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THE BOUNDARY REGULARITY OF A WEAK SOLUTION OF
THE NAVIER-STOKES EQUATION AND ITS CONNECTION
TO THE INTERIOR REGULARITY OF PRESSURE*

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Abstract. We assume that \( \mathbf{v} \) is a weak solution to the non-steady Navier-Stokes initial-boundary value problem that satisfies the strong energy inequality in its domain and the Prodi-Serrin integrability condition in the neighborhood of the boundary. We show the consequences for the regularity of \( \mathbf{v} \) near the boundary and the connection with the interior regularity of an associated pressure and the time derivative of \( \mathbf{v} \).

Keywords: Navier-Stokes equations, regularity

MSC 2000: 35Q30, 76D05

1. Introduction

Suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^\infty \) boundary \( \partial \Omega \) such that \( \Omega \) is locally on one side of \( \partial \Omega \). Let \( T > 0 \) and \( Q_T = \Omega \times (0, T) \). We deal with the Navier-Stokes initial-boundary value problem

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} &= -\nabla p + \nu \Delta \mathbf{v} \quad \text{in } Q_T, \\
\nabla \cdot \mathbf{v} &= 0 \quad \text{in } Q_T, \\
\mathbf{v} &= \mathbf{0} \quad \text{on } \partial \Omega \times (0, T), \\
\mathbf{v}\big|_{t=0} &= \mathbf{v}_0
\end{align*}
\]

where \( \mathbf{v} = (v_1, v_2, v_3) \) and \( p \) denote the velocity and the pressure and \( \nu > 0 \) is the viscosity coefficient. We will assume for simplicity that \( \nu = 1 \).

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We deal with a weak solution \( v \) of the problem (1)–(4) that satisfies a strong energy inequality. (Such a solution can be constructed.) The notion of a weak solution of the problem (1)–(4) is well known. The readers can find the definition and a survey of important properties e.g. in [3]. Let us only recall that \( v \in L^2(0, T; W_0^{1,2}(\Omega)^3) \cap L^\infty(0, T; L^2(\Omega)^3) \). The associated pressure is a scalar function \( p \) such that \( v \) and \( p \) satisfy equation (1) in \( QT \) in the sense of distributions. \( p \) is defined a.e. in \( QT \), it is determined modulo an additive function of time and can be chosen so that it belongs to \( L^{3/2}((\varepsilon, T) \times \Omega) \) for each \( \varepsilon \in (0, T) \) (see [13]).

A point \( (x,t) \in \overline{\Omega} \times (0,T) \) is called a regular point of the weak solution \( v \) if there exists a neighborhood \( U \) of \( (x,t) \) such that \( v \) is essentially bounded in \( U \cap QT \). The points of \( \overline{\Omega} \times (0,T) \) which are not regular are called singular.

The following lemma gives more information on interior regularity of the weak solution \( v \) of the problem (1)–(4).  \( t_1 \) and \( t_2 \) will always denote instants of time such that \( 0 \leq t_1 < t_2 \leq T \).

**Lemma 1.** Let \( \Omega_1 \) be a subdomain of \( \Omega \) and let at least one of the conditions 

(i) \( v \in L^a(t_1,t_2; L^b(\Omega_1)^3) \) for some \( a \in [2, +\infty), \ b \in (3, +\infty) \) such that \( 2/a + 3/b = 1 \), 

(i)' \( v \in L^\infty(t_1,t_2; L^3(\Omega_1)^3) \) and the norm of \( v \) in \( L^\infty(t_1,t_2; L^3(\Omega_1)^3) \) is sufficiently small be satisfied. Let \( \Omega_2 \) be a subdomain of \( \Omega_1 \) such that \( \overline{\Omega_2} \subset \Omega_1 \) and let \( \zeta \) be a positive number such that \( t_1 + \zeta < t_2 - \zeta \). Then

a) \( v \) and its space derivatives of arbitrary orders belong to \( L^\infty(\Omega_2 \times (t_1 + \zeta, t_2 - \zeta)^3 \) and

b) \( \nabla p \) and \( \partial v/\partial t \) and their space derivatives of arbitrary orders belong to \( L^\alpha(t_1 + \zeta, t_2 - \zeta; L^\infty(\Omega_2)^3) \) for each \( \alpha \in [1,2) \).

Statement a) follows from [11], while b) is proved e.g. in [10].

Regularity up to the boundary of a weak solution \( v \) of the problem (1)–(4) was studied by S. Takahashi [14]. S. Takahashi worked with a domain \( \Omega_1 \) of the form \( \Omega_1 = U_\delta(x_0) \cap \Omega \) for some \( x_0 \in \partial \Omega \) under the assumption that \( \partial \Omega_1 \cap \partial \Omega \) is part of a plane. He has shown that if \( v \) satisfies condition (i) or condition (i)' then it has no singular points in \( U_\delta'(x_0) \cap \overline{\Omega} \) in the time interval \( (t_1 + \zeta, t_2 - \zeta) \) for all \( \zeta \in (0, (t_2 - t_1)/2) \) and \( \delta' < \delta \).

We shall use the following notation:

- \( n \) is the outer normal vector on \( \partial \Omega \).
- \( L^2_0(\Omega)^3 \) is the closure of \( \{ \Phi \in C_0^\infty(\Omega)^3; \ \nabla \cdot \Phi = 0 \text{ in } \Omega \} \) in \( L^2(\Omega)^3 \). Functions from \( L^2_0(\Omega)^3 \) have the normal derivative on \( \partial \Omega \) equal to zero in the sense of traces and \( [L^2_0(\Omega)^3]^\perp = \{ \nabla \varphi \in L^2(\Omega)^3; \ \varphi \in W^{1,2}_{loc}(\Omega) \} \) (see e.g. [3], Chap. III).
• \( \| \cdot \|_q \) and \( \| \cdot \|_{s,q} \), will denote the norm in \( L^q(\Omega) \) and in \( W^{s,q}(\Omega) \), respectively. The norms of vector-valued or tensor-valued functions will be denoted in the same way as the norms of scalar-valued functions.

