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Numerical solution of boundary value problems for selfadjoint differential equations of $2n$th order

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NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS FOR SELFADJOINT DIFFERENTIAL EQUATIONS OF $2n$th ORDER*

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Abstract. The paper is devoted to solving boundary value problems for self-adjoint linear differential equations of $2n$th order in the case that the corresponding differential operator is self-adjoint and positive semidefinite. The method proposed consists in transforming the original problem to solving several initial value problems for certain systems of first order ODEs. Even if this approach may be used for quite general linear boundary value problems, the new algorithms described here exploit the special properties of the boundary value problems treated in the paper. As a consequence, we obtain algorithms that are much more effective than similar ones used in the general case. Moreover, it is shown that the algorithms studied here are numerically stable.

Keywords: ODE, two-point boundary value problem, transfer of boundary conditions, self-adjoint differential equation, numerical solution, Riccati differential equation

MSC 2000: 65L10, 34B05

1. INTRODUCTION

The paper is concerned with the numerical solution of boundary value problems for self-adjoint linear differential equations of $2n$th order such that the corresponding differential operator is self-adjoint and positive semidefinite. Our approach consists in transforming the boundary value problem to be solved to the solution of a sequence of initial value problems of a special structure. This makes it possible to employ the standard software for solving initial value problems to solve boundary value problems

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of this type. One of the possibilities how to transform the boundary value problems to solving initial value problems is to use the method of transfer of conditions as described in [6], [7] or the method of invariant imbedding [3], [5]. The principles of these methods could be used for the problems studied here directly, however, they would lead to algorithms that would be unnecessarily elaborate.

In particular, applying the methods of the above type that were devised for general boundary value problems we cannot guarantee, in general, the global solvability of the initial value problems obtained as the result of the above mentioned transformation. Therefore, to be able to obtain algorithms based on the transfer of boundary conditions that would be practicable and to achieve their numerical stability in the general case it is necessary to take certain elaborate measures [6], [7]. This is unnecessary and even superfluous to do with the algorithms devised here for the special boundary value problems considered in the paper. This result is not due to that fact that the original methods designed for general linear boundary value problems would possess the required properties when applied to the special problem considered. It is due to the fact that we have constructed new special methods of the transfer of conditions for the special boundary value problems considered that make use of all the additional information at our disposal here (self-adjointness, positive semidefiniteness).

Hence, the present paper exploits the special form of the problem under consideration and the numerical method proposed is chosen in order to use the additional information, namely all the symmetry and some sign properties. As a consequence, the resulting method is very advantageous as to the effort expended and fulfils all of the numerical stability requirements. In particular, we prove that the initial value problems for the matrix Riccati differential equation that occur in our algorithms have a bounded solution on the whole interval of question. In fact, all the entries of the matrix solution are shown to lie in $[0, 1]$. This is one of the most important results of the paper. To prove the properties of the transfer of conditions in the particular case studied here we need to prove some properties of the general transfer of conditions that have not yet been published and that apply to the special problem considered. We also note that contrary to the invariant imbedding method the coefficients of the equations are not required to be smooth here.

The structure of the paper is as follows. Section 2 contains a survey of the fundamental results of the methods based on the idea of the transfer of conditions for general two-point boundary value problems for systems of first order ODEs. In Section 3, we define the boundary value problem for a self-adjoint differential equation of the $2n$th order studied in the paper and introduce necessary notation and concepts. We also give necessary and sufficient conditions for the problem studied to be self-adjoint and positive semidefinite. These conditions were proved in [1]. Section 4 represents the essential part of the paper. Namely, it shows how the original boundary value problem is transformed to initial value problems and studies the properties of the resulting initial value problems. Theorems 4 and 5 contained there are the main results. Section 5 has an algorithmic character and describes algorithms based on the special transfer of boundary conditions introduced in Section 4. In Section 6, numerical stability of the algorithms is discussed. It is shown that all the algorithms of Section 5 are feasible and stable numerically. Theorem 6 of this section is another of the main results of the paper. Section 6 contains a short conclusion summarizing the results obtained.

2. Preliminaries

For the convenience of the reader, we start with a survey of the fundamental results of the methods based on the idea of the transfer of conditions for a general two-point boundary value problem for a system of N linear ordinary differential equations as discussed in [6], [7]. All the proofs of the statements quoted here may be found in [6], [7]. The general theory of the transfer of conditions deals with the system

$$x'(t) + A(t)x(t) = f(t) \quad \text{a.e. in} \quad (a, b), \tag{1}$$

where $x(t)$ and $f(t)$ are $N \times 1$ vectors, $A(t)$ is an $N \times N$ matrix, and (a, b) is a bounded interval. We suppose the entries of the matrix $A(t)$ and the components of the vector $f(t)$ to be Lebesgue-integrable functions. The boundary conditions are supposed to be separated of the form

$$Ux(a) = u, \tag{2}$$

$$Vx(b) = v, \tag{3}$$

where U and V are in general rectangular matrices, with the number of columns equal to N.

Definition 1. The following problem will be called Problem ψ: A vector $x(t)$ absolutely continuous on $[a, b]$ is sought that satisfies the following requirements:

1. $x'(t) + A(t)x(t) = f(t)$ a.e. in (a, b),
2. $Ux(a) = u$ and $Vx(b) = v$.

Now we formulate theorems on the transfer of the conditions for Problem ψ and define basic algorithms for the solution of the problem. For the sake of definiteness let the matrices U and V have n_1 and n_2 rows, respectively, and suppose that the matrices U and V have the maximum rank. Typically, $n_1 + n_2 = N$, which will be the case applied later in the paper.

Theorem 1. *Let $D(t)$ be an absolutely continuous $n_1 \times N$ matrix and $d(t)$ an absolutely continuous vector with n_1 components satisfying the equations*

$$D'(t) = D(t)A(t) + Z_1(t, D(t), d(t))D(t) \quad \text{a.e. in } (a,b), \tag{4}$$

$$d'(t) = D(t)f(t) + Z_1(t, D(t), d(t))d(t) \quad \text{a.e. in } (a,b), \tag{5}$$

and the initial conditions

$$D(a) = K_1 U, \tag{6}$$

$$d(a) = K_1 u, \tag{7}$$

where $Z_1(t, D, d)$ is an $n_1 \times n_1$ matrix such that $Z_1(t, D(t), d(t)) \in \mathcal{L}(a,b)$ and K_1 is a nonsingular matrix of order n_1.
Then

$$D(t)x(t) = d(t) \quad \text{for every } t \in [a,b] \tag{8}$$

for any function $x(t)$ satisfying (1), (2), *i.e., also for every solution of Problem ψ.*

This theorem brings us to the idea of the transfer of the left boundary condition (2) to the whole interval $[a, b]$. Equation (8) is called the *transferred condition* (2). We say that the matrix $D(t)$ and the vector $d(t)$ realize the transfer of the condition (2). Analogously we can formulate the theorem on the transfer of the right boundary condition.

Theorem 2. *Let $C(t)$ be an absolutely continuous $n_2 \times N$ matrix and $c(t)$ an absolutely continuous vector with n_2 components satisfying the equations*

$$C'(t) = C(t)A(t) + Z_2(t, C(t), c(t))C(t) \quad \text{a.e. in } (a,b), \tag{9}$$

$$c'(t) = C(t)f(t) + Z_2(t, C(t), c(t))c(t) \quad \text{a.e. in } (a,b), \tag{10}$$

and the initial conditions (this time at the point b)

$$C(b) = K_2 V, \tag{11}$$

$$c(b) = K_2 v, \tag{12}$$

where $Z_2(t, C, c)$ is an $n_2 \times n_2$ matrix such that $Z_2(t, C(t), c(t)) \in \mathcal{L}(a,b)$ and K_2 is a nonsingular matrix of order n_2.

Then

$$C(t)x(t) = c(t) \quad \textit{for every } t \in [a,b] \tag{13}$$

for any function $x(t)$ satisfying (1), (3), i.e., also for every solution of Problem ψ.

The transfer of conditions is not determined uniquely. There is some arbitrariness in the choice of the matrices $Z_1(t, D(t), d(t))$, $Z_2(t, C(t), c(t))$, K_1, and K_2. Just this gives us a variety of possible transfers and a variety of methods among which we look for the proper ones from the point of view of their numerical realization. Let us pay attention to the connection between two different transfers of the left condition. Let the matrix $D(t)$ and the vector $d(t)$ fulfil the assumptions of Theorem 1. Let matrix $^\circ D(t)$ and a vector $^\circ d(t)$ satisfy the assumptions of Theorem 1 except that in (4) and (5), the matrix $Z_1(t, D(t), d(t))$ is replaced by a generally different matrix $^\circ Z_1(t, ^\circ D(t), ^\circ d(t))$ and in (6) and (7), the matrix K_1 is replaced by a matrix $^\circ K_1$. Then the following lemma holds.

Lemma 1. *There exists an absolutely continuous and nonsingular matrix $K(t)$ on $[a,b]$ such that*

$$^\circ D(t) = K(t)D(t) \quad \textit{for } t \in [a,b], \tag{14}$$

$$^\circ d(t) = K(t)d(t) \quad \textit{for } t \in [a,b]. \tag{15}$$

This lemma implies that the transferred condition (8) is equivalent to the condition

$$^\circ D(t)x(t) = {}^\circ d(t).$$

To be able to study the solvability of the algebraic equations resulting from the transfer of conditions we will use the following lemma. We denote by $\operatorname{rank}(A)$ the rank of the matrix A.

Lemma 2. *The following statements hold on the interval $[a,b]$:*

1. $\operatorname{rank}(D(t)) = \text{const.}$
2. $\operatorname{rank}(D(t), d(t)) = \text{const.}$
3. $\operatorname{rank}(C(t)) = \text{const.}$
4. $\operatorname{rank}(C(t), c(t)) = \text{const.}$
5. $\operatorname{rank}\begin{pmatrix} D(t) \\ C(t) \end{pmatrix} = \text{const.}$
6. $\operatorname{rank}\begin{pmatrix} D(t), d(t) \\ C(t), c(t) \end{pmatrix} = \text{const.}$

Theorems 1 and 2 imply that every solution of Problem ψ satisfies the equation

$$\begin{pmatrix} D(t) \\ C(t) \end{pmatrix} x(t) = \begin{pmatrix} d(t) \\ c(t) \end{pmatrix} \quad \text{for } t \in [a, b]. \tag{16}$$

Lemma 2 (in particular assertions 5 and 6) implies that system (16) has a solution in the whole interval $[a, b]$ provided it has a solution at a single point of the interval $[a, b]$. Similarly, system (16) has a unique solution in the whole interval $[a, b]$ provided it has a unique solution at a single point of the interval $[a, b]$. Moreover, the following statement may be proved.

Theorem 3.

1. *Every solution of Problem ψ satisfies* (16) *for any $t \in [a, b]$.*
2. *System* (16) *has a solution at any $t \in [a, b]$ if and only if Problem ψ has a solution.*
3. *System* (16) *has exactly one solution for any $t \in [a, b]$ if and only if Problem ψ has exactly one solution.*

Provided Problem ψ has a unique solution, this solution may be found at any $t \in [a, b]$ by solving (16). However, there are additional algorithms of the transfer of conditions that may be preferable in particular situations. Now we will introduce two algorithms to be applied later to the boundary value problem for a self-adjoint equation of $2n$th order. At this moment, we will give a general form of these algorithms. In Section 4 we will present proper modifications useful and effective for solving the boundary value problems treated in this paper.

Algorithm A ([6], [7]). Let Problem ψ have a solution. We choose an absolutely continuous matrix $R(t)$ such that the matrix $\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}$ is nonsingular for all $t \in [a, b]$. We look for a vector $r(t)$ solving the differential equation

$$r'(t) = R(t)f(t) + (R'(t) - R(t)A(t)) \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} \begin{pmatrix} d(t) \\ r(t) \end{pmatrix} \tag{17}$$

from the right to the left with the initial condition at the point b given as

$$r(b) = R(b)p, \tag{18}$$

where p is a solution of the equation

$$\begin{pmatrix} D(b) \\ V \end{pmatrix} p = \begin{pmatrix} d(b) \\ v \end{pmatrix} \tag{19}$$

(the system (19) has a solution according to Theorem 3).

