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# FINITE ELEMENT APPROXIMATION OF A CONTACT VECTOR EIGENVALUE PROBLEM

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*Abstract.* We consider a nonstandard elliptic eigenvalue problem of second order on a two-component domain consisting of two intervals with a contact point. The interaction between the two domains is expressed through a coupling condition of nonlocal type, more specifically, in integral form. The problem under consideration is first stated in its variational form and next interpreted as a second-order differential eigenvalue problem. The aim is to set up a finite element method for this problem. The error analysis involved is shown to be affected by the nonlocal condition, which requires a suitable modification of the vector Lagrange interpolant on the overall finite element mesh. Nevertheless, we arrive at optimal error estimates. In the last section, an illustrative numerical example is given, which confirms the theoretical results.

*Keywords*: eigenvalue problem, nonlocal coupling condition, finite elements

*MSC 2000*: 65N30

## 1. Problem setting

### 1.1. Introduction.

In this paper we deal with a variational eigenvalue problem (EVP) on a two-component domain which consists of two intervals, touching each other at a contact point. The problem involves a nonlocal coupling condition between the components of the vector valued eigenfunction, defined on the respective contacting subdomains. The type of contact EVP considered here arises e.g. when setting up Fourier's expansion method for some transient boundary value problems modelling the heat exchange between two contacting rods, see e.g. [1].

Let $\Omega_1 = ]-1,1[$ and $\Omega_2 = ]0,1[$, as shown in Fig. 1. In $\Omega_1$, we will also consider a subinterval $D_\varepsilon = ]-\varepsilon,\varepsilon[$, where $\varepsilon \in ]0,1[$ is fixed. Next, we introduce the product space $H = L_2(\Omega_1) \times L_2(\Omega_2)$, endowed with its natural innerproduct

$$(u, v)_H = (u_1, v_1)_{L_2(\Omega_1)} + (u_2, v_2)_{L_2(\Omega_2)}$$

and the associated product norm $|\cdot|_H$, for vector-valued functions $u = [u_1, u_2]$ and $v = [v_1, v_2]$. In what follows, we will as well use the spaces $H^s(\Omega_1) \times H^s(\Omega_2)$, denoted as $\widehat{H^s(\Omega)}$, with their natural product norm $\|\cdot\|_{s,\Omega}$ and semi-norm $|\cdot|_{s,\Omega}$, $s \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Hence, the norm $|\cdot|_H$ may also be denoted as $\|\cdot\|_{0,\Omega}$.

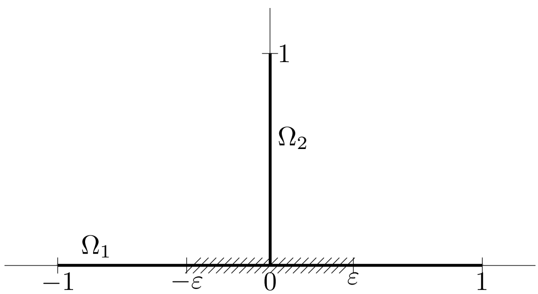


Figure 1. The subdomains $\Omega_1$ and $\Omega_2$.

To fix the ideas, we will consider the following model problem:

$$\text{(1)} \qquad \text{Find } [\lambda, u] \in \mathbb{R} \times V,\ \ u \neq 0\colon\ a(u, v) = \lambda(u, v)_H \ \ \forall v \in V,$$

where the space of trial and test functions is

$$\text{(2)} \qquad V = \Big\{ v = [v_1, v_2] \in V_1 \times V_2 \mid v_2(0) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} v_1 \, \mathrm{d}x \Big\}, \quad \varepsilon \in\, ]0, 1[ \ \text{ fixed},$$

with

$$V_1 = H_0^1(\Omega_1), \quad V_2 = \{ w \in H^1(\Omega_2) \mid w(1) = 0 \},$$

and the bilinear form is given by

$$\text{(3)} \qquad a(u, v) = \int_{-1}^{1} (p_1 u_1' v_1' + q_1 u_1 v_1) \, \mathrm{d}x + \int_0^1 (p_2 u_2' v_2' + q_2 u_2 v_2) \, \mathrm{d}x$$

with

$$\text{(4)} \qquad p_i, q_i \in L_\infty(\Omega_i), \quad \exists \tilde{p}_i > 0\colon\ p_i > \tilde{p}_i \ \text{ and } \ q_i \geqslant 0 \ \text{ a.e. in } \ \Omega_i, \quad i = 1, 2.$$

An outline of the paper is as follows.

