Heinrich Voss

A maxmin principle for nonlinear eigenvalue problems with application to a rational spectral problem in fluid-solid vibration

Applications of Mathematics, Vol. 48 (2003), No. 6, 607--622

Persistent URL: http://dml.cz/dmlcz/134554

48 (2003) APPLICATIONS OF MATHEMATICS No. 6, 607–622

A MAXMIN PRINCIPLE FOR NONLINEAR EIGENVALUE PROBLEMS WITH APPLICATION TO A RATIONAL SPECTRAL PROBLEM IN FLUID-SOLID VIBRATION

Heinrich Voss, Hamburg

Abstract. In this paper we prove a maxmin principle for nonlinear nonoverdamped eigenvalue problems corresponding to the characterization of Courant, Fischer and Weyl for linear eigenproblems. We apply it to locate eigenvalues of a rational spectral problem in fluid-solid interaction.

Keywords: nonlinear eigenvalue problem, variational characterization, maxmin principle, fluid structure interaction

MSC 2000: 35P30

1. Introduction

In this paper we consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \tag{1}$$

where $T(\lambda)$, $\lambda \in J$, is a selfadjoint and bounded operator on a real Hilbert space H, and J is a real open interval which may be unbounded. As in the linear case $T(\lambda) = \lambda I - A$ the parameter $\lambda \in J$ is called an eigenvalue of problem (1) if the equation (1) has a nontrivial solution $x \neq 0$.

For a wide class of linear selfadjoint operators $A\colon H \to H$ the eigenvalues of the linear eigenvalue problem (1) can be characterized by three fundamental variational principles, namely by Rayleigh's principle [13], by Poincaré's minmax characterization [12], and by the maxmin principle of Courant [3], Fischer [6] and Weyl [21].

These variational principles were generalized to the nonlinear eigenvalue problem (1) where the Rayleigh quotient $R(x) := \langle Ax, x\rangle / \langle x, x\rangle$ of a linear problem

$Ax = \lambda x$ was replaced by the so called Rayleigh functional p, which is a homogeneous functional defined by the equation $\langle T(p(x))x, x\rangle = 0$ for $x \neq 0$. Notice that in the linear case $T(\lambda) := \lambda I - A$, this is exactly the Rayleigh quotient.

If the Rayleigh functional p is defined on the entire space $H \setminus \{0\}$ then the eigenproblem (1) is called overdamped. This term is motivated by the quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda Cx + Kx = 0 \tag{2}$$

governing the damped free vibrations of a system where M, C and K are symmetric and positive definite matrices corresponding to the mass, the damping and the stiffness of the system, respectively.

Assume that the damping $C = \alpha\tilde{C}$ depends on a parameter $\alpha \geqslant 0$. Then for $\alpha = 0$ the system has purely imaginary eigenvalues corresponding to harmonic vibrations. Increasing α the eigenvalues move into the left half plane as conjugate complex pairs corresponding to damped vibrations. Ultimately they reach the negative real axis as double eigenvalues where they immediately split and move in opposite directions. When eventually all eigenvalues have become real, and all eigenvalues going right are to the right of all eigenvalues moving to the left the system is called overdamped. In this case the two solutions

$$p_\pm(x) = \frac{-\alpha\langle\tilde{C}x, x\rangle \pm \sqrt{\alpha^2\langle\tilde{C}x, x\rangle^2 - 4\langle Mx, x\rangle\langle Kx, x\rangle}}{2\langle Mx, x\rangle}$$

of the quadratic equation

$$\langle T(p(x))x, x\rangle = \lambda^2\langle Mx, x\rangle + \lambda\alpha\langle\tilde{C}x, x\rangle + \langle Kx, x\rangle = 0 \tag{3}$$

are real, and they satisfy $\sup\limits_{x\neq 0} p_-(x) < \inf\limits_{x\neq<0} p_+(x)$. Hence, equation (3) defines two Rayleigh functionals p_- and p_+ corresponding to the intervals $J_- := \big(-\infty, \inf\limits_{x\neq 0} p_+(x)\big)$ and $J_+ := \big(\sup\limits_{x\neq 0} p_-(x), 0\big)$.

For overdamped systems Hadeler [7], [8] generalized Rayleigh's principle proving that the eigenvectors are orthogonal with respect to the generalized scalar product

$$[x, y] := \begin{cases} \dfrac{\langle(T(p(x)) - T(p(y)))x, y\rangle}{p(x) - p(y)}, & \text{if } p(x) \neq p(y) \\ \langle T'(p(x))x, y\rangle, & \text{if } p(x) = p(y) \end{cases}$$

which is symmetric, definite and homogeneous, but in general not bilinear.

For finite dimensional overdamped problems Duffin [4] proved Poincaré's minmax characterization for quadratic eigenproblems and Rogers [15] for the general nonlinear eigenproblem. Infinite dimensional overdamped problems were treated by Hadeler [8], [9], Langer [10], Rogers [16], Turner [17], [18], and Werner [20], who proved generalizations of both characterizations, the minmax principle of Poincaré and the maxmin principle of Courant, Fischer and Weyl.

