0,T],V)} = 0.$$

We define functions $v_{h\tau}\colon [0,T] \to V_h$ by

$$v_{h\tau} = u_h - \tilde{u}^{h\tau}, \quad h \in (0, h_0), \ \ \tau \in (0, \tau_0). \tag{23}$$

Taking into account the relations (2), (17) we obtain the identity

$$\langle A^h(0) v_{h\tau}(t) + (A^h)' * v_{h\tau}(t), v_{h\tau}(t)\rangle = \langle \omega_{h\tau}(t), v_{h\tau}(t)\rangle, \quad t \in [0,T], \tag{24}$$

where

$$\begin{aligned}\omega_{h\tau}(t) = {} & A^h(0)u_h(t) - A(0)u(t) + (A^h)' * u_h(t) - A' * u(t) \\ & \int_t^{t_i} (A^h)'(t_i - s)\tilde{u}^{h\tau}(s)\,\mathrm{d}s - \int_0^t [(A^h)'(t-s) + (A^h)'(t_i - s)]\tilde{u}^{h\tau}(s)\,\mathrm{d}s \\ & + A^h(0)[\tilde{u}_h(t) - \overline{u}_h(t)] + f(t) - \overline{f}^{h\tau}(t).\end{aligned}$$

The uniform coercivity (6) and the assumption $A' \in L^\infty(0,T;\mathcal{B})$ imply the inequality

$$\|v_{h\tau}(t)\| \leqslant \|\omega_{h\tau}(t)\|_* + C_3(T)\int_0^t \|v_{h\tau}(s)\|\,\mathrm{d}s \quad \forall t \in [0,T].$$

The estimate

$$\|v_{h\tau}(t)\| \leqslant \|\omega_{h\tau}(t)\|_* \exp TC_3(T) \quad \forall t \in [0,T] \tag{25}$$

follows due to Gronwall's lemma. The previous assumptions and estimates imply the convergence

$$\omega_{h\tau} \to 0 \ \text{ in } \ C([0,T];V^*) \ \text{ as } \ h \to 0+, \ \ \tau \to 0+.$$

The uniform convergence (14) then follows from (25) and the proof is complete. □

2. A maximization problem and its approximation

Let us assume the compact subset $\mathcal{U}_{\mathrm{ad}} \subset \mathcal{U}$ of operator functions $A\colon [0,T] \to \mathcal{B}$ such that $A(0)$ fulfil uniform positive definiteness (1). The functional $\Phi\colon \mathcal{U} \times C([0,T];V) \to \mathbb{R}$ fulfils

$$\begin{aligned} & A_n \in \mathcal{U}_{\mathrm{ad}}, \ \{A_n, u_n\} \to \{A, u\} \ \text{ in } \ \mathcal{U} \times C([0,T];V) \ \text{ as } \ n \to \infty \\ & \qquad \Longrightarrow \lim_{n\to\infty} \sup \Phi(A_n, u_n) \leqslant \Phi(A, u).\end{aligned} \tag{26}$$

We formulate

Maximization problem:

$$A_* = \arg \max_{A\in\mathcal{U}_{\rm ad}} \Phi(A, u(A)), \tag{27}$$

where $u(A)$ is a solution of the integral equation (2).

Theorem 2.1. *Let the assumptions of Theorem* 1.1 *be fulfilled. Let the functional* Φ *satisfy* (26).

Then the Maximization Problem (27) *has at least one solution.*

Proof. Let $\{A_n\}$ be a maximizing sequence in $\mathcal{U}_{\rm ad}$ for the problem (27), i.e.,

$$\lim_{n\to\infty} \Phi(A_n, u(A_n)) = \sup_{A\in\mathcal{U}_{\rm ad}} \Phi(A, u(A)). \tag{28}$$

There exists its subsequence (again denoted by $\{A_n\}$) and $A_* \in \mathcal{U}_{\rm ad}$ such that

$$A_n \to A_* \ \text{ in } \mathcal{U}. \tag{29}$$

The corresponding sequence $\{u_n\}$ fulfils the equations

$$A_n(0)u_n + A_n' * u_n = f, \quad n = 1, 2, \ldots \tag{30}$$

Let us denote by $u_* \in C([0,T], V) \equiv u(A_*)$ the unique solution of the equation

$$A_*(0)u_* + A_*' * u_* = f. \tag{31}$$

If

$$u_n \to u_* \ \text{ in } C([0,T], V), \tag{32}$$

then the property (26) implies the relation (27).