In the next subsection the variational EVP is shown to fit into the general framework of abstract elliptic EVPs for symmetric, bounded and (strongly) coercive bilinear forms in Hilbert spaces, studied e.g. in [2], from which we directly infer the existence of exact eigenpairs, showing some suitable properties. In §1.3 we derive the underlying differential EVP for the eigenvalue $\lambda$ and the components of the corresponding vector valued eigenfunction $u$, being formally equivalent to the variational EVP above.

In §2, we pass to a proper FE-approximation, introducing first a suitable approximation space $V_h \subset V$. Due to the nonlocal coupling condition, entering the space $V$ of trial and test functions, an essential difficulty arises: the vector piecewise Lagrange interpolant of a test function with smooth components will, in general, not belong to $V_h$. Hence the classical interpolation theory, underlying the usual convergence and error analysis of the FEMs, is no longer applicable. To overcome this problem, a suitable "modified" or "imperfect" interpolant is introduced by the adaptation of one single nodal value; by construction, this imperfect interpolant will belong to the space $V_h$. The corresponding error estimates are established, from which it follows that the space $V_h$ shows a suitable approximation property which, together with a density result of $V$, implies that the error estimates holding for the eigenpairs of standard EVPs remain valid in the present case.

Finally, in §3, an illustrative numerical example is given, the exact eigenpairs of which can be found. The results obtained confirm our theoretical framework.

**1.2. Existence of eigenpairs.**

The space $V$, endowed with the product norm $\|\cdot\|_{1,\Omega}$, is directly seen to be a closed subspace of $V_1 \times V_2$ and hence is itself a Hilbert space. Evidently, $V$ is also compactly embedded in the product Hilbert space $H$. In addition, we show

**Lemma 1.** *The space $V$ is dense in $H$.*

Proof. Take $v = [v_1, v_2] \in H$ and $\zeta > 0$ arbitrarily. By the density of $V_i$ in $L_2(\Omega_i)$, $i = 1, 2$, there exists a couple $v^* = [v_1^*, v_2^*] \in V_1 \times V_2$, such that

$$\|v_i - v_i^*\|_{0,\Omega_i} < \frac{\zeta}{2\sqrt{2}} \quad \text{and hence} \quad \|v - v^*\|_{0,\Omega} < \frac{\zeta}{2}.$$

Let

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} v_1^* \,\mathrm{d}x = a \quad \text{and} \quad v_2^*(0) = b$$

and suppose that $a \neq b$. Assume first that $b \neq 0$. As $H_0^1(\Omega_2)$ is dense in $L_2(\Omega_2)$, there exists a function $w_2 \in H_0^1(\Omega_2)$, such that

$$\|w_2 - v_2^*\|_{0,\Omega_2} < \frac{\zeta}{2(1 - \frac{a}{b})}.$$

For the function

$$v_2^{**} = \frac{a}{b} v_2^* + \Big(1 - \frac{a}{b}\Big) w_2$$

we have that $v_2^{**} \in H^1(\Omega_2)$, $v_2^{**}(0) = a$ by construction, and $\|v_2^{**} - v_2^*\|_{0,\Omega_2} < \frac{\zeta}{2}$. Hence, putting $v^{**} = [v_1^*, v_2^{**}]$, we conclude that $v^{**} \in V$ and $\|v - v^{**}\|_{0,\Omega} < \zeta$.