• \( P_\sigma \) is the orthogonal projector of \( L^2(\Omega)^3 \) onto \( L^2_\sigma(\Omega)^3 \). Put \( Q_\sigma = I - P_\sigma \). If \( w \) is smooth enough, i.e. if \( \nabla \cdot w \in L^2(\Omega)^3 \), then \( Q_\sigma w \) has the form \( \nabla \varphi \) where \( \varphi \) satisfies the Neumann problem

\[
\Delta \varphi = \nabla \cdot w \quad \text{in} \quad \Omega, \quad \frac{\partial \varphi}{\partial n} \bigg|_{\partial \Omega} = (w \cdot n) \bigg|_{\partial \Omega}.
\]

Using the assumption about the smoothness of \( \partial \Omega \), one can deduce from the results on the regularity of solutions of this problem (see e.g. [5], p. 15) that \( P_\sigma \) and \( Q_\sigma \) are continuous linear operators in \( W^{s,q}(\Omega)^3 \) for all \( s \geq 0 \) and \( q \geq 2 \).

• \( A = -P_\sigma \circ \Delta \) with \( D(A) = W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega)^3 \). \( A \) is a selfadjoint positive operator in \( L^2_\sigma(\Omega)^3 \). It was proved in [1] and [4] that the domain of the fractional power \( A^s \) \((0 \leq s \leq 1)\) is \( D(A^s) = D((\nabla \Delta)^s) \cap L^2_\sigma(\Omega)^3 \) where \( -\Delta \) is considered to be the operator in \( L^2(\Omega)^3 \) with the domain \( D(\nabla \Delta) = W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \). Since \( D((-\Delta)^{1/2}) = W^{1,2}_0(\Omega)^3 \) and consequently \( D((-\Delta)^s) \) is the interpolation space \( [L^2(\Omega)^3, W^{1,2}_0(\Omega)^3]_{2s} = W^{2s,2}(\Omega)^3 \) \((0 \leq s < \frac{1}{4})\), we have \( D(A^s) = W^{2s,2}(\Omega)^3 \cap L^2_\sigma(\Omega)^3 \) \((0 \leq s < \frac{1}{4})\). It can be also deduced from [4] that \( A^s \) is a continuous operator from \( W^{2s,q}(\Omega)^3 \) into \( L^q(\Omega)^3 \) \((0 \leq s \leq 1, q \geq 2)\).

• \( U_r^* = U_r(\partial \Omega) \cap \Omega \) \(\text{for} \ r > 0\).

We shall further use the conditions

(ii) \( v \in L^a(t_1, t_2; L^b(U^*_r)^3) \) for some \( r > 0 \) and \( a \in [2, +\infty), b \in (0, +\infty) \) satisfying \( 2/a + 3/b = 1 \),

(ii)' \( v \in L^\infty(t_1, t_2; L^3(U^*_r)^3) \) and the norm of \( v \) in \( L^\infty(t_1, t_2; L^3(U^*_r)^3) \) is sufficiently small.

Both the conditions (ii) and (ii)' are obviously fulfilled if \( v \) has no singular points on \( \partial \Omega \) in the time interval \([t_1, t_2]\). The main results of this paper are given by the next two theorems.

**Theorem 1.** Let condition (ii) or condition (ii)' be fulfilled and let \( \zeta > 0 \) be such a number that \( t_1 + \zeta < t_2 - \zeta \). Then \( v \in L^\infty(t_1 + \zeta, t_2 - \zeta; W^{2+\delta,2}(U^*_r)^3) \) and both \( \partial v / \partial t \) and \( \nabla p \) belong to \( L^\infty(t_1 + \zeta, t_2 - \zeta; W^{\delta,2}(U^*_r)^3) \) for each \( \delta \in ([0, \frac{1}{4}) \) and \( r \in (0, r) \).

Let us note that statement b) of Lemma 1 holds with \( \alpha = +\infty \) in the case when \( \Omega = \mathbb{R}^3 \). (This will easily follow from Lemma 2 and the identity \( p^{II} = 0 \). It was also independently proved by P. Kučera and Z. Skalák—see [6] and [12], where this question and other related topics are also discussed.) Thus, a challenging question arises about the influence of the boundary of \( \Omega \) on the interior regularity of pressure.
and the time derivative of velocity, even if $\partial \Omega$ is arbitrarily far from the considered domains $\Omega_1$ and $\Omega_2$. Theorem 2 shows that conditions (ii) or (ii)' enable us to obtain the same result as in the case when $\Omega = \mathbb{R}^3$.

**Theorem 2.** Let $\Omega_1$ and $\Omega_2$ be subdomains of $\Omega$ such that $\overline{\Omega_2} \subset \Omega_1$ and let $\zeta$ be a positive number such that $t_1 + \zeta < t_2 - \zeta$. Suppose that at least one of the conditions (i) and (i)' and at least one of the conditions (ii) and (ii)' are satisfied. Then $\nabla p$, $\partial v/\partial t$ and their space derivatives of arbitrary orders belong to $L^\infty(\Omega_2 \times (t_1 + \zeta, t_2 - \zeta))^3$.