Comparing (30) and (31) we arrive at the equation

$$A_n(0)(u_n - u_*)(t) + \int_0^t (A_n')(t-s)(u_n - u_*)(s)\,\mathrm{d}s = \omega_n(t), \quad t \in [0,T] \tag{33}$$

with

$$\omega_n(t) = [A_*(0) - A_n(0)]u_*(t) + \int_0^t (A_* - A_n)'(t-s)u_*(s)\,\mathrm{d}s.$$

We have

$$\lim_{n\to\infty} \|\omega_n\|_{C([0,T],V^*)} = 0 \tag{34}$$

due to the convergence (29). The equation (33) implies, due to the uniform coercivity and boundedness of $\{A_n\}$ in $\mathcal{U}$, the inequality

$$\|(u_n - u_*)(t)\| \leqslant M \int_0^t \|(u_n - u_*)(s)\| \, \mathrm{d}s + \|\omega_n(t)\|_* \quad \forall t \in [0,T].$$

Applying Gronwall's lemma we arrive at the estimate

$$\|(u_n - u_*)(t)\| \leqslant C_4(M,T)\|\omega_n(t)\|_* \quad \forall t \in [0,T]$$

and the uniform convergence (32) follows due to (34). The convergence (28), (29), (32) together with the property (26) implies that a function $A_* \in \mathcal{U}_{\mathrm{ad}}$ solves the Maximization problem (27). $\square$

We continue with an approximate maximization problem. We assume that the assumptions of Theorem 1.2 hold.

Let $\mathcal{U}^h_{\mathrm{ad}} \subset \mathcal{U}_{\mathrm{ad}}$, $h \in (0,h_0)$ be such compact subsets that for all $A \in \mathcal{U}_{\mathrm{ad}}$ and $h \in (0,h_0)$ there exist approximating operator functions $A_h \in \mathcal{U}^h_{\mathrm{ad}}$ fulfilling the convergence (7). Let $\tau \in (0,\tau_0)$. We assume that the functional $\Phi\colon \mathcal{U} \times C([0,T],V) \to \mathbb{R}$ fulfils the continuity property

$$\begin{aligned} (35) \quad & A_k \in \mathcal{U}_{\mathrm{ad}},\ u_k \in V,\ \{A_k,u_k\} \to \{A,u\} \text{ in } \mathcal{U} \times C([0,T],V) \text{ as } k \to \infty \\ & \Longrightarrow \lim_{k\to\infty} \Phi(A_k,u_k) = \Phi(A,u). \end{aligned}$$

For $A \in \mathcal{U}^h_{\mathrm{ad}}$ we denote by $u^{h\tau}(A) \in W^{1,\infty}(0,T;V)$ a solution belonging to the approximating problem (9), (10).

The approximate maximization problem $\mathbf{P}^h$:

$$A^{h\tau}_* = \arg \max_{A \in \mathcal{U}^h_{\mathrm{ad}}} \Phi(A, u^{h\tau}(A)). \tag{36}$$

Theorem 2.2. *Let $f \in W^{1,\infty}(0,T;V^*)$ and let the admissible sets $\mathcal{U}_{\mathrm{ad}}$, $\mathcal{U}^h_{\mathrm{ad}}$ satisfy the assumptions stated above. Then there exists a solution $A^{h\tau}_* \in \mathcal{U}^h_{\mathrm{ad}}$ of Problem* (36).

Let the assumption (13) *be fulfilled for every $A \in \mathcal{U}$. If $\{h_n,\tau_n\}$ is such a sequence that*

$$h_n > 0, \quad \tau_n > 0, \quad h_n \to 0, \quad \tau_n \to 0,$$

then there exists its subsequence $\{h_k,\tau_k\}$ fulfilling

$$A^{h_k\tau_k}_* \to A_* \text{ in } \mathcal{U} \text{ for } k \to \infty, \tag{37}$$

$$u^{h_k\tau_k}(A^{h_k\tau_k}_*) \to u(A_*) \text{ in } C([0,T],V) \text{ for } k \to \infty, \tag{38}$$

$$\lim_{k\to\infty} \Phi(A^{h_k,\tau_k}_*, u^{h_k\tau_k}(A^{h_k\tau_k}_*)) = \Phi(A_*, u(A_*)), \tag{39}$$

where A_* is a solution of Problem (27), $u^{h_k\tau_k}(A_*^{h_k\tau_k})$ *are solutions of the approximate problem* (9), (10) *with* $h := h_k$, $\tau := \tau_k$, $A := A_*^{h_k\tau_k}$ *and* $u(A_*)$ *is a solution of the state equation* (2) *with* $A := A_*$.