Barston [1] characterized a subset of the real eigenvalues of a nonoverdamped quadratic eigenproblem of finite dimension, and Werner and the author [19] studied the general nonoverdamped case and proved a minmax principle generalizing the characterization of Poincaré. In this paper we add a maxmin characterization for nonoverdamped problems. Both principles are used to locate the eigenvalues of a rational eigenvalue problem governing the vibrations of interacting fluid-solid structures.

2. Characterization of eigenvalues

We consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \tag{4}$$

where $T(\lambda)$ is a selfadjoint and bounded operator on a real Hilbert space H for every λ in an open real interval J.

We assume that

$$f\colon \begin{cases} J \times H \to \mathbb{R} \\ (\lambda, x) \mapsto \langle T(\lambda)x, x\rangle \end{cases} \tag{5}$$

is continuously differentiable, and that for every fixed $x \in H^0 := H \setminus \{0\}$ the real equation

$$f(\lambda, x) = 0 \tag{6}$$

has at most one solution in J. Then equation (6) implicitly defines a functional p on some subset D of H^0, which we call the Rayleigh functional.

We assume that

$$\frac{\partial}{\partial\lambda} f(\lambda, x)\Big|_{\lambda = p(x)} > 0 \quad \text{for every } x \in D. \tag{7}$$

Then it follows from the implicit function theorem that D is an open set and that p is continuously differentiable on D.

In this section we first summarize the results of [19] concerning the minmax characterization of the nonlinear eigenvalue problem (4) corresponding to the Poincaré principle and amend them by the complementary Courant-Fischer-Weyl type characterizations.

We denote by H_n the set of all n-dimensional subspaces of H and by $V_1 := \{v \in V : \|v\| = 1\}$ the unit sphere of the subspace V of H.

For nonoverdamped problems the natural enumeration for which the smallest eigenvalue is the first, the second smallest is the second, etc. is not appropriate. This is easily seen if we consider a linear eigenvalue problem $T(\lambda) = \lambda I - A$ and make it nonlinear by restricting it to an open interval J which does not contain the smallest eigenvalue of A. Then $\inf_{x \in D} p(x) \notin J$, and the smallest eigenvalue of T, i.e. the smallest eigenvalue of A in J, is not the minimum of the Rayleigh functional.

For nonlinear eigenvalue problems the number of an eigenvalue λ of problem (4) is inherited from the location of the eigenvalue 0 in the spectrum of the operator $T(\lambda)$. We use the following notation

$$\mu_V(\lambda) := \min_{v \in V_1} \langle T(\lambda)v, v\rangle, \tag{8}$$

$$\mu_j(\lambda) := \sup_{V \in H_j} \mu_V(\lambda), \tag{9}$$

and we assume that the following conditions hold:

(A1) if $\mu_n(\lambda) = 0$ for some $n \in \mathbb{N}$ and $\lambda \in J$, then $\mu_j(\lambda) = \max_{V \in H_j} \mu_V(\lambda)$ for $j = 1, \ldots, n$ (i.e. the sup is attained by some $V \in H_j$), and $\mu_1(\lambda) \geqslant \mu_2(\lambda) \geqslant \ldots \geqslant \mu_n(\lambda)$ are the n largest eigenvalues of $T(\lambda)$;

(A2) if $\mu = 0$ is an eigenvalue of $T(\lambda)$, then there exists $n \in \mathbb{N}$ with $\mu_n(\lambda) = 0$.

It is well known (cf. Rektorys [14]) from the maxmin principle of Poincaré that (A1) and (A2) hold if for every fixed $\lambda \in J$ there exists $\eta(\lambda) > 0$ such that the linear operator $T(\lambda) + \eta(\lambda)I$ is completely continuous. For more general results see Dunford and Schwartz [5], p. 1543.

If $\lambda \in J$ is an eigenvalue of $T(\cdot)$ then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $n \in \mathbb{N}$ such that $\mu_n(\lambda) = 0$. In this case we call λ the n-th eigenvalue of (4). If conversely $\mu_n(\lambda) = 0$ for some $n \in \mathbb{N}$ and $\lambda \in J$ then (A1) implies that λ is the n-th eigenvalue of (4).

With this enumeration the following minmax characterization of the eigenvalues of the nonlinear eigenproblem (4) was proved in [19]:

Theorem 1. *Assume that the real equation* (6) *for every fixed* $x \in H \setminus \{0\}$ *has at most one solution* $\lambda = p(x)$ *in* J, *the condition* (7) *is satisfied, and suppose conditions* (A1) *and* (A2). *Then the following assertions hold:*

(i) *For every* $n \in \mathbb{N}$ *there is at most one* n*-th eigenvalue of problem* (4) *which can be characterized by*

$$\lambda_n = \min_{\substack{V \in H_n \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v). \tag{10}$$

The set of eigenvalues of (4) *is at most countable.*

(ii) *If*

$$\lambda_n = \inf_{\substack{V \in H_n \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v) \in J \tag{11}$$

for some $n \in \mathbb{N}$ *then* λ_n *is the* n*-th eigenvalue of* (4) *and* (10) *holds, i.e. the infimum is attained by some space* $V \in H_n$.