Next, let $b = 0$. As $v_2^* \in H^1(\Omega_2)$, we have, recalling the continuous embedding $H^1(\Omega_2) \hookrightarrow C^0(\overline{\Omega_2})$, that $v_2^*$ is continuous on $\Omega_2$. Hence, taking $0 < \delta < a$, there exists a $p_0 > 0$ such that

$$0 \leqslant x < p_0 \Longrightarrow |v_1^*(x) - v_1^*(0)| = |v_1^*(x)| < \delta.$$

Let $p < \min\big(\frac{\zeta^2}{16a^2}, p_0\big)$ and consider the function

$$v_2^{**} = \begin{cases} v_2^* & \text{in } \ ]p, 1], \\ \dfrac{v_2^*(p) - a}{p} x + a & \text{in } \ [0, p]. \end{cases}$$

We have that $v_2^{**} \in H^1(\Omega_2)$, $v_2^{**}(0) = a$ by construction, and

$$\begin{aligned} \|v_2^{**} - v_2^*\|_{0,\Omega_2}^2 &= \int_0^p (v_1^{**} - v_1^*)^2 \,\mathrm{d}x \\ &\leqslant \int_0^p \Big(\frac{v_1^*(p) - a}{p} x + a + \delta\Big)^2 \,\mathrm{d}x \leqslant 4pa^2 < \frac{\zeta^2}{4}. \end{aligned}$$

Hence, also in this case, we find $v^{**} = [v_1^*, v_2^{**}] \in V$ with $\|v - v^{**}\|_{0,\Omega} < \zeta$. □

Finally, the symmetric bilinear form $a\colon V \times V \to \mathbb{R}$, defined by (3), is easily found to be bounded and strongly coercive on account of the assumptions on the coefficient functions $p_1$, $q_1$, $p_2$ and $q_2$.

The above mentioned properties of the space $V$ and of the bilinear form $a(\cdot,\cdot)$ ensure that the EVP (1) fits into the framework of abstract elliptic EVPs in Hilbert spaces, studied e.g. in [2, §6.2]. Thus, one has

**Theorem 1.1.** *The EVP* (1) *has an infinite sequence of eigenvalues with no finite accumulation point. All eigenvalues are strictly positive and have finite multiplicity; we arrange them as*

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \to +\infty,$$

*where each eigenvalue occurs as many times as given by its multiplicity. Amongst the corresponding eigenfunctions a Hilbert basis* $(u_l)_{l=1}^\infty$ *of* $V$*, orthonormal w.r.t.* $a(\cdot,\cdot)$*, can be chosen; furthermore, the set* $(\sqrt{\lambda_l} u_l)_{l=1}^\infty$ *then constitutes a Hilbert orthonormal basis of* $H$*.*

### 1.3. Interpretation in differential form.

The aim of this section is to formulate a differential EVP which is, at least formally, equivalent to the model problem in variational form stated above.

To this end, consider the 2nd order EVP for $[\lambda, u_1, u_2] \in \mathbb{R} \times \widehat{H^2}(\Omega)$, consisting of the differential equations (DEs)

$$
-\frac{\mathrm{d}}{\mathrm{d}x}\left[p_1 \frac{\mathrm{d}u_1}{\mathrm{d}x}\right] + q_1 u_1 = \lambda u_1 + \frac{1}{2\varepsilon} p_2(0) u_2'(0) \chi_{]-\varepsilon,\varepsilon[} \quad \text{in } \Omega_1, \tag{5}
$$

$$
-\frac{\mathrm{d}}{\mathrm{d}x}\left[p_2 \frac{\mathrm{d}u_2}{\mathrm{d}x}\right] + q_2 u_2 = \lambda u_2 \quad \text{in } \Omega_2 \tag{6}
$$

along with the classical Dirichlet conditions

$$
u_1(-1) = u_1(1) = 0, \tag{7.a}
$$

$$
u_2(1) = 0, \tag{7.b}
$$

and accompanied with the nonlocal coupling condition

$$
u_2(0) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_1 \,\mathrm{d}x. \tag{8}
$$

Here, we assume that $p_1 \in H^1(\Omega_1)$ and $p_2 \in H^1(\Omega_2)$. Apart from that, the data fulfil the remaining conditions encompassed in (4). In (5), $\chi_{]-\varepsilon,\varepsilon[}$ denotes the characteristic function of $]-\varepsilon, \varepsilon[$.

In order to recover the DEs (5)-(6) from the EVP (1)—at least formally, it means when $u_1$, $u_2$ and the data are assumed to be sufficiently smooth, we will need the following lemma.