2. Proofs of Theorem 1 and Theorem 2

The problem (1)–(4) can be localized to $U^*_r$ in a standard way: Let $\varrho \in (0, r)$ and let $\eta$ be a $C^\infty$ cut-off function such that $\eta(x) = 1$ for $x \in U^*_\varrho$, $0 \leq \eta(x) \leq 1$ for $x \in U^*_\varrho - U^*_{(r+2\varrho)/3}$ and $\eta(x) = 0$ if $x \in \Omega - U^*_\varrho - U^*_{(r+2\varrho)/3}$. Put $u = \eta v - V$ where $\nabla \cdot V = \nabla \eta \cdot v$. Function $V$ can be constructed so that it has a compact support in $[U^*_\varrho - U^*_{\varrho/2}] \times [t_1, t_2]$ and

\[ \|\nabla^{m+1} V\|_2 \leq c(m)\|\nabla^m v\|_2 \]

for all $m \in \mathbb{N}$. (See e.g. [2], Theorem 3.2, Chap. III.3.) $u$ satisfies the equations

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla[\eta(p - \overline{p})] + \Delta u + h \quad \text{in} \quad \Omega \times (t_1, t_2), \]

\[ \nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (t_1, t_2) \]

where

\[ \overline{p}(t) = \int_{U^*_\varrho - U^*_{\varrho/2}} p(x, t) \, dx, \]

\[ h = -\frac{\partial V}{\partial t} - (V \cdot \nabla)(\eta v) - ((\eta v) \cdot \nabla)V + (V \cdot \nabla)V + (\eta v \cdot \nabla)\eta v \]
\[ - \eta(1 - \eta)(v \cdot \nabla)v - 2\nabla \eta \cdot \nabla v - v\Delta \eta + \Delta V + (p - \overline{p})\nabla \eta. \]

Note that supp $h \subset (U^*_\varrho - U^*_{\varrho/2}) \times [t_1, t_2]$. $u$ satisfies the boundary condition

\[ u = 0 \quad \text{on} \quad \partial \Omega \times (t_1, t_2). \]

An analysis of the system (6)–(8) requires some information about regularity of the function $h$, which is closely connected with the interior regularity of functions $p$ and
the time derivative of \( \mathbf{v} \). \( p \) can be written as a sum \( p^I + p^{II} \) where \( \nabla p^I = -Q_\sigma (\mathbf{v} \cdot \nabla) \mathbf{v} \) and \( \nabla p^{II} = Q_\sigma \Delta \mathbf{v} \). Then for a.a. \( t \in (t_1, t_2) \) one has

\[
(9) \quad \Delta p^I = -v_{i,j} v_{j,i} \quad \text{in} \ \Omega, \quad \frac{\partial p^I}{\partial \mathbf{n}}(x, t) \bigg|_{x \in \partial \Omega} = 0, \\
(10) \quad \Delta p^{II} = 0 \quad \text{in} \ \Omega, \quad \frac{\partial p^{II}}{\partial \mathbf{n}}(x, t) \bigg|_{x \in \partial \Omega} = (\Delta \mathbf{v}(x, t) \cdot \mathbf{n}) \bigg|_{x \in \partial \Omega}.
\]

The harmonic part \( p^{II} \) of pressure is connected with velocity only through the behavior of \( \Delta \mathbf{v} \) on the boundary. This is also observed and discussed in [9], pp. 83–85.

**Lemma 2.** Let \( \Omega_1 \) be a subdomain of \( \Omega \) and let at least one of the conditions (i) and (i)' be satisfied. Let \( \Omega_2 \) be a subdomain of \( \Omega_1 \) such that \( \overline{\Omega_2} \subset \Omega_1 \) and let \( \zeta \) be a positive number such that \( t_1 + \zeta < t_2 - \zeta \). Then \( \nabla p^I \) and its space derivatives of arbitrary orders belong to \( L^\infty(\Omega_2 \times (t_1 + \zeta, t_2 - \zeta)) \).

**Proof.** A solution \( \mathbf{v} \) can have singularities only at time instants \( t \in \Gamma \) where the set \( \Gamma \) is closed in \((0, T)\) and its measure is zero. Moreover, \( \mathbf{v} \) is of class \( C^\infty \) on \( \overline{\Omega} \times ((0, T) - \Gamma) \). (See e.g. [3].) Suppose that \( t \in (t_1 + \zeta, t_2 - \zeta) - \Gamma \) and \( \mathbf{a} \) is a unit vector. Let \( \mu \) be a \( C^\infty \) cut-off function such that \( \mu(x) = 1 \) for \( x \in \Omega_2 \), \( 0 \leq \mu(x) \leq 1 \) for \( x \in \Omega_1 - \Omega_2 \) and \( \mu(x) = 0 \) if \( x \not\in \Omega_1 \). Let \( x \in \Omega_2 \). Then

\[
a \cdot \mu(x) \nabla p^I(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} a \cdot \nabla \left[ \frac{\mu(y) \nabla_y p^I(y, t)}{|y - x|} \right] \, dy
\]

\[
= \frac{1}{4\pi} \int_\Omega \Delta_y \left( \frac{a \mu(y)}{|y - x|} \right) \cdot \nabla_y p^I(y, t) \, dy + \frac{a}{4\pi} \cdot \int_\Omega \frac{\mu(y)}{|y - x|} \nabla_y [v_{i,j}(y, t) v_{j,i}(y, t)] \, dy
\]

\[
= \frac{1}{4\pi} \int_\Omega \nabla_y \varphi^{x,a}(y) \cdot \nabla_y p^I(y, t) \, dy + \frac{a}{4\pi} \cdot I(x, t)
\]

where the integral \( I \) belongs to \( L^\infty(\Omega_1 \times (t_1 + \zeta, t_2 - \zeta)) \) (due to Lemma 1) and \( \nabla_y \varphi^{x,a}(y) = Q_\sigma \Delta_y (a \mu(y)/|y - x|) \). One can derive that