(iii) *If the* m*-th and* n*-th eigenvalues* λ_m *and* λ_n *exist in* J *and* $m < n$ *then* J *for every* $k = m, m+1, \ldots, n$ *contains the* k*-th eigenvalue* λ_k *and*

$$\inf J < \lambda_m \leqslant \lambda_{m+1} \leqslant \ldots \leqslant \lambda_n < \sup J.$$

(iv) *If* $\lambda_1 \in J$ *and* $\lambda_n \in J$ *for some* $n \in \mathbb{N}$ *then every* $V \in H_j$ *with* $V \cap D \neq \emptyset$ *and* $\lambda_j = \sup_{u \in V \cap D} p(u)$ *is contained in* D, *and the characterization* (10) *can be replaced by*

$$\lambda_j = \min_{\substack{V \in H_j \\ V_1 \subset D}} \max_{v \in V_1} p(v) \quad j = 1, \ldots, n. \tag{12}$$

The characterization of the eigenvalues in Theorem 1 is a generalization of the minmax principle of Poincaré for linear eigenvalue problems.

In a similar way as in [19] we now generalize the maxmin characterization of Courant, Fischer and Weyl to nonlinear and nonoverdamped eigenproblems. To this end we need the following two lemmas.

Lemma 2. *Let*

$$\nu_V(\lambda) := \max_{v \in V_1^{\perp}} \langle T(\lambda)v, v\rangle = 0 \tag{13}$$

where V is a finite dimensional subspace of H, and

$$V_1^{\perp} := \{w \in H \colon \ \langle v, w\rangle = 0 \text{ for every } v \in V, \ \|w\| = 1\}.$$

Then

$$V^{\perp} \cap D \neq \emptyset \quad \text{and} \quad \lambda = \min_{v \in V^{\perp} \cap D} p(v). \tag{14}$$

P r o o f. There exists $u \in V_1^{\perp}$ such that

$$\langle T(\lambda)v, v\rangle \leqslant \langle T(\lambda)u, u\rangle = 0 \quad \text{for every } \ v \in V_1^{\perp}.$$

Hence $u \in D$ with $p(u) = \lambda$, and it follows from assumption (7) that

$$p(u) \leqslant p(v) \quad \text{for every } \ v \in D \cap V^{\perp}.$$

$\square$

Lemma 3. *Let $\lambda \in J$ and let V be a finite dimensional subspace of H such that $V^{\perp} \cap D \neq \emptyset$. Then*

$$\lambda \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \inf_{v \in V^{\perp} \cap D} p(v) \Longleftrightarrow \max_{v \in V_1^{\perp}} \langle T(\lambda)v, v\rangle \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 0. \tag{15}$$

P r o o f. Suppose that

$$\lambda = \inf_{v \in V^{\perp} \cap D} p(v) \in J.$$

Let $P_{V^{\perp}}$ denote the orthogonal projection of H to $V^{\perp}$, and let $S(\lambda) := P_{V^{\perp}} T(\lambda) P_{V^{\perp}}$. Then $S(\lambda)$ satisfies the conditions of Theorem 1 where its Rayleigh functional $\tilde{p}$ is the restriction of p to $V^{\perp} \cap D$. Hence there exists $\tilde{v} \in V^{\perp} \cap D$ such that

$$p(\tilde{v}) = \min_{v \in V^{\perp} \cap D} p(v) = \lambda.$$

From

$$p(v) \geqslant p(\tilde{v}) = \lambda \quad \text{for every } \ v \in V^{\perp} \cap D$$

and from assumption (7) we get

$$\langle T(\lambda)v, v\rangle \leqslant \langle T(\lambda)\tilde{v}, \tilde{v}\rangle = 0 \quad \text{for every } v \in V^{\perp} \cap D, \tag{16}$$

and

$$\sup_{v \in V_1^{\perp}} \langle T(\lambda)v, v\rangle \geqslant 0.$$

Assume that

$$\sup_{v \in V_1^{\perp}} \langle T(\lambda)v, v\rangle > 0.$$

Then there exists $w \in V^{\perp}$ such that $\langle T(\lambda)w, w\rangle > 0$ and replacing w by $-w$ we can assume $\langle T(\lambda)\tilde{v}, w\rangle \geqslant 0$. Thus

$$\langle T(\lambda)(\tilde{v} + tw), \tilde{v} + tw\rangle = 2t\langle T(\lambda)\tilde{v}, w\rangle + t^2\langle T(\lambda)w, w\rangle > 0 \quad \text{for } t > 0,$$

contradicting (16) and the fact that D is an open set. Hence we have proved

$$\lambda = \inf_{v \in V^{\perp} \cap D} p(v) \Rightarrow \max_{v \in V_1^{\perp}} \langle T(\lambda)v, v\rangle = 0. \tag{17}$$