Then the solution $x(t)$ of Problem ψ is found from the system

$$\begin{pmatrix} D(t) \\ R(t) \end{pmatrix} x(t) = \begin{pmatrix} d(t) \\ r(t) \end{pmatrix} \quad \text{for all } t \in [a,b]. \tag{20}$$

Algorithm B ([6], [7]). Let Problem ψ have a solution. We choose an absolutely continuous matrix $R(t)$ such that the matrix $\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}$ is nonsingular for all $t \in [a,b]$. We look for a vector $r(t)$ solving the differential equation

$$r'(t) = R(t)f(t) + (R'(t) - R(t)A(t)) \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} \begin{pmatrix} r(t) \\ c(t) \end{pmatrix} \tag{21}$$

from the left to the right with the initial condition at the point a given as

$$r(a) = R(a)p, \tag{22}$$

where p is a solution of the equation

$$\begin{pmatrix} U \\ C(a) \end{pmatrix} p = \begin{pmatrix} u \\ c(a) \end{pmatrix} \tag{23}$$

(the system (23) has a solution according to Theorem 3).

Then the solution $x(t)$ of Problem ψ is found from the system

$$\begin{pmatrix} R(t) \\ C(t) \end{pmatrix} x(t) = \begin{pmatrix} r(t) \\ c(t) \end{pmatrix} \quad \text{for all } t \in [a,b]. \tag{24}$$

In order to apply Algorithms A and B efficiently, it is requirable that the matrices

$$\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1}, \quad \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1}$$

in Eqs. (17), (21) be given by simple expressions (this will be the case in Algorithms 2, 3, 5, and 6 below) or that they would not occur in the respective expressions at all (Algorithms 1 and 4).

3. On a selfadjoint boundary value problem for a differential equation of $2n$th order

Consider the boundary value problem for a self-adjoint differential equation of $2n$th order

$$\ell y \equiv \sum_{i=0}^{n} (-1)^i (p_{n-i}(t) y^{(i)}(t))^{(i)} = q(t) \quad \text{a.e. in } (a,b), \tag{25}$$

where the coefficients of the equation satisfy the following requirements: $1/p_0(t) \in \mathcal{L}(a,b)$, $p_i(t) \in \mathcal{L}(a,b)$ for $i = 1, \ldots, n$, and $q(t) \in \mathcal{L}^2(a,b)$. This last requirement that enables us to treat ℓ as an operator in $\mathcal{L}^2$ will be weakened later as this assumption will not be necessary for the application of our algorithms.

First of all let us introduce the concept of quasiderivatives in order to be able to formulate the boundary conditions for our problem.

Definition 2. We say that a function $y(t)$ has all quasiderivatives up to the $2n$th order provided that the following $2n$ functions exist:

$y^{[k]}(t) = y^{(k)}(t)$ for $k = 1, 2, \ldots, n-1$,
$y^{[n]}(t) = p_0(t) y^{(n)}(t)$,
$y^{[n+j]}(t) = p_j(t) y^{(n-j)}(t) - (y^{[n+j-1]}(t))'$ for $j = 1, \ldots, n$.

The last $n+1$ equations are assumed to hold almost everywhere and the first $2n-1$ quasiderivatives are assumed to be absolutely continuous.

In addition, we define $y^{[0]}(t) = y(t)$ and put $x_i(t) = y^{[i-1]}(t)$ for $i = 1, \ldots, 2n$. Let us introduce the vector $x(t) = (x_1(t), \ldots, x_{2n}(t))^T$. Consider the boundary condition for the differential equation (25) in the form

$$W_1 x(a) + W_2 x(b) = w, \tag{26}$$

where W_1 and W_2 are square matrices of order $2n$ and the vector w has $2n$ components. Let S be the set of all functions $y(t)$ whose quasiderivatives $y^{[k]}(t)$, $k = 0, 1, \ldots, 2n-1$, are absolutely continuous and $y^{[2n]}(t) \in \mathcal{L}^2$. Then S obviously is the largest linear set for which the operation ℓy has a natural sense and the operator ℓ can be considered as an operator in $\mathcal{L}^2$. The matrices W_1 and W_2 have to satisfy certain conditions in order that the boundary problem be self-adjoint. Let us formulate these conditions. For that reason we divide the matrices W_1 and W_2 into blocks:

$$W_1 = (B_1, B_2), \quad W_2 = (B_3, B_4),$$

where the matrices B_i $(i = 1, \ldots, 4)$ have n columns. Let T be the square matrix of order n defined as

$$T = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \quad \text{for } n \geqslant 2$$

and $T = 1$ for $n = 1$ (i.e., the matrix has 1's on the adjoint diagonal and zeros everywhere else).

Now we can formulate two known lemmas (see [1]).

Lemma 3. *A necessary and sufficient condition for problem* (25), (26) *to be self-adjoint is*

$$B_1TB_2^T - B_2TB_1^T = B_3TB_4^T - B_4TB_3^T \quad \text{and} \quad \operatorname{rank}(W_1, W_2) = 2n. \tag{27}$$

Lemma 4. *Let $p_i(t) \geqslant 0$ a.e. in (a, b) for $i = 0, \ldots, n$. A necessary and sufficient condition for the self-adjoint problem* (25), (26) *to be positive semidefinite is that the matrix*

$$B_1TB_2^T - B_3TB_4^T$$

is negative semidefinite.

In the following we will consider only boundary value problems with separated conditions, i.e., we will assume that the boundary conditions are of the form

$$Ux(a) = u, \tag{28}$$

$$Vx(b) = v, \tag{29}$$

where U and V are $n \times 2n$ matrices and the vectors u and v have n components. Let us divide the matrices U and V into blocks

$$U = (U_1, U_2), \qquad V = (V_1, V_2),$$

where U_1, U_2, V_1, and V_2 are square matrices.

Condition (27) of Lemma 3 turns into two equations

$$U_1TU_2^T = U_2TU_1^T, \tag{30}$$

$$V_1TV_2^T = V_2TV_1^T \tag{31}$$

and the requirement that $\operatorname{rank}(U) = \operatorname{rank}(V) = n$. The condition of Lemma 4 turns into the conditions that the matrix $U_1 T U_2^T$ is negative semidefinite and the matrix $V_1 T V_2^T$ is positive semidefinite.

4. Transfer of the conditions for a selfadjoint positive semidefinite boundary value problem

Throughout the remainder of the paper we suppose only (because of the techniques of the transfer of conditions used):

1. $1/p_0(t), q(t), p_i(t) \in \mathcal{L}(a,b)$, $i = 1, \ldots, n$. (Note that we do not need to assume $q(t) \in \mathcal{L}^2$ in what follows.)
2. $p_i(t) \geqslant 0$ a.e. in (a,b) for $i = 0, 1, \ldots, n$.
3. The matrix $U_1 T U_2^T$ is symmetric and negative semidefinite.
   The matrix $V_1 T V_2^T$ is symmetric and positive semidefinite.
4. The ranks of the matrices U and V equal n.

First we will replace the equation of the $2n$th order by the system of $2n$ equations of the first order in a standard way. The definition of quasiderivatives implies that the introduced vector $x(t)$ satisfies the differential equation

$$(32)\qquad x'(t) + \begin{bmatrix} 0 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -1 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & -1/p_0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -p_1 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & -p_2 & 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -p_{n-1} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ -p_n & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -q(t) \end{bmatrix}$$

a.e. in (a,b).

Now we will treat problem (25), (28), (29) as Problem ψ, where equation (32) corresponds to the differential equation (1) and conditions (28) and (29) correspond to the boundary conditions. That also defines the class in which we seek the solution of our problem: we demand that the vector $x(t)$ be absolutely continuous, i.e., we seek such a function $y(t)$ that it is absolutely continuous, together with all its quasiderivatives up to the order $2n-1$, and such that the $2n$ quasiderivative satisfies the equation

$$y^{[2n]}(t) = q(t) \quad \text{a.e. in } (a,b), \tag{33}$$

where owing to Definition 2 we have

$$y^{[2n]}(t) = \sum_{i=1}^{n} (-1)^i \big(p_{n-i}(t) y^{(i)}(t)\big)^{(i)}. \tag{34}$$

Definition 3. Let the above assumptions 1 to 4 be satisfied. Then the following problem will be called Problem ψ_{2n}: A vector $x(t)$ absolutely continuous on $[a,b]$ is sought that satisfies (32) and boundary conditions (28) and (29).

Equation (32) is of the form (1). Let the matrix $A(t)$ stand for the corresponding matrix of equation (32) and the vector $f(t)$ for the corresponding right-hand side. We will divide the matrix $A(t)$ into blocks as follows:

$$A(t) = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix},$$

where $A_i(t)$ $(i = 1,\dots,4)$ are $n \times n$ matrices. The vector $f(t)$ will be divided into two vectors,

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

such that each $f_i(t)$ $(i = 1,2)$ has n components. Equation (32) implies that $f_1(t) = 0$.

We have

$$A_4 = -A_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

for $n > 1$ and $A_1 = A_4 = 0$ for $n = 1$.

First, let us examine the properties of the matrices that realize the general transfer of the conditions for a self-adjoint and positive definite problem. Then we will deduce from these properties a certain particular transfer that will make full use of the symmetry property as well as the sign properties of our problem.

Let us divide the matrix $D(t)$ from Theorem 1 into blocks

$$D(t) = (D_1(t), D_2(t)),$$

where $D_1(t)$ and $D_2(t)$ are square matrices. Analogously, we divide the matrix $C(t)$ from Theorem 2 into blocks

$$C(t) = (C_1(t), C_2(t)),$$

where $C_1(t)$ and $C_2(t)$ are square matrices.

Differential equation (4) can now be rewritten as

$$D_1'(t) = D_1(t)A_1(t) + D_2(t)A_3(t) + Z_1(t, D(t), d(t))D_1(t), \tag{35}$$

$$D_2'(t) = D_1(t)A_2(t) + D_2(t)A_4(t) + Z_1(t, D(t), d(t))D_2(t), \tag{36}$$

and similarly for the differential equation (9) we have

$$C_1'(t) = C_1(t)A_1(t) + C_2(t)A_3(t) + Z_2(t, C(t), c(t))C_1(t), \tag{37}$$

$$C_2'(t) = C_1(t)A_2(t) + C_2(t)A_4(t) + Z_2(t, C(t), c(t))C_2(t). \tag{38}$$

Lemma 5. *Under the above assumptions the following statements hold:*

1. *The matrix $D_1(t)TD_2^T(t)$ is symmetric and negative semidefinite for all $t \in [a, b]$.*
2. *The matrix $C_1(t)TC_2^T(t)$ is symmetric and positive semidefinite for all $t \in [a, b]$.*

Proof. It is sufficient to carry out the proof of the first assertion only for the case $Z_1(t, D, d) = 0$ because according to Lemma 1, we obtain any other transfer of conditions by left-multiplying the matrix $D_1(t)TD_2^T(t)$ by a nonsingular matrix $K(t)$ and right-multiplying it by a matrix $K^T(t)$. These operations do not change the properties mentioned in Lemma 5. Let thus $Z_1(t, D, d) = 0$ for the purpose of the proof. Then

$$\begin{aligned}(D_1(t)TD_2^T(t))' = {} & D_1(t)A_1(t)TD_2^T(t) + D_2(t)A_3(t)TD_2^T(t) \\ & + D_1(t)TA_2^T(t)D_1^T(t) + D_1(t)TA_4^T(t)D_2^T(t)\end{aligned} \tag{39}$$

holds. Obviously $TA_4^T(t) = -A_1(t)T$ and therefore (39) reduces to

$$(D_1(t)TD_2^T(t))' = D_2(t)A_3(t)TD_2^T(t) + D_1(t)TA_2^T(t)D_1^T(t). \tag{40}$$

The matrices $A_3(t)T$ and $TA_2^T(t)$ are symmetric and negative semidefinite and thus the right-hand side of (40) is also a symmetric and negative semidefinite matrix. The initial condition (6) implies

$$D_1(a)TD_2^T(a) = K_1U_1TU_2^TK_1^T. \tag{41}$$

Then, equations (40) and (41) imply the symmetry property of the matrix $D_1(t) \times TD_2^T(t)$. To prove that this matrix is negative semidefinite, let us consider an arbitrary constant vector w. Then we obtain

$$(w^TD_1(t)TD_2^T(t)w)' \leqslant 0 \quad \text{a.e. in } (a,b). \tag{42}$$

Under our assumptions we have

$$w^TD_1(a)TD_2^T(a)w = w^TK_1U_1TU_2^TK_1^Tw \leqslant 0. \tag{43}$$

Using inequalities (42) and (43), we find that

$$w^TD_1(t)TD_2^T(t)w \leqslant 0$$

for all $t \in [a,b]$ and for all vectors w. Thus we have proved the first assertion of the lemma. The second assertion can be proved in an analogous way. □

Lemma 6. *The matrices $D_1(t) - D_2(t)T$ and $C_1(t) + C_2(t)T$ are nonsingular for all $t \in [a,b]$.*

P r o o f. The rank of the matrix $D(t)$ equals n, and therefore the matrix

$$D(t)D^T(t) = D_1(t)D_1^T(t) + D_2(t)D_2^T(t)$$

is positive definite. Consider the product

$$\begin{aligned}
&(D_1(t) - D_2(t)T)(D_1(t) - D_2(t)T)^T \\
&\quad = D_1(t)D_1^T(t) + D_2(t)D_2^T(t) - D_1(t)TD_2^T(t) - D_2(t)TD_1^T(t) \\
&\quad = D(t)D^T(t) + (-2D_1(t)TD_2^T(t)).^1
\end{aligned}$$

This product is a positive definite matrix because it is the sum of a positive definite and a positive semidefinite matrix. Thus the matrix $D_1(t) - D_2(t)T$ is nonsingular.