\[
\varphi^{x,a}(y) = a \cdot \left[ \nabla_y \frac{\mu(y)}{|y - x|} - \frac{\mu(x)}{|y - x|} \right] + w^x(y)
\]

where

\[
\Delta_y w^x(y) = 0 \quad \text{in} \ \Omega, \quad \frac{\partial w^x(y)}{\partial y} \bigg|_{y \in \partial \Omega} = \left( -\frac{n}{|y - x|^3} + 3 (y - x) \cdot \frac{n}{|y - x|^5} (y - x) \right) \bigg|_{y \in \partial \Omega}.
\]
Then we have
\[
\alpha \cdot \mu(x) \nabla p^I(x, t) = -\frac{1}{4\pi} \int_{\Omega} \varphi_{x,a}^I(y) v_{i,j}(y, t) v_{j,i}(y, t) \, dy + \alpha \cdot I(x, t)
\]
\[
= -\frac{1}{4\pi} \int_{\Omega} \varphi_{x,a}^I(y) v_i(y, t) v_j(y, t) \, dy + \alpha \cdot I(x, t).
\]
This shows that \( \nabla p^I \) belongs to \( L^\infty(\Omega_2 \times (t_1 + \zeta, t_2 - \zeta))^3 \). The same statement about the space derivatives of \( \nabla p^I \) can be obtained analogously, provided we deal with \( D_x^{|k|} \nabla p^I \) (where \( D_x^{|k|} = \partial^{|k|} / \partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3} \), \( k = (k_1, k_2, k_3) \) is a multiindex) instead of \( \nabla p^I \).

**Lemma 3.** Let \( \Omega_2 \) be a subdomain of \( \Omega \) such that \( \overline{\Omega_2} \subset \Omega \). Let \( \partial v / \partial n \in L^\beta(t_1, t_2; L^1(\partial \Omega)^3) \) (where \( \beta \geq 1 \)) and let \( \zeta \) be a positive number such that \( t_1 + \zeta < t_2 - \zeta \). Then \( \nabla p^{II} \) and its space derivatives of arbitrary orders belong to \( L^\beta(t_1 + \zeta, t_2 - \zeta; L^\infty(\Omega_2)^3) \).

**Proof.** Let \( \Omega_1 \) be a domain in \( \Omega \) such that \( \overline{\Omega_2} \subset \Omega_1 \subset \Omega \). Suppose that \( t, x, \alpha, \varphi, \mu, a \) have the same meaning as in the proof of Lemma 2. Then
\[
\alpha \cdot \mu(x) \nabla p^{II}(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \alpha \cdot \frac{\Delta_y \mu(y) \nabla y p^{II}(y, t)}{|y - x|} \, dy
\]
\[
= \frac{1}{4\pi} \int_{\Omega} \Delta_y \left( \frac{\alpha \mu(y)}{|y - x|} \right) \cdot \nabla y p^{II}(y, t) \, dy
\]
\[
= \frac{1}{4\pi} \int_{\Omega} \Delta_y \left( \frac{\alpha \mu(y)}{|y - x|} \right) \cdot Q^y \Delta_y v(y, t) \, dy
\]
\[
= \frac{1}{4\pi} \int_{\Omega} Q^y \Delta_y \left( \frac{\alpha \mu(y)}{|y - x|} \right) \cdot \Delta_y v(y, t) \, dy
\]
\[
= \frac{1}{4\pi} \int_{\Omega} \nabla_y \varphi(x,a)(y) \cdot \Delta_y v(y, t) \, dy
\]
\[
= \frac{1}{4\pi} \int_{\partial \Omega} \nabla_y \varphi(x,a)(y) \cdot \frac{\partial v}{\partial n}(y, t) \, dy + S
\]
\[
- \frac{1}{4\pi} \int_{\Omega} \varphi_{i,j}(y) v_{i,j}(y, t) \, dy
\]
\[
= \frac{1}{4\pi} \int_{\partial \Omega} \nabla_y \varphi(x,a)(y) \cdot \frac{\partial v}{\partial n}(y, t) \, dy + S
\]
\[
+ \frac{1}{4\pi} \int_{\Omega} \nabla_y \Delta_y \varphi(x,a)(y) \cdot v(y, t) \, dy.
\]
This proves the statement about \( \nabla p^{II} \). The same statement about the space derivatives of \( \nabla p^{II} \) can be obtained analogously.

The conclusions of Lemma 2 and Lemma 3 imply that if at least one of the conditions (i), (i)' is fulfilled and \( \partial v / \partial n \in L^\beta(t_1, t_2; L^1(\partial \Omega)^3) \) for some \( \beta \geq 2 \) then \( \nabla p \) has all space derivatives in \( L^\beta(t_1 + \zeta, t_2 - \zeta; L^\infty(\Omega_2)^3) \). Using also Lemma 1 and equation (1), one can obtain the same statement about \( \partial v / \partial t \).
Thus, conditions (ii) or (ii)', Lemma 1 (used with \( \Omega_1 = U^\ast \) and \( \Omega_2 = U^\ast_{(2r + \varrho)/3} \)), the assumption that \( \partial \mathbf{v}/\partial \mathbf{n} \in L^\beta(t_1, t_2; L^1(\partial \Omega)^3) \) for some \( \beta \geq 2 \) and inequality (5) imply that the function \( \mathbf{h} \) has all space derivatives in \( L^\beta(t_1 + \zeta, t_2 - \zeta; L^\infty(\Omega)^3) \).