Proof. Let $\{A_n\} \subset \mathcal{U}_{\rm ad}^h$ be a maximizing sequence for the problem (36), i.e.

$$\lim_{n\to\infty} \Phi(A_n, u^{h\tau}(A_n)) = \sup_{A\in\mathcal{U}_{\rm ad}^h} \Phi(A, u^{h\tau}(A)). \tag{40}$$

There exists its subsequence (again denoted by $\{A_n\}$) and $\tilde{A} \in \mathcal{U}_{\rm ad}^h$ such that

$$A_n \to \tilde{A} \text{ in } \mathcal{U}. \tag{41}$$

The corresponding sequence $\{u_n^{h\tau}\}$, $u_n^{h\tau} = u^{h\tau}(A_n)$ fulfils the relation analogous to (17):

$$\begin{aligned}\Big\langle A_n(0)\overline{u}_n^{h\tau}(t) + A_n' * \tilde{u}_n^{h\tau}(t) + \int_t^{t_i} (A_n)'(t_i - s)\tilde{u}_n^{h\tau}(s)\,\mathrm{d}s, v\Big\rangle \\ = \Big\langle \int_0^t [(A_n)'(t-s) - (A_n)'(t_i - s)]\tilde{u}_n^{h\tau}(s)\,\mathrm{d}s + \overline{f}^{h\tau}(t), v\Big\rangle \\ \text{for all } v \in V_h,\ t \in (t_{i-1}, t_i],\ i = 1, \ldots, N.\end{aligned} \tag{42}$$

Let $u_*^{h\tau}$ be a solution of the approximated scheme corresponding to $\tilde{A}$:

$$\begin{aligned}\Big\langle \tilde{A}(0)\overline{u}_*^{h\tau}(t) + (\tilde{A}' * \tilde{u}_*^{h\tau})(t) + \int_t^{t_i} \tilde{A}'(t_i - s)\tilde{u}_*^{h\tau}(s)\,\mathrm{d}s, v\Big\rangle \\ = \Big\langle \int_0^t [\tilde{A}'(t-s) - \tilde{A}'(t_i - s)]\tilde{u}_*^{h\tau}(s)\,\mathrm{d}s + \overline{f}^{h\tau}(t), v\Big\rangle \\ \text{for all } v \in V_h,\ t \in (t_{i-1}, t_i],\ i = 1, \ldots, N.\end{aligned} \tag{43}$$

The following estimates can be verified in the same way as in the proof of Theorem 1.2:

$$\begin{aligned}&\|u_n^{h\tau}\|_{W^{1,\infty}(0,T;V)} \leqslant C_5(T),\\ &\|u_n^{h\tau}(t) - \overline{u}_n^{h\tau}(t)\| \leqslant \tau C_6(T),\\ &\|u_n^{h\tau}(t) - \tilde{u}_n^{h\tau}(t)\| \leqslant \tau C_6(T),\\ &\|u_*^{h\tau}\|_{W^{1,\infty}(0,T;V)} \leqslant C_5(T),\\ &\|u_*^{h\tau}(t) - \overline{u}_*^{h\tau}(t)\| \leqslant \tau C_6(T),\\ &\|u_*^{h\tau}(t) - \tilde{u}_*^{h\tau}(t)\| \leqslant \tau C_6(T),\ \ n > n_0,\ \ h \in (0, h_0),\ \ \tau \in (0, \tau_0).\end{aligned} \tag{44}$$

Let us denote

(45) $$v_n^{h\tau} = \tilde{u}_n^{h\tau} - u_*^{h\tau}, \quad n > n_0,\ \ h \in (0, h_0),\ \ \tau \in (0, \tau_0).$$

We obtain from (42), (43) the identity

(46) $$\langle A_n(0) v_n^{h\tau}(t) + A_n' * v_n^{h\tau}(t), v_n^{h\tau}(t)\rangle = \langle \omega_n^{h\tau}(t), v_n^{h\tau}(t)\rangle$$

with $\omega_n^{h\tau} \in C([0,T], V^*)$ fulfilling

(47) $$\lim_{n\to\infty} \|\omega_n^{h\tau}\|_{C([0,T],V^*)} = 0.$$

Applying the uniform coercivity of the operators $\{A_n\}$ and Gronwall's lemma to (46) we obtain due to (44), (45), (47) the convergence

(48) $$u_n^{h\tau} \to u_*^{h\tau} \quad \text{in } \ C([0,T], V).$$

The property (26) of the functional Φ and the convergence (40), (41), (48) then imply that $\tilde{A} \equiv A_*^{h\tau}$ is a solution of Approximate Maximization Problem (36).