Analogously we can prove that the matrix $C_1(t) + C_2(t)T$ is nonsingular. We consider the product $(C_1(t) + C_2(t)T)(C_1(t) + C_2(t)T)^T$ to this end. □

[1] We use the equality $T^2 = I$ here.

Let us put

$$G(t) = (D_1(t) - D_2(t)T)^{-1}D_1(t), \tag{44}$$

$$g(t) = (D_1(t) - D_2(t)T)^{-1}d(t), \tag{45}$$

$$H(t) = (C_1(t) + C_2(t)T)^{-1}C_1(t), \tag{46}$$

$$h(t) = (C_1(t) + C_2(t)T)^{-1}c(t). \tag{47}$$

Even if the construction of the above matrices and vectors is based on a fixed choice of the transfer, i.e., on the choice of the matrices K_1, K_2, $Z_1(t, D(t), d(t))$ and $Z_2(t, C(t), c(t))$, neither the matrix $G(t)$ nor the vector $g(t)$ depend on the choice of the matrices K_1 and $Z_1(t, D(t), d(t))$. This is implied by Lemma 1 trivially. Analogously, the matrix $H(t)$ and the vector $h(t)$ do not depend on the choice of the matrices K_2 and $Z_2(t, C(t), c(t))$. In fact, we have defined a unique canonical transfer of conditions for the problem discussed here.

Lemma 7. *For all $t \in [a, b]$, the matrices $G(t)$, $I - G(t)$, $H(t)$, and $I - H(t)$ are symmetric and positive semidefinite. Moreover, the following equalities hold:*

$$(D_1(t) - D_2(t)T)^{-1}D_2(t) = (G(t) - I)T, \tag{48}$$

$$(C_1(t) + C_2(t)T)^{-1}C_2(t) = (I - H(t))T. \tag{49}$$

Proof. The matrix $(D_1(t) - D_2(t)T)D_1^T(t)$ is obviously symmetric and positive semidefinite. We can write the matrix $G(t)$ in the form

$$G(t) = (D_1(t) - D_2(t)T)^{-1}[D_1(t)(D_1(t) - D_2(t)T)^T]((D_1(t) - D_2(t)T)^{-1})^T,$$

hence the matrix $G(t)$ is also symmetric and positive definite for all $t \in [a, b]$. The matrix $(D_1(t) - D_2(t)T)TD_2^T(t)$ is symmetric and negative semidefinite. We have

$$\begin{aligned}&(D_1(t) - D_2(t)T)^{-1}D_2(t)T \\ &\quad = (D_1(t) - D_2(t)T)^{-1}[D_2(t)T(D_1(t) - D_2(t)T)^T]((D_1(t) - D_2(t)T)^{-1})^T\end{aligned}$$

and thus the matrix $(D_1(t) - D_2(t)T)^{-1}D_2(t)T$ is symmetric and negative semidefinite. We will prove relation (48) in the following way. Obviously we have

$$(D_1(t) - D_2(t)T)^{-1}(D_1(t) - D_2(t)T) = I,$$

hence

$$(D_1(t) - D_2(t)T)^{-1}D_1(t) - (D_1(t) - D_2(t)T)^{-1}D_2(t)T = I.$$

The last identity implies (48) immediately. Analogously we can prove the assertion on the matrices $H(t)$ and $I - H(t)$ together with relation (49). □

Remark. Lemma 7 says that the eigenvalues of the matrices $G(t)$ and $H(t)$ lie in the interval $[0,1]$. As a matter of fact, this gives us an estimate of the norm of the matrices $G(t)$ and $H(t)$, which will be of use in the investigations of the numerical stability questions.

Now, left-multiplying the transferred condition (8) by the matrix $(D_1(t) - D_2(t)T)^{-1}$ and doing some simple modification, we obtain

$$(G(t), (G(t) - I)T)x(t) = g(t) \quad \text{for } t \in [a,b]. \tag{50}$$

We can determine the value of the matrix $G(t)$ and of the vector $g(t)$ at the point a knowing the matrix U and the vector u only. If we knew the differential equations satisfied by the matrix $G(t)$ and the vector $g(t)$, we would construct the transferred condition (41) directly, solving a certain initial value problem. Let us compute the derivative of the matrix $G(t)$,

$$\begin{aligned} G'(t) = & -(D_1(t) - D_2(t)T)^{-1}(D_1'(t) - D_2'(t)T)(D_1(t) - D_2(t)T)^{-1}D_1(t) \\ & + (D_1(t) - D_2(t)T)^{-1}D_1'(t). \end{aligned}$$

Substituting, in accordance with equations (35) and (36), for the derivatives of the matrices $D_1(t)$ and $D_2(t)$, we obtain the matrix Riccati equation

$$\begin{aligned} G'(t) = {} & G(t)A_2(t)TG(t) - (G(t) - I)TA_3(t)(G(t) - I) - G(t)A_1(G(t) - I) \\ & - (G(t) - I)A_1^TG(t) \quad \text{a.e. in } (a,b). \end{aligned} \tag{51}$$

Analogously we have for the vector $g(t)$

$$\begin{aligned} g'(t) = & -G(t)(A_1 - A_2(t)T)g(t) - (G(t) - I)T(A_3(t) - A_4T)g(t) \\ & + (G(t) - I)Tf_2(t) \quad \text{a.e. in } (a,b). \end{aligned} \tag{52}$$

The initial conditions for these differential equations are

$$G(a) = (U_1 - U_2T)^{-1}U_1, \tag{53}$$

$$g(a) = (U_1 - U_2T)^{-1}u. \tag{54}$$

□

Thus we have proved the following theorem.

Theorem 4. *Under the above assumptions, there exist an absolutely continuous matrix $G(t)$ and an absolutely continuous vector $g(t)$ which are the unique solutions* *of the initial value problems for differential equations* (51) *and* (52) *with initial conditions* (53) *and* (54), *respectively. Any solution $x(t)$ of Problem ψ_{2n} satisfies the transferred condition* (50).

Analogously, a theorem can be proved also for the following particular transfer of the right boundary condition.

Theorem 5. *Under the above assumptions there exist an absolutely continuous matrix $H(t)$ and an absolutely continuous vector $h(t)$ which are the unique solutions of the initial value problems for the differential equations*

$$\begin{aligned}(55)\qquad H'(t) = &-H(t)A_2(t)TH(t) - (I - H(t))TA_3(t)(I - H(t))\\ &+ H(t)A_1(I - H(t)) + (I - H(t))A_1^T H(t) \quad \text{a.e. } in\ (a,b),\\ (56)\qquad h'(t) = &-H(t)(A_1 + A_2(t)T)h(t) - (I - H(t))T(A_3(t) + A_4T)h(t)\\ &+ (I - H(t))Tf_2(t) \quad \text{a.e. } in\ (a,b),\end{aligned}$$

with the initial conditions

$$H(b) = (V_1 + V_2T)^{-1}V_1, \tag{57}$$

$$h(b) = (V_1 + V_2T)^{-1}v, \tag{58}$$

respectively. Any solution $x(t)$ of Problem ψ_{2n} satisfies the transferred condition

$$(H(t), (I - H(t))T)x(t) = h(t) \quad for\ t \in [a,b]. \tag{59}$$

Theorems 4 and 5 define a particular transfer of conditions that will be called the *canonical transfer* and that exploits the given symmetry and sign properties. The matrix differential equation (4) represents the system of $2n^2$ equations whereas the matrix equation (51) represents the system of $(n^2 + n)/2$ equations only. Further advantages of this canonical transfer will be pointed out later.

5. Particular methods for the canonical transfer

In this section, the transfer of conditions will mean the canonical transfer treated by Theorems 4 and 5. Therefore, in Algorithms A and B, we have

$$D(t) = (G(t), (G(t) - I)T), \tag{60}$$

$$d(t) = g(t), \tag{61}$$

$$C(t) = (H(t), (I - H(t))T), \tag{62}$$

$$c(t) = h(t). \tag{63}$$

Now we will describe three algorithms that come from Algorithm A by a particular choice of the matrix $R(t)$.

Algorithm 1. Let our problem have a unique solution. We choose $R(t) = C(t)$ in Algorithm A. That can be done because the matrix $\begin{pmatrix} D(t) \\ C(t) \end{pmatrix}$ is nonsingular according to Theorem 3. Without any difficulty we determine that $r(t) = c(t)$. Therefore, the solution $x(t)$ of our problem is found from the system of $2n$ equations

$$\begin{pmatrix} D(t) \\ C(t) \end{pmatrix} x(t) = \begin{pmatrix} d(t) \\ c(t) \end{pmatrix}. \tag{64}$$

The matrix $(D(t), d(t))$ is built up by solving the initial value problem (51), (52), (53) and (54). Analogously, the matrix $(C(t), c(t))$ is built up by solving the initial value problem (55), (56), (57) and (58). It is sufficient to store these matrices only at the points where we are interested in the solution (i.e. at the points when the solution is output). System (64) is now of the particular form

$$\begin{pmatrix} G(t), & (G(t)-I)T \\ H(t), & (I-H(t))T \end{pmatrix} x(t) = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}.$$

The solution of this system may be transformed to solving the system of n linear algebraic equations with the matrix

$$G(t) + H(t) - 2G(t)H(t).$$

Algorithm 2. We put $R(t) = (I, T)$ in Algorithm A. Then

$$\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} = \begin{pmatrix} G(t), & (G(t)-I)T \\ I, & T \end{pmatrix}^{-1} = \begin{pmatrix} I, & I-G(t) \\ -T, & TG(t) \end{pmatrix}. \tag{65}$$

Differential equation (17) for the function $r(t)$ in Algorithm A acquires the form

$$r'(t) = Tf_2(t) - (I, T)\begin{pmatrix} A_1 & A_2(t) \\ A_3(t) & A_4 \end{pmatrix}\begin{pmatrix} I, & I-G(t) \\ -T, & TG(t) \end{pmatrix}\begin{pmatrix} g(t) \\ r(t) \end{pmatrix}. \tag{66}$$

The solution $x(t)$ is obtained from the equation

$$x(t) = \begin{pmatrix} I, & I-G(t) \\ -T, & TG(t) \end{pmatrix}\begin{pmatrix} g(t) \\ r(t) \end{pmatrix}. \tag{67}$$

Equation (66) is solved from the right to the left. Therefore, we must store the matrix $G(t)$ and the vector $g(t)$ during the solution of the initial value problems at as many points as are needed for the numerical integration of equation (66). Usually we have incomparably more such points than the points where the solution is output. The following Algorithm 3 avoids this unpleasant fact. The matrix $R(t)$ is chosen in such a way that the equation (17) becomes a quadrature and thus can be solved simultaneously with equations (51) and (52) from the left to the right with an initial condition. In the end, the values of the function $r(t)$ can be corrected by adding a constant at the points where we are interested in the solution. Hence this algorithm will inherit the character of Algorithm 1.

Algorithm 3. We choose $R(t) = (Q(t), Q(t)T)$ in Algorithm A, where the matrix $Q(t)$ will be constructed by solving a certain differential equation but from the left to the right. For our algorithm to be practicable the matrix $Q(t)$ has to be nonsingular. Then

$$(68)\qquad \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} = \begin{pmatrix} G(t), & (G(t)-I)T \\ Q(t), & Q(t)T \end{pmatrix}^{-1} = \begin{pmatrix} I, & (I-G(t))Q^{-1}(t) \\ -T, & TG(t)Q^{-1}(t) \end{pmatrix}.$$

Requiring equation (17) to be a quadrature, the following equation has to hold

$$\begin{aligned} &(R'(t) - R(t)A(t)) \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} \begin{pmatrix} d(t) \\ r(t) \end{pmatrix} \\ &= (R'(t) - R(t)A(t)) \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} \begin{pmatrix} d(t) \\ 0 \end{pmatrix} \quad \text{a.e. in } (a,b). \end{aligned}$$

This condition is fulfilled provided that the matrix $Q(t)$ satisfies the differential equation

$$(69)\quad Q'(t) = Q(t)(A_2(t)TG(t) - A_1^TG(t) - A_1G(t) - TA_3(t)G(t) + A_1 + TA_3(t)).$$

This is a homogeneous linear differential equation and the condition of nonsingularity is satisfied if we choose

$$(70)\qquad Q(a) = I.$$

Let us sum up: We seek matrices $G(t)$ and $Q(t)$ and vectors $g(t)$ and $r(t)$ when solving the above mentioned differential equations from the left to the right. Then, at the points where we are interested in the solution, the function $r(t)$ is corrected by adding a certain constant in such a way that the condition (18) is satisfied.