We shall further assume that (ii) or (ii)' holds. At the beginning, we do not have sufficient information on the integrability of \( \partial \mathbf{v}/\partial \mathbf{n} \) on \( \partial \Omega \times (t_1, t_2) \) and we can only derive by means of Lemma 1 that \( \mathbf{h} \) has all space derivatives in \( L^\alpha(t_1 + \zeta, t_2 - \zeta; L^\infty(\Omega)^3) \) for each \( \alpha \in [1, 2) \). However, this enables us to prove a higher smoothness of \( \mathbf{u} \) in \( \Omega \times (t_1 + \zeta, t_2 - \zeta) \) (Lemma 4). It implies certain integrability of \( \partial \mathbf{v}/\partial \mathbf{n} \) on \( \partial \Omega \times (t_1 + \zeta, t_2 - \zeta) \) (see estimate (13) which further makes it possible (by means of Lemmas 1, 2 and 3) to improve the information on function \( \mathbf{h} \), etc. This procedure will be repeated several times.

In the sequel, \( c \) will denote a generic constant, i.e. a constant whose value may change from line to line. It will depend on the function \( \mathbf{u} \), but it will be always independent of time.

**Lemma 4.** Let condition (ii) or condition (ii)' be satisfied and let \( \zeta > 0 \) be such a number that \( t_1 + \zeta < t_2 - \zeta \). Then \( A^{1/2} \mathbf{u} \in L^\infty(t_1 + \zeta, t_2 - \zeta; L^2(\Omega)^3) \) and \( A \mathbf{u} \in L^2(t_1 + \zeta, t_2 - \zeta; L^2(\Omega)^3) \).

**Proof.** Assume that e.g. condition (ii) holds. (The case of (ii)' could be treated analogously.) Suppose that \( t \in (t_1 + \zeta, t_2 - \zeta) \) - \( \Gamma \) (\( \Gamma \) is the set from the proof of Lemma 2.) If we multiply equation (6) by \( A \mathbf{u} \) and integrate over \( \Omega \), we obtain

\[
(11) \quad \frac{d}{dt} \frac{1}{2} \int_\Omega |A^{1/2} \mathbf{u}|^2 \, dx + \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot A \mathbf{u} \, dx + \int_\Omega |A \mathbf{u}|^2 \, dx = \int_\Omega \mathbf{h} \cdot A \mathbf{u} \, dx
\]

where

\[
\left| \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot A \mathbf{u} \, dx \right| \leq \frac{1}{8} \int_\Omega |A \mathbf{u}|^2 \, dx + c \int_\Omega |\mathbf{u}|^2 |\nabla \mathbf{u}|^2 \, dx \\
\leq \frac{1}{8} \int_\Omega |A \mathbf{u}|^2 \, dx + c \left( \int_\Omega |\mathbf{u}|^b \, dx \right)^{2/b} \left( \int_\Omega |\nabla \mathbf{u}|^2 \, dx \right)^{1/b} \left( \int_\Omega |\nabla \mathbf{u}|^6 \, dx \right)^{1/6} \\
\leq \frac{1}{8} \|A \mathbf{u}\|_2^2 + \delta \left( \int_\Omega |\nabla \mathbf{u}|^6 \, dx \right)^{1/3} + c(\delta) \left( \int_\Omega |\mathbf{u}|^b \, dx \right)^{a/b} \left( \int_\Omega |\nabla \mathbf{u}|^2 \, dx \right)^{3/2} \\
\leq \frac{1}{4} \|A \mathbf{u}\|_2^2 + c \left( \int_\Omega |\mathbf{u}|^b \, dx \right)^{a/b} \|A^{1/2} \mathbf{u}\|_2^2.
\]

(\( \delta \) is an appropriate positive number.) Let \( 0 \leq s < 1/4 \). Then \( D(A^s) = W^{2s,2}(\Omega)^3 \cap L^2(\Omega)^3 \) (see Sec. 1). Thus, \( P_s \mathbf{h}(\cdot, t) \in D(A^s) \). Let us further choose \( \gamma \in (0, 1) \) and
\( q \geq 2 \) so that \( 2 - \gamma \leq q \) and \( 3\gamma/4 \leq s \). Then \( 2q(1-\gamma)/(q-\gamma) \leq q \) and

\[
\left| \int_{\Omega} h \cdot Au \, dx \right| = \left| \int_{\Omega} A^s P_\sigma h \cdot A^{1-s} u \, dx \right| \leq \int_{\Omega} \left| A^s P_\sigma h \right|^{\gamma} \left| A^s P_\sigma h \right|^{1-\gamma} |A^{1-s} u| \, dx \\
\leq \left\| A^s P_\sigma h \right\|_q^\gamma \left( \int_{\Omega} |A^s P_\sigma h|^{2q(1-\gamma)/(q-\gamma)} \, dx + \int_{\Omega} |A^{1-s} u|^{\frac{2q}{3(\gamma-1)}} \, dx \right)^{q/(q-\gamma)}
\leq c \left\| h \right\|_{2s,q}^{2\gamma} \left\| A^s P_\sigma h \right\|_{2q(1-\gamma)/(q-\gamma)} + c \left\| h \right\|_{2s,q}^{\gamma} \left\| A^{1-s} u \right\|_{3\gamma/2q,2}^2
\leq c \left\| h \right\|_{2s,q}^{2-\gamma} + c \left\| h \right\|_{2s,q}^{\gamma} \left\| A^{1-s} u \right\|_{3\gamma/2q,2}^2
\leq c \left\| h \right\|_{2s,q}^{2-\gamma} + c \left\| h \right\|_{2s,q}^{\gamma} \left\| A^{1/2} u \right\|_{2}^{4s-3\gamma/4q} \left\| A u \right\|_{2}^{2-4s+3\gamma/q}
\leq c \left\| h \right\|_{2s,q}^{2-\gamma} + \frac{1}{4} \left\| A u \right\|_{2}^2 + c \left\| h \right\|_{2s,q}^{2\gamma q/(4s-3\gamma)} \left\| A^{1/2} u \right\|_{2}^2.
\]