If conversely

$$\langle T(\lambda)v, v\rangle \leqslant \langle T(\lambda)\tilde{v}, \tilde{v}\rangle = 0 \quad \text{for every } v \in V^{\perp}$$

and some $\tilde{v} \in V^{\perp}$ then it follows from Lemma 2 that $V^{\perp} \cap D \neq \emptyset$ and $\lambda = \min\limits_{v \in V^{\perp} \cap D} p(v)$. Thus

$$\lambda = \inf_{v \in V^{\perp} \cap D} p(v) \Longleftrightarrow \max_{v \in V_1^{\perp}} \langle T(\lambda)v, v\rangle = 0. \tag{18}$$

If $\max\limits_{v \in V^{\perp}} \langle T(\lambda)v, v\rangle < 0$ then

$$\langle T(\lambda)v, v\rangle < 0 \quad \text{for every } v \in V^{\perp}.$$

Condition (7) implies

$$\lambda < p(v) \quad \text{for every } v \in V^{\perp} \cap D,$$

and (18) yields

$$\lambda < \inf_{v \in V^{\perp} \cap D} p(v).$$

Finally, let

$$\max_{v \in V^{\perp}} \langle T(\lambda)v, v\rangle > 0.$$

Then

$$\langle T(\lambda)\tilde{v}, \tilde{v}\rangle > 0 \quad \text{for some } \tilde{v} \in V^{\perp}.$$

If $\lambda < \inf_{v \in V^{\perp} \cap D} p(v)$ then $\lambda < p(u)$ for some $u \in V^{\perp} \cap D$, and $\langle T(\lambda)u, u\rangle < 0$. Let

$$w(t) := tu + (1-t)\tilde{v} \quad \text{and} \quad g(t) := \langle T(\lambda)w(t), w(t)\rangle.$$

Then

$$g(0) = \langle T(\lambda)\tilde{v}, \tilde{v}\rangle > 0 > g(1) = \langle T(\lambda)u, u\rangle,$$

and there exists $\tilde{t} \in (0,1)$ such that $g(\tilde{t}) = 0$. Hence

$$w(\tilde{t}) \in V^{\perp} \cap D \quad \text{and} \quad p(w(\tilde{t})) = \lambda,$$

contradicting $\lambda < \inf_{v \in V^{\perp} \cap D} p(v)$. □

We are now in the position to prove the generalization of the maxmin characterization of Courant, Fischer and Weyl.

Theorem 4. *If the n-th eigenvalue $\lambda_n \in J$ of problem* (4) *exists, then*

$$\lambda_n = \max_{\substack{V \in H_{n-1} \\ V^{\perp} \cap D \neq \emptyset}} \inf_{v \in V^{\perp} \cap D} p(v), \tag{19}$$

and the maximum is attained by $W := \operatorname{span}\{u_1, \ldots, u_{n-1}\}$ where u_j denotes an eigenvector corresponding to the j-largest eigenvalue $\mu_j(\lambda_n)$ of $T(\lambda_n)$.

Proof. Since λ_n is the n-th eigenvalue of (4) the n largest eigenvalue of $T(\lambda_n)$ is $\mu_n(\lambda_n) = 0$. By the maxmin characterization of Courant, Fischer and Weyl there exists an $(n-1)$-dimensional subspace W of H such that

$$\mu_n(\lambda_n) = \min_{V \in H_{n-1}} \max_{v \in V_1^{\perp}} \langle T(\lambda_n)v, v\rangle = \max_{v \in W_1^{\perp}} \langle T(\lambda_n)v, v\rangle.$$

Hence $W^{\perp} \cap D \neq \emptyset$, and by Lemma 3

$$\lambda_n = \inf_{v \in W^{\perp} \cap D} p(v).$$

For $V \in H_{n-1}$ we have

$$\max_{v \in V_1^{\perp}} \langle T(\lambda_n)v, v\rangle \geqslant 0$$

and again Lemma 3 yields

$$\lambda_n \geqslant \inf_{v \in V^{\perp} \cap D} p(v).$$

That W is an invariant subspace corresponding to the n largest eigenvalues of $T(\lambda_n)$ is known from the linear theory. □

3. A rational eigenvalue problem in fluid structure interaction

In this section we apply the variational characterizations to locate eigenvalues of a rational eigenvalue problem governing free vibrations of a tube bundle immersed in a slightly compressible fluid under the following simplifying assumptions: The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they cannot move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity. Considering small vibrations of the fluid (and the tubes) around the state of rest, it can also be assumed that the fluid is irrotational.

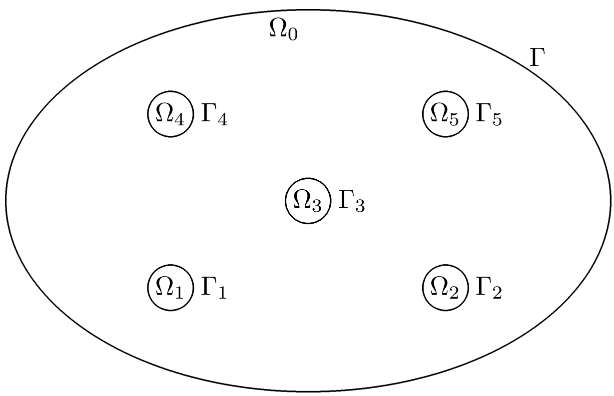

Figure 1.