We continue with the convergence of the method. Let $h_n \to 0$, $\tau_n \to 0$. The sequence $\{A_*^{h_n\tau_n}\}$ belongs to the compact set $\mathcal{U}_{\rm ad} \subset \mathcal{U}$. Hence there exist its subsequence $\{A_*^{h_k\tau_k}\}$ and an operator function $\hat{A} \in \mathcal{U}_{\rm ad}$ fulfilling

(49) $$A_*^{h_k\tau_k} = \arg \max_{A \in \mathcal{U}_{\rm ad}^{h_k}} \Phi(A, u^{h_k\tau_k}(A)),$$

(50) $$A_*^{h_k\tau_k} \to \hat{A} \ \text{ in } \mathcal{U}.$$

Let $\hat{u} \equiv u(\hat{A})$ be a unique solution of the state equation

(51) $$\hat{A}(0)\hat{u} + \hat{A} * \hat{u} = f$$

and $u_*^{h_k\tau_k} \equiv u^{h_k\tau_k}(A_*^{h_k\tau_k})$, $k = 1, 2\dots$ a unique solution of the approximate problem

$$\Big\langle A_*^{h_k\tau_k}(0)\overline{u}_*^{h_k\tau_k}(t) + (A_*^{h_k\tau_k})' * \tilde{u}_*^{h_k\tau_k}(t) + \int_t^{t_i} (A_*^{h_k\tau_k})'(t_i - s)\tilde{u}_*^{h_k\tau_k}(s)\,\mathrm{d}s, v\Big\rangle$$
$$= \Big\langle \int_0^t [(A_*^{h_k\tau_k})'(t-s) - (A_*^{h_k\tau_k})'(t_i - s)]\tilde{u}_*^{h_k\tau_k}(s)\,\mathrm{d}s + \overline{f}^{h_k\tau_k}(t), v\Big\rangle$$
$$\text{for all } \ v \in V_{h_k},\ \ t \in (t_{i-1}^k, t_i^k],\ \ t_i^k = i\tau_k,\ \ i = 1, \dots, N_k.$$

Using the same approach as in the proof of Theorem 1.2 we obtain the convergence

(52) $$u_*^{h_k\tau_k}(A_*^{h_k\tau_k}) \to \hat{u} \equiv u(\hat{A}) \ \text{ in } \ C([0,T], V) \ \text{ as } \ k \to \infty.$$

For an arbitrary $A \in \mathcal{U}_{\rm ad}$ there exists a sequence $\{\tilde{A}_k\} \in \mathcal{U}_{\rm ad}^{h_k}$ fulfilling $\tilde{A}_k \to A$ in $\mathcal{U}$. At the same time we have

$$u^{h_k\tau_k}(\tilde{A}^k) \to u(A) \text{ in } C([0,T],V) \text{ as } k \to \infty.$$

The relations (49)–(52) and the assumption (35) then imply the relations

$$\begin{aligned}\Phi(\hat{A}, u(\hat{A})) &= \lim_{k\to\infty} \Phi(A_*^{h_k\tau_k}, u_*(A_*^{h_k\tau_k})) \\ &\geqslant \lim_{k\to\infty} \Phi(\tilde{A}_k, u^{h_k\tau_k}(\tilde{A}_k)) = \Phi(A, u(A)).\end{aligned} \tag{53}$$

Hence we conclude that $\hat{A} \equiv A_*$ is a solution of Maximization Problem (27). Moreover, the convergence (37), (52), (53) holds and the proof is complete. □

3. Applications to maximization problems for viscoelastic bodies

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ with open in $\partial\Omega$ parts Γ_0, Γ_1, $\text{meas}(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and the unit outward normal vector $\mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \partial\Omega$. We assume a quasistationary state of a viscoelastic body occupying Ω and acting upon the body forces $\mathbf{f}(\mathbf{x},t)$, $\mathbf{x} \in \Omega$, $t \in [0,T]$ and surface tractions $\mathbf{g}(\mathbf{x},t)$, $\mathbf{x} \in \Gamma_1$, $t \in [0,T]$. Considering the Boltzman type anisotropic long memory material ([3]) we obtain the equilibrium equations

$$-\operatorname{div}\sigma(\mathbf{u};\mathbf{x},t) = \mathbf{f}(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \ t \in [0,T] \tag{54}$$

with boundary conditions

$$\mathbf{u}(\mathbf{x},t) = \mathbf{0}, \ \mathbf{x} \in \Gamma_0, \quad \sigma(\mathbf{u};\mathbf{x},t)\mathbf{n} = \mathbf{g}(\mathbf{x},t), \ \mathbf{x} \in \Gamma_1, \tag{55}$$

and stress-strain relations

$$\sigma_{ij}(\mathbf{u};\mathbf{x},t) = A_{ijkl}(\mathbf{x},0)\varepsilon_{kl}(\mathbf{u}(t)) + \int_0^t \frac{\partial}{\partial t} A_{ijkl}(\mathbf{x},t-s)\varepsilon_{kl}(\mathbf{u}(s))\,\mathrm{d}s, \tag{56}$$

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = 1,2,3. \tag{57}$$

We assume the components of fourth order tensor functions $A_{ijkl}(\cdot,\cdot)\colon\ \Omega\times[0,T] \to \mathbb{R}$ to fulfil

$$A_{ijkl} \in W^{(0,1)}(\Omega \times (0,T)) := \{a \in L^\infty(\Omega \times (0,T)); a' \in L^\infty(\Omega \times (0,T))\}. \tag{58}$$