Substituting this to (11), we have

\[
(12) \quad \frac{d}{dt} \left\| A^{1/2} u \right\|_{2}^2 + \left\| A u \right\|_{2}^2 \leq c(\left\| u \right\|_{b}^{a} + \left\| h \right\|_{2s,q}^{2\gamma q/(4s-3\gamma)}) \left\| A^{1/2} u \right\|_{2}^2 + c \left\| h \right\|_{2s,q}^{2-\gamma}.
\]

\( \left\| u \right\|_{b}^{a} \) is, due to condition (ii), an integrable function of \( t \) on \((t_1, t_2)\). We can choose \( \gamma \in (0, 1) \) so small and \( q > 2 \) so large that \((1+3\gamma)/q < 4s \). Then \( 2\gamma q/(4s-3\gamma) < 2 \) and therefore \( \left\| h \right\|_{2s,q}^{2\gamma q/(4s-3\gamma)} \) and \( \left\| h \right\|_{2s,q}^{2-\gamma} \) are integrable functions of \( t \) on \([t_1 + \zeta, t_2 - \zeta] \).

The number \( \zeta \) can be chosen not only arbitrarily small, but also such that \( t_1 + \zeta \notin \Gamma \), i.e. \( \left\| A^{1/2} u(\cdot, t_1 + \zeta) \right\|_2 < +\infty \).

Recall that inequality (12) holds for \( t \in (t_1 + \zeta, t_2 - \zeta) \). It implies that \( A^{1/2} u \) and \( A u \) satisfy the statement of the lemma if \( \left\| A^{1/2} u \right\|_2 \) is a left-lower and right-upper semi-continuous function of \( t \) at instants of time \( t \in \Gamma \). (Or in other words, unless \( \left\| A^{1/2} u \right\|_2 \) has jumps up at the time instants \( t \in \Gamma \).) This would be an easy consequence of classical results about the Navier-Stokes equations (see e.g. [3] or [7]) if \( h \), in addition to its space regularity, were at least square integrable in time. However, we actually know that the function \( h \) is only integrable in time with an arbitrary exponent \( \alpha \) \( \in [1, 2] \). Nevertheless, we can exclude the jumps up by means of the following argument: Let \( t' \in (t_1 + \zeta, t_2 - \zeta) \). We can choose \( t'_0 < t' \) arbitrarily close to \( t' \) and construct a local in time strong solution \( u' \) to the problem (6)–(8) on a time interval \((t'_0, t'_0 + T')\) overlapping \((t'_0, t']\), such that \( u'(t'_0) = u(t'_0) \). The existence of a local in time strong solution is well known—see e.g. [3] or [7] for details. In fact, we only need \( u' \) to satisfy the energy inequality and the norm \( \left\| \nabla u' \right\|_2 \) to have no jumps up and such a solution can be constructed even if \( h \) is integrable in time only with an exponent strictly less than two, but arbitrarily close to two. Since \( u \) satisfies the Prodi-Serrin integrability condition, \( u \) coincides with \( u' \) on the interval \((t'_0, t'_0 + T')\) and therefore its norm \( \left\| A^{1/2} u \right\|_2 \) has no jump up at the time instant \( t' \). \( \square \)
The theorem on traces now implies that
\[
\left( \int_{\partial \Omega} |\nabla u| \, dS \right)^4 \leq c\|u\|^{3/2}_{3/2,2} \leq c\|A^{3/4}u\|_2^4 + c \leq c\|A^{1/2}u\|_2^2 \|Au\|_2^2 + c
\]
\[
\leq c\|Au\|_2^2 + c.
\]
Since the right hand side is an integrable function of time on \((t_1 + \zeta, t_2 - \zeta)\) and \(u\) coincides with \(u\) on \(\partial \Omega \times (t_1, t_2)\), we also have \(\partial v/\partial n \in L^1(t_1 + \zeta, t_2 - \zeta; L^1(\partial \Omega)^3)\). Due to Lemma 2 and Lemma 3, \(\nabla p\) and \(\partial v/\partial t\) have all space derivatives in \(L^4(t_1 + \zeta, t_2 - \zeta; L^\infty(\Omega)^3)\) (where \(\Omega_2 = U_{(2r+\varepsilon)/3} - U_{\varepsilon/2}\)). Hence \(h\) and all its space derivatives belong to \(L^4(t_1 + \zeta, t_2 - \zeta; L^\infty(\Omega)^3)\).

**Lemma 5.** Let condition (ii) or condition (ii)' be fulfilled, \(0 < \varepsilon \leq 1\) and \(t_1 + \zeta < t_2 - \zeta\). Then \(A^{1-\varepsilon}u \in L^\infty(t_1 + \zeta, t_2 - \zeta; L^2(\Omega)^3)\).