The mathematical model governing the eigenfrequencies and eigenmodes was deduced by Planchard, Conca and Vanninathan (cf. [11], [2]), and it reads as follows. Let $\Omega \subset \mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ. We assume that there exists a family $\Omega_j \neq \emptyset$, $j = 1, \ldots, K$ (the sections of the tubes) of simply connected open sets such that $\overline{\Omega}_j \subset \Omega$ for every j, $\overline{\Omega}_j \cap \overline{\Omega}_i = \emptyset$ for $j \neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With this notation we set $\Omega_0 := \Omega \setminus \bigcup\limits_{j=1}^{K} \overline{\Omega}_j$. Then the boundary of Ω_0 consists of $K+1$ connected components which are Γ and Γ_j, $j = 1, \ldots, K$.

We denote by $H^1(\Omega_0) = \{u \in L^2(\Omega_0)\colon \nabla u \in L^2(\Omega_0)^2\}$ the standard Sobolev space equipped with the usual scalar product. Then the eigenfrequencies and the eigenmodes of the fluid-solid structure are governed by the following variational eigenvalue problem (cf. [11], [2]):

Find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega_0)$, $u \neq 0$ such that for every $v \in H^1(\Omega_0)$

$$c^2 \int_{\Omega_0} \nabla u \cdot \nabla v \,\mathrm{d}x = \lambda \int_{\Omega_0} uv \,\mathrm{d}x + \sum_{j=1}^{K} \frac{\lambda \varrho_0}{k_j - \lambda m_j} \int_{\Gamma_j} un \,\mathrm{d}s \cdot \int_{\Gamma_j} vn \,\mathrm{d}s. \tag{20}$$

Here u is the potential of the velocity of the fluid, c denotes the speed of sound in the fluid, ϱ_0 is the specific density of the fluid, k_j represents the stiffness constant of the spring system supporting tube j, m_j is the mass per unit length of the tube j, and n is the outward unit normal on the boundary of Ω_0.

Obviously $\lambda = 0$ is an eigenvalue of (20) with an eigenfunction $u = \text{const}$. We reduce the eigenproblem (20) to the space

$$H := \left\{ u \in H^1(\Omega_0) \colon \int_{\Omega_0} u(x) \,\mathrm{d}x = 0 \right\}$$

and consider the scalar product

$$\langle u, v \rangle := \int_{\Omega_0} \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x$$

on H which is known to define a norm on H which is equivalent to the norm induced by $(\cdot, \cdot)$.

By the Lax-Milgram lemma the variational eigenvalue problem (20) is equivalent to the nonlinear eigenvalue problem

Determine λ and $u \in H$, $u \neq 0$ such that

$$T(\lambda)u := \left(-c^2 I + \lambda A + \sum_{j=1}^{K} \frac{\varrho_0 \lambda}{k_j - \lambda m_j} B_j \right) u = 0 \tag{21}$$

where the linear symmetric operators A and B_j are defined by

$$\langle Au, v \rangle := \int_{\Omega_0} uv \,\mathrm{d}x \quad \text{for every } u, v \in H, \tag{22}$$

$$\langle B_j u, v \rangle := \left(\int_{\Gamma_j} un \,\mathrm{d}s \right) \cdot \left(\int_{\Gamma_j} vn \,\mathrm{d}s \right) \quad \text{for every } u, v \in H. \tag{23}$$

A is completely continuous by Rellich's embedding theorem and $w := B_j u$, $j = 1, \ldots, K$, is the weak solution in H of the elliptic problem

$$\Delta w = 0 \text{ in } \Omega_0, \quad \frac{\partial}{\partial n} w = 0 \text{ on } \partial\Omega_0 \setminus \Gamma_j, \quad \frac{\partial}{\partial n} w = n \cdot \int_{\Gamma_j} un \,\mathrm{d}s \text{ on } \Gamma_j.$$

By the continuity of the trace the operator B_j is continuous, and since the range of B_j is twodimensional spanned by the solutions $w_i \in H$ of

$$\Delta w_i = 0 \text{ in } \Omega_0, \quad \frac{\partial}{\partial n} w = 0 \text{ on } \partial\Omega_0 \setminus \Gamma_j, \quad \frac{\partial}{\partial n} w = n_i \text{ on } \Gamma_j, \ i = 1, 2,$$

it is even completely continuous. Hence the general conditions of Section 2 are satisfied.