In addition, we will describe three algorithms that come from Algorithm B of Section 1 by a particular choice of the matrix $R(t)$.

Algorithm 4. Let our problem have a unique solution. We choose $R(t) = D(t)$ in Algorithm B. That can be done because the matrix $\begin{pmatrix} D(t) \\ C(t) \end{pmatrix}$ is nonsingular according to Theorem 3. Without any difficulty we determine that $r(t) = d(t)$. Therefore, the solution $x(t)$ of our problem is found from the system of $2n$ equations

$$\begin{pmatrix} D(t) \\ C(t) \end{pmatrix} x(t) = \begin{pmatrix} d(t) \\ c(t) \end{pmatrix}. \tag{71}$$

As the result of this choice we obtain the same algorithm as Algorithm 1.

Algorithm 5. We put $R(t) = (I, -T)$ in Algorithm B. Then

$$\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} = \begin{pmatrix} I, & -T \\ H(t), & (I - H(t))T \end{pmatrix}^{-1} = \begin{pmatrix} I - H(t), & I \\ -TH(t), & T \end{pmatrix}. \tag{72}$$

The differential equation for the function $r(t)$ acquires the form

$$r'(t) = -Tf_2(t) - (I, -T) \begin{pmatrix} A_1 & A_2(t) \\ A_3(t) & A_4 \end{pmatrix} \begin{pmatrix} I - H(t), & I \\ -TH(t), & T \end{pmatrix} \begin{pmatrix} r(t) \\ h(t) \end{pmatrix}. \tag{73}$$

The solution $x(t)$ is obtained from the equation

$$x(t) = \begin{pmatrix} I - H(t), & I \\ -TH(t), & T \end{pmatrix} \begin{pmatrix} r(t) \\ h(t) \end{pmatrix}. \tag{74}$$

Equation (73) is solved from the left to the right. Therefore, we must store the matrix $H(t)$ and the vector $h(t)$ at as many points as are needed for the numerical integration of equation (73). Usually we have incomparably more such points than the points where the solution is output. The following algorithm avoids this unpleasant fact. The matrix $R(t)$ is chosen in such a way that equation (21) becomes a quadrature and thus can be solved simultaneously with equations (55) and (56) from the right to the left with an initial condition. In the end, the values of the function $r(t)$ can be corrected by adding a constant at the points where we are interested in the solution. Hence, this algorithm will inherit the character of Algorithm 4.

Algorithm 6. We choose $R(t) = (Q(t), -Q(t)T)$ in Algorithm B, where the matrix $Q(t)$ will be constructed by solving a certain differential equation from the right to the left. For our algorithm to be practicable, the matrix $Q(t)$ has to be nonsingular. Then

$$\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} = \begin{pmatrix} Q(t), & -Q(t)T \\ H(t), & (I - H(t))T \end{pmatrix}^{-1} = \begin{pmatrix} (I - H(t))Q^{-1}(t), & I \\ -TH(t)Q^{-1}(t), & T \end{pmatrix}. \tag{75}$$

Requiring equation (21) to be a quadrature, the following equation has to hold:

$$(R'(t) - R(t)A(t)) \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} \begin{pmatrix} r(t) \\ c(t) \end{pmatrix}$$
$$= (R'(t) - R(t)A(t)) \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ c(t) \end{pmatrix} \quad \text{a.e. in } (a, b).$$

This condition is fulfilled provided the matrix $Q(t)$ satisfies the differential equation

$$(76)\quad Q'(t) = Q(t)(-A_2(t)TH(t) - A_1^T H(t) - A_1 H(t) + TA_3(t)H(t) + A_1 - TA_3(t)).$$

This is a homogeneous linear differential equation and the condition of nonsingularity is satisfied if we choose

$$(77)\qquad Q(b) = I.$$

Let us sum up: We seek matrices $H(t)$ and $Q(t)$ and vectors $h(t)$ and $r(t)$ when solving the above mentioned differential equations from the right to the left. Then, at the points where we are interested in the solution, the function $r(t)$ will be corrected by adding a certain constant in such a way that condition (22) is satisfied.

6. Numerical stability

The method of transfer of conditions as described in the previous part of this paper, in particular in Section 1, is simple and rather graceful. It also possesses the advantage that it not only gives a procedure for solving the boundary value problems in question numerically but also provides an answer to the theoretical question about the existence of the boundary value problem treated.

The reader who has followed our discussion of various algorithms of the method up to this point would probably ask why not use the simplest variant of the method of transfer and choose just $Z_1(t, D(t), d(t)) = 0$ or $Z_2(t, C(t), c(t)) = 0$ in the respective formulas of the method. However, this choice is quite often inappropriate from the numerical realization point of view in finite-precision arithmetic. The cause of the trouble here is in fact the same as are the well known numerical problems connected with the shooting method. It is easy to see that the above choice of zero matrices for Z_1, Z_2 results in our solving essentially the same differential equations as are those of the boundary value problem to be solved but with initial conditions. It is well known, however, that the conditioning (or stability) of a boundary value problem for a differential equation may be quite different from that of an initial value problem for the same equation where the growth of the solution modes is not limited by all the boundary conditions. There are many examples from real world practice where initial value problems for the differential equations forming parts of well-conditioned boundary value problems are highly unstable so that the classical shooting method is inapplicable. We even dare say that the more a given problem is stable as a boundary value problem the less stable the corresponding initial value problems may be.

The magnitudes of the entries in the matrices that realize the transfer of conditions may grow or decrease very fast. The main problem here is in the unbearable sensitivity of the solutions to the differential equations realizing the transfer in the case of zero Z_1, Z_2 to the initial conditions. This shortcoming can be avoided by our requiring that the matrices $D(t)$, $C(t)$ that realize the transfer of conditions be "normalized" by which we mean that their norms and the norms of their pseudoinverses be bounded by constants that do not depend on the coefficients of the differential equation in question or on the length of the interval $[a, b]$.

In monographs [6], [7], the analysis of numerical stability of algorithms A and B has been performed. It has been shown that realizing the algorithms for solving the initial value problems involved numerically we bring into being certain inaccuracies, both in the solution of the differential equations and in the computation of the initial conditions for these differential equations. These inaccuracies are considered in the following theorem proved in [6] where also the estimates of the magnitudes of these perturbations in terms of the errors in solving the initial value problems are given.

Theorem 6. *Let the norm of the matrices $D(t)$ and $D^T(t)(D(t)D^T(t))^{-1}$ be bounded by an a priori given constant. Then all inaccuracies made in the realization of the method can be represented as perturbations in the coefficients of the primary Problem ψ whose magnitude can be estimated using the above mentioned constant. An analogous statement is valid also for the transfer of conditions from the right to the left.*

The investigation of numerical stability of the algorithm with an auxiliary matrix $R(t)$ leads to the requirements that the norms of the matrices $R(t)$, $R^T(t)(R(t)R^T(t))^{-1}$ and $\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1}$ (or $\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1}$) be bounded by an a priori given constant.

In what follows, we will show that the canonical transfer of conditions for Problem ψ_{2n} satisfies the assumptions of the previous theorem and thus the transfer is numerically stable. In our further exposition we will estimate norms of certain matrices that are rectangular in general. To this end we will employ the spectral norm that is induced by the Euclidean vector norm (2-norm). The spectral matrix norm of a general matrix A may be expressed as (see e.g. [2], Section 6.2)

$$\|A\| = \sqrt{\varrho(A^*A)} = \sigma_{\max}(A),$$

where ϱ is the spectral radius defined as

$$\varrho(B) = \max\{|\lambda|\colon \det(B - \lambda I) = 0\},$$

and where $\sigma_{\max}(A)$ denotes the largest singular value of A. As this norm is unitarily invariant, it has the property that $\|A^*\| = \|A\|$ and we will exploit this fact in what follows. Therefore, to compute the norm $\|\Phi(t)\|$ of a matrix $\Phi(t)$ of an arbitrary type, we will use any of the formulae

$$\begin{aligned}\|\Phi(t)\| &= \sqrt{\text{max. eigenvalue } (\Phi^T(t)\Phi(t))} \\ &= \sqrt{\text{max. eigenvalue } (\Phi(t)\Phi^T(t))}.\end{aligned} \tag{78}$$

We obtain

$$D(t)D^T(t) = [G(t),\ (G(t) - I)T]\begin{bmatrix} G(t) \\ T(G(t) - I) \end{bmatrix} = G^2(t) + (G(t) - I)^2 \tag{79}$$

and

$$C(t)C^T(t) = [H(t),\ (I - H(t))T]\begin{bmatrix} H(t) \\ T(I - H(t)) \end{bmatrix} = H^2(t) + (I - H(t))^2. \tag{80}$$

Thus

$$\|D(t)\| \leqslant \sqrt{\max_{0 \leqslant \lambda \leqslant 1} (\lambda^2 + (\lambda - 1)^2)} = 1 \tag{81}$$

and

$$\|D^T(t)(D(t)D^T(t))^{-1}\| \leqslant \sqrt{\max_{0 \leqslant \lambda \leqslant 1} \frac{1}{\lambda^2 + (\lambda - 1)^2}} = \sqrt{2}. \tag{82}$$

Analogously we obtain estimate for the norms of the matrices $C(t)$ and $C^T(t) \times (C(t)C^T(t))^{-1}$:

$$\|C(t)\| \leqslant 1 \quad \text{and} \quad \|C^T(t)(C(t)C^T(t))^{-1}\| \leqslant \sqrt{2}. \tag{83}$$

As a result of these bounds, Algorithm 1 (or 4) is numerically stable.

Let us now investigate the norms of the matrices $\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1}$ and $\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1}$ from Algorithms 2 and 5. Equations (65) and (72) imply

$$\left\| \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} \right\| \leqslant \sqrt{\|(I, I - G(t))\|^2 + \|(-T, TG(t))\|^2} \leqslant 2$$

and

$$\left\| \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} \right\| \leqslant \sqrt{\|(I - H(t), I)\|^2 + \|(-TH(t), T)\|^2} \leqslant 2.$$

These results and Theorem 6 imply the numerical stability of Algorithms 2 and 5.

In Algorithms 3 and 6, the norms of matrices $Q(t)$ and $Q^{-1}(t)$ cannot be a priori estimated. They depend on the value of coefficients and on the length of the interval $[a, b]$. However, the numerical experiments performed have given good results.

The structure of the matrices $A_i(t)$, $i = 1, \ldots, 4$, is tremendously simple and the complexity of the above mentioned matrix differential equations does not present any difficulty for the numerical solution.

7. Conclusion

We have presented several methods of solution of self-adjoint boundary value problems for differential equations of the $2n$th order by transforming them to solving sequences of initial value problems. First, we have studied a method where the boundary conditions are transferred both from the left and the right independently while the transfer of the conditions from one of the end-points represents the solution of an initial value problem for one matrix Riccati equation and the solution of an initial value problem for a system of linear differential equations. Then, the solution of our boundary value problem may be obtained by solving a system of linear algebraic equations at those points where we want to compute it.

Second, we described methods where the boundary conditions are transferred from one side only and the solution to the boundary value problem under consideration is obtained in such a way that we construct an auxiliary vector function by solving a system of linear ordinary differential equations and the final solution is then obtained by evaluating just a simple expression.

We have presented algorithms that lead to solving the Riccati differential equations and have shown that for a series of problems these equations possess a unique solution on the whole interval in question and, moreover, that these equations are represented by symmetric matrices whose eigenvalues lie in $[0, 1]$. The methods used so far had to check whether the solutions to the Riccati equations exist on the whole interval in question or whether they do not exceed some a priori given barriers [4], which is not necessary in our canonical transfer of conditions.

As a consequence of our results, we know that the solutions of the initial value problems used in our algorithms exist and are bounded on the whole interval $[a, b]$. Therefore, the algorithms presented in the paper are always feasible under the assumptions given. Finally, we note that the class of boundary value problems discussed here is quite often applicable in the physical science and technology.