**Proof.** We can assume without loss of generality that \(\zeta\) is chosen such that \(t_1 + \zeta \leq 0\), i.e. \(\|Au(\cdot, t_1 + \zeta)\|_2 < +\infty\). Let \(t \in (t_1 + \zeta, t_2 - \zeta)\). We will denote \(t_0 = t_1 + \zeta\) for simplicity. We can obviously deal only with \(\varepsilon \in (0, \frac{1}{2})\). Using the integral representation of \(u(\cdot, t)\) by means of the semigroup \(e^{At}\), we have
\[
A^{1-\varepsilon}u(\cdot, t) = A^{1-\varepsilon}e^{A(t-t_0)}u(\cdot, t_0) + \int_{t_0}^t A^{1-\varepsilon}e^{A(t-\tau)}P_\sigma h(\cdot, \tau) \, d\tau
\]
\[
- \int_{t_0}^t A^{1-\varepsilon}e^{A(t-\tau)}P_\sigma (u(\cdot, \tau) \cdot \nabla)u(\cdot, \tau) \, d\tau.
\]
Let us choose a number \(\xi \in [0, \frac{1}{4}]\) such that \(\varepsilon + \xi > \frac{1}{4}\). Then \(4(1 - \varepsilon - \xi) < 1\) and \(P_\sigma h(\cdot, \tau) \in D(A^\xi)\) for a.a. \(\tau \in (t_0, t)\). Thus, we obtain
\[
\left\| \int_{t_0}^t A^{1-\varepsilon}e^{A(t-\tau)}P_\sigma h(\cdot, \tau) \, d\tau \right\|_2
\]
\[
= \left\| \int_{t_0}^t A^{1-\varepsilon-\xi}e^{A(t-\tau)}A^\xi P_\sigma h(\cdot, \tau) \, d\tau \right\|_2 \leq c \int_{t_0}^t \|A^\xi P_\sigma h(\cdot, \tau)\|_2 (t - \tau)^{1-\varepsilon-\xi} \, d\tau
\]
\[
\leq c \left( \int_{t_0}^t \frac{d\tau}{(t - \tau)^{4(1-\varepsilon-\xi)/3}} \right)^{3/4} \left( \int_{t_0}^t \|h(\cdot, \tau)\|_{\xi,2}^4 \, d\tau \right)^{1/4} \leq c.
\]
Suppose that \(\varepsilon = \frac{1}{4} + \kappa\) where \(\kappa \in (0, \frac{1}{4}]\) for a while. (Hence \(4(1 - \varepsilon)/3 < 1\).) Using the results of Lemma 4, we can derive that
\[
\left\| \int_{t_0}^t A^{1-\varepsilon}e^{A(t-\tau)}P_\sigma (u(\cdot, \tau) \cdot \nabla)u(\cdot, \tau) \, d\tau \right\|_2
\]
\[
\leq \int_{t_0}^t \frac{c}{(t - \tau)^{1-\varepsilon}} \|Au(\cdot, \tau)\|_2^{1/2} \, d\tau
\]
\[
\leq c \left( \int_{t_0}^t \frac{d\tau}{(t - \tau)^{4(1-\varepsilon)/3}} \right)^{3/4} \left( \int_{t_0}^t \|Au(\cdot, \tau)\|_2^2 \, d\tau \right)^{1/4} \leq c.
\]
Inequalities (15) and (16), together with Lemma 4 and identity (14), imply that 
\[ A^{1-\varepsilon}u = A^{3/4-\kappa}u \in L^\infty(t_1 + \zeta, t_2 - \zeta; L^2(\Omega)^3). \]

Let \( \varepsilon \in (0, \frac{1}{2}) \) now. Let us choose \( \kappa > 0 \) so small that \( 1 - \varepsilon < (1 + 2\kappa)/(1 + 4\kappa) \). Using the above information on \( A^{3/4-\kappa}u \), we can replace estimates (16) by

\[
\begin{align*}
(17) \quad \left\| \int_{t_0}^{t} A^{1-\varepsilon} e^{A(t-\tau)} P_\sigma(u(\cdot, \tau) \cdot \nabla) u(\cdot, \tau) \, d\tau \right\|_2 \\
&\leq c \int_{t_0}^{t} \frac{1}{(t-\tau)^{1-\varepsilon}} \| A^{3/4} u(\cdot, \tau) \|_2 \, d\tau \\
&\leq c \int_{t_0}^{t} \frac{1}{(t-\tau)^{1-\varepsilon}} \| A^{3/4-\kappa} u(\cdot, \tau) \|^{1/(1+4\kappa)} \| A u(\cdot, \tau) \|^{4\kappa/(1+4\kappa)} \, d\tau \\
&\leq c \left( \int_{t_0}^{t} \frac{d\tau}{(t-\tau)^{1-\varepsilon}} \right)^{\frac{1+2\kappa}{1+4\kappa}} \left( \int_{t_0}^{t} \| A u(\cdot, \tau) \|_2^2 \, d\tau \right)^{\frac{2\kappa}{1+4\kappa}} 
&\leq c.
\end{align*}
\]

The statement of the lemma follows from Lemma 4, (14), (15) and (17).

We can now proceed similarly as after the proof of Lemma 4: We have

\[
(18) \quad \int_{\partial \Omega} |\nabla u| \, dS \leq c \| u \|_{3/2, 2} \leq c \| A^{3/4} u \|_2 + c.
\]

This estimate and Lemma 5 imply that \( \partial v / \partial n \in L^\infty(t_1 + \zeta, t_2 - \zeta; L^1(\partial \Omega)^3) \). Thus, \( \nabla p \) and \( \partial v / \partial t \) have all space derivatives in \( L^\infty((U^*_d + \rho^*)/3 - \rho^*/2) \times (t_1 + \zeta, t_2 - \zeta))^3 \) and consequently, \( h \) and all its space derivatives belong to \( L^\infty(\Omega \times (t_1 + \zeta, t_2 - \zeta))^3 \).

**Lemma 6.** Let \( g \in L^\infty(t_1 + \zeta, t_2 - \zeta; W^{2-\xi,2}(\Omega)^3) \) for some \( \xi \in [0, \frac{1}{2}) \). Then the operator \( B_t w = (g(\cdot, t) \cdot \nabla) w \) is for a.a. \( t \in (t_1 + \zeta, t_2 - \zeta) \) and for \( 0 \leq s \leq 1 \) a continuous linear operator from \( W^{s+1,2}(\Omega)^3 \) into \( W^{s,2}(\Omega)^3 \) and the estimate

\[
(19) \quad \| B_t w \|_{s, 2} \leq c \| w \|_{s+1, 2}
\]

holds uniformly for a.a. \( t \in (t_1 + \zeta, t_2 - \zeta) \).