Rayleigh functionals corresponding to problem (21) are defined by the real function

$$\begin{aligned} f(\lambda, u) &:= \langle T(\lambda)u, u\rangle \\ &= -c^2 \int_{\Omega_0} |\nabla u|^2 \, \mathrm{d}x + \lambda \int_{\Omega_0} u^2 \, \mathrm{d}x + \sum_{j=1}^{K} \frac{\varrho_0 \lambda}{k_j - \lambda m_j} \left| \int_{\Gamma_j} un \, \mathrm{d}s \right|^2 . \end{aligned} \tag{24}$$

Since

$$\frac{\partial}{\partial \lambda} f(\lambda, u) = \int_{\Omega_0} u^2 \, \mathrm{d}x + \sum_{j=1}^{K} \frac{\varrho_0 k_j}{(k_j - \lambda m_j)^2} \left| \int_{\Gamma_j} un \, \mathrm{d}s \right|^2 > 0 \tag{25}$$

there exists a Rayleigh functional corresponding to the eigenvalue problem (21) for every interval $J \subset \mathbb{R}$ such that $k_j/m_j \notin J$ for $j = 1, \dots, K$, and the results of Section 2 apply. Hence the rational eigenproblem (21) has (at most) a countable set of eigenvalues.

In particular, we consider the intervals

$$J_1 := \Big(-\infty, \min_{j=1,\dots,K} \frac{k_j}{m_j}\Big) \quad \text{and} \quad J_2 := \Big(\max_{j=1,\dots,K} \frac{k_j}{m_j}, \infty\Big),$$

and comparing the nonlinear problem with the linear eigenproblem

$$\begin{gathered} \text{Determine } \lambda \text{ and } u \in H, \ u \neq 0 \text{ such that} \\ c^2 \int_{\Omega_0} \nabla u \cdot \nabla v \, \mathrm{d}x = \mu \int_{\Omega_0} uv \, \mathrm{d}x \quad \text{for every } v \in H \end{gathered} \tag{26}$$

(which is obtained from (20) by neglecting the rational terms) we prove inclusion results for the eigenvalues of (20) in these intervals.

We denote the Rayleigh functionals of problem (21) corresponding to J_1 and J_2 by p_1 and p_2 and their domains of definition by D_1 and D_2, respectively.

Lemma 5. *For $u \in H$, $u \neq 0$ let*

$$R(u) := c^2 \int_{\Omega_0} |\nabla u|^2 \,\mathrm{d}x \Big/ \int_{\Omega_0} u^2 \,\mathrm{d}x$$

be the Rayleigh quotient of the linear eigenvalue problem (26). *Then:*

(i) *If $R(u) \in J_1$ then $u \in D_1$ and*

$$p_1(u) \leqslant R(u).$$

(ii) *If $R(u) \in J_2$ then $u \in D_2$ and*

$$p_2(u) \geqslant R(u).$$

P r o o f. If $R(u) < \min\limits_{j=1,\dots,K} k_j/m_j$ then

$$f(R(u),u) = \sum_{j=1}^{K} \frac{\varrho_0 R(u)}{k_j - R(u)m_j} \left| \int_{\Gamma_j} un \,\mathrm{d}s \right|^2 \geqslant 0$$

and

$$f(0,u) = -c^2 \int_{\Omega_0} |\nabla u|^2 \,\mathrm{d}x < 0.$$

Hence, $u \in D_1$ and $p_1(u) \leqslant R(u)$.

Similarly for $R(u) > \max\limits_{j=1,\dots,K} k_j/m_j$ we have

$$f(R(u),u) = \sum_{j=1}^{K} \frac{\varrho_0 R(u)}{k_j - R(u)m_j} \left| \int_{\Gamma_j} un \,\mathrm{d}s \right|^2 \leqslant 0$$

and

$$\lim_{\lambda\to\infty} f(\lambda,u) = -c^2 \int_{\Omega_0} |\nabla u|^2 \,\mathrm{d}x - \sum_{j=1}^{K} \frac{\varrho_0}{m_j} \left| \int_{\Gamma_j} un \,\mathrm{d}s \right|^2 + \lim_{\lambda\to\infty} \left(\lambda \int_{\Omega_0} u^2 \,\mathrm{d}x \right) \to \infty,$$

and therefore $u \in D_2$ and $p_2(u) \geqslant R(u)$. $\square$

For the lower part of the spectrum of problem (20) we obtain the following existence result and bounds.

Theorem 6. *Let*

$$\mu_r := \min_{V\in H_r}\max_{u\in V_0} R(u) < \min_{j=1\dots,K}\frac{k_j}{m_j}. \tag{27}$$

Then there are (at least) r eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_r$ of the nonlinear eigenvalue problem (21) *in J_1 and*

$$0 < \lambda_j \leqslant \mu_j, \quad j = 1,\ldots,r. \tag{28}$$

P r o o f. From $p_1(u) > 0$ for every $u \in D_1$ we get $\inf\limits_{u\in D_1} p_1(u) \in J_1$. Hence we obtain the existence of r eigenvalues $\lambda_1 \leqslant \ldots \leqslant \lambda_r$ from Theorem 1 (iv) provided

$$\inf_{\substack{\dim V=r\\ V\cap D_1\neq\emptyset}} \sup_{u\in V\cap D} p_1(u) \in J_1. \tag{29}$$

Let

$$\mu_r = \min_{V\in H_j}\max_{u\in V_0} R(u) = \max_{u\in W_0} R(u), \quad \dim W = r.$$

Then

$$R(u) \leqslant \mu_r < \min\frac{k_j}{m_j} \quad \text{for every } u\in W_0,$$

and Lemma 5 implies $W_0 \subset D_1$ and $p_1(u) \leqslant R(u) < \sup J_1$ for every $u \in W_0$. In particular, (29) holds.