**Lemma 1.2.** *Let $I$ be an open interval in $\mathbb{R}$ and let*

$$
\mathcal{D}^0(I) = \left\{ \varphi \in C^\infty(I) \mid \int_I \varphi \,\mathrm{d}x = 0 \right\}.
$$

*Then, for an arbitrary function $f \in L_2(I)$, the following statements are equivalent:*

(i) $\int_I f\varphi \,\mathrm{d}x = 0$ *for all* $\varphi \in \mathcal{D}^0(I)$,

(ii) $f$ *is constant in* $I$.

Actually, we have

**Theorem 1.2.** *The EVPs* (1) *and* (5)–(8) *are* formally *equivalent.*

P r o o f. In order to recover from (1) the differential equation (5) on $\Omega_1 \backslash D_\varepsilon$, we take a test function $v$ to be $[v_1, 0]$, where

$$v_1 = \begin{cases} \varphi_1 \in \mathcal{D}(\Omega_1 \backslash D_\varepsilon) & \text{on } \Omega_1 \backslash D_\varepsilon, \\ 0 & \text{on } D_\varepsilon, \end{cases}$$

apply partial integration and invoke the density of $\mathcal{D}(\Omega_1 \backslash D_\varepsilon)$ in $L_2(\Omega_1 \backslash D_\varepsilon)$. Analogously, taking $v = [0, \varphi_2]$ with $\varphi_2 \in \mathcal{D}(\Omega_2)$, we arrive at the differential equation (6) on $\Omega_2$.

Next, to recover the differential equation (5) on $D_\varepsilon$, we again take $v = [v_1, 0]$, however now with

$$v_1 = \begin{cases} 0 & \text{on } \Omega_1 \backslash D_\varepsilon, \\ \tilde{\varphi}_1 \in \mathcal{D}(D_\varepsilon) & \text{on } D_\varepsilon, \end{cases}$$

with

$$\int_{D_\varepsilon} \tilde{\varphi}_1 \, \mathrm{d}x = 0.$$

Invoking the previous lemma, we infer that

$$-\frac{\mathrm{d}}{\mathrm{dx}}\left[p_1 \frac{\mathrm{d}u_1}{\mathrm{d}x}\right] + q_1 u_1 = \lambda u_1 + C \quad \text{on } D_\varepsilon,$$

where it remains to determine the constant $C$. To this end, we reconsider the variational problem (1) for an arbitrary test function in $V$. We apply partial integration to both integrals and invoke the boundary conditions incorporated in $V$, as well as the differential equations which we already have recovered. We arrive at

$$C \int_{D_\varepsilon} v_1 \, \mathrm{d}x - \frac{1}{2\varepsilon} p_2(0) u_2'(0) \int_{D_\varepsilon} v_1 \, \mathrm{d}x = 0,$$

from where it directly follows that $C = \frac{1}{2\varepsilon} p_2(0) u_2'(0)$. □

## 2. Finite element approximations

### 2.1. The approximation space $V_h$.

For simplicity we consider an FE-mesh with first degree polynomials on both $\Omega_1$ and $\Omega_2$.

Let $(x_i)_{i=0}^{2n+2l}$ denote the set of nodes in $\overline{\Omega_1}$, where $x_0 = -1$, $x_n = -\varepsilon$, $x_{n+l} = 0$, $x_{n+2l} = \varepsilon$ and $x_{2n+2l} = 1$. We introduce spaces

$$\begin{aligned} X_{1h_1} &= \bigl\{v_1 \in C^0(\overline{\Omega_1}) \mid v_1|_{[x_{i-1}, x_i]} \in P_1([x_{i-1}, x_i]),\ i = 1, \ldots, 2n + 2l\bigr\}, \\ V_{1h_1} &= \{v_1 \in X_{1h_1} \mid v_1(-1) = v_1(1) = 0\}, \end{aligned}$$

where $h_1$ is the corresponding mesh parameter. The canonical basis of $X_{1h_1}$ is denoted by $(\varphi_i)_{i=0}^{2n+2l}$.

Similarly, $(z_j)_{j=0}^{m}$ denotes the set of nodes in $\overline{\Omega_2}$, where $z_0 = 0$ and $z_m = 1$, and we consider

$$\begin{aligned} X_{2h_2} &= \bigl\{v_2 \in C^0(\overline{\Omega_2}) \mid v_2|_{[z_{j-1},z_j]} \in P_1([z_{j-1},z_j]),\ \ j = 1,\dots,m\bigr\},\\ V_{2h_2} &= \{v_2 \in X_{2h_2} \mid v_2(1) = 0\}, \end{aligned}$$

with the mesh parameter $h_2$. Here, the canonical basis of $X_{2h_2}$ is denoted by $(\psi_j)_{j=0}^{m}$.