The fourth order tensors $A_{ijkl}(\mathbf{x},0)$ are assumed to be uniformly positive definite:

$$A_{ijkl}(\mathbf{x},0)\varepsilon_{ij}\varepsilon_{kl} \geqslant c_0\varepsilon_{ij}\varepsilon_{ij},\ \ c_0>0,\ \text{ a.e. in } \Omega,\ \ \forall\{\varepsilon_{ij}\}\in\mathbb{R}^{3\times 3}_{\text{sym}}, \tag{59}$$

where $\mathbb{R}^{3\times 3}_{\text{sym}}$ is the space of all symmetric tensors $\{\varepsilon_{ij}\}\in\mathbb{R}^{3\times 3}$.

In order to simplify the admissible class of operator functions $A(t)$, $t\in[0,T]$, we will consider 6×6 matrix functions A $(\equiv (A_{ij}(\mathbf{x},t))_{ij=1}^{6}$ instead of tensor functions $\{A_{ijkl}\}$.

We set

$$V=\{\mathbf{v}\in H^1(\Omega)^3\colon\ \mathbf{v}(\mathbf{x})=\mathbf{0},\ \ \mathbf{x}\in\Gamma_0\},$$

the Hilbert space of displacements vectors $\mathbf{v}\colon\ \Omega\to\mathbb{R}^3$.

The operator function $A\in\mathcal{U}$ appearing in the state equation (1) has then the form

$$\langle A(t)\mathbf{u},\mathbf{v}\rangle=\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u})A(\mathbf{x},t)\boldsymbol{\varepsilon}^T(\mathbf{v})\,\mathrm{d}x,\qquad \mathbf{u},\mathbf{v}\in V, \tag{60}$$

where we set

$$\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_6),\quad \varepsilon_i=\varepsilon_{ii},\ \ i=1,2,3,\ \ \varepsilon_4=\varepsilon_{12},\ \ \varepsilon_5=\varepsilon_{13},\ \ \varepsilon_6=\varepsilon_{23}.$$

The operator $A(0)\colon\ V\to V^*$ is positive definite with some constant $\alpha_0>0$ due to the uniform positive-definiteness of the tensor function $\{A_{ijkl}(\cdot,0)\}$ or the matrix $A(\cdot,0)$ and Korn's inequality, verified in ([13]).

If we define a functional $f(t)\in V^*$, $t\in[0,T]$ by

$$\langle f(t),\mathbf{v}\rangle=\int_\Omega \mathbf{f}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}x+\int_{\Gamma_1}\mathbf{g}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r})\,\mathrm{d}r,\ \ v\in V \tag{61}$$

we can identify the state integral equation (2) with a weak formulation of the boundary value problem (54)–(57). Applying Theorem 1.1 we obtain

Theorem 3.1. *Let* $\mathbf{f}\in C([0,T],L^2(\Omega)^3)$, $\mathbf{g}\in C([0,T],L^2(\Gamma_1)^3)$. *Then there exists a unique weak solution* $\mathbf{u}\in C([0,T],V)$ *of the problem* (54)–(57).

Let us introduce the following spaces of matrix functions:

$$\mathcal{U}=[W^{(0,1)}(\Omega\times(0,T))]^{6\times 6} \tag{62}$$

and

$$\mathcal{V}=[W^{1,\infty}(0,T;W^{1,p}(\Omega))\cap W^{2,r}(0,T;L^1(\Omega))]^{6\times 6},\quad p>3,\ \ r>1. \tag{63}$$

In order to carry out the compactness analysis we recall the part of Corollary 4 devoted to compactness in L^∞ spaces from the extensive paper of Simon ([16]).

Lemma 3.2. *Let $X \subset B \subset Y$ be Banach spaces with compact imbedding $X \to B$. Let F be a bounded set of functions in $L^\infty(0,T;Y)$ and let $\partial F/\partial t = \{\partial f/\partial t\colon\ f \in F\}$ be bounded in $L^r(0,T;Y)$, $r > 1$. Then F is relatively compact in $C(0,T;B)$.*

Setting $X := W^{1,p}(\Omega)$, $p > 3$, $B = L^\infty(\Omega)$, $Y = L^1(\Omega)$ we obtain

Lemma 3.3. *The set $\mathcal{V}$ is relatively compact in $\mathcal{U}$.*

We have applied the compact imbedding $W^{1,q}(\Omega) \subset\subset L^\infty(\Omega)$ ([12]) and the expression $f(t) = f(0) + \int_0^t f'(s)\,\mathrm{d}s$, a.e. in $[0,T]$ for any function $f \in W^{1,1}(0,T;Z)$, Z a Banach space.