Moreover, Lemma 5 yields

$$\lambda_j = \min_{\substack{V\in H_j\\ V_0\subset D}}\max_{u\in V_0} p_1(u) \leqslant \min_{\substack{V\in H_j\\ V_0\subset D}}\max_{u\in V_0} R(u) \leqslant \min_{V\in H_j}\max_{u\in V_0} R(u) = \mu_j \quad \text{for } j=1,\ldots,r.$$

□

The following theorem contains bounds for eigenvalues in the upper part of the spectrum of problem (20).

Theorem 7. *Let*

$$\mu_s := \min_{V\in H_s}\max_{u\in V_0} R(u) > \max_{j=1\dots,K}\frac{k_j}{m_j}. \tag{30}$$

Then the s-th eigenvalue λ_s of the nonlinear eigenvalue problem (21) *exists in J_2 and*

$$\mu_s \leqslant \lambda_s \leqslant \mu_{s+2K}. \tag{31}$$

P r o o f. For $V \in H_s$ let $u \in V$ be such that $R(u) = \max_{v \in V_0} R(v)$. Then

$$R(u) \geqslant \min_{W \in H_s} \max_{w \in W_0} R(w) = \mu_s > \max_{j=1\dots,K} \frac{k_j}{m_j}.$$

Hence by Lemma 5 (ii) $u \in D_2$, i.e. $V \cap D_2 \neq \emptyset$ for every $V \in H_s$, and $p_2(u) \geqslant R(u)$ from which we obtain

$$\max_{v \in V \cap D_2} p_2(v) \geqslant p_2(u) \geqslant R(u) = \max_{v \in V_0} R(v).$$

Thus

$$\inf_{W \in H_s} \max_{v \in W \cap D_2} p_2(v) \geqslant \inf_{W \in H_s} \max_{v \in W_0} R(v) = \mu_s.$$

The upper bound μ_{s+2K} of λ_s is obtained from Theorem 4. Let $u_1, \dots, u_{s-1}$ be eigenvectors corresponding to the $s-1$ largest eigenvalues of $T(\lambda_n)$, and denote by v_{2j-1}, v_{2j} a basis of the range of B_j for $j = 1, \dots, K$. Let $W := \operatorname{span}\{u_1, \dots, u_{s-1}, v_1, \dots, v_{2K}\}$ and $\tilde{W} := \operatorname{span}\{u_1, \dots, u_{s-1}\}$. Then the Rayleigh quotient R and the Rayleigh functional p_2 coincide on W, and the maxmin characterization of Courant, Fischer and Weyl yields

$$\mu_{s+2K} = \max_{V \in H_{s+2K-1}} \min_{v \in V_1^{\perp}} R(v) \geqslant \min_{v \in W_1^{\perp}} R(v) = \min_{v \in W_1^{\perp}} p_2(v) \geqslant \min_{v \in \tilde{W}_1^{\perp}} p_2(v) = \lambda_s.$$

□

Conca, Planchard and Vanninathan [2] proved that the nonlinear eigenvalue problem (20) has a countable set of eigenvalues using methods from linear functional analysis. To this end they transformed the rational eigenvalue problem to a linear compact eigenproblem on a Hilbert space which is nonselfadjoint, but can be symmetrized easily. The existence of countably many eigenvalues of problem (20) follows from Theorem 7 as well, since the linear eigenproblem (26) clearly has countably many eigenvalues.

Moreover, they proved an inclusion theorem for the eigenvalues. In addition to the comparison problem (26) they considered the rational eigenproblem

$$\text{Find } \theta \in \mathbb{R},\ \theta > 0 \text{ and } u \in H,\ u \neq 0 \text{ such that}$$

$$\int_{\Omega_0} \nabla u \cdot \nabla v \,\mathrm{d}x = \sum_{j=1}^{K} \frac{\varrho_0 \theta}{k_j - \theta m_j} \int_{\Gamma_j} un \,\mathrm{d}s \cdot \int_{\Gamma_j} vn \,\mathrm{d}s \quad \text{for every } v \in H \tag{32}$$

which corresponds to the vibrations of the fluid-solid structure when the fluid is assumed to be incompressible. This problem was studied by Planchard [11] who proved that there exist $2K$ eigenvalues θ_j satisfying $0 < \theta_j < \max_j k_j/m_j$.