The family of partitions of $\Omega_i$, $i = 1,2$, is assumed to be quasi-uniform in the sense of [3]. Furthermore, we assume that

$$\frac{h_1}{h_2} < \beta,$$

with $\beta > 0$ independent of the mesh. Finally, we put $h = \max(h_1, h_2)$ and consider the space

$$V_h = \left\{ v_h = [v_{1h}, v_{2h}] \in V_{1h_1} \times V_{2h_2} \mid v_{2h}(0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} v_{1h}\,\mathrm{d}x \right\} \subset V. \tag{9}$$

The corresponding approximate EVP of (1) reads

$$\text{Find } [\lambda_h, u_h] \in \mathbb{R} \times V_h,\ \ u_h \neq 0\colon\ a(u_h, v) = \lambda_h (u_h, v)_H \ \ \forall v \in V_h. \tag{10}$$

For this discrete EVP a counterpart of Theorem 1.2 holds.

**Theorem 2.1.** *Denote the dimension of the finite element space $V_h$ by $I \equiv I(h)$. Then the EVP* (10) *has $I$ positive eigenvalues, arranged as*

$$0 < \lambda_{1h} \leqslant \lambda_{2h} \leqslant \ldots \leqslant \lambda_{Ih}.$$

*Moreover, amongst the corresponding approximate eigenfunctions, a basis of $V_h$ orthonormal w.r.t. $a(\cdot,\cdot)$ can be chosen. We denote it as $(u_{mh})_{m=1}^{I}$.*

**2.2. Imperfect interpolation.**

Let $v \in V$ $\bigl(\hookrightarrow C^0(\overline{\Omega_1}) \times C^0(\overline{\Omega_2})\bigr)$. Its vector piecewise Lagrange interpolant on the mesh is $\Pi_h v \equiv [\Pi_{1h_1} v_1, \Pi_{2h_2} v_2] \in V_{1h} \times V_{2h}$ with

$$\begin{aligned} (\Pi_{1h_1} v_1)(x_i) &= v_1(x_i), \quad i = 0,\dots,2n+2l,\\ (\Pi_{2h_2} v_2)(z_j) &= v_2(z_j), \quad j = 0,\dots,m. \end{aligned} \tag{11}$$

In general, by the nonlocal coupling condition incorporated in $V_h$ one has that $\Pi_h v \notin V_h$. Consequently, standard interpolation theory, usually underlying the error analysis of the FEMs, is no longer applicable. To overcome this difficulty, we introduce the following imperfect interpolant.

**Definition 2.1.** Let $v = [v_1, v_2] \in V$. We define its imperfect piecewise Lagrange interpolant on the mesh as $\widetilde{\Pi}_h v \equiv \big[\Pi_{1h_1} v_1, \widetilde{\Pi_{2h_2}} v_2\big] \in V_{1h} \times V_{2h}$ with

$$(12) \qquad \big(\widetilde{\Pi_{2h_2}} v_2\big)(z_j) = v_2(z_j), \;\; j = 1, \ldots, m, \quad \big(\widetilde{\Pi_{2h_2}} v_2\big)(z_0) = \sum_{i=n}^{n+2l} \alpha_i v_1(x_i),$$

where

$$\alpha_i = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \varphi_i \, \mathrm{d}x \quad i = n, \ldots, n+2l.$$

By the deliberately chosen value of $\widetilde{\Pi_{2h_2}} v_2$ at $z = 0$, one has

**Proposition 2.1.** $\widetilde{\Pi}_h v \in V_h$, $\forall v \in V \cap \widehat{H^2}(\Omega)$.

It remains to establish the error $v - \widetilde{\Pi}_h v$ committed in the imperfect interpolation. This is achieved in the following proposition.