The set of admissible matrix functions

$$\begin{aligned}\mathcal{U}_{\mathrm{ad}} = \{(A_{ij})_{ij=1}^6 \in \mathcal{V}\colon\ \|(A_{ij})_{ij=1}^6\|_{\mathcal{V}} \leqslant c_1,\\ \boldsymbol{\varepsilon} A(\mathbf{x},t)\boldsymbol{\varepsilon}^T \geqslant c_0\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T,\quad c_0>0,\ \forall \mathbf{x}\in\Omega,\ \forall\boldsymbol{\varepsilon}\in\mathbb{R}^6\}\end{aligned} \tag{64}$$

is then compact in the Banach space $\mathcal{U}$.

Instead of the set $\mathcal{U}_{\mathrm{ad}}$ we can consider its arbitrary convex closed (in $\mathcal{V}$) subset. Lower and upper estimates can be imposed on the matrix members $A_{ij}(\mathbf{x},0)$, $A'_{ij}(\mathbf{x},t)$ in a similar way as in [6].

Most of viscoelastic materials are described by coefficients fulfilling an exponential decrease of their time derivatives. In that case we can consider as the admissible set

$$\mathcal{U}^1_{\mathrm{ad}} = \{(A_{ij}) \in \mathcal{U}_{\mathrm{ad}}\colon\ \|(A'_{ij}(\cdot,t))\|_{L^\infty(\Omega)^{6\times6}} \leqslant c_2\mathrm{e}^{-\beta t},\ \ \beta>0,\ \ \forall t\in[0,T]\}.$$

A very important special case of the set $\mathcal{U}^1_{\mathrm{ad}}$ is the set of coefficients in the exponential form

$$A_{ij}(t) = B_{ij}^{\langle 0\rangle} + \sum_{m=1}^{M} B_{ij}^{\langle m\rangle}\mathrm{e}^{-\beta_m t},\qquad \beta_m>0,\ \ m=1,\dots,M$$

with a positive definite matrix $(A_{ij})_{ij=1}^6(0)$. Precisely, the admissible set has the form

$$\begin{aligned}\mathcal{U}^2_{\mathrm{ad}} = \Bigg\{[(\{B_{ij}^{\langle n\rangle}), \{\beta_m\}] \in [W^{1,q}(\Omega)^{6\times6}\times\mathbb{R}]^M,\ \ q>3\colon\\ \sum_{n=0}^{M}\boldsymbol{\varepsilon}(B_{ij})^{\langle n\rangle}(\mathbf{x})\boldsymbol{\varepsilon}^T \geqslant \alpha_0\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T,\ \ \alpha_0>0,\ \ \forall\mathbf{x}\in\Omega,\ \ \forall\boldsymbol{\varepsilon}\in\mathbb{R}^6,\\ \|(B_{ij})^{\langle n\rangle}\|_{W^{1,q}(\Omega)^{6\times6}} \leqslant c_m,\ \ n=0,\dots,M;\\ 0<\gamma_m\leqslant\beta_m\leqslant\delta_m,\quad m=1,\dots,M\Bigg\}.\end{aligned}$$

Let

$$\overline{\Omega} = \bigcup_{m=1}^{M} \overline{\Omega}_m, \quad \Omega_i \cap \Omega_j = \emptyset, \text{ for } i \neq j.$$

We assume the coefficients to be constant with respect to $\mathbf{x}$ on the subsets Ω_m, $m = 1, \dots, M$. The admissible set has then the form

$$\begin{gathered} \mathcal{U}^3_{\mathrm{ad}} = \{(A_{ij}) \in \mathcal{U}\colon\ A_{ij}|_{\Omega_m}(\mathbf{x}, t) = A^{\langle m \rangle}_{ij}(t), \\ \boldsymbol{\varepsilon}(A^{\langle m \rangle}_{ij}(0))\boldsymbol{\varepsilon}^T \geqslant c_0 \boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T,\ \ c_0 > 0,\ \ \forall \boldsymbol{\varepsilon} \in \mathbb{R}^6, \\ A^{\langle m \rangle}_{ij} \in W^{2,p}(0,T),\ \ p > 1,\ \ \|A^{\langle m \rangle}_{ij}\|_{W^{2,p}(0,T)} \leqslant c_m,\ \ m = 1, \dots, M\}. \end{gathered}$$

We can formulate

Maximization problem $\mathcal{P}$:

$$\mathcal{A}_* = \arg \max_{\mathcal{A} \in \mathcal{U}_{\mathrm{ad}}} \Phi(\mathcal{A}, \mathbf{u}(\mathcal{A})), \quad \mathcal{A} = \{A_{ijkl}\}$$

with goal functionals $\Phi_i\colon \mathcal{U} \times C([0,T]; V) \to \mathbb{R}$, $i = 1, 2$ fulfilling the assumptions (26).