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NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS FOR SELFADJOINT DIFFERENTIAL EQUATIONS OF $2n$th ORDER*

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Abstract. The paper is devoted to solving boundary value problems for self-adjoint linear differential equations of $2n$th order in the case that the corresponding differential operator is self-adjoint and positive semidefinite. The method proposed consists in transforming the original problem to solving several initial value problems for certain systems of first order ODEs. Even if this approach may be used for quite general linear boundary value problems, the new algorithms described here exploit the special properties of the boundary value problems treated in the paper. As a consequence, we obtain algorithms that are much more effective than similar ones used in the general case. Moreover, it is shown that the algorithms studied here are numerically stable.

Keywords: ODE, two-point boundary value problem, transfer of boundary conditions, self-adjoint differential equation, numerical solution, Riccati differential equation

MSC 2000: 65L10, 34B05

1. Introduction

The paper is concerned with the numerical solution of boundary value problems for self-adjoint linear differential equations of $2n$th order such that the corresponding differential operator is self-adjoint and positive semidefinite. Our approach consists in transforming the boundary value problem to be solved to the solution of a sequence of initial value problems of a special structure. This makes it possible to employ the standard software for solving initial value problems to solve boundary value problems

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of this type. One of the possibilities how to transform the boundary value problems to solving initial value problems is to use the method of transfer of conditions as described in [6], [7] or the method of invariant imbedding [3], [5]. The principles of these methods could be used for the problems studied here directly, however, they would lead to algorithms that would be unnecessarily elaborate.

In particular, applying the methods of the above type that were devised for general boundary value problems we cannot guarantee, in general, the global solvability of the initial value problems obtained as the result of the above mentioned transformation. Therefore, to be able to obtain algorithms based on the transfer of boundary conditions that would be practicable and to achieve their numerical stability in the general case it is necessary to take certain elaborate measures [6], [7]. This is unnecessary and even superfluous to do with the algorithms devised here for the special boundary value problems considered in the paper. This result is not due to that fact that the original methods designed for general linear boundary value problems would possess the required properties when applied to the special problem considered. It is due to the fact that we have constructed new special methods of the transfer of conditions for the special boundary value problems considered that make use of all the additional information at our disposal here (self-adjointness, positive semidefiniteness).

Hence, the present paper exploits the special form of the problem under consideration and the numerical method proposed is chosen in order to use the additional information, namely all the symmetry and some sign properties. As a consequence, the resulting method is very advantageous as to the effort expended and fulfils all of the numerical stability requirements. In particular, we prove that the initial value problems for the matrix Riccati differential equation that occur in our algorithms have a bounded solution on the whole interval of question. In fact, all the entries of the matrix solution are shown to lie in $[0, 1]$. This is one of the most important results of the paper. To prove the properties of the transfer of conditions in the particular case studied here we need to prove some properties of the general transfer of conditions that have not yet been published and that apply to the special problem considered. We also note that contrary to the invariant imbedding method the coefficients of the equations are not required to be smooth here.

The structure of the paper is as follows. Section 2 contains a survey of the fundamental results of the methods based on the idea of the transfer of conditions for general two-point boundary value problems for systems of first order ODEs. In Section 3, we define the boundary value problem for a self-adjoint differential equation of the $2n$th order studied in the paper and introduce necessary notation and concepts. We also give necessary and sufficient conditions for the problem studied to be self-adjoint and positive semidefinite. These conditions were proved in [1]. Section 4 represents the essential part of the paper. Namely, it shows how the original boundary value problem is transformed to initial value problems and studies the properties of the resulting initial value problems. Theorems 4 and 5 contained there are the main results. Section 5 has an algorithmic character and describes algorithms based on the special transfer of boundary conditions introduced in Section 4. In Section 6, numerical stability of the algorithms is discussed. It is shown that all the algorithms of Section 5 are feasible and stable numerically. Theorem 6 of this section is another of the main results of the paper. Section 6 contains a short conclusion summarizing the results obtained.

2. Preliminaries

For the convenience of the reader, we start with a survey of the fundamental results of the methods based on the idea of the transfer of conditions for a general two-point boundary value problem for a system of N linear ordinary differential equations as discussed in [6], [7]. All the proofs of the statements quoted here may be found in [6], [7]. The general theory of the transfer of conditions deals with the system

$$x'(t) + A(t)x(t) = f(t) \quad \text{a.e. in } (a, b), \tag{1}$$

where $x(t)$ and $f(t)$ are $N \times 1$ vectors, $A(t)$ is an $N \times N$ matrix, and (a, b) is a bounded interval. We suppose the entries of the matrix $A(t)$ and the components of the vector $f(t)$ to be Lebesgue-integrable functions. The boundary conditions are supposed to be separated of the form

$$Ux(a) = u, \tag{2}$$

$$Vx(b) = v, \tag{3}$$

where U and V are in general rectangular matrices, with the number of columns equal to N.

Definition 1. The following problem will be called Problem ψ: A vector $x(t)$ absolutely continuous on $[a, b]$ is sought that satisfies the following requirements:

1. $x'(t) + A(t)x(t) = f(t) \quad$ a.e. in $\quad (a, b)$,
2. $Ux(a) = u$ and $Vx(b) = v$.

Now we formulate theorems on the transfer of the conditions for Problem ψ and define basic algorithms for the solution of the problem. For the sake of definiteness let the matrices U and V have n_1 and n_2 rows, respectively, and suppose that the matrices U and V have the maximum rank. Typically, $n_1 + n_2 = N$, which will be the case applied later in the paper.

Theorem 1. *Let $D(t)$ be an absolutely continuous $n_1 \times N$ matrix and $d(t)$ an absolutely continuous vector with n_1 components satisfying the equations*

$$D'(t) = D(t)A(t) + Z_1(t, D(t), d(t))D(t) \quad \text{a.e. in } (a,b), \tag{4}$$

$$d'(t) = D(t)f(t) + Z_1(t, D(t), d(t))d(t) \quad \text{a.e. in } (a,b), \tag{5}$$

and the initial conditions

$$D(a) = K_1 U, \tag{6}$$

$$d(a) = K_1 u, \tag{7}$$

where $Z_1(t, D, d)$ is an $n_1 \times n_1$ matrix such that $Z_1(t, D(t), d(t)) \in \mathcal{L}(a,b)$ and K_1 is a nonsingular matrix of order n_1.
Then

$$D(t)x(t) = d(t) \quad \text{for every } t \in [a,b] \tag{8}$$

for any function $x(t)$ satisfying (1), (2), *i.e., also for every solution of Problem ψ.*

This theorem brings us to the idea of the transfer of the left boundary condition (2) to the whole interval $[a,b]$. Equation (8) is called the *transferred condition* (2). We say that the matrix $D(t)$ and the vector $d(t)$ realize the transfer of the condition (2). Analogously we can formulate the theorem on the transfer of the right boundary condition.

Theorem 2. *Let $C(t)$ be an absolutely continuous $n_2 \times N$ matrix and $c(t)$ an absolutely continuous vector with n_2 components satisfying the equations*

$$C'(t) = C(t)A(t) + Z_2(t, C(t), c(t))C(t) \quad \text{a.e. in } (a,b), \tag{9}$$

$$c'(t) = C(t)f(t) + Z_2(t, C(t), c(t))c(t) \quad \text{a.e. in } (a,b), \tag{10}$$

and the initial conditions (this time at the point b)

$$C(b) = K_2 V, \tag{11}$$

$$c(b) = K_2 v, \tag{12}$$

where $Z_2(t, C, c)$ is an $n_2 \times n_2$ matrix such that $Z_2(t, C(t), c(t)) \in \mathcal{L}(a,b)$ and K_2 is a nonsingular matrix of order n_2.

Then

$$C(t)x(t) = c(t) \quad \textit{for every } t \in [a,b] \tag{13}$$

for any function $x(t)$ satisfying (1), (3)*, i.e., also for every solution of Problem ψ.*

The transfer of conditions is not determined uniquely. There is some arbitrariness in the choice of the matrices $Z_1(t, D(t), d(t))$, $Z_2(t, C(t), c(t))$, K_1, and K_2. Just this gives us a variety of possible transfers and a variety of methods among which we look for the proper ones from the point of view of their numerical realization. Let us pay attention to the connection between two different transfers of the left condition. Let the matrix $D(t)$ and the vector $d(t)$ fulfil the assumptions of Theorem 1. Let matrix ${}^{\circ}D(t)$ and a vector ${}^{\circ}d(t)$ satisfy the assumptions of Theorem 1 except that in (4) and (5), the matrix $Z_1(t, D(t), d(t))$ is replaced by a generally different matrix ${}^{\circ}Z_1(t, {}^{\circ}D(t), {}^{\circ}d(t))$ and in (6) and (7), the matrix K_1 is replaced by a matrix ${}^{\circ}K_1$. Then the following lemma holds.

Lemma 1. *There exists an absolutely continuous and nonsingular matrix $K(t)$ on $[a,b]$ such that*

$$ {}^{\circ}D(t) = K(t)D(t) \quad \textit{for } t \in [a,b], \tag{14}$$

$$ {}^{\circ}d(t) = K(t)d(t) \quad \textit{for } t \in [a,b]. \tag{15}$$

This lemma implies that the transferred condition (8) is equivalent to the condition

$$ {}^{\circ}D(t)x(t) = {}^{\circ}d(t).$$

To be able to study the solvability of the algebraic equations resulting from the transfer of conditions we will use the following lemma. We denote by $\operatorname{rank}(A)$ the rank of the matrix A.

Lemma 2. *The following statements hold on the interval $[a,b]$:*

1. $\operatorname{rank}(D(t)) = \text{const.}$
2. $\operatorname{rank}(D(t), d(t)) = \text{const.}$
3. $\operatorname{rank}(C(t)) = \text{const.}$
4. $\operatorname{rank}(C(t), c(t)) = \text{const.}$
5. $\operatorname{rank}\begin{pmatrix} D(t) \\ C(t) \end{pmatrix} = \text{const.}$
6. $\operatorname{rank}\begin{pmatrix} D(t), d(t) \\ C(t), c(t) \end{pmatrix} = \text{const.}$

Theorems 1 and 2 imply that every solution of Problem ψ satisfies the equation

$$\begin{pmatrix} D(t) \\ C(t) \end{pmatrix} x(t) = \begin{pmatrix} d(t) \\ c(t) \end{pmatrix} \quad \text{for } t \in [a,b]. \tag{16}$$

Lemma 2 (in particular assertions 5 and 6) implies that system (16) has a solution in the whole interval $[a,b]$ provided it has a solution at a single point of the interval $[a,b]$. Similarly, system (16) has a unique solution in the whole interval $[a,b]$ provided it has a unique solution at a single point of the interval $[a,b]$. Moreover, the following statement may be proved.

Theorem 3.

1. *Every solution of Problem ψ satisfies* (16) *for any $t \in [a,b]$.*
2. *System* (16) *has a solution at any $t \in [a,b]$ if and only if Problem ψ has a solution.*
3. *System* (16) *has exactly one solution for any $t \in [a,b]$ if and only if Problem ψ has exactly one solution.*

Provided Problem ψ has a unique solution, this solution may be found at any $t \in [a,b]$ by solving (16). However, there are additional algorithms of the transfer of conditions that may be preferable in particular situations. Now we will introduce two algorithms to be applied later to the boundary value problem for a self-adjoint equation of $2n$th order. At this moment, we will give a general form of these algorithms. In Section 4 we will present proper modifications useful and effective for solving the boundary value problems treated in this paper.

Algorithm A ([6], [7]). Let Problem ψ have a solution. We choose an absolutely continuous matrix $R(t)$ such that the matrix $\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}$ is nonsingular for all $t \in [a,b]$. We look for a vector $r(t)$ solving the differential equation

$$r'(t) = R(t)f(t) + (R'(t) - R(t)A(t)) \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} \begin{pmatrix} d(t) \\ r(t) \end{pmatrix} \tag{17}$$

from the right to the left with the initial condition at the point b given as

$$r(b) = R(b)p, \tag{18}$$

where p is a solution of the equation

$$\begin{pmatrix} D(b) \\ V \end{pmatrix} p = \begin{pmatrix} d(b) \\ v \end{pmatrix} \tag{19}$$

(the system (19) has a solution according to Theorem 3).