**Proposition 2.2.** *There exists a constant $C > 0$, independent of $h$, such that*

$$(13) \qquad \big|v - \widetilde{\Pi}_h v\big|_{m,\Omega} \leqslant C h^{2-m} \|v\|_{2,\Omega}, \quad \forall v \in V \cap \widehat{H^2}(\Omega).$$

P r o o f. Take $v \in V \cap \widehat{H^2}(\Omega)$ arbitrarily. From the definitions (11)–(12) we get

$$\Pi_{2h_2} v_2 - \widetilde{\Pi_{2h_2}} v_2 = \big[(\Pi_{2h_2} v_2(0) - \widetilde{\Pi_{2h}} v_2(0)\big] \psi_0 \quad \text{in } \Omega_2,$$

from which we infer, taking into account the coupling condition between $v_1$ and $v_2$ and the definition of the imperfect interpolant,

$$\big|\Pi_{2h_2} v_2 - \widetilde{\Pi_{2h_2}} v_2\big|_{m,\Omega_2} \leqslant \frac{1}{2\varepsilon} \left[\int_{-\varepsilon}^{\varepsilon} |v_1 - \Pi_{1h_1} v_1| \, \mathrm{d}x\right] |\psi_0|_{m,\Omega_2}, \quad m = 0, 1.$$

Next, we need to derive upper bounds for the two factors in the right-hand side of this inequality. Invoking a classical error estimate for piecewise linear Lagrange interpolation, see [3], we obtain that

$$\int_{-\varepsilon}^{\varepsilon} |v_1 - \Pi_{1h_1} v_1| \, \mathrm{d}x \leqslant C(\varepsilon) \|v_1 - \Pi_{1h_1} v_1\|_{0,\Omega_1} \leqslant C \|v_1\|_{2,\Omega_1}.$$

Furthermore, by a straightforward calculation, we find

$$|\psi_0|_{m,\Omega_2} \leqslant C h^{\frac{1}{2} - m}, \quad m = 0, 1.$$

Combining these results, we arrive at

$$\left|\Pi_{2h_2}v_2 - \widetilde{\Pi_{2h_2}}v_2\right|_{m,\Omega_2} < Ch^{\frac{5}{2}-m}|v_1|_{2,\Omega_1}, \quad m = 0, 1.$$

Writing $v_2 - \widetilde{\Pi_{2h_2}}v_2 = (v_2 - \Pi_{2h_2}v_2) + \left(\Pi_{2h_2}v_2 - \widetilde{\Pi_{2h_2}}v_2\right)$ leads us to

$$\left|v_2 - \widetilde{\Pi_{2h_2}}v_2\right|_{m,\Omega_2} < Ch^{2-m}\|v\|_{2,\Omega}, \quad m = 0, 1,$$

from which the estimate (13) readily follows provided we again invoke classical interpolation estimates, see [3]. □

For the error analysis of the FEMs used, the estimate (13) is crucial. In particular, it underlies the following approximation property for the space $V_h$ relative to $V$.

**Proposition 2.3.** *Let the spaces $V_h$ and $V$ be given by* (9) *and* (2)*, respectively. Then we have*

$$\inf_{w\in V_h}\left\{|v-w|_{0,\Omega} + h|v-w|_{1,\Omega}\right\} \leqslant Ch^2\|v\|_{2,\Omega}, \quad \forall v = [v_1, v_2] \in V \cap \widehat{H^2}(\Omega), \tag{14}$$

*where the constant $C$ is independent of $h$.*

Moreover, on account of (14), the elliptic projection operator $P\colon V \to V_h$, given by

$$a(v - Pv, w) = 0, \quad \forall v \in V,\ \forall w \in V_h,$$

retains its classical properties, see e.g. [4].

Hence, there remains only one auxiliary result to be proved in order to be able to adapt the well-known method of [2] to the present case:

**Proposition 2.4.** *The space $\widehat{H^2}(\Omega) \cap V$ is dense in $V$.*

P r o o f. This result is shown in a constructive way, very similar to the proof of Lemma 1.1. □