Let $\Omega_j \subset \Omega$ and let $I_j \subset [0,T]$, $j = 1, \dots, J$ be intervals

1) $\Phi_1(\mathcal{A}, \mathbf{u}(\mathcal{A})) = \max\limits_{1 \leqslant j \leqslant J} \psi_j(\mathbf{u}(\mathcal{A}))$ with
   a) $\psi_j(\mathbf{u}(\mathcal{A})) = (\operatorname{meas} \Omega_j)^{-1} \max\limits_{1 \leqslant k \leqslant 3} \int_{\Omega_j} u_k(\mathcal{A})(t_*)\,\mathrm{d}x$, $t_* \in (0,T]$, or
   b) $\psi_j(\mathbf{u}(\mathcal{A})) = (\operatorname{meas} I_j)^{-1} (\operatorname{meas} \Omega_j)^{-1} \max\limits_{1 \leqslant k \leqslant 3} \int_{I_j} \int_{\Omega_j} u_k(\mathcal{A})\,\mathrm{d}t\,\mathrm{d}x$.
2) $\Phi_2(\mathcal{A}, \mathbf{u}(\mathcal{A})) = \int_0^T \int_\Omega \kappa(\mathcal{A}, \mathbf{u}(\mathcal{A}))\,\mathrm{d}t\,\mathrm{d}x$,
   $\kappa(\mathcal{A}, \mathbf{u}(\mathcal{A})) = \sum\limits_{i \neq j} [a_{ij}(\sigma_{ii} - \sigma_{jj})^2 + b_{ij}\sigma_{ij}^2], \quad a_{ij} > 0,\ \ b_{ij} > 0,$
   $\sigma_{ij} \equiv \sigma_{ij}(\mathcal{A}, \mathbf{u}(\mathcal{A}))(t) = A_{ijkl}(0)\varepsilon_{kl}(\mathbf{u}(t)) + (A'_{ijkl} * \varepsilon_{kl}(\mathbf{u}))(t)$.

The functional Φ_2 expresses the intensity of the shear stresses.

It can be verified using the standard methods that Maximization problem $\mathcal{P}$ fulfils for all above mentioned choices of admissible sets and goal functions the conditions of the general theory and has at least one solution $\mathcal{A}_* = \{A^*_{ijkl}\}$.

We continue with the finite element approximation of Problem $\mathcal{P}$. We assume the polyhedral region Ω divided regularly (see [4] for the details) by tetrahedron $\{G_i\}$:

$$\overline{\Omega} = \bigcup_{i=1}^{I(h)} \overline{G}_i,\ \ G_i \cap G_j = \emptyset,\ \ i \neq j,\ \ h = \operatorname{diam} G_i,\ \ i = 1, \dots, I(h).$$

The partition is consistent with the partition $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Let

$$V_h = \{\mathbf{v} \in V \cap C(\overline{\Omega})^3\colon\ \mathbf{v}|_{G_i} \in \mathbf{P}_1\},$$

where $\mathbf{P}_1 \subset \mathbb{R}^3$ is the space of vector polynomials of the first degree. Let us assume the admissible set $\mathcal{U}_{\rm ad}$ defined in (64). In order to fulfil the regularity of coefficients $\{A_{ijkl}\}$ we shall consider the Hermitian interpolation with respect to the time variable. The method of Galerkin space-time discretization used in [15] can be used in final numerical algorithms.

For $\tau > 0$ we recall the division of the interval $[0, T]$ by

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T, \quad t_m = m\tau, \ \ m = 0, 1, \ldots, N(\tau)$$

and the approximation $\mathbf{u}^{h\tau} \in C([0, T], V_h)$ of a weak solution $\mathbf{u}$ of (54)–(57) by

$$\mathbf{u}^{h\tau}(t) = \mathbf{u}^{h\tau}_{m-1} + \frac{t - t_{m-1}}{\tau}(\mathbf{u}^{h\tau}_m - \mathbf{u}^{h\tau}_{m-1}), \quad t \in [t_{m-1}, t_m].$$