Then the solution $x(t)$ of Problem ψ is found from the system

$$\begin{pmatrix} D(t) \\ R(t) \end{pmatrix} x(t) = \begin{pmatrix} d(t) \\ r(t) \end{pmatrix} \quad \text{for all } t \in [a,b]. \tag{20}$$

A l g o r i t h m B ([6], [7]). Let Problem ψ have a solution. We choose an absolutely continuous matrix $R(t)$ such that the matrix $\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}$ is nonsingular for all $t \in [a,b]$. We look for a vector $r(t)$ solving the differential equation

$$r'(t) = R(t)f(t) + (R'(t) - R(t)A(t)) \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} \begin{pmatrix} r(t) \\ c(t) \end{pmatrix} \tag{21}$$

from the left to the right with the initial condition at the point a given as

$$r(a) = R(a)p, \tag{22}$$

where p is a solution of the equation

$$\begin{pmatrix} U \\ C(a) \end{pmatrix} p = \begin{pmatrix} u \\ c(a) \end{pmatrix} \tag{23}$$

(the system (23) has a solution according to Theorem 3).

Then the solution $x(t)$ of Problem ψ is found from the system

$$\begin{pmatrix} R(t) \\ C(t) \end{pmatrix} x(t) = \begin{pmatrix} r(t) \\ c(t) \end{pmatrix} \quad \text{for all } t \in [a,b]. \tag{24}$$

In order to apply Algorithms A and B efficiently, it is requirable that the matrices

$$\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1}, \quad \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1}$$

in Eqs. (17), (21) be given by simple expressions (this will be the case in Algorithms 2, 3, 5, and 6 below) or that they would not occur in the respective expressions at all (Algorithms 1 and 4).

3. On a selfadjoint boundary value problem for a differential equation of $2n$th order

Consider the boundary value problem for a self-adjoint differential equation of $2n$th order

$$\ell y \equiv \sum_{i=0}^{n} (-1)^i (p_{n-i}(t) y^{(i)}(t))^{(i)} = q(t) \quad \text{a.e. in } (a,b), \tag{25}$$

where the coefficients of the equation satisfy the following requirements: $1/p_0(t) \in \mathcal{L}(a,b)$, $p_i(t) \in \mathcal{L}(a,b)$ for $i = 1, \ldots, n$, and $q(t) \in \mathcal{L}^2(a,b)$. This last requirement that enables us to treat ℓ as an operator in $\mathcal{L}^2$ will be weakened later as this assumption will not be necessary for the application of our algorithms.

First of all let us introduce the concept of quasiderivatives in order to be able to formulate the boundary conditions for our problem.

Definition 2. We say that a function $y(t)$ has all quasiderivatives up to the $2n$th order provided that the following $2n$ functions exist:

$y^{[k]}(t) = y^{(k)}(t) \quad \text{for } k = 1, 2, \ldots, n-1,$
$y^{[n]}(t) = p_0(t) y^{(n)}(t),$
$y^{[n+j]}(t) = p_j(t) y^{(n-j)}(t) - (y^{[n+j-1]}(t))' \quad \text{for } j = 1, \ldots, n.$

The last $n+1$ equations are assumed to hold almost everywhere and the first $2n-1$ quasiderivatives are assumed to be absolutely continuous.

In addition, we define $y^{[0]}(t) = y(t)$ and put $x_i(t) = y^{[i-1]}(t)$ for $i = 1, \ldots, 2n$. Let us introduce the vector $x(t) = (x_1(t), \ldots, x_{2n}(t))^T$. Consider the boundary condition for the differential equation (25) in the form

$$W_1 x(a) + W_2 x(b) = w, \tag{26}$$

where W_1 and W_2 are square matrices of order $2n$ and the vector w has $2n$ components. Let S be the set of all functions $y(t)$ whose quasiderivatives $y^{[k]}(t)$, $k = 0, 1, \ldots, 2n-1$, are absolutely continuous and $y^{[2n]}(t) \in \mathcal{L}^2$. Then S obviously is the largest linear set for which the operation ℓy has a natural sense and the operator ℓ can be considered as an operator in $\mathcal{L}^2$. The matrices W_1 and W_2 have to satisfy certain conditions in order that the boundary problem be self-adjoint. Let us formulate these conditions. For that reason we divide the matrices W_1 and W_2 into blocks:

$$W_1 = (B_1, B_2), \quad W_2 = (B_3, B_4),$$

where the matrices B_i $(i = 1, \dots, 4)$ have n columns. Let T be the square matrix of order n defined as

$$T = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \quad \text{for } n \geqslant 2$$

and $T = 1$ for $n = 1$ (i.e., the matrix has 1's on the adjoint diagonal and zeros everywhere else).

Now we can formulate two known lemmas (see [1]).

Lemma 3. *A necessary and sufficient condition for problem* (25), (26) *to be self-adjoint is*

$$B_1TB_2^T - B_2TB_1^T = B_3TB_4^T - B_4TB_3^T \qquad (27)$$
$$\textit{and}\ \operatorname{rank}(W_1, W_2) = 2n.$$

Lemma 4. *Let* $p_i(t) \geqslant 0$ *a.e. in* (a, b) *for* $i = 0, \dots, n$. *A necessary and sufficient condition for the self-adjoint problem* (25), (26) *to be positive semidefinite is that the matrix*

$$B_1TB_2^T - B_3TB_4^T$$

is negative semidefinite.

In the following we will consider only boundary value problems with separated conditions, i.e., we will assume that the boundary conditions are of the form

$$Ux(a) = u, \qquad (28)$$
$$Vx(b) = v, \qquad (29)$$

where U and V are $n \times 2n$ matrices and the vectors u and v have n components. Let us divide the matrices U and V into blocks

$$U = (U_1, U_2), \qquad V = (V_1, V_2),$$

where U_1, U_2, V_1, and V_2 are square matrices.

Condition (27) of Lemma 3 turns into two equations

$$U_1TU_2^T = U_2TU_1^T, \qquad (30)$$
$$V_1TV_2^T = V_2TV_1^T \qquad (31)$$

and the requirement that $\operatorname{rank}(U) = \operatorname{rank}(V) = n$. The condition of Lemma 4 turns into the conditions that the matrix $U_1TU_2^T$ is negative semidefinite and the matrix $V_1TV_2^T$ is positive semidefinite.

4. Transfer of the conditions for a selfadjoint positive semidefinite boundary value problem

Throughout the remainder of the paper we suppose only (because of the techniques of the transfer of conditions used):

1. $1/p_0(t), q(t), p_i(t) \in \mathcal{L}(a,b)$, $i = 1,\dots,n$. (Note that we do not need to assume $q(t) \in \mathcal{L}^2$ in what follows.)
2. $p_i(t) \geqslant 0$ a.e. in (a,b) for $i = 0,1,\dots,n$.
3. The matrix $U_1TU_2^T$ is symmetric and negative semidefinite.
   The matrix $V_1TV_2^T$ is symmetric and positive semidefinite.
4. The ranks of the matrices U and V equal n.

First we will replace the equation of the $2n$th order by the system of $2n$ equations of the first order in a standard way. The definition of quasiderivatives implies that the introduced vector $x(t)$ satisfies the differential equation

$$(32)\qquad x'(t) + \begin{bmatrix} 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1/p_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -p_1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -p_2 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -p_{n-1} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -p_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -q(t) \end{bmatrix}$$

a.e. in (a,b).

Now we will treat problem (25), (28), (29) as Problem ψ, where equation (32) corresponds to the differential equation (1) and conditions (28) and (29) correspond to the boundary conditions. That also defines the class in which we seek the solution of our problem: we demand that the vector $x(t)$ be absolutely continuous, i.e., we seek such a function $y(t)$ that it is absolutely continuous, together with all its quasiderivatives up to the order $2n-1$, and such that the $2n$ quasiderivative satisfies the equation

$$y^{[2n]}(t) = q(t) \quad \text{a.e. in } (a,b), \tag{33}$$

where owing to Definition 2 we have

$$y^{[2n]}(t) = \sum_{i=1}^{n} (-1)^i \big(p_{n-i}(t) y^{(i)}(t)\big)^{(i)}. \tag{34}$$

Definition 3. Let the above assumptions 1 to 4 be satisfied. Then the following problem will be called Problem ψ_{2n}: A vector $x(t)$ absolutely continuous on $[a,b]$ is sought that satisfies (32) and boundary conditions (28) and (29).

Equation (32) is of the form (1). Let the matrix $A(t)$ stand for the corresponding matrix of equation (32) and the vector $f(t)$ for the corresponding right-hand side. We will divide the matrix $A(t)$ into blocks as follows:

$$A(t) = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix},$$

where $A_i(t)$ $(i = 1, \ldots, 4)$ are $n \times n$ matrices. The vector $f(t)$ will be divided into two vectors,

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

such that each $f_i(t)$ $(i = 1, 2)$ has n components. Equation (32) implies that $f_1(t) = 0$.

We have

$$A_4 = -A_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

for $n > 1$ and $A_1 = A_4 = 0$ for $n = 1$.

First, let us examine the properties of the matrices that realize the general transfer of the conditions for a self-adjoint and positive definite problem. Then we will deduce from these properties a certain particular transfer that will make full use of the symmetry property as well as the sign properties of our problem.

Let us divide the matrix $D(t)$ from Theorem 1 into blocks

$$D(t) = (D_1(t), D_2(t)),$$

where $D_1(t)$ and $D_2(t)$ are square matrices. Analogously, we divide the matrix $C(t)$ from Theorem 2 into blocks

$$C(t) = (C_1(t), C_2(t)),$$

where $C_1(t)$ and $C_2(t)$ are square matrices.

Differential equation (4) can now be rewritten as

$$D_1'(t) = D_1(t)A_1(t) + D_2(t)A_3(t) + Z_1(t, D(t), d(t))D_1(t), \tag{35}$$

$$D_2'(t) = D_1(t)A_2(t) + D_2(t)A_4(t) + Z_1(t, D(t), d(t))D_2(t), \tag{36}$$

and similarly for the differential equation (9) we have

$$C_1'(t) = C_1(t)A_1(t) + C_2(t)A_3(t) + Z_2(t, C(t), c(t))C_1(t), \tag{37}$$

$$C_2'(t) = C_1(t)A_2(t) + C_2(t)A_4(t) + Z_2(t, C(t), c(t))C_2(t). \tag{38}$$

Lemma 5. *Under the above assumptions the following statements hold:*

1. *The matrix $D_1(t)TD_2^T(t)$ is symmetric and negative semidefinite for all $t \in [a, b]$.*
2. *The matrix $C_1(t)TC_2^T(t)$ is symmetric and positive semidefinite for all $t \in [a, b]$.*

P r o o f. It is sufficient to carry out the proof of the first assertion only for the case $Z_1(t, D, d) = 0$ because according to Lemma 1, we obtain any other transfer of conditions by left-multiplying the matrix $D_1(t)TD_2^T(t)$ by a nonsingular matrix $K(t)$ and right-multiplying it by a matrix $K^T(t)$. These operations do not change the properties mentioned in Lemma 5. Let thus $Z_1(t, D, d) = 0$ for the purpose of the proof. Then

$$\begin{aligned}(D_1(t)TD_2^T(t))' = D_1(t)A_1(t)TD_2^T(t) + D_2(t)A_3(t)TD_2^T(t)\\ + D_1(t)TA_2^T(t)D_1^T(t) + D_1(t)TA_4^T(t)D_2^T(t)\end{aligned} \tag{39}$$

holds. Obviously $TA_4^T(t) = -A_1(t)T$ and therefore (39) reduces to

$$(D_1(t)TD_2^T(t))' = D_2(t)A_3(t)TD_2^T(t) + D_1(t)TA_2^T(t)D_1^T(t). \tag{40}$$

The matrices $A_3(t)T$ and $TA_2^T(t)$ are symmetric and negative semidefinite and thus the right-hand side of (40) is also a symmetric and negative semidefinite matrix. The initial condition (6) implies

$$D_1(a)TD_2^T(a) = K_1U_1TU_2^TK_1^T. \tag{41}$$

Then, equations (40) and (41) imply the symmetry property of the matrix $D_1(t) \times TD_2^T(t)$. To prove that this matrix is negative semidefinite, let us consider an arbitrary constant vector w. Then we obtain

$$(w^TD_1(t)TD_2^T(t)w)' \leqslant 0 \quad \text{a.e. in } (a,b). \tag{42}$$

Under our assumptions we have

$$w^TD_1(a)TD_2^T(a)w = w^TK_1U_1TU_2^TK_1^Tw \leqslant 0. \tag{43}$$

Using inequalities (42) and (43), we find that

$$w^TD_1(t)TD_2^T(t)w \leqslant 0$$

for all $t \in [a,b]$ and for all vectors w. Thus we have proved the first assertion of the lemma. The second assertion can be proved in an analogous way. □

Lemma 6. *The matrices $D_1(t) - D_2(t)T$ and $C_1(t) + C_2(t)T$ are nonsingular for all $t \in [a,b]$.*

Proof. The rank of the matrix $D(t)$ equals n, and therefore the matrix

$$D(t)D^T(t) = D_1(t)D_1^T(t) + D_2(t)D_2^T(t)$$

is positive definite. Consider the product

$$\begin{aligned}
&(D_1(t) - D_2(t)T)(D_1(t) - D_2(t)T)^T \\
&\quad = D_1(t)D_1^T(t) + D_2(t)D_2^T(t) - D_1(t)TD_2^T(t) - D_2(t)TD_1^T(t) \\
&\quad = D(t)D^T(t) + (-2D_1(t)TD_2^T(t)).^{[1]}
\end{aligned}$$

This product is a positive definite matrix because it is the sum of a positive definite and a positive semidefinite matrix. Thus the matrix $D_1(t) - D_2(t)T$ is nonsingular.