Theorem 8 (Conca, Planchard, Vanninathan [2]). *Let* $0 < \sigma_1 \leqslant \sigma_2 \leqslant \ldots$ *denote the union of eigenvalues* μ_j *of problem* (26) *and* θ_j *of problem* (32) *ordered by magnitude and regarding their multiplicity, and let* $0 < \lambda_1' \leqslant \lambda_2' \leqslant \ldots$ *be the eigenvalues of the rational eigenproblem* (20) *ordered in the natural way. Then the following interlacing inequalities hold:*

(33) $$\lambda_j' \leqslant \sigma_{j+2K} \quad \text{for } j = 1, \ldots, 2K,$$

(34) $$\sigma_{j-2K} \leqslant \lambda_j' \leqslant \sigma_{j+2K} \quad \text{for } j \geqslant 2K+1.$$

Since the enumerations of the eigenvalues in Theorems 6 and 7 on the one hand and in Theorem 8 on the other hand are different the bounds do not compare directly.

However, since problem (32) has exactly $2K$ eigenvalues, it is obvious that $\mu_j \leqslant \sigma_{j+2K}$, and therefore the upper bounds given in Theorem 6 for $\lambda_1, \ldots, \lambda_r$ are tighter than those given in Theorem 8. Moreover, it follows from $\theta_j < k/m := \max_j k_j/m_j$ for $j = 1, \ldots, 2K$ that no eigenvalue of problem (32) appears in $\big(k/m, \infty\big)$. Hence, although counted in different ways, the upper bounds in Theorem 7 and in (34) are identical, but the lower bounds μ_j improve the ones in (34).

References

[1] *E. M. Barston*: A minimax principle for nonoverdamped systems. Internat. J. Engrg. Sci. *12* (1974), 413–421.

[2] *C. Conca, J. Planchard and M. Vanninathan*: Existence and location of eigenvalues for fluid-solid structures. Comput. Methods Appl. Mech. Engrg. *77* (1989), 253–291.

[3] *R. Courant*: Über die Eigenwerte bei den Differentialgleichungen der mathematischen Physik. Math. Z. *7* (1920), 1–57.

[4] *R. J. Duffin*: A minmax theory for overdamped networks. J. Ration. Mech. Anal. *4* (1955), 221–233.

[5] *N. Dunford, J. T. Schwartz*: Linear Operators, Part II. Wiley, New York-London, 1963.

[6] *E. Fischer*: Über quadratische Formen mit reellen Koeffizienten. Monatshefte für Math. und Phys. *16* (1905), 234–249.

[7] *K. P. Hadeler*: Mehrparametrige und nichtlineare Eigenwertaufgaben. Arch. Ration. Mech. Anal. *27* (1967), 306–328.

[8] *K. P. Hadeler*: Variationsprinzipien bei nichtlinearen Eigenwertaufgaben. Arch. Ration. Mech. Anal. *30* (1968), 297–307.

[9] *K. P. Hadeler*: Nonlinear Eigenvalue Problems. In: Numerische Behandlung von Differentialgleichungen (R. Ansorge, L. Collatz, G. Hämmerlin and W. Törnig, eds.). Birkhäuser, Stuttgart, 1975, pp. 111–129.

[10] *H. Langer*: Über stark gedämpfte Scharen im Hilbertraum. J. Math. Mech. *17* (1968), 685–705.

[11] *J. Planchard*: Eigenfrequencies of a tube bundle placed in a confined fluid. Comput. Methods Appl. Mech. Engrg. *30* (1982), 75–93.

[12] *H. Poincaré*: Sur les équations aux dérivées partielles de la physique mathématique. Amer. J. Math. *12* (1890), 211–294.

[13] *Rayleigh*: Some general theorems relating to vibrations. Proc. London Math. Soc. *4* (1873), 357–368.
[14] *K. Rektorys*: Variationsmethoden in Mathematik, Physik und Technik. Carl Hanser Verlag, München, 1984.
[15] *E. H. Rogers*: A minmax theory for overdamped systems. Arch. Ration. Mech. Anal. *16* (1964), 89–96.
[16] *E. H. Rogers*: Variational properties of nonlinear spectra. J. Math. Mech. *18* (1968), 479–490.
[17] *R. E. L. Turner*: Some variational principles for nonlinear eigenvalue problems. J. Math. Anal. Appl. *17* (1967), 151–160.
[18] *R. E. L. Turner*: A class of nonlinear eigenvalue problems. J. Funct. Anal. *7* (1968), 297–322.
[19] *H. Voss, B. Werner*: A minimax principle for nonlinear eigenvalue problems with applications to nonoverdamped systems. Math. Methods Appl. Sci. *4* (1982), 415–424.
[20] *B. Werner*: Das Spektrum von Operatorenscharen mit verallgemeinerten Rayleighquotienten. Arch. Ration. Mech. Anal. *42* (1971), 223–238.
[21] *H. Weyl*: Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). Math. Ann. *71* (1912), 441–479.

Author's address: *Heinrich Voss*, Department of Mathematics, Technical University of Hamburg-Harburg, D-21071 Hamburg, Germany, e-mail: `voss@tu-harburg.de`.