**Proof.** It can be verified that

\[
\begin{align*}
\| B_t w \|_2 \leq \| g(\cdot, t) \|_{2-\xi, 2} \| w \|_{1, 2} &\leq c \| w \|_{1, 2}, \\
\| B_t w \|_{1, 2} \leq c(\| g(\cdot, t) \|_{2-\xi, 2} + \| g(\cdot, t) \|_{1, 2}) \| w \|_{2, 2} &\leq c \| w \|_{2, 2}
\end{align*}
\]

uniformly for a.a. \( t \in [t_1 + \zeta, t_2 - \zeta] \). Hence \( B_t \) is a linear continuous operator from \( [W^{2,2}(\Omega)^3, W^{1,2}(\Omega)^3]_{1-s} \equiv W^{s+1,2}(\Omega)^3 \) into \( [W^{1,2}(\Omega)^3, L^2(\Omega)^3]_{1-s} \equiv W^{s,2}(\Omega)^3 \) and the norm of this operator can be estimated by a constant which is independent of \( t \) for a.a. \( t \in [t_1 + \zeta, t_2 - \zeta] \). (This can be deduced e.g. from [8], p. 27.)

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Lemma 5 implies that $u \in L^\infty(t_1 + \zeta, t_2 - \zeta; W^{2-\xi,2}(\Omega)^3)$ for each $\xi \in (0, \frac{1}{2})$. Hence we can use Lemma 6 with $g = u$ and $w = u(\cdot, t)$ and obtaining the estimate
\[(20) \quad \| (u(\cdot, t) \cdot \nabla) u(\cdot, t) \|_{s, 2} \leq c \| u(\cdot, t) \|_{s+1, 2} \leq c \| A^{(s+1)/2} u(\cdot, t) \|_2 \]
for a.a. $t \in [t_1 + \zeta, t_2 - \zeta]$. (Of course $c$ depends on $u$, but it does not matter because we work only with just one function $u$.)

Proof of Theorem 1. Put $\varepsilon = \delta/2$. We can assume without loss of generality that $t_1 + \zeta \notin \Gamma$, i.e. $\| A^{1+\varepsilon} u(\cdot, t_1 + \zeta) \|_2 < +\infty$. Let $t \in (t_1 + \zeta, t_2 - \zeta)$ and $t_0 = t_1 + \zeta$. Then
\[(21) \quad A^{1+\varepsilon} u(\cdot, t) = A^{1+\varepsilon} e^{A(t-t_0)} u(\cdot, t_0) + \int_{t_0}^{t} A^{1+\varepsilon} e^{A(t-\tau)} P_\tau h(\cdot, \tau) \, d\tau \]
\[- \int_{t_0}^{t} A^{1+\varepsilon} e^{A(t-\tau)} P_\tau (u(\cdot, \tau) \cdot \nabla) u(\cdot, \tau) \, d\tau.\]
Let us choose $\xi$ such that $\varepsilon < \xi < \frac{1}{4}$. Then $P_\tau h(\cdot, \tau) \in D(A^\xi)$ and $P_\tau (u(\cdot, \tau) \cdot \nabla) u(\cdot, \tau) \in D(A^\xi)$ for a.a. $\tau \in (t_0, t)$ and
\[
\left\| \int_{t_0}^{t} A^{1+\varepsilon} e^{A(t-\tau)} P_\tau (u(\cdot, \tau) \cdot \nabla) u(\cdot, \tau) \, d\tau \right\|_2 \\
= \left\| \int_{t_0}^{t} A^{1+\varepsilon-\xi} e^{A(t-\tau)} A^\xi P_\tau (u(\cdot, \tau) \cdot \nabla) u(\cdot, \tau) \, d\tau \right\|_2 \\
\leq \int_{t_0}^{t} \frac{c}{(t-\tau)^{1+\varepsilon-\xi}} \| (u(\cdot, \tau) \cdot \nabla) u(\cdot, \tau) \|_{2\xi, 2} \, d\tau \\
\leq \int_{t_0}^{t} \frac{c}{(t-\tau)^{1+\varepsilon-\xi}} \| A^{(2\xi+1)/2} u(\cdot, \tau) \|_2 \, d\tau \\
\leq \int_{t_0}^{t} \frac{c}{(t-\tau)^{1+\varepsilon-\xi}} \, d\tau \leq c, \\
\left\| \int_{t_0}^{t} A^{1+\varepsilon} e^{A(t-\tau)} P_\tau h(\cdot, \tau) \, d\tau \right\|_2 = \left\| \int_{t_0}^{t} A^{1+\varepsilon-\xi} e^{A(t-\tau)} A^\xi P_\tau h(\cdot, \tau) \, d\tau \right\|_2 \\
\leq \int_{t_0}^{t} \frac{c}{(t-\tau)^{1+\varepsilon-\xi}} \| A^\xi P_\tau h(\cdot, \tau) \|_2 \, d\tau \\
\leq \int_{t_0}^{t} \frac{c}{(t-\tau)^{1+\varepsilon-\xi}} \, d\tau \leq c.\]

The statement of Theorem 1 about $v$ now follows from these estimates, (21) and the relation between the solutions $u$ and $v$. The statements about $\partial v/\partial t$ and $\nabla p$ further follow from equation (6). \[\square\]
Proof of Theorem 2. Lemma 5, estimate (18) and the coincidence of $u$ and $v$ in the neighborhood of $\partial \Omega$ imply that $\partial u / \partial n \in L^\infty(t_1 + \zeta, t_2 - \zeta; L^1(\partial \Omega)^3)$. The statement of Theorem 2 is now an easy consequence of Lemma 2 and Lemma 3. □

References


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