Analogously we can prove that the matrix $C_1(t) + C_2(t)T$ is nonsingular. We consider the product $(C_1(t) + C_2(t)T)(C_1(t) + C_2(t)T)^T$ to this end. □

[1] We use the equality $T^2 = I$ here.

Let us put

$$G(t) = (D_1(t) - D_2(t)T)^{-1}D_1(t), \tag{44}$$

$$g(t) = (D_1(t) - D_2(t)T)^{-1}d(t), \tag{45}$$

$$H(t) = (C_1(t) + C_2(t)T)^{-1}C_1(t), \tag{46}$$

$$h(t) = (C_1(t) + C_2(t)T)^{-1}c(t). \tag{47}$$

Even if the construction of the above matrices and vectors is based on a fixed choice of the transfer, i.e., on the choice of the matrices K_1, K_2, $Z_1(t, D(t), d(t))$ and $Z_2(t, C(t), c(t))$, neither the matrix $G(t)$ nor the vector $g(t)$ depend on the choice of the matrices K_1 and $Z_1(t, D(t), d(t))$. This is implied by Lemma 1 trivially. Analogously, the matrix $H(t)$ and the vector $h(t)$ do not depend on the choice of the matrices K_2 and $Z_2(t, C(t), c(t))$. In fact, we have defined a unique canonical transfer of conditions for the problem discussed here.

Lemma 7. *For all $t \in [a, b]$, the matrices $G(t)$, $I - G(t)$, $H(t)$, and $I - H(t)$ are symmetric and positive semidefinite. Moreover, the following equalities hold:*

$$(D_1(t) - D_2(t)T)^{-1}D_2(t) = (G(t) - I)T, \tag{48}$$

$$(C_1(t) + C_2(t)T)^{-1}C_2(t) = (I - H(t))T. \tag{49}$$

Proof. The matrix $(D_1(t) - D_2(t)T)D_1^T(t)$ is obviously symmetric and positive semidefinite. We can write the matrix $G(t)$ in the form

$$G(t) = (D_1(t) - D_2(t)T)^{-1}[D_1(t)(D_1(t) - D_2(t)T)^T]((D_1(t) - D_2(t)T)^{-1})^T,$$

hence the matrix $G(t)$ is also symmetric and positive definite for all $t \in [a, b]$. The matrix $(D_1(t) - D_2(t)T)TD_2^T(t)$ is symmetric and negative semidefinite. We have

$$\begin{aligned}&(D_1(t) - D_2(t)T)^{-1}D_2(t)T\\ &\qquad = (D_1(t) - D_2(t)T)^{-1}[D_2(t)T(D_1(t) - D_2(t)T)^T]((D_1(t) - D_2(t)T)^{-1})^T\end{aligned}$$

and thus the matrix $(D_1(t) - D_2(t)T)^{-1}D_2(t)T$ is symmetric and negative semidefinite. We will prove relation (48) in the following way. Obviously we have

$$(D_1(t) - D_2(t)T)^{-1}(D_1(t) - D_2(t)T) = I,$$

hence

$$(D_1(t) - D_2(t)T)^{-1}D_1(t) - (D_1(t) - D_2(t)T)^{-1}D_2(t)T = I.$$

The last identity implies (48) immediately. Analogously we can prove the assertion on the matrices $H(t)$ and $I - H(t)$ together with relation (49). $\square$

R e m a r k. Lemma 7 says that the eigenvalues of the matrices $G(t)$ and $H(t)$ lie in the interval $[0,1]$. As a matter of fact, this gives us an estimate of the norm of the matrices $G(t)$ and $H(t)$, which will be of use in the investigations of the numerical stability questions.

Now, left-multiplying the transferred condition (8) by the matrix $(D_1(t) - D_2(t)T)^{-1}$ and doing some simple modification, we obtain

$$(G(t),(G(t)-I)T)x(t) = g(t) \quad \text{for } \ t \in [a,b]. \tag{50}$$

We can determine the value of the matrix $G(t)$ and of the vector $g(t)$ at the point a knowing the matrix U and the vector u only. If we knew the differential equations satisfied by the matrix $G(t)$ and the vector $g(t)$, we would construct the transferred condition (41) directly, solving a certain initial value problem. Let us compute the derivative of the matrix $G(t)$,

$$\begin{aligned} G'(t) = & -(D_1(t)-D_2(t)T)^{-1}(D_1'(t)-D_2'(t)T)(D_1(t)-D_2(t)T)^{-1}D_1(t) \\ & + (D_1(t)-D_2(t)T)^{-1}D_1'(t). \end{aligned}$$

Substituting, in accordance with equations (35) and (36), for the derivatives of the matrices $D_1(t)$ and $D_2(t)$, we obtain the matrix Riccati equation

$$\begin{aligned} G'(t) = G(t)A_2(t)TG(t) &- (G(t)-I)TA_3(t)(G(t)-I) - G(t)A_1(G(t)-I) \\ &- (G(t)-I)A_1^TG(t) \quad \text{a.e. in } \ (a,b). \end{aligned} \tag{51}$$

Analogously we have for the vector $g(t)$

$$\begin{aligned} g'(t) = & -G(t)(A_1 - A_2(t)T)g(t) - (G(t)-I)T(A_3(t)-A_4T)g(t) \\ & + (G(t)-I)Tf_2(t) \quad \text{a.e. in } \ (a,b). \end{aligned} \tag{52}$$

The initial conditions for these differential equations are

$$G(a) = (U_1 - U_2T)^{-1}U_1, \tag{53}$$

$$g(a) = (U_1 - U_2T)^{-1}u. \tag{54}$$

□

Thus we have proved the following theorem.

Theorem 4. *Under the above assumptions, there exist an absolutely continuous matrix $G(t)$ and an absolutely continuous vector $g(t)$ which are the unique solutions* *of the initial value problems for differential equations* (51) *and* (52) *with initial conditions* (53) *and* (54), *respectively. Any solution* $x(t)$ *of Problem* ψ_{2n} *satisfies the transferred condition* (50).

Analogously, a theorem can be proved also for the following particular transfer of the right boundary condition.

Theorem 5. *Under the above assumptions there exist an absolutely continuous matrix* $H(t)$ *and an absolutely continuous vector* $h(t)$ *which are the unique solutions of the initial value problems for the differential equations*

$$\begin{aligned}(55)\qquad H'(t) = &-H(t)A_2(t)TH(t) - (I-H(t))TA_3(t)(I-H(t))\\ &+H(t)A_1(I-H(t)) + (I-H(t))A_1^TH(t) \quad \text{a.e. in } (a,b),\\ (56)\qquad h'(t) = &-H(t)(A_1+A_2(t)T)h(t) - (I-H(t))T(A_3(t)+A_4T)h(t)\\ &+(I-H(t))Tf_2(t) \quad \text{a.e. in } (a,b),\end{aligned}$$

with the initial conditions

$$H(b) = (V_1+V_2T)^{-1}V_1, \tag{57}$$

$$h(b) = (V_1+V_2T)^{-1}v, \tag{58}$$

respectively. Any solution $x(t)$ *of Problem* ψ_{2n} *satisfies the transferred condition*

$$(H(t),(I-H(t))T)x(t) = h(t) \quad for \ \ t\in[a,b]. \tag{59}$$

Theorems 4 and 5 define a particular transfer of conditions that will be called the *canonical transfer* and that exploits the given symmetry and sign properties. The matrix differential equation (4) represents the system of $2n^2$ equations whereas the matrix equation (51) represents the system of $(n^2+n)/2$ equations only. Further advantages of this canonical transfer will be pointed out later.

5. Particular methods for the canonical transfer

In this section, the transfer of conditions will mean the canonical transfer treated by Theorems 4 and 5. Therefore, in Algorithms A and B, we have

$$D(t) = (G(t),(G(t)-I)T), \tag{60}$$

$$d(t) = g(t), \tag{61}$$

$$C(t) = (H(t),(I-H(t))T), \tag{62}$$

$$c(t) = h(t). \tag{63}$$

Now we will describe three algorithms that come from Algorithm A by a particular choice of the matrix $R(t)$.

Algorithm 1. Let our problem have a unique solution. We choose $R(t) = C(t)$ in Algorithm A. That can be done because the matrix $\begin{pmatrix} D(t) \\ C(t) \end{pmatrix}$ is nonsingular according to Theorem 3. Without any difficulty we determine that $r(t) = c(t)$. Therefore, the solution $x(t)$ of our problem is found from the system of $2n$ equations

$$\begin{pmatrix} D(t) \\ C(t) \end{pmatrix} x(t) = \begin{pmatrix} d(t) \\ c(t) \end{pmatrix}. \tag{64}$$

The matrix $(D(t), d(t))$ is built up by solving the initial value problem (51), (52), (53) and (54). Analogously, the matrix $(C(t), c(t))$ is built up by solving the initial value problem (55), (56), (57) and (58). It is sufficient to store these matrices only at the points where we are interested in the solution (i.e. at the points when the solution is output). System (64) is now of the particular form

$$\begin{pmatrix} G(t), & (G(t) - I)T \\ H(t), & (I - H(t))T \end{pmatrix} x(t) = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}.$$

The solution of this system may be transformed to solving the system of n linear algebraic equations with the matrix

$$G(t) + H(t) - 2G(t)H(t).$$

Algorithm 2. We put $R(t) = (I, T)$ in Algorithm A. Then

$$\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} = \begin{pmatrix} G(t), & (G(t) - I)T \\ I, & T \end{pmatrix}^{-1} = \begin{pmatrix} I, & I - G(t) \\ -T, & TG(t) \end{pmatrix}. \tag{65}$$

Differential equation (17) for the function $r(t)$ in Algorithm A acquires the form

$$r'(t) = Tf_2(t) - (I, T) \begin{pmatrix} A_1 & A_2(t) \\ A_3(t) & A_4 \end{pmatrix} \begin{pmatrix} I, & I - G(t) \\ -T, & TG(t) \end{pmatrix} \begin{pmatrix} g(t) \\ r(t) \end{pmatrix}. \tag{66}$$

The solution $x(t)$ is obtained from the equation

$$x(t) = \begin{pmatrix} I, & I - G(t) \\ -T, & TG(t) \end{pmatrix} \begin{pmatrix} g(t) \\ r(t) \end{pmatrix}. \tag{67}$$

Equation (66) is solved from the right to the left. Therefore, we must store the matrix $G(t)$ and the vector $g(t)$ during the solution of the initial value problems at as many points as are needed for the numerical integration of equation (66). Usually we have incomparably more such points than the points where the solution is output. The following Algorithm 3 avoids this unpleasant fact. The matrix $R(t)$ is chosen in such a way that the equation (17) becomes a quadrature and thus can be solved simultaneously with equations (51) and (52) from the left to the right with an initial condition. In the end, the values of the function $r(t)$ can be corrected by adding a constant at the points where we are interested in the solution. Hence this algorithm will inherit the character of Algorithm 1.

Algorithm 3. We choose $R(t) = (Q(t), Q(t)T)$ in Algorithm A, where the matrix $Q(t)$ will be constructed by solving a certain differential equation but from the left to the right. For our algorithm to be practicable the matrix $Q(t)$ has to be nonsingular. Then

$$(68)\qquad \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} = \begin{pmatrix} G(t), & (G(t)-I)T \\ Q(t), & Q(t)T \end{pmatrix}^{-1} = \begin{pmatrix} I, & (I-G(t))Q^{-1}(t) \\ -T, & TG(t)Q^{-1}(t) \end{pmatrix}.$$

Requiring equation (17) to be a quadrature, the following equation has to hold

$$(R'(t) - R(t)A(t)) \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} \begin{pmatrix} d(t) \\ r(t) \end{pmatrix}$$
$$= (R'(t) - R(t)A(t)) \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} \begin{pmatrix} d(t) \\ 0 \end{pmatrix} \quad \text{a.e. in } (a,b).$$

This condition is fulfilled provided that the matrix $Q(t)$ satisfies the differential equation

$$(69)\quad Q'(t) = Q(t)(A_2(t)TG(t) - A_1^TG(t) - A_1G(t) - TA_3(t)G(t) + A_1 + TA_3(t)).$$

This is a homogeneous linear differential equation and the condition of nonsingularity is satisfied if we choose

$$(70)\qquad Q(a) = I.$$

Let us sum up: We seek matrices $G(t)$ and $Q(t)$ and vectors $g(t)$ and $r(t)$ when solving the above mentioned differential equations from the left to the right. Then, at the points where we are interested in the solution, the function $r(t)$ is corrected by adding a certain constant in such a way that the condition (18) is satisfied.