Approximate maximization problem $\mathcal{P}^{h\tau}$:

$$\mathcal{A}^{h\tau}_* = \arg \max_{\mathcal{A} \in \mathcal{U}^{h\tau}_{\rm ad}} \Phi(\mathcal{A}, \mathbf{u}^{h\tau}(\mathcal{A})), \quad \mathcal{A} = (A_{ij})^6_{ij=1}$$

with the approximate admissible set of 6×6 matrix functions

$$\begin{aligned}
\mathcal{U}^{h\tau}_{\rm ad} = \Big\{ \mathcal{A} \in \mathcal{U}_{\rm ad} \colon \ & \mathcal{A}(t) = \mathcal{A}^{\langle 0 \rangle}_{m-1} \varphi_0\Big(\frac{t - t_{m-1}}{\tau}\Big) \\
& + \mathcal{A}^{\langle 1 \rangle}_{m-1} \varphi_1\Big(\frac{t - t_{m-1}}{\tau}\Big) + \mathcal{A}^{\langle 0 \rangle}_{m} \varphi_0\Big(\frac{t - t_{m}}{\tau}\Big) + \mathcal{A}^{\langle 1 \rangle}_{m} \varphi_1\Big(\frac{t - t_{m}}{\tau}\Big), \\
\mathcal{A}^{\langle 0 \rangle}_m = \mathcal{A}^{\langle 0 \rangle}_0 + \tau \sum_{n=1}^{m} \mathcal{A}^{\langle 1 \rangle}_n, \ \ & t_{m-1} \leqslant t \leqslant t_m, \ \ m = 1, \ldots, N(\tau) \Big\}, \\
\mathcal{A}^{\langle r \rangle}_m := (A^{\langle r \rangle}_{ij,m}), \ \ & (A^{\langle r \rangle}_{ij,m}) \in U_h, \ \ r = 0, 1, \\
U_h = \{\varphi \in C(\overline{\Omega}) \colon \ & \varphi|_{G_n} \in P_1, \ \ n = 1, \ldots, I(h)\}.
\end{aligned}$$

The Hermitian basic functions φ_0, φ_1 have the form

$$\varphi_0(x) = \begin{cases} 1 - 3x^2 - 2x^3, & -1 \leqslant x \leqslant 0, \\ 1 - 3x^2 + 2x^3, & 0 \leqslant x \leqslant 1, \end{cases}$$

$$\varphi_1(x) = \begin{cases} x + 2x^2 + x^3, & -1 \leqslant x \leqslant 0, \\ x - 2x^2 + x^3, & 0 \leqslant x \leqslant 1. \end{cases}$$

The discrete values of $\mathbf{u}^{h\tau}$ are determined by variational equations

(65)
$$\langle A^{\langle 0 \rangle}_0 \mathbf{u}^{h\tau}_0, \mathbf{v} \rangle = \langle f^h_0, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V_h,$$

$$\Big\langle A^{\langle 0 \rangle}_0 \mathbf{u}^{h\tau}_m + \sum_{n=0}^{m-1} \tau A^{\langle 1 \rangle}_{m-n} \mathbf{u}^{h\tau}_n, \mathbf{v} \Big\rangle = \langle f^h_m, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in V_h, \ \ m = 1, \ldots, N(\tau)$$

with operators $A_m^{\langle r\rangle}\colon V \to V^*$ defined by

$$\langle A_m^{\langle r\rangle}\mathbf{u},\mathbf{v}\rangle = \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u})(A_{ij,m}^{\langle r\rangle})(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{v})^T \,\mathrm{d}x, \quad \mathbf{u},\mathbf{v}\in V,$$

and finite element approximations f_m^h of the functionals $f(t_m)\in V^*$, $m=1,\dots,N(\tau)$ defined in (61).

Using the approach similar to the proof of Theorem 2.2 the convergence of a subsequence of $\{\mathcal{A}^{h\tau}\}$ as $h\to 0+$, $\tau\to 0+$ can be verified.

R e m a r k 3.4. The maximization problem (27) can be formulated also for the bending problem of a viscoelastic plate of variable thickness made of a long memory material. The deflections of the middle surface Ω are elements of the Hilbert space

$$V = \Bigl\{v\in H^2(\Omega)\colon\ v|_{\Gamma_0} = \frac{\partial v}{\partial \mathbf{n}}\Big|_{\Gamma_0} = 0,\ \ v|_{\Gamma_1} = 0\Bigr\},$$

if the part Γ_0 of the boundary $\partial\Omega$ is clamped and Γ_1 is simply supported.

The functionals $A(t)\colon V\to V^*$ are of the form

$$\langle A(t),v\rangle = \int_\Omega e^3(x)A_{ijkl}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial^2 v}{\partial x_j\partial x_k}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

with the tensors $\{A_{ijkl}(x,t)\}$, $i,j,k,l\in\{1,2\}$, $x=(x_1,x_2)$ fulfilling the positive definiteness for $t=0$. The variable thickness $e\colon\overline{\Omega}\to\mathbb{R}$ can play the role of control parameters in a similar way as in [1], [2], where a minimization problem for a short memory material was investigated.

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Authors' addresses: *Igor Bock*, Department of Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, 812 19 Bratislava, Slovakia, e-mail: bock@kmat.elf.stuba.sk; *Ján Lovíšek*, Department of Mechanics, Faculty of Civil Engineering, Slovak University of Technology, 813 68 Bratislava, Slovakia, e-mail: lovisek@svf.stuba.sk.