**2.3. An error estimate.**

Leaning upon the above results, similarly as in [2, §6.5] we arrive at

**Theorem 2.2.** *Let the data satisfy the conditions* (4)*; let $([\lambda_m, u_m])_{m\geqslant 1}$ be the eigenpairs from Theorem* 1.2 *and let $([\lambda_{mh}, u_{mh}])_{m=1}^{I}$ be the corresponding eigenpairs from Theorem* 2.1*. First of all, we have*

$$\lim_{h\to 0} |\lambda_{mh} - \lambda_m| = 0.$$

*Moreover, if* $\operatorname{span}\{u_1, \ldots, u_m\} \subset \widehat{H^2}(\Omega)$*, one has*

$$|\lambda_{mh} - \lambda_m| \leqslant Ch^2, \quad m = 1, \ldots, I.$$

*In case* $\lambda_m$ *is a simple eigenvalue of* (1)*, one gets*

$$\lim_{h \to 0} \|u_{mh} - u_m\|_{1,\Omega} = 0$$

*and, again if* $\operatorname{span}\{u_1, \ldots, u_m\} \subset \widehat{H^2}(\Omega)$*, also the estimate*

$$\|u_{mh} - u_m\|_{1,\Omega} \leqslant Ch.$$

*Furthermore, if the* BVP *associated with* (1) *is* regular *in the sense of* [3]*, one has the order of convergence* $h^2$ *in the* $\|\cdot\|_{0,\Omega}$*-norm.*

For error estimates in the case of a multiple eigenvalue, also the operator method of [4] can be adapted to the present situation.

Let us mention that, in practice, the forms $a(\cdot,\cdot)$ and $(\cdot,\cdot)_H$ will be approximated by suitable numerical quadrature formulas. This will not affect the orders of convergence mentioned in the above theorem, provided that the quadrature used (e.g. Gauss-Legendre for $a(\cdot,\cdot)$ and Lobatto for $(\cdot,\cdot)_H$) is sufficiently precise and that the data of the problem are sufficiently regular.

**2.4. Computational aspects.**

The main task is to identify a suitable basis (of vector valued functions) for the approximation space $V_h$, (9). Recall that the canonical basis of $X_{1h_1}$ is denoted by $(\varphi_i)_{i=0}^{2n+2l}$, while the canonical basis of $X_{2h_2}$ is $(\psi_j)_{j=0}^{m}$. Notice that a basis for $V_{1h_1} \times V_{2h_2}$ is given by

$$\{[\varphi_i, 0] \mid i = 1, \ldots, 2n + 2l - 1\} \cup \{[0, \psi_j] \mid j = 0, \ldots, m - 1\}. \tag{15}$$

Clearly, the dimension of $V_{1h_1} \times V_{2h_2}$ is $2n + 2l + m - 1$, directly implying that the dimension of $V_h$ is $2n + 2l + m - 2$.

In order to construct a basis for $V_h$, we first identify those functions from (15) which fulfil (in a trivial way) the nonlocal coupling condition, and hence belong to $V_h$. It concerns a set of $2n + m - 3$ functions, given by

- $(\varphi_i, 0)$, $i = 1, \ldots, n - 1$,
- $(\varphi_i, 0)$, $i = n + 2l + 1, \ldots, 2n + 2l - 1$,
- $(0, \psi_j)$, $j = 1, \ldots, m - 1$.

Consequently, we are still looking for $2l+1$ functions which fulfil the coupling condition in a nontrivial way; clearly, we may take

$$(\varphi_i, \alpha_i \psi_0), \quad i = n, \ldots, n+2l.$$

Introducing a consistent and clear numbering for these function, i.e.

$$\begin{aligned}
&\Phi_k = (\varphi_k, 0), && k = 1, \ldots, n-1, \\
&\Phi_k = (\varphi_k, \alpha_k \psi_0), && k = 1, \ldots, n+2l, \\
&\Phi_k = (\varphi_k, 0), && k = \ +2l+1, \ldots, 2n+2l-1, \\
&\Phi_k = (0, \psi_{k-2n-2l+1}), && k = 2n+2l, \ldots, 2n+2l+m-2
\end{aligned}$$

we have

**Proposition 2.5.** *The set of $I = 2n+2l+m-2$ vector valued functions $(\Phi_k)_{k=1}^I$ constitutes a basis for $V_h$.*

Putting

$$u_h = \sum_{k=1}^{I} c_k \Phi_k$$

the EVP (10) may be rewritten as a generalized algebraic EVP, viz.

$$\text{Find } [\lambda_h, c_h] \in \mathbb{R} \times \mathbb{R}^I, \;\; c_h \neq 0\colon \; \mathcal{K} c_h = \lambda_h \mathcal{M} c_h,$$

where $c_h = [c_1, \ldots, c_I]^T$ and $\mathcal{K}$ and $\mathcal{M}$ are the stiffness and mass matrix, respectively, defined in the usual way.