In addition, we will describe three algorithms that come from Algorithm B of Section 1 by a particular choice of the matrix $R(t)$.

A l g o r i t h m 4. Let our problem have a unique solution. We choose $R(t) = D(t)$ in Algorithm B. That can be done because the matrix $\begin{pmatrix} D(t) \\ C(t) \end{pmatrix}$ is nonsingular according to Theorem 3. Without any difficulty we determine that $r(t) = d(t)$. Therefore, the solution $x(t)$ of our problem is found from the system of $2n$ equations

$$\begin{pmatrix} D(t) \\ C(t) \end{pmatrix} x(t) = \begin{pmatrix} d(t) \\ c(t) \end{pmatrix}. \tag{71}$$

As the result of this choice we obtain the same algorithm as Algorithm 1.

A l g o r i t h m 5. We put $R(t) = (I, -T)$ in Algorithm B. Then

$$\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} = \begin{pmatrix} I, & -T \\ H(t), & (I - H(t))T \end{pmatrix}^{-1} = \begin{pmatrix} I - H(t), & I \\ -TH(t), & T \end{pmatrix}. \tag{72}$$

The differential equation for the function $r(t)$ acquires the form

$$r'(t) = -Tf_2(t) - (I, -T) \begin{pmatrix} A_1 & A_2(t) \\ A_3(t) & A_4 \end{pmatrix} \begin{pmatrix} I - H(t), & I \\ -TH(t), & T \end{pmatrix} \begin{pmatrix} r(t) \\ h(t) \end{pmatrix}. \tag{73}$$

The solution $x(t)$ is obtained from the equation

$$x(t) = \begin{pmatrix} I - H(t), & I \\ -TH(t), & T \end{pmatrix} \begin{pmatrix} r(t) \\ h(t) \end{pmatrix}. \tag{74}$$

Equation (73) is solved from the left to the right. Therefore, we must store the matrix $H(t)$ and the vector $h(t)$ at as many points as are needed for the numerical integration of equation (73). Usually we have incomparably more such points than the points where the solution is output. The following algorithm avoids this unpleasant fact. The matrix $R(t)$ is chosen in such a way that equation (21) becomes a quadrature and thus can be solved simultaneously with equations (55) and (56) from the right to the left with an initial condition. In the end, the values of the function $r(t)$ can be corrected by adding a constant at the points where we are interested in the solution. Hence, this algorithm will inherit the character of Algorithm 4.

A l g o r i t h m 6. We choose $R(t) = (Q(t), -Q(t)T)$ in Algorithm B, where the matrix $Q(t)$ will be constructed by solving a certain differential equation from the right to the left. For our algorithm to be practicable, the matrix $Q(t)$ has to be nonsingular. Then

$$\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} = \begin{pmatrix} Q(t), & -Q(t)T \\ H(t), & (I - H(t))T \end{pmatrix}^{-1} = \begin{pmatrix} (I - H(t))Q^{-1}(t), & I \\ -TH(t)Q^{-1}(t), & T \end{pmatrix}. \tag{75}$$

Requiring equation (21) to be a quadrature, the following equation has to hold:

$$(R'(t) - R(t)A(t)) \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} \begin{pmatrix} r(t) \\ c(t) \end{pmatrix}$$
$$= (R'(t) - R(t)A(t)) \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ c(t) \end{pmatrix} \quad \text{a.e. in } (a,b).$$

This condition is fulfilled provided the matrix $Q(t)$ satisfies the differential equation

$$Q'(t) = Q(t)(-A_2(t)TH(t) - A_1^T H(t) - A_1 H(t) + TA_3(t)H(t) + A_1 - TA_3(t)). \tag{76}$$

This is a homogeneous linear differential equation and the condition of nonsingularity is satisfied if we choose

$$Q(b) = I. \tag{77}$$

Let us sum up: We seek matrices $H(t)$ and $Q(t)$ and vectors $h(t)$ and $r(t)$ when solving the above mentioned differential equations from the right to the left. Then, at the points where we are interested in the solution, the function $r(t)$ will be corrected by adding a certain constant in such a way that condition (22) is satisfied.

6. Numerical stability

The method of transfer of conditions as described in the previous part of this paper, in particular in Section 1, is simple and rather graceful. It also possesses the advantage that it not only gives a procedure for solving the boundary value problems in question numerically but also provides an answer to the theoretical question about the existence of the boundary value problem treated.

The reader who has followed our discussion of various algorithms of the method up to this point would probably ask why not use the simplest variant of the method of transfer and choose just $Z_1(t, D(t), d(t)) = 0$ or $Z_2(t, C(t), c(t)) = 0$ in the respective formulas of the method. However, this choice is quite often inappropriate from the numerical realization point of view in finite-precision arithmetic. The cause of the trouble here is in fact the same as are the well known numerical problems connected with the shooting method. It is easy to see that the above choice of zero matrices for Z_1, Z_2 results in our solving essentially the same differential equations as are those of the boundary value problem to be solved but with initial conditions. It is well known, however, that the conditioning (or stability) of a boundary value problem for a differential equation may be quite different from that of an initial value problem for the same equation where the growth of the solution modes is not limited by all the boundary conditions. There are many examples from real world practice where initial value problems for the differential equations forming parts of well-conditioned boundary value problems are highly unstable so that the classical shooting method is inapplicable. We even dare say that the more a given problem is stable as a boundary value problem the less stable the corresponding initial value problems may be.

The magnitudes of the entries in the matrices that realize the transfer of conditions may grow or decrease very fast. The main problem here is in the unbearable sensitivity of the solutions to the differential equations realizing the transfer in the case of zero Z_1, Z_2 to the initial conditions. This shortcoming can be avoided by our requiring that the matrices $D(t)$, $C(t)$ that realize the transfer of conditions be "normalized" by which we mean that their norms and the norms of their pseudoinverses be bounded by constants that do not depend on the coefficients of the differential equation in question or on the length of the interval $[a, b]$.

In monographs [6], [7], the analysis of numerical stability of algorithms A and B has been performed. It has been shown that realizing the algorithms for solving the initial value problems involved numerically we bring into being certain inaccuracies, both in the solution of the differential equations and in the computation of the initial conditions for these differential equations. These inaccuracies are considered in the following theorem proved in [6] where also the estimates of the magnitudes of these perturbations in terms of the errors in solving the initial value problems are given.

Theorem 6. *Let the norm of the matrices $D(t)$ and $D^T(t)(D(t)D^T(t))^{-1}$ be bounded by an a priori given constant. Then all inaccuracies made in the realization of the method can be represented as perturbations in the coefficients of the primary Problem ψ whose magnitude can be estimated using the above mentioned constant. An analogous statement is valid also for the transfer of conditions from the right to the left.*

The investigation of numerical stability of the algorithm with an auxiliary matrix $R(t)$ leads to the requirements that the norms of the matrices $R(t)$, $R^T(t)(R(t)R^T(t))^{-1}$ and $\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1}$ (or $\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1}$) be bounded by an a priori given constant.

In what follows, we will show that the canonical transfer of conditions for Problem ψ_{2n} satisfies the assumptions of the previous theorem and thus the transfer is numerically stable. In our further exposition we will estimate norms of certain matrices that are rectangular in general. To this end we will employ the spectral norm that is induced by the Euclidean vector norm (2-norm). The spectral matrix norm of a general matrix A may be expressed as (see e.g. [2], Section 6.2)

$$\|A\| = \sqrt{\varrho(A^*A)} = \sigma_{\max}(A),$$

where ϱ is the spectral radius defined as

$$\varrho(B) = \max\{|\lambda|:\ \det(B - \lambda I) = 0\},$$

and where $\sigma_{\max}(A)$ denotes the largest singular value of A. As this norm is unitarily invariant, it has the property that $\|A^*\| = \|A\|$ and we will exploit this fact in what follows. Therefore, to compute the norm $\|\Phi(t)\|$ of a matrix $\Phi(t)$ of an arbitrary type, we will use any of the formulae

$$\begin{aligned} \|\Phi(t)\| &= \sqrt{\text{max. eigenvalue }\ (\Phi^T(t)\Phi(t))} \\ &= \sqrt{\text{max. eigenvalue }\ (\Phi(t)\Phi^T(t))}. \end{aligned} \tag{78}$$

We obtain

$$D(t)D^T(t) = [G(t),\ (G(t) - I)T]\begin{bmatrix} G(t) \\ T(G(t) - I) \end{bmatrix} = G^2(t) + (G(t) - I)^2 \tag{79}$$

and

$$C(t)C^T(t) = [H(t),\ (I - H(t))T]\begin{bmatrix} H(t) \\ T(I - H(t)) \end{bmatrix} = H^2(t) + (I - H(t))^2. \tag{80}$$

Thus

$$\|D(t)\| \leqslant \sqrt{\max_{0\leqslant\lambda\leqslant1}(\lambda^2 + (\lambda - 1)^2)} = 1 \tag{81}$$

and

$$\|D^T(t)(D(t)D^T(t))^{-1}\| \leqslant \sqrt{\max_{0\leqslant\lambda\leqslant1}\frac{1}{\lambda^2 + (\lambda - 1)^2}} = \sqrt{2}. \tag{82}$$

Analogously we obtain estimate for the norms of the matrices $C(t)$ and $C^T(t) \times (C(t)C^T(t))^{-1}$:

$$\|C(t)\| \leqslant 1 \quad \text{and} \quad \|C^T(t)(C(t)C^T(t))^{-1}\| \leqslant \sqrt{2}. \tag{83}$$

As a result of these bounds, Algorithm 1 (or 4) is numerically stable.

Let us now investigate the norms of the matrices $\begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1}$ and $\begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1}$ from Algorithms 2 and 5. Equations (65) and (72) imply

$$\left\| \begin{pmatrix} D(t) \\ R(t) \end{pmatrix}^{-1} \right\| \leqslant \sqrt{\|(I, I - G(t))\|^2 + \|(-T, TG(t))\|^2} \leqslant 2$$

and

$$\left\| \begin{pmatrix} R(t) \\ C(t) \end{pmatrix}^{-1} \right\| \leqslant \sqrt{\|(I - H(t), I)\|^2 + \|(-TH(t), T)\|^2} \leqslant 2.$$

These results and Theorem 6 imply the numerical stability of Algorithms 2 and 5.

In Algorithms 3 and 6, the norms of matrices $Q(t)$ and $Q^{-1}(t)$ cannot be a priori estimated. They depend on the value of coefficients and on the length of the interval $[a, b]$. However, the numerical experiments performed have given good results.

The structure of the matrices $A_i(t)$, $i = 1, \ldots, 4$, is tremendously simple and the complexity of the above mentioned matrix differential equations does not present any difficulty for the numerical solution.

7. Conclusion

We have presented several methods of solution of self-adjoint boundary value problems for differential equations of the $2n$th order by transforming them to solving sequences of initial value problems. First, we have studied a method where the boundary conditions are transferred both from the left and the right independently while the transfer of the conditions from one of the end-points represents the solution of an initial value problem for one matrix Riccati equation and the solution of an initial value problem for a system of linear differential equations. Then, the solution of our boundary value problem may be obtained by solving a system of linear algebraic equations at those points where we want to compute it.

Second, we described methods where the boundary conditions are transferred from one side only and the solution to the boundary value problem under consideration is obtained in such a way that we construct an auxiliary vector function by solving a system of linear ordinary differential equations and the final solution is then obtained by evaluating just a simple expression.

We have presented algorithms that lead to solving the Riccati differential equations and have shown that for a series of problems these equations possess a unique solution on the whole interval in question and, moreover, that these equations are represented by symmetric matrices whose eigenvalues lie in $[0, 1]$. The methods used so far had to check whether the solutions to the Riccati equations exist on the whole interval in question or whether they do not exceed some a priori given barriers [4], which is not necessary in our canonical transfer of conditions.

As a consequence of our results, we know that the solutions of the initial value problems used in our algorithms exist and are bounded on the whole interval $[a, b]$. Therefore, the algorithms presented in the paper are always feasible under the assumptions given. Finally, we note that the class of boundary value problems discussed here is quite often applicable in the physical science and technology.

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