Due to the particular form and numbering of the basis functions, the stiffness and mass matrix will have a transparent structure, reflected in Fig. 2.

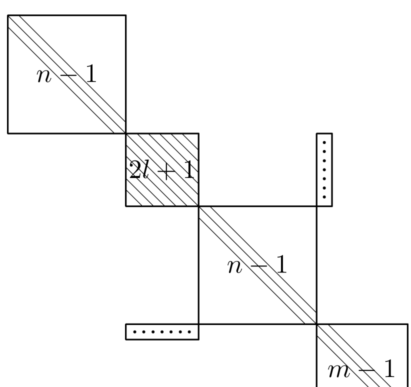


Figure 2. Structure of the stiffness and mass matrix.

## 3. An illustrative example

To illustrate the above analysis by a numerical example, the exact eigenpairs of which can be determined, we resort to a simple model problem, viz. the EVP (1) with $p_i \equiv 1$, $q_i \equiv 0$, $i = 1, 2$.

The exact eigenvalues of this EVP are found to be

$$\lambda = 4k^2\pi^2, \quad k \in \mathbb{N}_0, \quad \text{and} \quad \lambda = (2k+1)^2\pi^2, \quad k \in \mathbb{N}$$

with respective eigenfunctions $[\sin(2k\pi x), 0]$ and $[\sin((2k+1)\pi x), 0]$, as well as

$$\lambda = (2k+1)^2\frac{\pi^2}{4}, \quad k \in \mathbb{N}$$

with eigenfunctions

$$(-1)^k\left[\cos\Big((2k+1)\frac{\pi}{2}x\Big), \frac{2}{(2k+1)\pi}\sin\Big((2k+1)\frac{\pi}{2}\varepsilon\Big)\cos\Big((2k+1)\frac{\pi}{2}z\Big)\right]$$

and finally, the solutions of the transcendental equation

$$\varepsilon\sqrt{\lambda}\cos\sqrt{\lambda} - 2\varepsilon^2\lambda\sin\sqrt{\lambda} - \cos\big(\sqrt{\lambda}(1-\varepsilon)\big)\sin\big(\sqrt{\lambda}\varepsilon\big).$$

Let $\varepsilon = 0.1$, then the first three eigenvalues are given by

$$\lambda_1 = 2.467401; \quad \lambda_2 = 9.869604; \quad \lambda_3 = 10.318218,$$

the third coming from the transcendental equation. We use uniform meshes, both in $\Omega_1$ and $\Omega_2$, where even $h_1 = h_2$; hence we have $\frac{l}{n} = \frac{1}{9}$ and $m = n + l$.

The results, shown below for the third exact eigenvalue, reveal a very good agreement between the exact and the approximated values. Moreover, they are in agreement with the theoretical $\mathcal{O}(h^2)$-convergence, as well as the expected approximation from above.

| $m$ | $\lambda_{3,h}$ | $R$ in % |
|---|---|---|
| 10 | 10.730126 | 3.9920 |
| 20 | 10.410824 | 0.8975 |
| 40 | 10.339301 | 0.2043 |
| 80 | 10.322658 | 0.0430 |

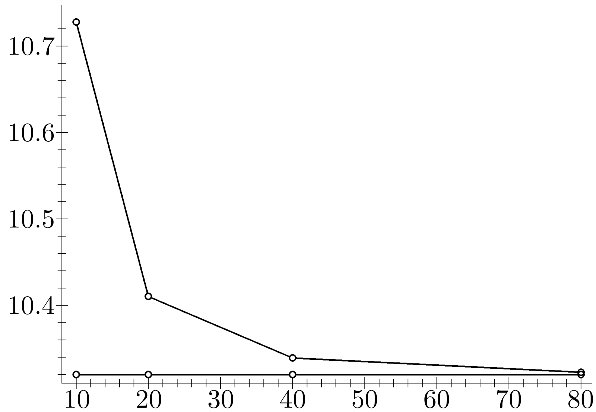

Figure 3. Approximation of $\lambda_3$ by subsequent mesh refinement.

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